Key Concepts: Fundamentals of Logic and Techniques for Mathematical Proofs*

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Additional Readings:

- *Analysis with an Introduction to Proof 3rd ed.* by Steven L Ray, Prentice Hall.

1 Some Elements of Logic

When proving mathematical theorems, we are attempting to demonstrating the truth of certain statements. In doing so, we make two main assumptions: 1) statement can be either true or false (Law of the excluded middle); 2) no statement is both true and false

Definition 1 *A* ***statement*** *is any logical expression that can be said to be true or false.*

Example 1 Which are statements?

1. Bob is human.
2. $2 + 2 = 5$
3. $x \geq 13$
4. If a number is even, then it is divisible by two.

*These notes are based in large part on three sources: *Proofs and Fundamentals. A First Course in Abstract Mathematics*by Ethan D. Bloch, Birkhäuser; *Analysis with an Introduction to Proof 3rd ed.* by Steven L Ray, Prentice Hall; and Lecture notes by Professor Lawrence Fearnley, Department of Mathematics, Brigham Young University.

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Let P and Q be statements

Definition 2 *The* ***conjunction*** *of P and Q, denoted $P \wedge Q$, is a statement that is true if both P and Q are true, and is false otherwise. We read $P \wedge Q$ as "P and Q"*

*This definition can be represented by the "**truth table**":*

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

This truth table shows whether the new statement is true or false for each possible combination of the truth or falsity of each P and Q.

Example 2 Let P="it is raining today" and let Q="it is cold today". The statement $P \wedge Q$ is "it is raining today and it is cold today"

Definition 3 *The* ***disjunction*** *of P and Q, denoted $P \vee Q$, is a statement that is true if either P is true or Q is true or both are true, and is false otherwise. We read $P \vee Q$ as "P or Q".*

The following truth table presents this definition:

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

The truth of the statement $P \vee Q$ means that at least one of P or Q is true.

Example 3 Let P="my car is red" and let Q="it will rain today". The statement $P \vee Q$ is "my car is red or it will rain today"

Definition 4 *The* ***negation*** *of P, denoted $\neg P$, is a statement that is true if P is false, and is false if P is true. We read $\neg P$ as "it is not the case that P" or "not P".*

The following truth table presents this definition:

P	$\neg P$
T	F
F	T

The negation of a statement $\neg A$ is a logical statement that minimally violates A.

Example 4 Negate the following statements (beware, it's not always as easy as it looks!):

1. Someone in this room is wearing a wristwatch.
2. Everyone in this room is wearing a wristwatch.
3. Either everyone in this room is wearing a wristwatch or no one is.
4. Bill and Bob are both wearing wristwatches.
5. Every element of set U is a number that is both rational and greater than 6.

Remark 1 Negation with "$\forall$" statements (here's where minimality comes into play): As in the above examples, the negation of any "$\forall$" statement will usually include an "$\exists$" in it. This produces the minimal violation of the statement and is thusly most general. For example, the statement "No one in this room is wearing a wristwatch," violates the second example above, but it is too restrictive in the sense that there are other statements which also produce a violation but are not covered by this statement. For example, "Ted is not wearing a wristwatch but Susan is," violates the original statement above, but it also violates the statement "No one in this room is wearing a wristwatch." Hence, the appropriate negation of the second example above is "$\exists$ someone in this classroom who is not wearing a wristwatch."

Remark 2 Negation with "and" and "or" statements: Note that many statements are made up of two or more simpler statements thrown together using an "and" or an "or" statement. In order to negate the statement "A and B are both true," we need only show that either A or B is false. On the other hand, in order to negate "either A or B are true," we must show that both are simultaneously false. Thus, the negation of an "and" statement will usually involve an "or" statement, and vice versa.

Definition 5 *The* ***conditional from*** P ***to*** Q*, denoted $P \to Q$, is a statement that is true if it is never the case that P is true and Q is false. We read $P \to Q$ as "if P then Q". P is called the "antecedent" and Q is called the "consequent".*

The following truth table presents this definition:

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

Example 5 Let P=“it rains today” and let Q=“I will see a movie this evening”. The statement $P \to Q$ is “if it rains today, then I will see a movie this evening”

We can also say:

If P,Q;
Q if P; ”I will see a movie this evening, if it rains today”
P only if Q;
Q provided that P;
Assuming that P, then Q;
Q given that P;
P is sufficient for Q;
Q is necessary for P.

Definition 6 *The **biconditional** from P to Q, denoted $P \leftrightarrow Q$, is a statement that is true if P and Q are both true or both false, and is false otherwise. We read $P \leftrightarrow Q$ as “P if and only if Q” or “P iff Q”.*

The following truth table presents this definition:

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Example 6 Let P=“I will go for a walk” and let Q=“Fred will join me”. The statement $P \leftrightarrow Q$ is “I will go for a walk if and only if Fred will join me”.

Sometimes we say ”P is necessary and sufficient for Q”. Note that $P \leftrightarrow Q$ is equivalent to $Q \leftrightarrow P$.

Example 7 Consider three statements: P, Q and R. We can form $P \vee (Q \to \neg R)$. Note the standard convention that negation $\neg$ takes precedence over the other four operations. The truth table is

P	Q	R	$\neg R$	$Q \to \neg R$	$P \vee (Q \to \neg R)$
T	T	T			
T	T	F			
T	F	T			
T	F	F			
F	T	T			
F	T	F			
F	F	T			
F	F	F			

Another way to write this table down:

P	Q	R	P	$\vee$	$(Q$	$\to$	$\neg R)$
T	T	T	T		T		
T	T	F	T		T		
T	F	T	T		F		
T	F	F	T		F		
F	T	T	F		T		
F	T	F	F		T		
F	F	T	F		F		
F	F	F	F		F		

Definition 7 *A* ***tautology*** *is a statement which is true under any circumstances, or necessarily true.*

Definition 8 *A* ***contradiction*** *is a statement which is necessarily false and is denoted by* $\to\leftarrow$.

Definition 9 *We say that* P ***implies*** Q *if the statement* $P \to Q$ *is a tautology. Notation: "*$P \Rightarrow Q$*"*

Example 8 Use the following truth table to prove that the statement $A = (\neg(P \to Q))$ implies the statement $B = (P \vee Q)$, or that $A \to B$ is a tautology: $A \to B$ is always true under any possible circumstance or contingency.

P	Q	$(\neg$	$(P$	$\to$	$Q)$	$\to$	$(P$	$\vee$	$Q)$
T	T		T		T		T		T
T	F		T		F		T		F
F	T		F		T		F		T
F	F		F		F		F		F

Example 9 (There is a hurricane.) $\Rightarrow$ (There are wind gusts in excess of 70mph.)

Definition 10 *Statement A* ***is implied by*** *B if A's negation implies B's negation. This relation is denoted by $A \Leftarrow B$, and it may be alternatively said that A is* ***necessary*** *for B.*

Definition 11 *It is said that $A \Leftarrow B$ is the* ***converse*** *of $A \Rightarrow B$.*

Definition 12 *It is said that $\neg A \Rightarrow \neg B$ is the* ***inverse*** *of $A \Rightarrow B$.*

Definition 13 *We say that P and Q are* ***equivalent*** *if the statement $P \leftrightarrow Q$ is a tautology. Notation: "$P \Leftrightarrow Q$". If P and Q are equivalent, then P is said to be <u>both necessary and sufficient</u> for Q. This relation is denoted by $P \Leftrightarrow Q$, and it can be alternatively said that P is true* ***if and only if*** *Q is true. This is commonly abbreviated by saying P iff Q.*

Exercise 1 Use the following truth table to show that the statement $A = P \rightarrow Q$ is true if and only if $B = \neg Q \rightarrow \neg P$ is true

P	Q	$(P$	$\rightarrow$	$Q)$	$\leftrightarrow$	$(\neg Q$	$\rightarrow$	$\neg P)$
T	T	T		T				
T	F	T		F				
F	T	F		T				
F	F	F		F				

Example 10 Are the following statement pairs equivalent?

1. A:$(x \in \mathbb{R})$ B:$(x^2 \geq 0)$
2. A:$(x \in \mathbb{Q})$ B:$(x = \frac{a}{b}$ for some $a, b \in \mathbb{Z})$
3. A:(Larry played in the NBA all–star game.) B:(Larry is a professional basketball player.)
4. A: ($f(x)$ is differentiable.) B:($f(x)$ is continuous.)

Exercise 2 Using a truth table, show that $A \vee \neg A$ is a tautology, $A \wedge \neg A$ is a contradiction and that A is a tautology iff $\neg A$ is a contradiction.

Exercise 3 Complete the truth table to find out whether the following statement is a tautology, a contradiction, or neither: $((P \wedge Q) \rightarrow R) \rightarrow (P \rightarrow (Q \rightarrow R))$.

P	Q	R	$((P$	$\wedge$	$Q)$	$\rightarrow$	$R)$	$\rightarrow$	$(P$	$\rightarrow$	$(Q$	$\rightarrow$	$R))$
T	T	T	T		T		T		T		T		T
T	T	F	T		T		F		T		T		F
T	F	T	T		F		T		T		F		T
T	F	F	T		F		F		T		F		F
F	T	T	F		T		T		F		T		T
F	T	F	F		T		F		F		T		F
F	F	T	F		F		T		F		F		T
F	F	F	F		F		F		F		F		F

Exercise 4 Complete the truth table to find out whether the following statement is a tautology, a contradiction, or neither: $(Q \rightarrow (P \wedge \neg Q)) \wedge Q$.

P	Q	$(Q$	$\rightarrow$	$(P$	$\wedge$	$\neg Q))$	$\wedge$	Q
T	T	T		T				T
T	T	T		T				T
T	F	F		T				F
T	F	F		T				F
F	T	T		F				T
F	T	T		F				T
F	F	F		F				F
F	F	F		F				F

Exercise 5 Negate the following statements:

1. The set of real numbers is finite.
2. Seven is prime or five is even.
3. If today is not Monday, then it is hot.
4. If it rains today, then the roof will leak.
5. If a matrix is non–singular then it is invertible.
6. If a function is differentiable then it is continuous.

Exercise 6 Negate the statement "All members of set $B \subset \mathbb{R}$ are rational," by using the definition of $\mathbb{Q}$, the set of rational numbers:

$$x \in \mathbb{Q} \quad \text{iff} \quad x = \frac{a}{b}, \quad \text{for} \quad a, b \in \mathbb{Z}.$$

Exercise 7 Negate the statement "the columns of matrix M are linearly independent."

Exercise 8 An **orthogonal matrix** is one in which the columns are all orthogonal to one another. That is, any two columns will have an inner product of zero. Negate the statement "M is an orthogonal matrix."

Exercise 9 An **orthonormal matrix** is an orthogonal matrix with the additional property that its columns all have a norm of 1. Negate the statement "M is an orthonormal matrix."

2 Techniques for Proving and Disproving statements

2.1 Disproving through Counterexamples

Recall that in order for a statement to be true, it must be true under *every possible circumstance*. Therefore, in order to demonstrate that a statement is false, we need only find one circumstance under which the statement is violated. This is called a **counterexample**.

Example 11 Ancient belief of the Pythagoreans: All real numbers can be expressed as the ratio of two integers. **COUNTEREXAMPLE:** $\sqrt{2}$ (proof in class).

Exercise 10 Find counterexamples for the following statements:

1. $\forall n \in \mathbb{N}$, the function $p(n) = n^2 + n + 17$ returns a prime number.
2. Every prime number is an odd number.
3. No rational number satisfies the equation $x^3 + (x-1)^2 = x^2 + 1$.

2.2 Proving Techniques

Definition 14 *A **direct proof** of a statement $A \Rightarrow B$ involves the construction of a string of statements such that*

$$A \Rightarrow R_1, R_1 \Rightarrow R_2, ..., R_n \Rightarrow B.$$

Example 12 Let a and b be integers. We say a divides b if there is some integer q such that $aq = b$. If a divides b, we write $a \mid b$, and we say that a is a factor of b, and that b is divisible by a.

Theorem: Any integer divides zero

Direct Proof: (done in class)

Theorem: Let a,b, and c be integers. If $a \mid b$ and $b \mid c$, then $a \mid c$.

Direct Proof: (done in class)

Definition 15 *A **contrapositive proof** of the statement $A \Rightarrow B$, is a direct proof of the inverse of its converse:*

$$\neg B \Rightarrow \neg A$$

Example 13 Let n be an integer. If n is odd, then $7n$ is odd.

Proof by Contrapositive: (done in class)

Example 14 If f is continuous $\int_0^1 f(x)dx \neq 0$, then $\exists x \in [0,1]$ such that $f(x) \neq 0$.

Proof by Contrapositive: (done in class)

Definition 16 *A **contradiction proof:** of the statement $A \Rightarrow B$ involves proving that*

$$\neg(A \Rightarrow B) \Rightarrow \rightarrow\leftarrow .$$

Example 15 **Claim:** If x is rational and y is not rational, then $x + y$ is not rational.

Proof by Contradiction: (done in class)

Example 16 **Claim:** If $x \in \mathbb{R}_{++}$ then $\frac{1}{x} > 0$.

Proof by Contradiction: (done in class)

Definition 17 *An **induction proof** is useful for proving statements with countably many cases. This method takes advantage of the well–ordering property of $\mathbb{N}$, and takes the following form:*

1. *Number all instances of the statement.*
2. *Show that it is true in the first instance.*
3. *Assume that it is true in the k^{th} instance and show that from there it follows that the statement must also be true in the $(k+1)^{st}$ instance as well.*

Example 17 **Claim:** $\forall n \in \mathbb{N}, \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

Proof by Induction: (done in class)

2.3 Cases, and If and Only If

Sometimes a statement $P \Rightarrow Q$ is difficult to prove all at once. In such cases, it is often useful to break up the proof into a number of cases (and subcases, and etc.). We use this method if we can write P in the form of $A \vee B$. Note here that statements A and B must be *exhaustive*; that is, they must cover all possible cases. It can be shown that $(A \vee B) \Rightarrow Q$ is equivalent to $(A \Rightarrow Q) \vee (B \Rightarrow Q)$ (verify this using a truth table). So we can prove that the statements $A \Rightarrow Q$ and $B \Rightarrow Q$ are true, and thus, $(A \vee B) \rightarrow Q$ is true as well.

Example 18 If a function $f(x)$ is differentiable, then it must be continuous.

2-Case Proof by Contradiction: (done in class)

Other types of theorems involve proving equivalence of two or more statements. In order to prove that $A \Leftrightarrow B$, we must prove both $A \Rightarrow B$ *and* $A \Leftarrow B$.

Example 19 Let m and n be integers. Then mn is odd iff both m and n are odd.

Proof: (done in class)

In other cases, we may have need to prove equivalence of more than just two statements, say A, B and C. At first glance one might think that in such cases it is necessary to prove each and every pairwise equivalence, but this is actually not needed. All that needs to be done is to prove that the statements follow a logical circuit:

$$A \Rightarrow B \Rightarrow C \Rightarrow A.$$

2.4 One Final Example:

Can you identify the type of proof employed in the following quote from *Pirates of the Caribbean: The Curse of the Black Pearl*?

Marine 1: "The Black Pearl doesn't exist!"

Marine 2: "Yes it does; I've seen it!"

Marine 1: “You’ve seen a ship with black sails, that’s crewed by the damned and captained by a man so evil that hell itself spat him back out?”
Marine 2: “No, But I have seen a ship with black sails.”
Marine 1: “Oh! And no ship not crewed by the damned and captained by a man so evil, hell itself spat him back out, could possibly have black sails, therefore couldn’t possibly be any other ship than The Black Pearl? Is that what you’re saying?”
Marine 2: No.