Fermat's Last Theorem Proved on a Single Page Revisited

"5% of the people think; 10% of the people think that they think; and the other 85% would rather die than think."----Thomas Edison

"The simplest solution is usually the best solution"---Albert Einstein

Abstract

Honorable Pierre de Fermat could have squeezed the proof of his last theorem into a page margin. Fermat's last theorem has been proved on a single page. Three similar versions of the proof are presented, using a single page for each version. The approach used in each proof is exemplified by the following system: If a system functions properly and one wants to determine if the same system will function properly with changes in the system, one will first determine the necessary conditions which allow the system to function properly, and then guided by the necessary conditions, one will determine if the changes will allow the system to function properly. So also, if one wants to prove that there are no solutions for the equation $c^n = a^n + b^n$ when $n > 2$, one should first determine why there are solutions when $n = 2$, and note the necessary conditions in the solution for $n = 2$. The necessary conditions in the solutions for $n = 2$ will guide one to determine if there are solutions when $n > 2$. For the first two versions, the proof is based on the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$, or $\sin^2 x + \cos^2 x = 1$; and for the third version, on $(a^2 + b^2)/c^2 = 1$, with $n = 2$; where (a, b, c) is a primitive Pythagorean triple. It is shown by contradiction that the uniqueness of the $n = 2$ identity excludes all other n-values, $n > 2$, from satisfying the equation $c^n = a^n + b^n$. One will first show that if $n = 2$, $c^n = a^n + b^n$ holds, noting the necessary conditions in the solution; followed by showing that if $n > 2$ (n an integer), $c^n = a^n + b^n$ does not hold. For the first version of the proof, the proof began with reference to a right triangle. The second version of the proof began with ratio terms without any reference to a geometric figure. The third version occupies about half of a page. The third version of the proof began without any reference to a geometric figure or ratio terms. The second and third versions confirmed the proof in the first version. Each proof version is very simple, and even high school students can learn it. The approach used in the proof has applications in science, engineering, medicine, research, business, and any properly working system when desired changes are to be made in the system. Perhaps, the proof in this paper is the proof that Fermat wished there were enough margin for it in his paper. With respect to prizes, if the prize for a 150-page proof were $715,000, then the prize for a single page proof (considering the advantages) using inverse proportion, would be $107,250,000.

Proof: Version 1

Given: $c^n = a^n + b^n$ (n an integer; $a, b,$and c are relatively prime positive integers)

Required: To prove that $c^n = a^n + b^n$ does not hold if $n > 2$.

Plan: One will first show that if $n = 2$, $c^n = a^n + b^n$ holds,

followed by showing that if $n > 2$ (n an integer), $c^n = a^n + b^n$ does not hold.

Proof: Let $a, b,$and c be three relatively prime positive integers which are the lengths of the sides of the right triangle in the figure below, where c is the length of the hypotenuse, and a and b are the lengths of the other two sides. Also, let θ denote the acute angle at vertex A.

Then $a = c\sin\theta$ (1)

$b = c\cos\theta$ (2)

$c^n = a^n + b^n$ (3)

$c^n = (c\sin\theta)^n + (c\cos\theta)^n$

$c^n = c^n\sin^n\theta + c^n\cos^n\theta$ (4)

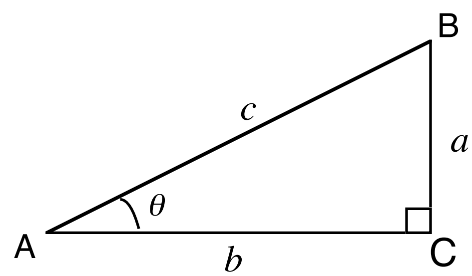
$c^n = c^n(\sin^n\theta + \cos^n\theta)$ (5).

Left-hand side (LHS) of equation (5) equals right-hand side (RHS) of (5) only if

$\sin^n\theta + \cos^n\theta = 1$

That is, a necessary condition for (5) to be true is that

$\boxed{\sin^n\theta + \cos^n\theta = 1}$



If $n = 2$, $c^2 = c^2(\sin^2\theta + \cos^2\theta)$ is true since $\sin^2\theta + \cos^2\theta = 1$.Therefore equations (5) and (3) are true. One will next show that if $n > 2$, the necessary condition $\sin^n\theta + \cos^n\theta = 1$ is never satisfied and, and consequently $c^n = a^n + b^n$ and $c^n = c^n\sin^n\theta + c^n\cos^n\theta$ are never satisfied.

Proof for $n > 2$ by contradiction

If $n = 2$ **,** $\sin^n x + \cos^n x = 1$ becomes $\sin^2 x + \cos^2 x = 1$, which is true. If $n > 2$, and one assumes that $\sin^n x + \cos^n x = 1$, then $\sin^n x + \cos^n x = \sin^2 x + \cos^2 x$ (B)
(By the transitive equality property, since $\sin^2 x + \cos^2 x = 1$).
From (B), and equating the exponents, $n > 2 = 2$ is false, since an integer greater than 2 cannot be equal to 2. Hence, the assumption that $\sin^n x + \cos^n x = 1$, if $n > 2$, is false. Therefore, $\sin^n x + \cos^n x$ is not equal to 1. ($\sin^n x + \cos^n x \neq 1$) if $n > 2$. Since the necessary condition $\sin^n x + \cos^n x = 1$, is not satisfied if $n > 2$, the equation $c^n = a^n + b^n$ has no solutions if $n > 2$.
Therefore $c^n = a^n + b^n$ has solutions only if $n = 2$ and does not have solutions if $n > 2$.
The proof is complete.

Conclusion

Fermat's last theorem has been proved in this paper. Perhaps, the proof in this paper is the proof that Fermat wished there were enough margin for it in his paper

Adonten

$a = c\sin\theta$ $b = c\cos\theta$ $c^n = a^n + b^n$ $c^n = (c\sin\theta)^n +$ $(c\cos\theta)^n$ $c^n = c^n\sin^n\theta +$ $c^n\cos^n\theta$ $c^n = .c^n(\sin^n\theta +$ $\cos^n\theta)$. Equation (5) is true only if $\sin^n\theta + \cos^n\theta = 1$ For (5) to be true $\sin^n\theta +$ $\cos^n\theta = 1$. If $n = 2$, $c^2 = c^2(\sin^2\theta +$ $\cos^2\theta)$ is true since $\sin^2\theta + \cos^2\theta = 1$ and therefore, equations (5) and (3) hold. **Proof for** $n > 2$ **bycontradiction** If $n = 2$, $\sin^n x + \cos^n x = 1$ becomes $\sin^2 x + \cos^2 x = 1$, which is true. If $n > 2$, and oneassumes that $\sin^n x + \cos^n x = 1$, then $\sin^n x + \cos^n x$ $= \sin^2 x + \cos^2 x$ (B) since $\sin^2 x + \cos^2 x = 1$ From (B), and equating the exponents, $n > 2 = 2$ is false, since an integer greater than 2 cannot be equal to 2). Hence ,the assumption that $\sin^n x + \cos^n x = 1$, if $n > 2$, is false. $\therefore \sin^n x + \cos^n x \neq 1$ if $n > 2$.Since the necessary condition is not satisfied if $n > 2$, $c^n = a^n + b^n$ has no solutions if $n > 2$ QED	**Proof: Version 1** **in the Margin** Fermat was truthful. He could have squeezed the proof into the page margin. If Fermat were reincarnated, he would be pleased.

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Proof: Version 2 (Using ratios)

Confirmation of Version 1 Proof

Given: $c^n = a^n + b^n$ (n an integer; a, b, and c are relatively prime positive integers)

Required: To prove that $c^n = a^n + b^n$ does not hold if $n > 2$

Plan: One will first show that if $n = 2$, $c^n = a^n + b^n$ holds, followed by showing that if $n > 2$ (n an integer), $c^n = a^n + b^n$ does not hold. One will begin by applying ratio terms.

$c^n = a^n + b^n$ (1) (Given)

$a^n + b^n = c^n$ (2) (rewriting)

$a^n = rc^n$ (3) (r is a ratio term)

$b^n = sc^n$ (4) (s is a ratio term) ($r + s = 1$)

$rc^n + sc^n = c^n$ (5) (substitute for a^n and b^n from (3) and (4)

$c^n(r + s) = c^n$ (6)

Now, by the substitution axiom, since $r + s = 1$, $r + s$ can be replaced by any quantity = 1. One can therefore replace $r + s$ by $\sin^2 x + \cos^2 x$, since $\sin^2 x + \cos^2 x = 1$. Then equation (6) becomes

$c^n(\sin^2 x + \cos^2 x) = c^n$ (7)

If $n = 2$, (7) becomes $c^2(\sin^2 x + \cos^2 x) = c^2$ (8)

$c^2 = c^2(\sin^2 x + \cos^2 x)$ (8) (rewriting)

Equation (8) is true since $\sin^2 x + \cos^2 x = 1$. Consequently, equations (8) and (1) hold. Therefore, if $n = 2$, $c^n = a^n + b^n$ has solutions.

Generalizing equation (7), one obtains $c^n(\sin^n x + \cos^n x) = c^n$ (9) in which the necessary condition for (9) to hold is $\sin^n x + \cos^n x = 1$.

One will next show that if $n > 2$, the condition $\sin^n \theta + \cos^n \theta = 1$ is never satisfied and consequently $c^n = a^n + b^n$ and $c^n = c^n \sin^n \theta + c^n \cos^n \theta$ are never satisfied.

Example on ratio terms
If $4 + 8 = 12$, and the ratio terms are $\frac{1}{3}$ and $\frac{2}{3}$, then
$4 = \frac{1}{3} \bullet 12$,
$8 = \frac{2}{3} \bullet 12$; and the sum of the ratio terms is
$\frac{1}{3} + \frac{2}{3} = 1$

Other equivalent identities
Note: "magic "number, 2.
$\sec^2 x - \tan^2 x = 1$
$\csc^2 x - \cot^2 x = 1$
$\cos 2x + 2\sin^2 x = 1$
$2\cos^2 x - \cos 2x = 1$

Elimination of the ratio terms r and s
The author was impressed and gratified by the substitution axiom which permitted the introduction of the much needed necessary condition $\sin^n x + \cos^n x = 1$ in Versions 1 of the proof.

Proof for $n > 2$ by contradiction

If $n = 2$, $\sin^n x + \cos^n x = 1$ becomes $\sin^2 x + \cos^2 x = 1$, which is true. If $n > 2$, and one assumes that $\sin^n x + \cos^n x = 1$, then $\sin^n x + \cos^n x = \sin^2 x + \cos^2 x$ (B) (By the transitive equality property, since $\sin^2 x + \cos^2 x = 1$).

From (B), and equating the exponents, $n > 2 = 2$ is false, since an integer greater than 2 cannot be equal to 2. Hence, the assumption that $\sin^n x + \cos^n x = 1$, if $n > 2$, is false. Therefore, $\sin^n x + \cos^n x$ is not equal to 1. ($\sin^n x + \cos^n x \neq 1$) if $n > 2$. Since the necessary condition $\sin^n x + \cos^n x = 1$, is not satisfied if $n > 2$, the equation $c^n = a^n + b^n$ has no solutions if $n > 2$.

Therefore $c^n = a^n + b^n$ has solutions only if $n = 2$ and does not have solutions if $n > 2$.
The proof is complete.

Conclusion: Fermat's last theorem has been proved in this paper. . **Adonten**

Proof: Version 3

This version begins without reference to any geometric figure or ratio terms.

$c^n = a^n + b^n$

$\frac{a^n + b^n}{c^n} = \frac{c^n}{c^n}$

$\boxed{\frac{a^n + b^n}{c^n} = 1}$ <-necessary condition for $c^n = a^n + b^n$ to be true, or to have solutions.

IIf $n = 2$, $\frac{a^n + b^n}{c^n} = \frac{a^2 + b^2}{c^2} = 1$ is true for a Pythagorean triple a, b, c, (For the integers $3, 4, 5$, $\frac{a^2 + b^2}{c^2} = 1$ $(3^2 + 4^2)/5^2 = 25/25 = 1)$.

Thus, if $n = 2$, the necessary condition , $(a^n + b^n)/c^n = 1$, is satisfied and $c^n = a^n + b^n$ is true or has solutions

One will next show that if $n > 2$, the necessary condition , $\frac{a^n + b^n}{c^n} = 1$, is never satisfied.

Proof for $n > 2$ by contradiction

If $n > 2$, and one assumes that $\frac{a^n + b^n}{c^n} = 1$, then $\frac{a^n + b^n}{c^n} = \frac{a^2 + b^2}{c^2}$ (A)

(By the transitive equality property, since $\frac{a^2 + b^2}{c^2} = 1$).

From (A) and equating the exponents, $n > 2 = 2$ is false, since an integer greater than 2 cannot be equal to 2. Hence, the assumption that $\frac{a^n + b^n}{c^n} = 1$, if $n > 2$, is false.

Therefore, $\frac{a^n + b^n}{c^n}$ is not equal to 1. $(\frac{a^n + b^n}{c^n} \neq 1)$ if $n > 2$. Since the necessary condition $\frac{a^n + b^n}{c^n} = 1$, is not satisfied if $n > 2$, the equation $c^n = a^n + b^n$ has no solutions if $n > 2$.

Therefore $c^n = a^n + b^n$ has solutions only if $n = 2$, and does not have solutions if $n > 2$.
The proof is complete

Question for a mathematics final exam for the 2016 Fall semester.
Bonus Question: Prove Fermat's Last Theorem.

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Discussion

About Version 2 of the proof (Using ratios)

From equation (6) in Version 2 proof, one could replace $r + s$ by each of the equivalent identities as shown below. Note that $r + s = 1$; $c^n(r + s) = c^n$ (6)

Alternatives for equation (6) in Version 2 proof	**Other equivalent identities** Note the number, 2 in various positions.
$c^n(\sec^2 x - \tan^2 x) = c^n$	$\sec^2 x - \tan^2 x = 1$
$c^n(\csc^2 x - \cot^2 x) = c^n$	$\csc^2 x - \cot^2 x = 1$
$c^n(\cos 2x + 2\sin^2 x) = c^n$	$\cos 2x + 2\sin^2 x = 1$
$c^n(2\cos^2 x - \cos 2x) = c^n$	$2\cos^2 x - \cos 2x = 1$

Comparison of necessary conditions for $c^n = a^n + b^n$ to be true.

Version 1	**Version 2**	**Version 3** (a,b,c of rt. Δ)
$\boxed{\sin^n \theta + \cos^n \theta = 1}$ **=**	$\boxed{\sin^n x + \cos^n x = 1}$ **=**	$\boxed{\dfrac{a^n + b^n}{c^n} = 1}$ **;** $\dfrac{a^2 + b^2}{c^2} = 1$
$\sin^2 \theta + \cos^2 \theta = 1$	$\sin^2 \theta + \cos^2 \theta = 1$	

Overall Conclusion

Three versions of the proof of Fermat's last theorem have been presented in this paper. In the first version of the proof, one began with reference to a right triangle; but in the second version of the proof, the proof construction began with ratio terms without reference to a triangle. The third version began without any reference to a geometric figure or ratio terms. For the second version, the ratio terms were later on "miraculously" and pleasantly eliminated from the equations. For the first two versions, the necessary condition for the relevant equations involved to be true is that $\sin^n x + \cos^n x = 1$ (or $\sin^n \theta + \cos^n \theta = 1$). Thus, if $c^n = c^n(\sin^n x + \cos^n x)$ and $c^n = a^n + b^n$ are to hold, $\sin^n x + \cos^n x = 1$ or $\sin^n \theta + \cos^n \theta = 1$ must be satisfied. For the third version $(a^n + b^n)/c^n = 1$ must be satisfied. First, the author determined, why the equation, $c^n = a^n + b^n$ is true if $n = 2$. These necessary conditions are satisfied only if $n = 2$, to produce $\sin^2 x + \cos^2 x = 1$ or $(a^2 + b^2)/c^2 = 1$. If $n > 2$, the necessary $\sin^n x + \cos^n x = 1$ or $\sin^n \theta + \cos^n \theta = 1$ or $(a^n + b^n)/c^n = 1$ is never satisfied. Therefore, $c^n = a^n + b^n$ has solutions only if $n = 2$, and does not have solutions if $n > 2$.

One should note above that versions 2 and 3 proofs confirmed the proof in version 1.

Perhaps, the proof in this paper is the proof that Fermat wished there were enough margin for it in his paper.

PS

Application of the approach used in proving Fermat's last theorem

If a 3-ton non-portable machine functions properly in environment number 2 and one wants to determine if the same machine will function properly in environments 3, 4, and, 5 up to 1000 different environments, one option is to dismantle the machine in environment number 2, and reassemble it in each of the new environments, up 1000 environments and test the machine. Another option, the better option, is to determine the necessary conditions which allow the machine to function properly in environment 2. If the necessary conditions are not available in environments 3, 4, 5, etc, the machine will not function properly in the new environments, and no efforts should be wasted in carrying the machine to the environments and be tested.

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