Tutorial Problems #10

MAT 292 – Calculus III – Fall 2014

SOLUTIONS

5.1 - # 15 Find the Laplace Transform of

$$f(t) = \begin{cases} 0, & 0 \leqslant t \leqslant 1 \\ 1, & 1 < t \leqslant 2 \\ 0, & 2 < t \end{cases}$$

Solution By direct computation we have

$$\mathcal{L}\{f(t)\} = \int_1^2 e^{-st} dt = \frac{-1}{s} e^{-st}\Big|_1^2 = \frac{e^{-s} - e^{-2s}}{s}$$

5.1 - # 16 Find the Laplace Transform of

$$f(t) = \begin{cases} 0, & 0 \leqslant t \leqslant 1 \\ e^{-t}, & 1 < t \end{cases}$$

Solution Again, by direct computation we have

$$\mathcal{L}\{f(t)\} = \int_1^\infty e^{-st} e^{-t} dt = \int_1^\infty e^{-(s+1)t} dt = \frac{-1}{s+1} e^{-(s+1)t}\Big|_1^\infty = \frac{e^{-(s+1)}}{s+1}$$

5.1- # 37 Consider the Laplace transform of t^p, where $p > -1$.

(a) Referring to Problem 36, show that

$$\mathcal{L}\{t^p\} = \int_0^\infty e^{-st} t^p dt = \frac{1}{s^{p+1}} \int_0^\infty e^{-x} x^p dx = \frac{\Gamma(p+1)}{s^{p+1}}. \quad s > 0$$

¶Proceed with the change of variables $x = st$, treating s as a positive constant (otherwise we'll have orientation issues). We obtain

$$\mathcal{L}\{t^p\} = \int_0^\infty e^{-st} t^p dt = \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^p \frac{1}{s} dx = \frac{1}{s^{p+1}} \int_0^\infty e^{-x} x^p dx$$

Following Problem 36 gives us the integral formulation for the Gamma function, hence

$$\mathcal{L}\{t^p\} = \frac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$$

(b) Let p be a positive integer n in a). Show that

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad s > 0$$

¶If p is a positive integer n, by Problem 36 we know that

$$\Gamma(p+1) = n!$$

Using our above formula, now shows that

$$\mathcal{L}\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}} = \frac{n!}{s^{n+1}}, \quad s > 0$$

(c) Show that

$$\mathcal{L}\{t^{-1/2}\} = \frac{2}{\sqrt{s}}\int_0^\infty e^{-x^2}dx = \sqrt{\frac{\pi}{2}}, \quad s > 0$$

¶Again, by problem 36, we know that $\Gamma(1/2) = \sqrt{\pi}$, hence if $p = -1/2$ we have

$$\mathcal{L}\{t^{-1/2}\} = \frac{\Gamma(1/2)}{s^{1/2}} = \sqrt{\frac{\pi}{s}}, \quad s > 0$$

(d) Show that

$$\mathcal{L}\{t^{1/2}\} = \frac{\sqrt{\pi}}{2s^{3/2}}$$

¶One more time by problem 36, we have that $p\Gamma(p) = \Gamma(p+1)$ for $p > 0$. So we have

$$\mathcal{L}\{t^{1/2}\} = \frac{\Gamma(3/2)}{s^{3/2}} = \frac{\Gamma(1/2)}{2s^{3/2}} = \frac{\sqrt{\pi}}{2s^{3/2}}, \quad s > 0$$

5.2- # 11 Let $F(s) = \mathcal{L}\{f(t)\}$, where $f(t)$ is piecewise continuous and of exponential order on $[0, \infty)$. Show that

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{1}{s}F(s)$$

Solution Recall that we have

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$$

via integration by parts since $f(t)$ is piecewise continuous and of exponential order. Define

$$g(t) = \int_0^t f(\tau)d\tau \implies g'(t) = f(t)$$

Hence, by plugging this into the above formula we have

$$\mathcal{L}\{f(t)\} = s\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} \implies \mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{F(s)}{s}$$

5.2- #28 The Laplace transforms of certain functions can be found conveniently form their Taylor series expansions. Using the Taylor series for $\sin t$

$$\sin t = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!}$$

and assuming that the Laplace transform of this series can be computed term by term, verify that

$$\mathcal{L}\{\sin t\} = \frac{1}{s^2+1}, \quad s > 1$$

Solution By linearity, we have that

$$\mathcal{L}\{\sin t\} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\mathcal{L}\{t^{2n+1}\}$$

Luckily, we previous did a question that gave us a nice formula for this! Recalling

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \implies \mathcal{L}\{t^{2n+1}\} = \frac{(2n+1)!}{s^{2n+2}}$$

Plugging this into the above form, we have

$$\mathcal{L}\{\sin t\} = \sum_{n=0}^{\infty} \frac{(-1)^n}{s^{2n+2}} = \frac{1}{s^2}\sum_{n=0}^{\infty}\left(-\frac{1}{s^2}\right)^n$$

Recall the geometric series formula

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

clearly, if $x = -1/s^2$, then

$$\frac{1}{1+s^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{s^{2n+2}} = \mathcal{L}\{\sin t\}$$

as desired.

5.2 - # 29 For each of the following initial value problems, use Theorem 5.2.4 to find the differential equation satisfied by $F(s) = \mathcal{L}\{f(t)\}$, where $y = f(t)$ is the solution of the given initial value problem.

$$y'' - ty = 0; \quad y(0) = 1, \quad y'(0) = 0$$

Solution Theorem 5.2.4 states that for f piecewise continuous of exponential order a, then

$$\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s), \quad s > a$$

where $F^{(n)} = d^n/dx^n \mathcal{L}\{f(t)\}$. Clearly we have

$$\mathcal{L}\{y''(t)\} = s^2F(s) - sf(0) - f'(0) = s^2F(s) - s$$

and

$$\mathcal{L}\{ty(t)\} = -F'(s)$$

Hence

$$\mathcal{L}\{y'' - ty\} = \mathcal{L}\{y''\} - \mathcal{L}\{ty\} = \boxed{s^2F(s) + F'(s) = s}$$

is the ODE we're looking for.

5.3 - # 17 Use the linearity of $\mathcal{L}^{-1}$ with partial fraction expansion and Table 5.3.1 to find the inverse Laplace transform of the given function:

$$F(s) = \frac{1-2s}{s^2+4s+5}$$

Solution Well, by glancing at the table, we see it's a pretty good idea to complete the square in the dominator, so we do.

$$s^2 + 4s + 5 = (s+2)^2 + 1$$

Hence we may rewrite $F(s)$ as

$$F(s) = \frac{1-2s}{s^2+4s+5} = 5\left(\frac{1}{(s+2)^2+1}\right) - 2\left(\frac{s+2}{(s+2)^2+1}\right)$$

Noting that

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+1}\right\} = e^{-2t}\sin(t) \quad \& \quad \mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+1}\right\} = e^{-2t}\cos(t)$$

by the table, we have

$$\mathcal{L}^{-1}\{F(s)\} = e^{-2t}(5\sin t - 2\cos t)$$

5.3 - # 24 Use the linearity of $\mathcal{L}^{-1}$ with partial fraction expansion and Table 5.3.1 to find the inverse Laplace transform of the given function:

$$F(s) = \frac{s^2+3}{(s^2+2s+2)^2}$$

Solution Rewrite the fraction as follows

$$F(s) = \frac{s^2+3}{(s^2+2s+2)^2} = \frac{(s+1)^2+1-2s+1}{((s+1)^2+1)^2} = \frac{1}{(s+1)^2+1} - \frac{2s-1}{((s+1)^2+1)^2}$$

Now the table manages the first term, and we see that the other term looks like some sort of derivative. So we try

$$-\frac{d}{ds}\left(A\frac{1}{(s+1)^2+1} + B\frac{s+1}{(s+1)^2+1}\right) = \frac{2A(s+1)+Bs(s+2)}{((s+1)^2+1)^2}$$

We have to complete the square on the top for B, so

$$B\frac{s(s+2)}{((s+1)^2+1)^2} = B\frac{(s+1)^2+1-2s}{((s+1)^2+1)^2} = B\frac{1}{(s+1)^2+1} - B\frac{2s}{(s+1)^2+1)^2}$$

Hence, comparing with our original expansion we see ($B = 3/2$ and $A = 1$)

$$F(s) = \left(\frac{5}{2} + \frac{d}{ds}\right)\left(\frac{1}{(s+1)^2+1}\right) + \frac{3}{2}\frac{d}{ds}\left(\frac{s+1}{(s+1)^2+1}\right)$$

Comparing with the table gives

$$\implies \mathcal{L}^{-1}\{F(s)\} = \left(\frac{5}{2} - t\right)e^{-t}\sin t - \frac{3}{2}te^{-t}\cos t$$