Growth of the number of simple closed geodesics on hyperbolic surfaces

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Contents

1 Introduction

This paper investigates the growth of $s_X(L)$, the number of simple closed geodesics of length $\leq L$ on a hyperbolic surface X. We prove that $s_X(L)$ is asymptotic to $n_X \cdot L^{6g-6+2n}$ as $L \to \infty$. We also study the frequencies of different types of simple closed geodesics on a hyperbolic surface and their relationship with Weil-Petersson volumes of moduli spaces of bordered Riemann surfaces.

Simple closed geodesics. Let $\mathcal{M}_{g,n}$ be the moduli space of hyperbolic Riemann surfaces of genus g with n cusps. For $X \in \mathcal{M}_{g,n}$, let $c_X(L)$ be the number of primitive closed geodesics on X of length $\leq L$. By work of Delsart, Huber, Selberg and Margulis, we have

$$c_X(L) \sim e^L/L$$

as $L \to \infty$. However, very few closed geodesics are *simple* [BS] and it is hard to discern them in $\pi_1(S_{g,n})$.

Counting problems. To understand the growth of $s_X(L)$, it proves fruitful to fix a simple closed curve $\gamma \in \mathcal{ML}_{g,n}(\mathbb{Z})$ and consider more generally the counting function

$$s_X(L,\gamma) = \#\{\alpha \in \mathrm{Mod}_{g,n} \cdot \gamma \mid \ell_\alpha(X) \leq L\}.$$

There are only finitely many isotopy classes of simple closed curves on $S_{g,n}$ up to the action of the mapping class group. The class of a simple closed curve γ is determined by the topology of $S_{g,n} - \gamma$, the surface that we get by cutting $S_{g,n}$ along γ. Therefore, to compute $s_X(L)$ it suffices to compute $s_X(L,\gamma)$ for each type of simple closed curve γ. In fact, we also get counting results for multicurves. In §6 we show :

Theorem 1.1 *For any $\gamma \in \mathcal{ML}_{g,n}(\mathbb{Z})$, we have*

$$\lim_{L\to\infty} \frac{s_X(L,\gamma)}{L^{6g-6+2n}} = n_\gamma(X),$$

where $n_\gamma(X)$ is a continuous proper function of $X \in \mathcal{M}_{g,n}$.

In the case of $\mathcal{M}_{1,1}$, this result was previously obtained by McShane and Rivin [MR]. Polynomial lower and upper bounds for $s_X(L)$ were found by I. Rivin. Explicitly, in [Ri] it is proved that for any $X \in \mathcal{T}_{g,n}$, there exists $c_X > 0$ such that

$$\frac{1}{c_X} L^{6g-6+n} \leq s_X(L) \leq c_X \cdot L^{6g-6+2n}.$$

Similar upper and lower bounds for the number of pants decompositions of length $\leq L$ on a hyperbolic surface X were obtained by M. Rees in [Rs].

Let B_X be the unit ball in the space of measured geodesic laminations with respect to the length function at X:

$$B_X = \{\lambda \,|\ell_\lambda(X) \leq 1\} \subset \mathcal{ML}_{g,n}.$$

In [Mirz1], we show that B_X is convex with respect to the piecewise linear structure of $\mathcal{ML}_{g,n}$. Let $B(X) = \mathrm{Vol}(B_X)$ with respect to the natural volume form in the Lebesgue measure class on $\mathcal{ML}_{g,n}$. We show that

$$b_{g,n} = \int\limits_{\mathcal{M}_{g,n}} B(X)\, dX$$

is a finite number in $\pi^{6g-6+2n} \cdot \mathbb{Q}$ which can be calculated in terms of the leading coefficients of the volume polynomials (§5). Next, we show that the contributions of X and γ to $n_\gamma(X)$ separate as follows:

Theorem 1.2 *For any $\gamma \in \mathcal{ML}_{g,n}(\mathbb{Z})$, there exists a rational number c_γ such that we have:*

$$n_\gamma(X) = \frac{c_\gamma \cdot B(X)}{b_{g,n}}.$$

Idea of the proof of Theorem 1.2**.** The crux of matter is to understand the density of $\mathrm{Mod}_{g,n} \cdot\gamma$ in $\mathcal{ML}_{g,n}(\mathbb{Z})$. This is similar to the problem of the density of relatively prime pairs (p, q) in $\mathbb{Z}^2$. Our approach is to use the moduli space $\mathcal{M}_{g,n}$ to understand the average of these densities. To prove Theorem 1.2, we:

(I): Apply Theorem 4.2 to show that the integral of $s_X(L, \gamma)$ over the moduli space

$$S(L,\gamma) = \int\limits_{\mathcal{M}_{g,n}} s_X(L,\gamma)\, dX$$

is well-behaved; in fact it is a polynomial in L (§5). Here the integral on $\mathcal{M}_{g,n}$ is taken with respect to the Weil-Petersson volume form. , and

(II): Use the ergodicity of the action of the mapping class group on the space $\mathcal{ML}_{g,n}$ of geodesic measured laminations on $S_{g,n}$ [Mas] to prove that these densities exist (§6) as follows.

Let μ^γ be the discrete measure supported on the orbit γ, that is

$$\mu^\gamma = \sum_{g \in \mathrm{Mod}_{g,n}} \delta_{g \cdot \gamma}.$$

Note that $\mathcal{ML}_{g,n}$ has a natural action of $\mathbb{R}_+$ by dilation. For $T \in \mathbb{R}_+$, Let $T^*(\mu^\gamma)$ denote the rescaling of μ^γ by factor T. Although the action of $\mathrm{Mod}_{g,n}$ on $\mathcal{ML}_{g,n}$ is not linear, it is homogeneous. We define the measure $\mu_{T,\gamma}$ by

$$\mu_{T,\gamma} = \frac{T^*(\mu^\gamma)}{T^{6g-6+2n}}.$$

The measure

$$\mu_{T,\gamma}(U) = \frac{\mu^\gamma(T \cdot U)}{T^{6g-6+2n}}$$

is also a $\mathrm{Mod}_{g,n}$ invariant measure. Also, we have

$$s_X(L,\gamma) = \mu^\gamma(L \cdot B_X) \ \ , \ \ \mu_{T,\gamma}(B_X) = \frac{s_X(T,\gamma)}{T^{6g-6+2n}}. \tag{1.1}$$

Therefore that understanding the asymptotic behavior of $s_X(L,\gamma)$ is closely related to the asymptotic behavior of the sequence $\mu = \{\mu_{T,\gamma}\}_T$.
Let $v_{g,n}$ denote the volume form on $\mathcal{ML}_{g,n}$. In §6, by using the ergodicity of the action of the mapping class group on $\mathcal{ML}_{g,n}$ ([Mas]), we prove that the sequence μ weakly converges to a $\mathrm{Mod}_{g,n}$ invariant measure in the Lebesgue measure class of $\mathcal{ML}_{g,n}$. In other words, as $T \to \infty$

$$\mu_{T,\gamma} \to c_{g,n} \cdot v_{g,n}.$$

Then Theorem 1.1 follows by an elementary lattice-counting argument §3.
Frequencies of different types of simple closed curves. From Theorem 1.2, it follows that the relative frequencies of different types of simple closed curves on X are universal rational numbers.

Corollary 1.3 *For $X \in \mathcal{M}_{g,n}$ and $\gamma_1, \gamma_2 \in \mathcal{ML}_{g,n}(\mathbb{Z})$, we have*

$$\lim_{L \to \infty} \frac{s_X(L,\gamma_1)}{s_X(L,\gamma_2)} = \frac{c_{\gamma_1}}{c_{\gamma_2}} \in \mathbb{Q}.$$

The limit is a positive rational number independent of X.

Remark. The same result holds when the surface X has variable negative curvature, and the rational numbers are independent of the metric (§6).

Example: For $i = 1, 2$, Let α_i be a curve on S_2 that cuts the surface into i connected parts. Then as $L \to \infty$

$$\frac{s_X(L, \alpha_1)}{s_X(L, \alpha_2)} \to 6.$$

In other words, a very long simple closed geodesic on a surface of genus 2 is 6 times more likely to be non-separating. For more examples see §6.

The frequency $c_\gamma \in \mathbb{Q}$ of a given simple closed curve can be described in a purely topological way as follows ([Mirz1]):

Theorem 1.4 *For any connected simple closed curve γ, we have*

$$\frac{\#(\{\lambda \in \mathcal{ML}_{g,n}(\mathbb{Z}) \mid i(\lambda, \gamma) \leq k\}/\operatorname{Stab}(\gamma))}{k^{6g-6+2n}} \to c_\gamma$$

as $k \to \infty$.

Note that $c_\gamma = c_\delta$ for all $\delta \in \operatorname{Mod}_{g,n} \cdot \gamma$. We can also calculate c_γ recursively using our recursive formula for $V_{g,n}(\ell)$.

Connection with intersection numbers of tautological line bundles. We can also calculate c_γ recursively using our recursive formula for $V_{g,n}(a)$. In [Mirz3], we relate the coefficients of $V_{g,n}(a)$ to the intersection numbers of tautological line bundles over $\overline{\mathcal{M}}_{g,n}$. Therefore, we can write the number c_γ in terms of the intersection numbers of tautological line bundles over the moduli space of Riemann surfaces of type $S_{g,n}(\gamma)$, the surface that we get by cutting $S_{g,n}$ along γ.

An alternative proof. In a sequel we give a different proof of the growth of the number of simple closed geodesics by using ergodic theory of the earthquake flow on $\mathcal{PM}_{g,n}$ the bundle of geodesic measured laminations of unit length over moduli space.

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2 Background material

In this section, We present some familiar concepts in a less familiar setting about the symplectic structure of the moduli space of bordered Riemann surfaces and the space of measured geodesic laminations. we also recall some basic facts and results on hyperbolic geometry.

Recall that a *symplectic structure* on a manifold M is a non-degenerate closed 2-form $\omega \in \Omega^2(M)$. The n-fold wedge product

$$\omega \wedge \cdots \wedge \omega$$

never vanishes and defines a volume form on M.

Teichmüller Space. Here we briefly summarize the background material on Teichmüller theory of Reimann surfaces with geodesic boundary components.

A point in the *Teichmüller space* $\mathcal{T}(S)$ is a complete hyperbolic surface X equipped with a diffeomorphism $f : S \to X$. The map f provides a *marking* on X by S. Two marked surfaces $f : S \to X$ and $g : S \to Y$ define the same point in $\mathcal{T}(S)$ if and only if $f \circ g^{-1} : Y \to X$ is isotopic to a conformal map. When ∂S is nonempty, consider hyperbolic Riemann surfaces homeomorphic to S with geodesic boundary components of fixed length. Let $A = \partial S$ and $L = (L_\alpha)_{\alpha \in A} \in \mathbb{R}_+^{|A|}$. A point $X \in \mathcal{T}(S, L)$ is a marked hyperbolic surface with geodesic boundary components such that for each boundary component $\beta \in \partial S$, we have

$$\ell_\beta(X) = L_\beta.$$

Let $S_{g,n}$ be an oriented connected surface of genus g with n boundary components $(\beta_1, \ldots, \beta_n)$. Then

$$\mathcal{T}_{g,n}(L_1, \ldots, L_n) = \mathcal{T}(S_{g,n}, L_1, \ldots, L_n),$$

denote the Teichmüller space of hyperbolic structures on $S_{g,n}$ with geodesic boundary components of length $L_1, \ldots, L_n$. By convention, a geodesic of length zero is a cusp and we have

$$\mathcal{T}_{g,n} = \mathcal{T}_{g,n}(0, \ldots, 0).$$

Let $\mathrm{Mod}(S)$ denote the mapping class group of S, or the group of isotopy classes of orientation preserving self homeomorphisms of S leaving each boundary component point wise fixed. The mapping class group $\mathrm{Mod}_{g,n} = \mathrm{Mod}(S_{g,n})$ acts on $\mathcal{T}_{g,n}(L)$ by changing the marking. The quotient space

$$\mathcal{M}_{g,n}(L) = \mathcal{M}(S_{g,n}, \ell_{\beta_i} = L_i) = \mathcal{T}_{g,n}(L_1, \ldots, L_n)/Mod_{g,n}$$

is the moduli space of Riemann surfaces homeomorphic to $S_{g,n}$ with n boundary components of length $\ell_{\beta_i} = L_i$. Also, we have

$$\mathcal{M}_{g,n} = \mathcal{M}_{g,n}(0, \ldots, 0).$$

For a disconnected surface $S = \bigcup_{i=1}^{k} S_i$ such that $A_i = \partial S_i \subset \partial S$, we have

$$\mathcal{M}(S, L) = \prod_{i=1}^{k} \mathcal{M}(S_i, L_{A_i}),$$

where $L_{A_i} = (L_s)_{s \in A_i}$.

The Weil-Petersson symplectic form. By work of Goldman [Gol], the space $\mathcal{T}_{g,n}(L_1, \ldots, L_n)$ carries a natural symplectic form invariant under the action of the mapping class group. This symplectic form is called *Weil-Petersson symplectic form,* and denoted by w or w_{wp}. In this thesis, we are interested in calculating the volume of the moduli space with respect to the volume form induced by the Weil-Petersson symplectic form. Note that when S is disconnected, we have

$$\mathrm{Vol}(\mathcal{M}(S, L)) = \prod_{i=1}^{k} \mathrm{Vol}(\mathcal{M}(S_i, L_{A_i})).$$

The Fenchel-Nielsen coordinates. A *pants decomposition* of S is a set of disjoint simple closed curves which decompose the surface into pairs of pants. Fix a system of pants decomposition of $S_{g,n}$, $\mathcal{P} = \{\alpha_i\}_{i=1}^{k}$, where $k = 6g-6+2n$. For a marked hyperbolic surface $X \in \mathcal{T}_{g,n}(L)$, the *Fenchel-Nielsen coordinates* associated with $\mathcal{P}$, $\{\ell_{\alpha_1}(X), \ldots, \ell_{\alpha_k}(X), \tau_{\alpha_1}(X), \ldots, \tau_{\alpha_k}(X)\}$, consists of the set of lengths of all geodesics used in the decomposition and the set of the *twisting* parameters used to glue the pieces. We have an isomorphism

$$\mathcal{T}_{g,n}(L) \cong \mathbb{R}_+^{\mathcal{P}} \times \mathbb{R}^{\mathcal{P}}$$

by the map

$$X \to (\ell_{\alpha_i}(X), \tau_{\alpha_i}(X)).$$

By work of Wolpert, over Teichmüller space the Weil-Petersson symplectic structure has a simple form in Fenchel-Nielsen coordinates [Wol].

Theorem 2.1 (Wolpert) *The Weil-Petersson symplectic form is given by*

$$\omega_{wp} = \sum_{i=1}^{k} d\ell_{\alpha_i} \wedge d\tau_{\alpha_i}.$$

Geodesic measured laminations. A *geodesic lamination* on a hyperbolic surface X is a closed subset of X which is disjoint union of simple geodesics.

A *measured geodesic lamination* is a geodesic lamination that carries a transverse invariant measure. Thurston showed that the positive real multiples of simple closed geodesics are dense in the set of measured geodesic laminations. In fact, the notion of geodesic measure lamination is independent of the geometry of X. Let $\mathcal{ML}_{g,n}$ be the space of compactly supported *measured laminations* on $S_{g,n}$ which is the closure of the set of weighted simple closed curves . For any two isotopy classes of essential simple closed curves on $S_{g,n}$ the intersection number $i(\alpha, \beta)$ is the minimum number of points in which transverse representatives of α and β must meet. The intersection pairing extends to a continuous map $i : \mathcal{ML}_{g,n} \times \mathcal{ML}_{g,n} \to \mathbb{R}$.

Train track coordinates. A *train track* in $S_{g,n}$ is an embedded 1-complex τ such that:

(i): Each edge(branch) of τ is a smooth path with a well-defined tangent vectors at the end points.That is, all edges at a given vertex(switch) are tangent.

(ii): For each component R of $S - \tau$ the double of R along the interior of edges of ∂R has negative Euler characteristic.

Any assignment of non negative numbers satisfying the switch conditions determines a unique geodesic lamination with transverse measure.

The space $\mathcal{ML}_{g,n}$ has a piecewise-linear structure and admits an atlas whose charts are associated to maximal trivalent train tracks in S. A geodesic lamination is carried on τ if there is a homotopy of S taking λ to a set of train routes. A train track is *recurrent* if there is a transverse measure which is positive on every branch. For a recurrent train track τ, let $P(\tau)$ denote the polyhedron of measures supported on τ satisfying the switch conditions. We can also think of $P(\tau)$ as a subset of $\mathcal{ML}_{g,n}$, and will occasionally blur the distinction between the two points of views. Let $U(\tau) = \mathrm{Int}(P(\tau))$ denote the set of weights on τ which are positive on every branch. This is an open cone in $\mathbb{R}_+^{6g-6}$.

Thurston volume form on $\mathcal{ML}_{g,n}$. Recall that $\mathcal{ML}_{g,n}$ has a natural volume form induced by the Euclidean volume form on train-track neighborhoods. Points of $\mathcal{ML}(\mathbb{Z}) \cap U(\tau)$ are exactly integral points of $P(\tau)$. Thus the transition functions preserve the Euclidean volume. Moreover, there is a natural symplectic form on $\mathcal{ML}_{g,n}$ preserved by the action of $\mathrm{Mod}_{g,n}$. See [HP] for more details.

Let $v = v(S)$ denote the volume form that we get this way. We will use the following result proved in [Mas] to study lengths of simple closed geodesics:

Theorem 2.2 (Masur) *The action of* $\mathrm{Mod}_{g,n}$ *on* $\mathcal{ML}_{g,n}$ *is ergodic with* *respect to the Lebesgue measure class.*

Length functions. The hyperbolic length $\ell_\gamma(X)$ of a simple closed geodesic γ on a hyperbolic surface $X \in \mathcal{T}(S_{g,n}, L)$ determines a real analytic function on the space and admits a unique continuous extension to $\mathcal{ML}_{g,n}$. For more details see [Th].

3 Counting multi-curves

In this section we study a simpler version of the counting problem and investigate the growth of the number of unions of simple closed geodesics on a hyperbolic Riemann surface. Each point $X \in \mathcal{T}_{g,n}(L)$ defines the length function $\ell(X)$ on $\mathcal{ML}(S_{g,n}, a)$ where $\ell_\lambda(X)$ denote the hyperbolic length of the corresponding measured lamination on X. Define $B_X \subset \mathcal{ML}(S_{g,n}, a)$ to be the unit ball of the length function, that is

$$B_X = \{\lambda \in \mathcal{ML}_{g,n} \mid \ell_X(\lambda) \leq 1\}.$$

The function $B(X)$ defined by

$$B(X) = \mathrm{Vol}(B_X)$$

plays an important role in this section. In fact, in [Mirz1] we show that B_X looks like a convex ball.

A *multi-curve* on $S_{g,n}$ is a union of disjoint essential simple closed curves on $S_{g,n}$ so that no component of it is homotopic to a boundary component. There is a one-to-one correspondence between the integral measured laminations, $\mathcal{ML}_{g,n}(\mathbb{Z})$, and the set of multi-curves, up to isotopy.

Counting Multi-curves. Define $b_X(L)$ by

$$b_X(L) = \#\{\gamma \in \mathcal{ML}_{g,n}(\mathbb{Z}) \mid \ell_\gamma(X) \leq L\}.$$

It is easy to prove that the asymptotic behavior of $b_X(L)$ is governed by $B(X) = \mathrm{Vol}_v(B_X)$.

Theorem 3.1 *For any $X \in \mathcal{T}_{g,n}(a)$, we have*

$$\frac{b_X(L)}{L^{6g-6+2n}} \to B(X)$$

as $L \to \infty$.

Proof of Theorem 3.1**.** For any train track τ, let

$$b_\tau(U,L) = \#(\mathcal{ML}(\mathbb{Z}) \cap L \cdot U \cap U_\tau).$$

Notice that U_τ has a Euclidean structure and the points in $U_\tau \cap \mathcal{ML}(\mathbb{Z})$ are exactly integral points. Therefore by usual lattice point estimate, we get

$$\frac{b_\tau(U,L)}{L^{6g-6+2n}} \to \mathrm{Vol}(U \cap U_\tau)$$

as $L \to \infty$. We can cover $\mathcal{ML}_{g,n}$ by train-track charts, and the transition functions are volume preserving, so we can use the inclusion-exclusion formula to get the result. $\square$

Note that the function B descends to a function over $\mathcal{M}_{g,n}(L)$. Next, we prove some basic properties about the variation of the function B over $\mathcal{M}_{g,n}(a)$. Since for any $\lambda \in \mathcal{ML}_{g,n}$, the length function

$$\ell_\lambda : \mathcal{T}_{g,n} \to \mathbb{R}_+$$

is smooth [Ker], it is easy to verify that:

Theorem 3.2 *The function $B : \mathcal{M}_{g,n}(a) \to \mathbb{R}_+$, defined as above, is continuous.*

Dehn's coordinates for multi-curves. Let

$$\mathcal{P} = \{\alpha_1, \ldots, \alpha_{3g-3+2n}\}$$

be a maximal system of simple closed curves on $S_{g,n}$. In order to prove that function B is a proper, integrable function over $\mathcal{M}_{g,n}(a)$, we use Dehn's coordinates for multi-curves to estimate the hyperbolic length in terms of the combinatorial length with respect to $\mathcal{P}$. For any pants decomposition $\mathcal{P}$ of $S_{g,n}$, define m_i and t_i as follows:

- Let $m_j = i(\gamma, \alpha_j) \in \mathbb{Z}_{\geq 0}$ be the intersection number of γ and α_j, and
- Let $t_j \in \mathbb{Z}$ to be the twisting number of γ around α_j.

Also, let $Z(\mathcal{P})$ be the set of $\{m_1, t_1, \ldots, m_k, t_k\} \in \mathbb{Z}^{2k}$ such that

1. $m_i \geq 0$ and if $m_i = 0$, then $t_i \geq 0$.
2. If α_{i_1}, α_{i_2},α_{i_3} bound an embedded pairs of pants in S, then $m_{i_1} + m_{i_2} + m_{i_3} \in 2\mathbb{Z}$.

Dehn's theorem asserts that these parameters uniquely determine a multicurve:

Theorem 3.3 (Dehn) *For any pants decomposition of $S_{g,n}$, the map*

$$DT : \mathcal{ML}_{g,n}(\mathbb{Z}) \to Z(\mathcal{P})$$

is a bijection.

See [HP] for more details.

For $\gamma \in \mathcal{ML}_{g,n}(\mathbb{Z})$ and $X \in \mathcal{T}_{g,n}$, let $L(X,\gamma)$ be the combinatorial length of γ on X with respect to a pants decomposition $\mathcal{P}$ defined by

$$L(X,\gamma) = \sum_{i=1}^{k} \left(m_i \cdot S(\ell_{\alpha_i}(X)) + |t_i| \cdot \ell_{\alpha_i}(X) \right),$$

where

$$S(x) = \operatorname{arcsinh}\left(\frac{1}{\sinh(x/2)} \right).$$

In fact, $S(\ell_\alpha(X))$ is the width of the collar neighborhood around α on X [Bus]. We show that

Theorem 3.4 *Let $\mathcal{P}$ be a pants decomposition of $S_{g,n}$. Given $L > 0$, there exists a constant $c(L)$ so that if $\ell_{\alpha_i}(X) \leq L$ for any $\alpha_i \in \mathcal{P}$ then we have*

$$\frac{1}{c(L)}\, L(X,\gamma) \leq \ell_\gamma(X) \leq c(L)\, L(X,\gamma), \tag{3.1}$$

where $c > 0$ is a constant independent of X.

Broken arcs. A *broken arc* in $\mathbb{H}^2$ is a sequence of oriented geodesic arcs,

$$V_1, H_1, \ldots, V_r, H_r, V_{r+1},$$

such that consecutive segments meet orthogonally and H_i and H_{i+1} are contained in opposite sides of V_{i+1}(See Figure 1). Let s_i and d_i denote the geodesic lengths of V_i and H_i respectively. Fixing the pants decomposition $\mathcal{P}$, we construct a broken arc associated to a closed geodesic γ as follows. Let p be a lift of one of the intersection points of γ with an element of $\mathcal{P}$ and $\tilde{\gamma}$ be the lift of γ through p. Let $\tilde{C}_1, \ldots, \tilde{C}_r$ be the lifts of the geodesics in $\mathcal{P}$ which are intersected in order by γ such that $\tilde{C}_1 \cap \tilde{\gamma} = p$ and $\tilde{C_{r+1}}$ is the image of $\tilde{C}_1$ under the covering translation corresponding to γ. Then $\tilde{C}_i$ and $\tilde{C_{i+1}}$ project to two boundary components of a unique pair of pants.

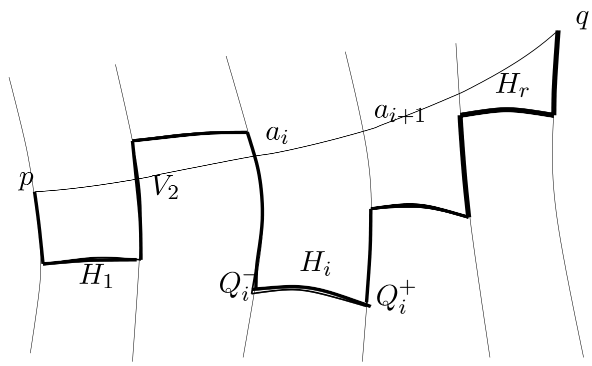

Figure 1.

Consider the common perpendicular segment to $\tilde{C}_i, \tilde{C_{i+1}}$, with end points denoted by Q_i^-,Q_i^+. Then we define $BA_\gamma(X)$ to be the broken arc with vertical arcs Q_i^+, Q_{i+1}^-, and horizontal arcs the segments $Q_i^+Q_i^-$. Then we have:

- $r = i(\mathcal{P}, \gamma)$, where $i(\mathcal{P}, \gamma) = \sum_{j=1}^{k} i(\alpha_j, \gamma)$;
- $d_j \geq S(\ell_{\alpha_{i_1}}) + S(\ell_{\alpha_{i_2}})$, where $\tilde{C}_j$ and $\tilde{C}_{j+1}$ are preimages of α_{i_1} and α_{i_2}, and finally
- The shift s_j is given by $s_j = |t_i\ \ell_{\alpha_i} + \tau_{\alpha_i} + e_j|$, where $e_j < \ell_j$.

Note that e_j and t_j are independent of the geometry of X and depend only on the topology of γ relative to the pants decomposition $\mathcal{P}$. See [DS] for more details.

Also, for any $L > 0$, there exists $C = C(L) > 0$ such that if $\ell_\alpha \leq L$ for $\alpha \in \mathcal{P}$ then

$$\frac{1}{C} \sum_{i=1}^{k}(H_i + V_i) \leq L(X, \gamma) \leq C \sum_{i=1}^{k}(H_i + V_i). \tag{3.2}$$

Sketch of the proof of Theorem 3.4**.** Without loss of generality, we can assume that

$$0 \leq \tau_i(X) \leq \ell_{\alpha_i}(X).$$

Note that if there exists $L > 0$ that $\ell_{\alpha_i}(X) \leq L$, then there exists a constant $D_L > 0$ such that $d_i \geq D_L$. Consider the broken arc corresponding to γ.

Let d_i, x_i,y_i and z_i be the hyperbolic length of $a_i a_{i+1}$, $a_i Q_i^-$, $Q_i^- Q_i^+$ and $Q_i^+ a_{i+1}$ respectively. Then we have

$$d_i \geq y_i,$$

and there exists $D(L) > 0$ such that

$$y_i \geq D(L).$$

Next, we will use the following basic properties of hyperbolic triangles [Bus]:

1. There exists $K(\theta)$ so that for any hyperbolic triangle with side lengths a, b, c and angle θ opposite to c, we have: $c > a + b - K(\theta)$.

2. For $D > 0$, there exists $\theta(D)$ so that any hyperbolic triangle with one side of length $d \geq D$ and angles $\pi/2, \theta$ we have $\theta \leq \theta(D)$.

Hence there exists $K = K(L)$ such that we have

$$d_i \geq x_i + y_i + z_i - K(L).$$

By a simple compactness argument,

$$\frac{d}{x + y + z}$$

is bounded below when $K(L) \geq y \geq D(L)$, otherwise we have

$$d \geq \frac{x + y + z}{2}.$$

Therefore, there exits $C(L) > 0$ such that we have

$$d_i \geq C(L)\ (x_i + y_i + z_i).$$

Now by adding these inequalities for $i = 1, \ldots, r$, we get

$$\sum_{i=1}^{r}(H_i + V_i) \geq \ell_\gamma(X) \geq C(L) \sum_{i=1}^{r}(H_i + V_i).$$

Now the result is immediate from 3.2. $\square$

Finding upper and lower bounds for $B(X)$. Next we find upper and lower bounds for the function $B(X)$ in terms of the lengths of small geodesics on X. Define $R : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$R(x) = \frac{1}{x\,|\log(x)|}.$$

Theorem 3.5 *For any $X \in \mathcal{T}_{g,n}(a)$, sufficiently small $\epsilon > 0$ and $1 \leq L$, there are constants $C_1, C_2 > 0$ such that*

$$C_1 \cdot \prod_{\gamma\colon \ell_\gamma(X) \leq \epsilon} R(\ell_\gamma(X)) \leq B(X), \tag{3.3}$$

and

$$\frac{b_X(L)}{L^{6g-6+2n}} \leq C_2 \cdot \prod_{\gamma\colon \ell_\gamma(X) \leq \epsilon} \left(R(\ell_\gamma(X)) + \frac{1}{\ell_\gamma(X)} \right), \tag{3.4}$$

where C_1, C_2 depends only on g, n and ϵ.

Sketch of the proof. We prove this theorem for $L = 0$. The proof for $L \neq 0$ is similar. Take ϵ small enough such that no two closed geodesic of length$\leq \epsilon$ on a hyperbolic surface meet. For $X \in \mathcal{T}_{g,n}$, let $\alpha_1, \dots, \alpha_s$ be the set of all simple closed geodesics of length $\leq \epsilon$ on X and

$$\mathcal{P}_X = \{\alpha_1, \dots, \alpha_s, \dots, \alpha_k\}$$

be a maximal set of disjoint simple closed geodesics such that $\ell_{\alpha_i}(X) \leq L_{g,n}$, where $L_{g,n}$ is the Bers' constant for $S_{g,n}$ (See [Bus]).

Consider the set $A_{x,y}(L)$ defined by

$$A_{x,y}(L) = \{(m,n) \in (\mathbb{Z}_+)^2 \mid mx + ny \leq L\}.$$

By basic lattice counting estimates, one can easily check that we have

$$\frac{1}{k} \left(\frac{L^2}{x \cdot y} - \frac{L}{\min\{x,y\}} \right) \leq |A_{x,y}(L)| \leq k \left(\frac{L^2}{x \cdot y} + \frac{L}{\min\{x,y\}} \right),$$

where $k > 0$ is a constant independent of X and L. This implies that there are constants $k_1, k_2 > 0$ such that

$$k_1 \cdot \left(\frac{L^2}{x \cdot y} \right) \leq |A_{x,y}(L)| \tag{3.5}$$

for big enough L, and

$$|A_{x,y}(L)| \leq k_2 \left(\frac{L^2}{x \cdot y} + \frac{L}{\min\{x,y\}} \right), \tag{3.6}$$

where k_2 is a constant independent of L.

As $b_X(L) = \#(\text{multi-curves of length} \leq L)$, we can apply (3.1) to use the combinatorial length of multi-curve geodesics instead of their geodesic length and estimate $b_X(L)$. Note that conditions (**i**) and (**ii**) do not effect the growth type of the number of $\{(m_i,t_i)\}_1^k$ such that $\ell_{\gamma(m_i,t_i)} \leq L$.

By setting $x_i = S(\ell_{\alpha_i}(X))$, $y_i = \ell_{\alpha_i}(X)$ we have

$$\frac{1}{c_1}\prod_{i=1}^{k} A_{x_i,y_i}(\frac{L}{k}) \leq b_X(L) \leq c_1 \prod_{i=1}^{k} A_{x_i,y_i}(L), \tag{3.7}$$

where $c_1 > 0$ is a constant independent of X and L.

On the other hand, it can be easily checked that

$$S(x) \sim \log(x)$$

as $x \to 0$ and there exists a constant c such that $1/c \leq x \cdot S(x) \leq c$, for $\epsilon \leq x \leq L_{g,n}$. So the result follows by applying (3.6) and (3.5) in (3.7). $\square$

Properness and integrability of the function B. In this part we show that the upper bound in (3.4) is actually an integrable proper function.

Theorem 3.6 *The function B is proper and integrable over $\mathcal{M}_{g,n}(a)$, namely*

$$b_{g,n}(a) = \int\limits_{\mathcal{M}_{g,n}(a)} B(X)\cdot dX < \infty.$$

Proof. Note that $\inf\{\ell_\gamma\}_\gamma \to 0$ as $X \to \infty$ in $\mathcal{M}_{g,n}$ and

$$R(\epsilon) \to \infty$$

as $\epsilon \to 0$. So (3.3) implies the function B is proper.

Also, when ℓ_γ is small $R(\ell_\gamma) \ll 1/\ell_\gamma$. Therefore, it suffices to prove that the function $F : \mathcal{M}_{g,n}(a) \to \mathbb{R}$ defined by

$$F(X) = \prod_{\gamma:\, \ell_\gamma \leq \epsilon} \frac{1}{\ell_\gamma(X)},$$

is integrable over $\mathcal{M}_{g,n}(a)$. Let $\mathcal{M}_{g,n}^{k,\epsilon} \subset \mathcal{M}_{g,n}$ be the subset consisting of elements with k simple closed geodesic of length $\leq \epsilon$. So the result is immediate by using (3.4) since the the set $\mathcal{M}_{g,n}^{k,\epsilon}$ can covered by finitely many open sets of the form

$$V_{\epsilon,k} = \{(x_i,y_i)_1^{3g-3+n} \,|\, 0 \leq x_1, \ldots x_k \leq \epsilon,\, x_i \leq L_{g,n},\, 0 \leq y_i \leq x_i\}.$$

$\square$

Later, we will calculate the $b_{g,n}(a)$'s in terms of the coefficients of the $V_{i,j}(a)$'s.

Define $f_L : \mathcal{M}_{g,n}(L) \to \mathbb{R}_+$ by

$$f_L(X) = \frac{b_X(L)}{L^{6g-g+2n}}.$$

The following is an immediate consequence of Theorem 3.5:

Corollary 3.7 *The sequence $\{f_L\}_L$, satisfies the hypothesis of the Lebesgue Dominated Convergence Theorem. That is $\{f_L\}_L$ is uniformly bounded by an integrable function.*

4 Integration over the moduli space of curves

In this section we recall the results obtained in [Mirz2] and [Mirz3] for integrating certain geometric functions over the moduli space of curves.

Simple closed geodesics on hyperbolic surfaces. Here we recall basic properties of simple closed geodesics and measured geodesic laminations on hyperbolic surfaces.

Symmetry group of a simple closed curve. For any set A of homotopy classes of simple closed curves on $S_{g,n}$, define $\mathrm{Stab}(A)$ by

$$\mathrm{Stab}(A) = \{g \in \mathrm{Mod}_{g,n} \ | g \cdot A = A\} \subset \mathrm{Mod}_{g,n}.$$

For $\gamma = \sum\limits_{i=1}^{k} c_i \gamma_i$, define the symmetry group of γ, $\mathrm{Sym}(\gamma)$, by

$$\mathrm{Sym}(\gamma) = \mathrm{Stab}(\gamma)/\cap_{i=1}^{k} \mathrm{Stab}(\gamma_i).$$

In fact, when γ has extra symmetry

$$\bigcap_{i=1}^{k} \mathrm{Stab}(\gamma_i) \neq \mathrm{Stab}(\gamma).$$

For any connected simple closed curve α, $|\,\mathrm{Sym}(\alpha)| = 1$. Also, if α_1 and α_2 bound a pair of pants with a boundary component of $S_{g,n}$, then $|\,\mathrm{Sym}(\alpha_1 + \alpha_2)| = 1$.

Splitting along a simple closed curve. Let γ

$$\gamma = \sum_{i=1}^{k} c_i \gamma_i,$$

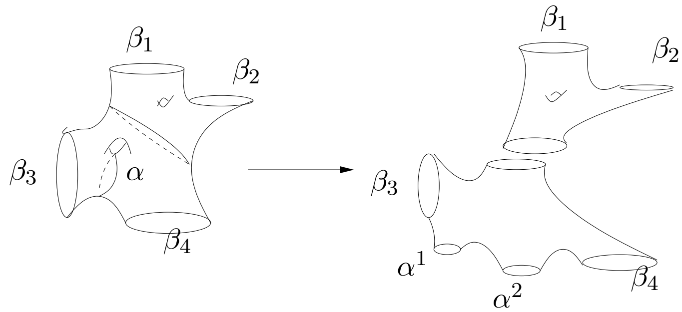

Figure 2. Cutting the surface

where $\gamma_1, \ldots$ and γ_k are distinct, disjoint simple closed curves, be the isotopy class of a multi curve on $S_{g,n}$.

Consider the surface $S_{g,n} - U_\gamma$, where U_γ is an open set homeomorphic to $\bigcup_1^k (0,1) \times \gamma_i$ around γ. We denote this surface by $S_{g,n}(\gamma)$, which is a (possibly disconnected) surface with $n + 2k$ boundary components and $s = s(\gamma)$ connected components. Each connected component γ_i of γ, gives rise to 2 boundary components, γ_i^1 and γ_i^2 on $S_{g,n}(\gamma)$. Namely,

$$\partial(S_{g,n}(\gamma)) = \{\beta_1, \ldots, \beta_n\} \cup \{\gamma_1^1, \gamma_1^2, \ldots, \gamma_k^1, \gamma_k^2\}.$$

Now for $\Gamma = (\gamma_1, \ldots, \gamma_k)$, $L = (L_1, \ldots, L_n)$ and $\mathbf{x} = (x_1, \ldots, x_k) \in \mathbb{R}_+^k$, let

$$\mathcal{M}(S_{g,n}(\gamma), \ell_\Gamma = \mathbf{x}, \ell_\beta = L)$$

be the moduli space of hyperbolic Riemann surfaces homeomorphic to $S_{g,n}(\gamma)$ such that $\ell_{\gamma_i} = x_i$ and $\ell_{\beta_i} = L_i$.

Simple closed curves on $X \in \mathcal{M}_{g,n}$. Let $[\gamma]$ denotes the homotopy class of a simple closed curve γ on $S_{g,n}$. Although there is no canonical simple closed geodesic on $X \in \mathcal{M}_{g,n}$ corresponding to $[\gamma]$, the set

$$\mathcal{O}_\gamma = \{[\alpha] |\ \alpha \in \mathrm{Mod} \cdot \gamma\},$$

of homotopy classes of simple closed curves in the $\mathrm{Mod}_{g,n}$-orbit of γ on X, is determined by γ. In other words, $\mathcal{O}_\gamma$ is the set of $[\phi(\gamma)]$ where $\phi : S_{g,n} \to X$ is a marking of X. Let $\ell_\alpha(X)$ denote the hyperbolic length of α on X. Here, we study functions of the form

$$f_\gamma : \mathcal{M}_{g,n} \to \mathbb{R}_+$$
$$X \to \sum_{\alpha \in \mathcal{O}_\gamma} f(\ell_\alpha(X)),$$

where $f : \mathbb{R} \to \mathbb{R}_+$.

Integration over the moduli space. In [Mirz2], we show that

Theorem 4.1 *For any $\gamma = \sum_{i=1}^{k} c_i \gamma_i$, the integral of f_γ over $\mathcal{M}_{g,n}(L)$ with respect to the Weil-Petersson volume form is given by*

$$\int_{\mathcal{M}_{g,n}(L)} f_\gamma(X)\, dX = \frac{1}{|\operatorname{Sym}(\gamma)|} \iint_{(\mathbf{x},t)\in V} f(|\mathbf{x}|)\, V_{g,n}(\Gamma, \mathbf{x}, \beta, L)\, \mathbf{x} \cdot d\mathbf{x}\, dt,$$

where $V = \{(\mathbf{x}, t) |\, 0 < t, \sum_{i=1}^{k} c_i \cdot x_i = t\}$, and $|\mathbf{x}| = \sum_{i=1}^{k} c_i\, x_i$.

Here $\mathbf{x} \cdot d\mathbf{x} = x_1 \cdots x_n \cdot dx_1 \wedge \cdots \wedge dx_n$. Also, for any $\mathbf{x} = (x_1, \ldots, x_k) \in \mathbb{R}^k_+$, $V_{g,n}(\Gamma, \mathbf{x}, \beta, L)$ is defined by

$$\operatorname{Vol}(\mathcal{M}(S_{g,n}(\gamma), \ell_\Gamma = \mathbf{x}, \ell_\beta = L)).$$

Sketch the proof of Theorem 4.1**.** Here we sketch the main idea of calculating the integral of f_γ over $\mathcal{M}_{g,n}$ with respect to the Weil-Petersson volume form when γ is a connected simple closed curve.

First, consider the covering space of $\mathcal{M}_{g,n}$

$$\pi^\gamma : \mathcal{M}^\gamma_{g,n} = \{(X, \alpha) \mid X \in \mathcal{M}_{g,n}, \text{and } \alpha \in \mathcal{O}_\gamma \text{ is a geodesic on } X\} \to \mathcal{M}_{g,n},$$

where $\pi^\gamma(X, \alpha) = X$. The hyperbolic length function descends to the function,

$$\ell : \mathcal{M}^\gamma_{g,n} \to \mathbb{R}$$

defined by $\ell(X, \eta) = \ell_\eta(X)$. Therefore, we have

$$\int_{\mathcal{M}_{g,n}} f_\gamma = \int_{\mathcal{M}^\gamma_{g,n}} f \circ \ell = \int_0^\infty f(t) \operatorname{Vol}(\ell^{-1}(t))\, dt,$$

where the volume is taken with respect to the volume form $- * d\ell$ on $\ell^{-1}(t)$.

The main idea for integrating over $\mathcal{M}^\gamma_{g,n}$ is that the decomposition of the surface along γ gives rise to a description of $\mathcal{M}^\gamma_{g,n}$ in terms of moduli spaces corresponding to simpler surfaces. This leads to formulas for the integral of f_γ in terms of the Weil-Petersson volumes of moduli spaces of bordered Riemann surfaces and the function f. We have a natural circle bundle

$$\begin{tikzcd} S^1 \arrow[r] & \ell^{-1}(t) \subset \mathcal{M}^{\gamma}_{g,n} \arrow[d] \\ & \mathcal{M}(S_{g,n}(\gamma), \ell_\gamma = t) \end{tikzcd}$$

We will study the S^1-action on the level set $\ell^{-1}(t) \subset \mathcal{M}^{\gamma}_{g,n}$ induced by twisting the surface along γ. The quotient space $\ell^{-1}(t)/S^1$ inherits a symplectic form from the Weil-Petersson symplectic form. On the other hand, $\mathcal{M}(S_{g,n}(\gamma), \ell_\gamma = t)$ is equipped with the Weil-Petersson symplectic form. By investigating these S^1-actions in more detail, we can prove that

$$\ell^{-1}(t)/S^1 \cong \mathcal{M}(S_{g,n}(\gamma), \ell_\gamma = t)$$

as symplectic manifolds. Therefore, we have

$$\mathrm{Vol}(\ell^{-1}(t)) = t \ \mathrm{Vol}(\mathcal{M}(S_{g,n}(\gamma), \ell_\gamma = t)).$$

Hence for any connected simple closed curve γ on $S_{g,n}$, we have

$$\int_{\mathcal{M}_{g,n}} f_\gamma(X)\, dX = \int_0^\infty f(t)\, t \ \mathrm{Vol}(\mathcal{M}(S_{g,n}(\gamma), \ell_\gamma = t))\, dt.$$

The Weil-Petersson volume of the moduli space. In [Mirz2], by using an identity for the lengths of simple closed geodesics on hyperbolic surfaces and using Theorem 4.1, we obtain a recursive method for calculating volume polynomials.

Theorem 4.2 *The volume $V_{g,n}(a_1, \dots, a_n) = \mathrm{Vol}_{wp}(\mathcal{M}(S_{g,n}, a_1, \dots, a_n))$ is a polynomial in $a_1, \dots, a_n$, namely we have:*

$$V_{g,n}(a) = \sum_{\substack{\alpha \\ |\alpha| \leq 3g-3+n}} C_\alpha \cdot a^{\,2\alpha},$$

where

$$C_\alpha = \frac{2^{|\alpha|}}{\alpha!} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{\alpha_1} \cdots \psi_n^{\alpha_n} \cdot \omega^{3g-3-|\alpha|},$$

where ψ_i is the first Chern class of the ith tautological line bundle, ω is the Weil-Petersson symplectic form, $\alpha! = \prod \alpha_i!$ and $|\alpha| = \sum \alpha_i$.

Here $a = (a_1, \ldots, a_n)$, and the exponent $\alpha = (\alpha_1, \ldots, \alpha_n)$ ranges over elements in $(\mathbb{Z}_{\geq 0})^n$, $a^\alpha = a_1^{\alpha_1} \cdots a_n^{\alpha_n}$, and $|\alpha| = \sum \alpha_i$.
Also, $C_\alpha > 0$ lies in $\pi^{6g-6+2n-|2\,\alpha|} \cdot \mathbb{Q}$.

See [Mirz2] and [Mirz3].

5 Counting curves and Weil-Petersson volumes

In this section we establish a relationship between $s_X(L, \gamma)$ and the Weil-Petersson volume of moduli spaces of bordered Riemann surfaces. We use this relationship to calculate $b_{g,n}$ in terms of the leading coefficients of the $V_{g,n}(a)$'s.

Let $P(L, \gamma, a)$ be the integral of $s_X(L, \gamma)$ over $\mathcal{M}_{g,n}(a)$, given by

$$P(L, \gamma, a) = \int\limits_{\mathcal{M}_{g,n}(a)} s_X(L, \gamma)\, dX.$$

Now by using Theorem 4.1 for $F = \chi(\Delta_L^k)$ where

$$\Delta_L^k = \{(x_1, \ldots, x_k) |\ \sum c_i\, x_i \leq L\},$$

we obtain the following result:

Theorem 5.1 *For any $\gamma = \sum\limits_{i=1}^{k} c_i\ \gamma_i \in \mathcal{ML}_{g,n}(\mathbb{Z})$, the integral of $s_X(L, \gamma)$ is given by*

$$P(L, \gamma, a) = \frac{1}{|\operatorname{Sym}(\gamma)|} \int\limits_0^L \int\limits_{\gamma \cdot \mathbf{x} = T} \operatorname{Vol}_{wp}(\mathcal{M}(S_{g,n}(\gamma), \ell_\Gamma = \mathbf{x}, \ell_\beta = a))\ \mathbf{x}\, d\mathbf{x}\, dT,$$

Here $\gamma \cdot \mathbf{x} = \sum\limits_{i=1}^{k} c_i \cdot x_i$.

Corollary 5.2 *For any $\gamma \in \mathcal{ML}_{g,n}(\mathbb{Z})$, $P(L, \gamma, a)$ is a polynomial of degree $6g-6+2n$ in L and a. The coefficient of $L^{6g-6+2n}$ is a positive rational coefficient which will be denoted by c_γ. In other words,*

$$c_\gamma = \lim_{L \to \infty} \frac{P(L, \gamma, a)}{L^{6g-6+2n}}$$

is a positive rational number independent of a.

For calculating c_γ, we have to find the leading coefficients of the volume polynomials of moduli spaces of bordered Reimann surfaces of type $S_{g,n}(\gamma)$, possibly disconnected surface that we get from $S_{g,n}$ by splitting along γ.

Calculation of $b_{g,n}(a)$. Now we can explicitly calculate the value of the integral of the function B over $\mathcal{M}_{g,n}$ in terms of the coefficients of the $V_{i,j}(a)$'s and symmetry groups of different types of simple closed curves on $S_{g,n}$.

For $\eta = \eta_1 \cup \ldots \cup \eta_k$, where γ_i are disjoint connected simple closed curves on $S_{g,n}$ and $X \in \mathcal{T}_{g,n}(a)$, let

$$\widetilde{s}_X(L,\eta) = \sum_{\mathbf{m}} s_X(L, \mathbf{m}\cdot\eta),$$

where $\mathbf{m}$ ranges over $\mathbb{N}^k$ and $\mathbf{m}\cdot\eta = \sum_{i=1}^{k} m_i \cdot \eta_i \in \mathcal{ML}(\mathbb{N})$. Then

- As a result of Corollary 3.7 and Theorem 3.1, we have

$$b_{g,n}(a) = \int\limits_{\mathcal{M}_{g,n}(a)} B(X) = \lim_{L\to\infty} \int\limits_{\mathcal{M}_{g,n}(a)} \frac{b_X(L)}{L^{6g-6+2n}}.$$

- For any $X \in \mathcal{T}_{g,n}(a)$, we have

$$b_X(L) = \sum_{\eta} \tilde{s}_X(L,\eta),$$

  and therefore,

$$\int\limits_{\mathcal{M}_{g,n}(a)} b_X(L) = \sum_{\eta} \int\limits_{\mathcal{M}_{g,n}(a)} \tilde{s}_X(L,\eta),$$

  where the sums are over all (finitely many) different types of (possibly disconnected) simple closed curves on $S_{g,n}$.

- Because $s_X(L,\eta) \leq b_X(L)$, Corollary 3.7 allows us to use the Lebesgue Dominated Convergence Theorem and change the order of integration and summation and get

$$\int\limits_{\mathcal{M}_{g,n}(a)} \widetilde{s}_X(L,\eta)\, dX = \sum_{\mathbf{m}\in\mathbb{N}^{\mathbf{k}}} \int\limits_{\mathcal{M}_{g,n}(a)} s_X(L, \mathbf{m}\cdot\eta)\ .$$

- Finally, using Theorem 5.1, we can calculate the integral of $s_X(L, \mathbf{m}\cdot\eta)$ and hence $b_X(L)$ over $\mathcal{M}_{g,n}(a)$ in terms of the volume polynomials.

See [Mirz1] for more details. Using Theorem 4.2, we get:

Corollary 5.3 *For any g, n, the integral $b_{g,n}(a)$ is a number in $\pi^{6g-6+2n} \cdot \mathbb{Q}_{>0}$ independent of a.*

6 Counting different types of simple closed curves

In this section we exploit the relation between $s_X(L,\gamma)$ and Weil-Petersson volumes of moduli space, and ergodicity of the action of the mapping class group on the space of measured laminations to establish the following results:

Theorem 6.1 *For any multi curve $\gamma \in \mathcal{ML}_{g,n}(\mathbb{Z})$ and $X \in \mathcal{T}_{g,n}(a)$ we have*

$$s_X(L,\gamma) \sim \frac{B(X)}{b_{g,n}}\, c_\gamma\, L^{6g-6+2n},$$

as $L \to \infty$.

Note that $b_{g,n}$ and c_γ are independent of X and L and only depend on the topological type of γ on $S_{g,n}$. Therefore, we get:

Corollary 6.2 *For any $X \in \mathcal{T}_{g,n}(a)$, as $L \to \infty$*

$$\frac{s_X(L,\gamma_1)}{s_X(L,\gamma_2)} \to \frac{c(\gamma_1)}{c(\gamma_2)}.$$

Since there are only finitely many isotopy classes of simple closed curves on $S_{g,n}$ up to the action of the mapping class group, the following result is immediate:

Corollary 6.3 *The number of simple closed geodesics of length $\leq L$ on $X \in \mathcal{M}_{g,n}(a)$ has the asymptotic behavior*

$$s_X(L) \sim n(X) L^{6g-6+2n}$$

as $L \to \infty$, where $n : \mathcal{M}_{g,n}(a) \to \mathbb{R}_+$ is proper and continuous.

Discrete measures on $\mathcal{ML}_{g,n}$. Any $\gamma \in \mathcal{ML}_{g,n}(\mathbb{Z})$, defines a sequence of discrete measure on $\mathcal{ML}_{g,n}$, $\{\mu_{T,\gamma}\}$, as follows. For any $U \subset \mathcal{ML}_{g,n}$ define $\mu_{T,\gamma}(U)$ by

$$\mu_{T,\gamma}(U) = \frac{\#(T \cdot U \cap \mathrm{Mod}_{g,n} \cdot \gamma)}{T^{6g-6+2n}}.$$

There is a close relation between the asymptotic behavior of this sequence of measures and counting different types of simple closed geodesics.

Theorem 6.4 *For any multi curve $\gamma \in \mathcal{ML}_{g,n}(\mathbb{Z})$, as $T \to \infty$*

$$\mu_{T,\gamma} \xrightarrow{w^*} \frac{c_\gamma}{b_{g,n}} \cdot v_{g,n},$$

where $v_{g,n}$ is the Thurston volume form on $\mathcal{ML}_{g,n}$.

Lemma 6.5 *The sequence $\{\mu_{T,\gamma}\}$ is weakly-normal, that is any subsequence contains a weakly-convergent subsequence.*

Proof. Using basic properties of weak convergence of measures, it suffices to prove that for any compact subset $K \subset \mathcal{ML}_{g,n}$

$$\sup_T \{\mu_{T,\gamma}\} < \infty.$$

Fix $X_0 \in \mathcal{T}_{g,n}$, without loss of generality we can assume that $K = L \cdot B_{X_0}$. Then using Theorem 3.5, we have:

$$\mu_{T,\gamma}(L \cdot B_{X_0}) = \frac{s_{X_0}(L \cdot T)}{T^k} \leq C(X_0, L),$$

where $C(X_0, L)$ is a constant depending only on X_0 and L, and in particular is independent of T. $\square$

Lemma 6.6 *Any weak limit of the sequence $\{\mu_{T,\gamma}\}$ is absolutely continuous with respect to the Lebesgue measure on $\mathcal{ML}_{g,n}$ and is invariant under the action of the mapping class group.*

Proof. Assume that

$$\mu_{T_i,\gamma} \to v$$

as $i \to \infty$. Using ergodicity of the action of the mapping class group, it suffices to prove that if $R \subset \mathcal{ML}_{g,n}$ has Lebesgue measure zero, then $v(R) = 0$. Using basic properties of weak-convergence and Theorem 3.1, we have:

$$v(U) \leq \liminf_{i \to \infty} \mu_{t_i,\gamma}(U) \leq \lim_{i \to \infty} \frac{b(t_i, U)}{t_i^k} = \mathrm{Vol}(U)$$

for any U open subset of $\mathcal{ML}_{g,n}$. The result is immediate since we can approximate R with open subsets of $\mathcal{ML}_{g,n}$. □

Proof of Theorem 6.4**.** By Lemma 6.5, it suffices to prove that if a subsequence $\{\mu_{T_i}\}_{i\in J}$ satisfies

$$\mu_{T_i,\gamma} \to v_J$$

as $i \in J \to \infty$, then $v_J = k_J \cdot v_{g,n}$ and k_J is independent of the subsequence. Note that by using Lemma 6.6 and ergodicity of the action of the mapping class group we can write the limit as $v_J = k_J \cdot v_{g,n}$.

On the other hand, it is easy to verify that for any $X \in \mathcal{T}_{g,n}(a)$ the set ∂B_X has Lebesgue measure zero. This combined with the fact that $s_X(T,\gamma) = \mu_{T,\gamma}(B_X)$ implies that for any $X \in \mathcal{T}_{g,n}$, we have:

$$\frac{s_X(T_i,\gamma)}{T_i^k} \to k_J \cdot B(X)$$

as $i \to \infty$. Now by Corollary 3.7 and the Lebesgue Dominated convergence Theorem the order of integration and limitation can be interchanged. As a result we get :

$$k_J \cdot b_{g,n} = k_J \cdot \int\limits_{\mathcal{M}_{g,n}(a)} B(X)\, dX = \int\limits_{\mathcal{M}_{g,n}(a)} \lim_{i\to\infty} \frac{s_X(T_i,\gamma)}{T_i^k} \cdot dX =$$

$$\lim_{i\to\infty} \int\limits_{\mathcal{M}_{g,n}(a)} \frac{s_X(T_i,\gamma)}{T_i^k} \cdot dX = \lim_{i\to\infty} \frac{P(T_i,\gamma,a)}{T_i^k} = c_\gamma.$$

Also, by Theorem 3.6, $b_{g,n} < \infty$ which implies that k_J is independent of J and a. Therefore we have

$$k_J = \frac{c_\gamma}{b_{g,n}},$$

and

$$\mu_{T,\gamma} \to \frac{c_\gamma}{b_{g,n}} \cdot v_{g,n}. \tag{6.1}$$

□

Proof of Theorem 6.1**.** Since ∂B_X has measure zero, we can use (6.1) to show that

$$\mu_{T,\gamma}(B_X) \to \frac{c_\gamma}{b_{g,n}} \cdot v_{g,n}(B_X).$$

On the other hand, we have

$$s_X(L,\gamma) = \#(L\cdot B_X \cap \mathrm{Mod}_{g,n}\cdot\gamma).$$

Now the result is immediate since we have

$$\mu_{T,\gamma}(B_X) = \frac{s_X(L,\gamma)}{L^{6g-6+2n}}.$$

□

Examples. We calculate the frequencies of different types of simple closed curves for some cases using the results in §5.

1. The simplest case is when γ is a maximal set of disjoint simple closed curves on $S_{g,n}$. That is when $S_{g,n}(\gamma)$ is a union of pairs of pants. Then since
$$\int\limits_{\sum_{i=1}^{k} c_i x_i = 1} x_1\cdots x_n dx_1\cdots dx_n = \frac{1}{(2n-1)!},$$
we have:
$$c_\gamma = \frac{1}{|\,\mathrm{Sym}(\gamma)|\,(6g-6+2n)!}.$$
Let α and β be two types of pants decomposition then as $L\to\infty$,
$$\frac{s_X(L,\beta)}{s_X(L,\alpha)} \to \frac{|\,\mathrm{Sym}(\beta)|}{|\,\mathrm{Sym}(\alpha)|}.$$

2. Let γ_{n_1} be a connected simple closed curve on $S_{0,n}$ such that we have
$$S_{0,n}(\gamma_{n_1}) \cong S_{0,n_1+1} \cup S_{0,n-n_1+1}.$$
By results obtained in [Mirz2], and Theorem 4.2 the coefficient of L_1^{2n-4} in $V_{0,n+1}(L_1,\ldots,L_n)$ equals
$$\frac{2^{n-2}}{(n-2)!}.$$
Since $|\,\mathrm{Sym}(\gamma_{n_1})| = 1$, we have
$$c(\gamma_{n_1}) = \frac{2^{n-4}}{(n_1-2)!\,(n-n_1-2)!\,(2n-7)}.$$
Then given $X\in\mathcal{T}_{0,n}(a)$, We have
$$\frac{s_X(L,\alpha_i)}{s_X(L,\alpha_j)} \to \frac{\binom{n-4}{i-2}}{\binom{n-4}{j-2}}$$
as $L\to\infty$.

3. Let γ_i be a connected simple closed curve on a surface of genus g that cuts the surface into two parts of genus i and $g - i$. Since

$$\int_{\overline{\mathcal{M}}_{g,1}} \psi_1^{3g-2} = \frac{1}{24^g g!},$$

(see [Pand]) by Theorem 4.2, the leading term of $V_{g,1}(L)$ is equal to

$$\frac{2^g}{(3g-2)!\; g!\; 24^g} L^{6g-4}.$$

Now since $|\,\mathrm{Sym}(\gamma_{g_1}| = 1$, we can use the results obtained in §5 and show that

$$c(\gamma_{g_1}) = \frac{1}{24^g\; g_1!(g-g_1)!\; (3g_1-2)!(3(g-g_1)-2)!}.$$

Hence we have

$$\frac{s_X(L,\gamma_1)}{s_X(L,\gamma_2)} \to \frac{\binom{3g-4}{3i-2}\binom{g}{i}}{\binom{3g-4}{3j-2}\binom{g}{j}}$$

as $L \to \infty$.

Remark. Let Y be a closed orientable surface of genus g with bounded negative curvature. Then each homotopy class of closed curves contains a unique geodesic. Consider the space $\mathcal{ML}(Y)$ of measured geodesic laminationson Y and let $\ell_\gamma(Y)$ be the geodesic length of γ on Y. Then $\mathcal{ML}(Y) \cong \mathcal{ML}_{g,n}$ and the length function extends to a continuous function on $\mathcal{ML}_{g,n}$. Since Theorem 6.4 is independent of the Riemannian mertic on the surface, the results of Theorem 6.1 and Corollary 6.2 hold for any Riemann surface with bounded negative curvature.

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