

3.3. TAYLOR SERIES

Having explored sequences, series, and power series, we are now ready to return to our original motivation: Taylor polynomials. Recall that the Taylor polynomial of degree n for $f(x)$ centered at $x = a$, $T_n(x)$, is the unique polynomial of degree n which matches $f(x)$ and its first n derivatives at $x = a$.

Back in Section 1.1, we observed that higher degree Taylor polynomials tended to give better approximations to $f(x)$. Then in Section 1.2 we made this observation precise with the Remainder Theorem.

The Remainder Theorem. Suppose that f is $n + 1$ times differentiable and let R_n denote the difference between $f(x)$ and the Taylor polynomial of degree n for $f(x)$ centered at a . Then

$$R_n(x) = f(x) - T_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x-a)^{n+1}$$

for some c between a and x .

Now that we have established the notion of series, we are ready to consider Taylor polynomials of “infinite degree”, which we will call *Taylor series*[†]. These are a special type of power series.

Taylor Series. Suppose that the function $f(x)$ is infinitely differentiable (smooth) at $x = a$. The Taylor series for $f(x)$ centered at $x = a$ is then

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n.$$

Assuming that we can compute all the derivatives of $f(x)$ at $x = a$, this definition is very concrete, but it leaves some unanswered questions:

Where does the Taylor series for $f(x)$ converge?

When the Taylor series for $f(x)$ does converge, does it converge to $f(x)$?

In general, these two questions must be answered on a function-by-function basis, and such answers revolve around the Remainder Theorem.

[†]Taylor series centered at $x = 0$ are sometimes referred to as *Maclaurin series* after the Scottish mathematician Colin Maclaurin (1698–1746).

For our first case study, we consider the function $f(x) = \sin x$, centered at $x = 0$. Since the derivatives of $\sin x$ repeat in the pattern $\cos x, -\sin x, -\cos x, \sin x$, the sequence of derivatives evaluated at $x = 0$ is $\{1, 0, -1, 0, \dots\}$. Therefore the Taylor series for $\sin x$ centered at $x = 0$ is:

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

We could apply the Ratio Test to this series (as in Section 3.1) to show that it converges for all x , but there is a more important question: Does this series converge to $\sin x$ for all x ?

By the Remainder Theorem, the difference between $\sin x$ and its Taylor polynomial of degree n centered at $x = 0$, $T_n(x)$, is

$$\sin x - T_n(x) = R_n(x) = \frac{\sin^{(n+1)}(c)}{(n+1)!} x^{n+1}$$

for some c between 0 and x . Since the derivatives of $\sin x$ only take on values between -1 and 1 , we have that

$$|R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!}.$$

We would like to show that the Taylor series for $\sin x$ converges to $\sin x$ for all x , which is equivalent to showing that the limit of the remainders $R_n(x)$ is 0 (as $n \rightarrow \infty$). This is really just an exercise from Section 2.1, although we have delayed it now.

Fact 1. For all values of x , $\lim_{n \rightarrow \infty} \frac{|x|^n}{n!} = 0$.

Proof. First let us consider a special case, where $|x| = 4$. In this case, we are interested in the sequence $\{4^n/n!\}$. For the first 4 terms, this sequence is actually increasing:

n	$4^n/n!$
0	1
1	4
2	$4^2/2! = 8$
3	$4^3/3! = 10.6666\dots$
4	$4^4/4! = 10.6666\dots$

But then the sequence starts decreasing. To get the $n = 5$ term, we multiply the $n = 4$ term by $4/5$; to get the $n = 6$ term we multiply this by $4/6$; to get the $n = 7$ term we multiply this by $4/7$, and so on. For $n \geq 5$, we therefore have the bound

$$\frac{4^n}{n!} \leq \left(\frac{4}{5}\right)^{n-4} \frac{4^4}{4!} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Our proof for generic x follows the same general approach. Choose an integer $m > |x|$. We have, for $n \geq m$,

$$\frac{|x|^n}{n!} = \frac{|x|^n}{\underbrace{n \cdot (n-1) \cdots (m+1) \cdot m \cdot (m-1)!}_{n-m \text{ terms, all } \geq m > |x|}} \leq \frac{|x|^n}{m^{n-m}(m-1)!}.$$

Ideally we would like to have m^n in the denominator, so we multiply top and bottom by m^m :

$$\frac{|x|^n}{n!} \leq \frac{|x|^n m^m}{m^n (m-1)!} = \frac{|x|^n}{m^n} \cdot \frac{m^m}{(m-1)!}.$$

Our goal is to prove that the sequence $\{|x|^n/n!\}$ converges to 0 by sandwiching it between the constant sequence $\{0\}$ and the sequence above.

Remember that m is not changing, so $m^m/(m-1)!$ is just a constant. Since $m > |x| \geq 0$, $|x|/m$ is between 0 and 1, so Example 5 of Section 2.1 shows that the geometric sequence $(|x|/m)^n \rightarrow 0$. Therefore, by the Sandwich Theorem, $|x|^n/n! \rightarrow 0$ as $n \rightarrow \infty$ for all values of x . \square

Our next example collects these observations to show that the Taylor series for $\sin x$ centered at $x = 0$ converges[†] to $\sin x$.

Example 2. Show that the Taylor series centered at $x = 0$ for $\sin x$ converges to $\sin x$ for all values of x .

Solution. Proving this statement is equivalent to showing that for any fixed value of x , the remainder term $R_n(x)$ tends to 0 as $n \rightarrow \infty$. Fix a value of x . By the Remainder Theorem, our previous bounds, and Fact 1,

$$|R_n(x)| = |\sin x - T_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!} \rightarrow 0 \text{ as } n \rightarrow \infty,$$

which is precisely what we wanted to show. \bullet

Example 2 is in some sense completely bizarre. Remember that we defined Taylor polynomials (and therefore series as well) by matching derivatives at $x = a$ (the center). Therefore, this example shows that if a power series matches all of the derivatives of $\sin x$ at $x = 0$, then the series is *equal* to $\sin x$ for *all* values of x . Why should $\sin x$ be completely determined by its derivatives at 0? With our limited tools, the best answer we will be able to provide here is that $\sin x$ is a very well-behaved function; the technical term for such functions is *analytic*.

The same approach would work for the function $f(x) = \cos x$, but there is an easier way. Now that we know that the Taylor series for $\sin x$ centered at $x = 0$ converges to $\sin x$

[†]This type of convergence is known as *pointwise convergence*, because the Taylor series converges to $\sin x$ at each individual point. There is a stronger, more global type of convergence known as *uniform convergence*.

for all values of x , we may differentiate it term-by-term to obtain:

$$\begin{aligned}
 \cos x &= \frac{d}{dx} \sin x \\
 &= \frac{d}{dx} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \\
 &= \sum_{n=0}^{\infty} (-1)^n (2n+1) \frac{x^{2n}}{(2n+1)!} \\
 &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \\
 &= 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots .
 \end{aligned}$$

For our next case study, we consider the function $f(x) = e^x$, again centered at $x = 0$. Since every derivative of e^x is e^x itself, $f^{(n)}(0) = 1$ for every n , and thus the Taylor series centered at $x = 0$ for e^x is

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots .$$

Example 3. Show that the Taylor series centered at $x = 0$ for e^x converges to e^x for all values of x .

Solution. We want to show that for fixed x , $|R_n(x)| \rightarrow 0$ as $n \rightarrow \infty$. The Remainder Theorem shows that

$$|R_n(x)| = |e^x - T_n(x)| = \left| \frac{e^c}{(n+1)!} x^{n+1} \right|$$

for some c between 0 and x (here we used the fact that the $n+1$ st derivative of e^x is e^x itself). We have to be a little careful here about giving a bound for e^c , since the bound changes depending on whether x is positive or not:

$$e^c \leq \begin{cases} e^x & \text{if } x \geq 0, \\ 1 & \text{if } x < 0. \end{cases}$$

Still, since we are considering a *fixed* value of x , our bound on e^c is just a constant, and Fact 1 shows that

$$\frac{|x|^{n+1}}{(n+1)!} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Therefore $|R_n(x)| \rightarrow 0$ as $n \rightarrow \infty$, proving that the Taylor series for e^x centered at $x = 0$ converges to e^x for all values of x . ●

Substituting $x = 1$ into the Taylor series from Example 3 gives an appealing identity:

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

Because $n!$ grows so quickly, this series converges to e exceptionally fast. For example, using just the first 10 terms, we get e correct to six decimal places:

$$\begin{aligned} e &= 2.7182818\dots \\ \sum_{n=0}^9 \frac{1}{n!} &= 2.7182815\dots \end{aligned}$$

The speed of this convergence (and it really is fast — at $n = 59$, we are adding the reciprocal of the estimated atoms in the universe, about 10^{-80}) allows us to prove that e is irrational in Exercises 50–52.

As with general power series, we are allowed to substitute into Taylor series and to multiply and divide them. This can be quite useful for expressing integrals as power series (our next example), evaluating limits (Example 5), computing derivatives (Example 6), and finding Taylor series for quotients (Example 7).

Example 4. Express $\int \sin x^2 dx$ as a power series.

Solution. We know from Example 2 that

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!},$$

so

$$\sin x^2 = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n+1)!}.$$

All that remains is to integrate this series term-by-term:

$$\int \sin x^2 = \int \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+3}}{(4n+3)(2n+1)!} + C.$$

It has been proved that the antiderivative of $\sin x^2 dx$ is not an “elementary function”, so this one of the nicest possible ways to express this integral. ●

Example 5. Evaluate $\lim_{x \rightarrow 0} \frac{\cos x - 1 + x^2/2}{x^4}$.

Solution. While this example could be solved with l'Hôpital's Rule, it is easier to use power series. We know that

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots,$$

so

$$\begin{aligned} \frac{\cos x - 1 + \frac{x^2}{2}}{x^4} &= \frac{\left(1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) - 1 + \frac{x^2}{2}}{x^4} \\ &= \frac{\frac{x^4}{4!} - \frac{x^6}{6!} + \cdots}{x^4} \\ &= \frac{1}{4!} - \frac{x^2}{6!} + \cdots. \end{aligned}$$

As $x \rightarrow 0$, this quantity approaches $1/4! = 1/24$, so this is the limit. ●

Example 6. Compute the 102nd derivative of $f(x) = e^{2x^3}$ at $x = 0$.

Solution. Example 3 shows that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

so by substitution, we have that

$$e^{2x^3} = \sum_{n=0}^{\infty} \frac{(2x^3)^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n x^{3n}}{n!}.$$

Since this is the Taylor series for $f(x) = e^{2x^3}$ centered at $x = 0$, by the definition of Taylor series, we know that it is equal to

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

Therefore if we are interested in $f^{(102)}(0)$, we need only look at the coefficient of x^{102} . We get the coefficient of x^{102} in

$$\sum_{n=0}^{\infty} \frac{2^n x^{3n}}{n!}$$

by substituting $n = 102/3 = 34$, so

$$\frac{2^{34}}{34!} = \frac{f^{102}(0)}{102!}.$$

Solving for $f^{(102)}(0)$ shows that

$$f^{(102)}(0) = \frac{2^{34}102!}{34!}.$$

We could have solved this problem simply by taking 102 derivatives, but finding the Taylor series is *much* easier. ●

Example 7. Find the first four nonzero terms of the Taylor series centered at $x = 0$ for the function $f(x) = \tan x$.

Solution. It would be possible to solve this problem by taking derivatives of $\tan x$, but this approach gets quite messy. An easier method is to use the series we have for $\sin x$ and $\cos x$ together with the Division Theorem from Section 3.2. Since

$$\tan x = \frac{\sin x}{\cos x} = \frac{x - x^3/3! + x^5/5! - x^7/7! + \dots}{1 - x^2/2! + x^4/4! - x^6/6! + \dots},$$

we can use long division

$$\begin{array}{r}
 x + x^3/3 + 2x^5/15 + 17x^7/315 + \dots \\
 1 - x^2/2 + x^4/24 - x^6/720 + \dots \mid x - x^3/6 + x^5/120 - x^7/5040 + \dots \\
 \underline{x - x^3/2 + x^5/24 - x^7/720 + \dots} \\
 x^3/3 - x^5/30 + x^7/840 - \dots \\
 \underline{x^3/3 - x^5/6 + x^7/72 - \dots} \\
 2x^5/15 - 4x^7/315 + \dots \\
 \underline{2x^5/15 - x^7/15 + \dots} \\
 17x^7/315 + \dots
 \end{array}$$

to find that the first four nonzero terms of the Taylor series centered at $x = 0$ for $\tan x$ are

$$x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315}.$$

This is also therefore the Taylor polynomial of $\tan x$ of degree 7, centered at $x = 0$. ●

We can also use the Taylor series for e^x to *define* the function e^x for more quantities than just real numbers. For example, we can define e^x for *complex numbers* x . Recall that a complex number is a number of the form $a + bi$ where a and b are real numbers, and i is the (imaginary) square-root of -1 . This leads to the following formula, which the Nobel Prize winning physicist Richard Feynman (1918–1988) referred to as “one of the most remarkable, almost astounding, formulas in all of mathematics”:

Example 8 (Euler’s Formula). Prove Euler’s Formula $e^{i\theta} = \cos \theta + i \sin \theta$.

Solution. We begin by substituting $i\theta$ into the Taylor series for e^x :

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{i^n \theta^n}{n!} = \sum_{n=0}^{\infty} \frac{i^{2n} \theta^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{i^{2n+1} \theta^{2n+1}}{(2n+1)!}.$$

Now we need to compute the powers of i :

$$\begin{aligned} i^2 &= -1, \\ i^3 &= i \cdot i^2 = -i, \\ i^4 &= i^2 \cdot i^2 = (-1) \cdot (-1) = 1, \\ i^5 &= i \cdot i^4 = i. \end{aligned}$$

This shows that the powers of i form a periodic sequence $\{i, -1, -i, 1, \dots\}$, which allows us to simplify our series for $e^{i\theta}$ above:

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n+1}}{(2n+1)!} = \cos \theta + i \sin \theta.$$

An alternative proof, using derivatives, is outlined in Exercise 45. ●

A particularly wondrous special case of Euler's Formula, known as Euler's Identity, is,

$$e^{\pi i} + 1 = 0,$$

an identity relating five of the most important numbers in all of mathematics. In Exercises 46–49 we use a similar approach to define e^M for a matrix M .

Euler's formula is also a very convenient way to prove various trigonometric identities. For example, the angle addition formulas, which show how to evaluate $\sin(\alpha + \beta)$ and $\cos(\alpha + \beta)$ in terms of $\sin \alpha$, $\sin \beta$, $\cos \alpha$, and $\cos \beta$, have a particularly straightforward derivation using Euler's formula. As the French mathematician Jacques Hadamard (1865–1963) wrote, “the shortest route between two truths in the real domain sometimes passes through the complex domain.”

Example 9. Derive the angle addition formulas.

Solution. We begin with the trigonometric functions we are interested in, convert to exponentials, simplify the expression, and then convert back to trigonometric functions:

$$\begin{aligned} \cos(\alpha + \beta) + i \sin(\alpha + \beta) &= e^{(\alpha+\beta)i} \\ &= e^{\alpha i} \cdot e^{\beta i} \\ &= (\cos \alpha + i \sin \alpha) (\cos \beta + i \sin \beta) \\ &= (\cos \alpha \cos \beta - \sin \alpha \sin \beta) + i (\sin \alpha \cos \beta + \cos \alpha \sin \beta). \end{aligned}$$

Now equate real and imaginary parts of both sides to see that

$$\begin{aligned}\cos(\alpha + \beta) &= \cos \alpha \cos \beta - \sin \alpha \sin \beta, \\ \sin(\alpha + \beta) &= \sin \alpha \cos \beta + \cos \alpha \sin \beta,\end{aligned}$$

as desired. ●

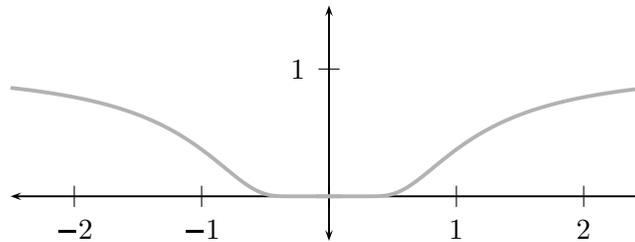
We know from Section 3.1 that power series define infinitely differentiable (smooth) functions when they converge, so not every function is equal to its Taylor series (in fact, a function must be infinitely differentiable for us to even define its Taylor series). Our success in this section and the last with finding Taylor series for infinitely differentiable functions suggests a final question:

Is every infinitely differentiable function equal to its Taylor series?

The answer is **no**. A counterexample is provided by the function

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

whose plot is shown below.



It is possible (using l'Hôpital's Rule) to establish that every derivative of this function at $x = 0$ is 0, so therefore its Taylor series is simply 0, which does not converge to this function. Therefore, infinite differentiability is a necessary condition for a function to equal its Taylor series, but it is not a sufficient condition.

EXERCISES FOR SECTION 3.3

Derive Taylor series for the functions in Exercises 1–8 at the specified centers.

1. $f(x) = \sin 2x$ centered at $x = 0$

2. $f(x) = \cos x$ centered at $x = \pi/2$

3. $f(x) = x^2 \cos(\sqrt{2}x^3)$ centered at $x = 0$

4. $f(x) = (e^x)^2$ centered at $x = 0$

5. $f(x) = 2xe^x$ centered at $x = 0$

6. $f(x) = 2xe^x$ centered at $x = 1$

7. $f(x) = \sinh x = \frac{e^x - e^{-x}}{2}$ centered at $x = 0$

8. $f(x) = \cosh x = \frac{e^x + e^{-x}}{2}$ centered at $x = 0$

Compute the limits in Exercises 9–16 using Taylor series.

9. $\lim_{x \rightarrow 0} \frac{\sin x - x}{x^3}$

10. $\lim_{x \rightarrow 0} \frac{x - \arctan x}{x^3}$

11. $\lim_{x \rightarrow 0} \frac{e^x - e^{-x}}{x}$

12. $\lim_{x \rightarrow 0} \frac{\sin x - \arctan x}{x^3}$

13. $\lim_{x \rightarrow 0} \frac{(2 - 2 \cos x)^3}{x^6}$

14. $\lim_{x \rightarrow 0} \frac{\ln(1 + x^3)}{x^3}$

15. $\lim_{x \rightarrow 0} \frac{(\sin x - x)^2}{(\cos 5x - 1)^3}$

16. $\lim_{x \rightarrow 0} \frac{(\sin 3x - 3x)^2}{(e^{2x} - 1 - 2x)^3}$

Use Taylor series of known functions to evaluate the sums in Exercises 17–20.

17. $\sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n}}{(2n)!}$

18. $\sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n+1}}{2^n (2n+1)!}$

19. $\sum_{n=0}^{\infty} (-1)^n \frac{\pi^n}{n!}$

20. $\sum_{n=0}^{\infty} (-7)^n \frac{\pi^{2n}}{(2n)!}$

Derive Taylor series for the integrals in Exercises 21–24 (centered at $x = 0$). Note that none of these integrals has an answer in terms of elementary functions.

21. $\int \frac{\cos(x^3) - 1}{x^2} dx$

22. $\int \frac{\sin x}{x} dx$

23. $\int e^{-x^2/2} dx$

24. $\int \frac{\ln(x+1)}{x} dx$

Use the Remainder Theorem in Exercises 25–30.

25. Give an upper bound on the error when using $T_3(x) = 1 + x + x^2/2 + x^3/6$ to approximate $f(x) = e^x$ for $0 \leq x \leq 1/2$.

26. Give an upper bound on the error when using $T_1(x) = x$ to approximate $f(x) = \tan x$ for $-\pi/4 \leq x \leq \pi/4$.

27. Give an upper bound on the error when using $T_3(x) = 1 - 2(x-1) + 3(x-1)^2 - 4(x-1)^3$ to approximate $f(x) = 1/x^2$ for $3/4 \leq x \leq 5/4$.

28. Give an upper bound on the error when using $T_2(x) = 2 + (x-4)/4 - (x-4)^2/64$ to approximate $f(x) = \sqrt{x}$ for $4 \leq x \leq 4.1$.

29. Give an upper bound on the error when using $T_3(x) = (x-1) + (x-1)^2/2 - (x-1)^3/6$ to approximate $f(x) = x \ln x$ for $1/2 \leq x \leq 3/2$.

30. Give an upper bound on the error when using $T_4(x) = x - x^2/2 + x^3/3 - x^4/4$ to approximate $f(x) = \ln 1 + x$ for $0 \leq x \leq 1$.

31. Prove that $x^n/n! \rightarrow 0$ as $n \rightarrow \infty$ for every fixed value of x . (Note that this is a bit stronger than Fact 1.)

For Exercises 32–35, give the Taylor polynomials of degree 8 (centered at $x = 0$) for the specified functions.

32. $f(x) = e^{\cos x^4}$

33. $f(x) = (1 + x + x^2) \sin x^3$

34. $f(x) = \frac{\sin x}{1 - 2x}$

35. $f(x) = \ln(\cos x)$

Exercises 36–39 investigate another way to bound the error in using Taylor polynomials to approximate functions. Consider first the example of using $T_3(x) = 1 + x + x^2/2 + x^3/6$ to approximate $e^{1/2}$. We could of course use the Remainder Theorem for this problem (as is requested in Exercise 25), but since we know that

$$e^{1/2} = 1 + 1/2 + \frac{(1/2)^2}{2} + \frac{(1/2)^3}{6} + \sum_{n=4}^{\infty} \frac{(1/2)^n}{n!},$$

another way to estimate this error is simply to bound the tail by comparing it to a geometric series:

$$\begin{aligned} \sum_{n=4}^{\infty} \frac{(1/2)^n}{n!} &= \frac{(1/2)^4}{4!} + \frac{(1/2)^5}{5!} + \frac{(1/2)^6}{6!} + \dots \\ &< \frac{(1/2)^4}{4!} + \frac{(1/2)^5}{4 \cdot 4!} + \frac{(1/2)^6}{4^2 \cdot 4!} + \dots \\ &= \frac{(1/2)^4}{4!} \left(1 + \frac{1/2}{4} + \frac{1/2}{4^2} + \dots \right) \\ &= \frac{(1/2)^4}{4!} \left(\frac{1}{1 - 1/4} \right). \end{aligned}$$

(This gives an error estimate of about 0.003, whereas the Remainder Theorem shows that the error is at most 0.004.)

36. Bound the error involved in using the Taylor polynomial of degree 5 to approximate $\sin x$ near $x = 1/2$ without using the Remainder Theorem.

37. Bound the error involved in using the Taylor polynomial of degree 5 to approximate

$$\int_0^1 e^{-x^2/2} dx$$

without using the Remainder Theorem.

38. Show that the Remainder Theorem cannot give any practical bounds on the error involved in approximating $f(x) = -\ln(1-x)$ at $x = 1/2$ by its Taylor polynomial of degree 3, $T_4(x) = x + x^2/2 + x^3/3$.

39. Give a bound on the error involved in the estimate in Exercise 38 by observing that

$$\begin{aligned} R_3(1/2) &= \frac{(1/2)^4}{4} + \frac{(1/2)^5}{5} + \frac{(1/2)^6}{6} + \dots \\ &< \frac{(1/2)^4}{4} (1 + (1/2) + (1/2)^2 + \dots). \end{aligned}$$

Exercises 40 and 41 concern the series

$$\sum_{n=2}^{\infty} \frac{\pi^n}{n!}.$$

40. Compute the 10th partial sum of this series.

41. Prove that the sum of this series is *not* 19.

42. Use Euler's formula to prove the two identities

$$\begin{aligned} \cos \theta &= \frac{e^{i\theta} + e^{-i\theta}}{2}, \\ \sin \theta &= \frac{e^{i\theta} - e^{-i\theta}}{2i}. \end{aligned}$$

43. Use Euler's formula to prove De Moivre's Formula,

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta.$$

for all integers n .

44. Give an example showing that De Moivre's Formula does not necessarily hold when n is not an integer.

45. Define

$$f(\theta) = (\cos \theta + i \sin \theta) e^{-i\theta}.$$

Compute the derivative of $f(\theta)$ and use this to give another proof of Euler's Formula.

As with complex numbers, we also use the Taylor series for e^x to define e^M for matrices square M as

$$I + M + \frac{M^2}{2!} + \frac{M^3}{3!} + \dots,$$

where I denotes the identity matrix (which has 1s along the diagonal and 0s everywhere else). Use

this to compute e^M for the matrices in Exercises 46–49.

$$46. M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$47. M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$48. M = \begin{pmatrix} -1 & -2 \\ 0 & 1 \end{pmatrix}$$

$$49. M = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Suppose that e were a rational number, so $e = a/b$ for positive integers a and b . In Exercises 50–52 we will draw a contradiction, thereby proving that e is irrational. This proof is often attributed to Charles Hermite (1822–1901).

50. Define

$$a_n = n! \left(e - \sum_{k=0}^n \frac{1}{k!} \right),$$

and prove that a_n is an integer for all $n \geq b$.

$$51. \text{ Show that } a_n = \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \cdots.$$

52. Show that $0 < a_n < 1$, and therefore a_n can not be an integer, contradicting our assumption that e is rational.

Is there an irrational number which raised to an irrational power is rational? Exercise 54 proves that there is, although it doesn't tell us precisely what it is. I thank Professor Steven Landsburg for making me aware of these problems.

53. Verify that $\sqrt{2}^{\sqrt{2}^{\sqrt{2}}} = 2$. (One method is to take the logarithm of both sides.)

54. Using the result of the previous exercise and the fact that $\sqrt{2}$ is irrational, prove that there is an irrational number which raised to an irrational number is rational.

Exercises 55–59 present Euler's original computation of $\sum 1/n^2$, from 1735. While this proof was considered a breakthrough at the time, Euler mistakenly assumed that he could apply a fact about

polynomials to power series (see Exercise 58). The fact he assumed, however, does not hold for arbitrary series, as Exercise 59 shows. It was not until about 150 years later that Karl Weierstrass (1815–1897) "corrected" this proof by proving the Weierstrass Factorization (or, Product) Theorem. It should also be noted that Euler gave two other correct proofs for this calculation in the same paper, and a fourth proof in 1741, but his "incorrect" proof is the one that is best remembered. As the American author Henry Mencken (1880–1956) wrote, "For every problem, there is one solution which is simple, neat, and wrong."

55. Find a series centered at 0 which is equal to

$$f(x) = \frac{\sin \sqrt{x}}{\sqrt{x}}$$

for $x > 0$.

56. Find all the roots of the power series from Exercise 55. This should involve two steps; first show that this series has no negative roots, and then find all the positive roots, which are (by the previous problem) also roots of

$$f(x) = \frac{\sin \sqrt{x}}{\sqrt{x}}.$$

57. If a polynomial of degree 3 has roots r_1 , r_2 , and r_3 , then it is given by $p(x) = c(x-r_1)(x-r_2)(x-r_3)$ for some constant c . By expanding this product, verify that

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = -\frac{\text{coefficient of } x}{\text{constant term}}.$$

(This generalizes to polynomials of any degree.)

58. Assume, as Euler did, that Exercise 57 holds for series to show that

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \cdots = \frac{1}{6}$$

where r_1, r_2, r_3, \dots are the roots from Exercise 56.

Conclude that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

59. The function $f(x) = 2 - 1/(1-x)$ has a single root, $x = 1/2$. Derive its power series (centered at $x = 0$) and conclude that, contrary to Euler's assumption, Exercise 57 cannot be applied to arbitrary series.

ANSWERS TO SELECTED EXERCISES, SECTION 3.3

1. $\sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n+1} x^{2n+1}}{(2n+1)!}$
3. $x^2 \sum_{n=0}^{\infty} (-1)^n \frac{(\sqrt{2}x^3)^{2n}}{(2n)!} = x^2 \sum_{n=0}^{\infty} (-1)^n \frac{2^n x^{6n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{2^n x^{6n+2}}{(2n)!}$
5. $2x \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} 2 \frac{x^{n+1}}{n!}$
7. $\frac{1}{2} \sum_{n=0}^{\infty} \frac{x^n}{n!} - \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$
9. The series is $\frac{-1}{3!} + \frac{x^2}{5!} - \frac{x^4}{7!} + \dots$, so the limit is $-1/3! = -1/6$.
11. The series is $2 + \frac{2x^2}{3!} + \frac{2x^4}{5!} + \dots$, so the limit is 2.
13. The series is $1 - \frac{x^2}{4} + \dots$, so the limit is 1.
15. The series is $\frac{-2}{140625} - \frac{41x^2}{468750} - \dots$, so the limit is $-2/140625$.
17. $\cos \pi = 0$
19. $e^{-\pi}$
21. $\int \sum_{n=1}^{\infty} (-1)^n \frac{x^{6n-2}}{(2n)!} dx = \sum_{n=1}^{\infty} (-1)^n \frac{x^{6n-1}}{(6n-1)(2n)!} + C$
23. $\int \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2^n n!} dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2^n (2n+1)n!} + C$