The Astrodynamics and Mechanics of Orbital Spaceflight

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1 Introduction to Rocketry

Before getting into the details of orbital mechanics, we must understand the fundamentals of rockets and the principles behind spaceflight.

1.1 Rocket Equation

The most important concept to grasp first is the rocket equation:

$$\Delta v = v_e \ln \frac{m_0}{m_1}$$

Derived by the Russian physicist Konstantin Tsiolkovsky in 1903, the rocket equation states that the change in velocity a spacecraft can produce is equal to: the product of it's exhaust velocity, and linear logarithm of the ratio of it's total mass to dry mass. Here 'exhaust velocity' means the average velocity of the rocket exhaust, and the ratio can be simplified as the 'mass ratio'.

This equation governs a critical quantity in spaceflight; **delta-v**, the change in velocity that a spacecraft can produce.

1.2 Exhaust Velocity

The exhaust velocity can be defined as the velocity at which propellant gases escape the rocket engine of the spacecraft. In the rocket equation, **effective exhaust velocity** is taken. This means the practical velocity of propellant particles, taking into account factors like atmospheric retardation. Exhaust velocity in it's simplest form is represented as:

$$v_e = g_0 I_{sp}$$

Where g_0 is the gravitation on the surface of the Earth (m/s), and I_{sp} is the specific impulse of the engine measured in seconds. This gives the exhaust velocity the unit of m/s.

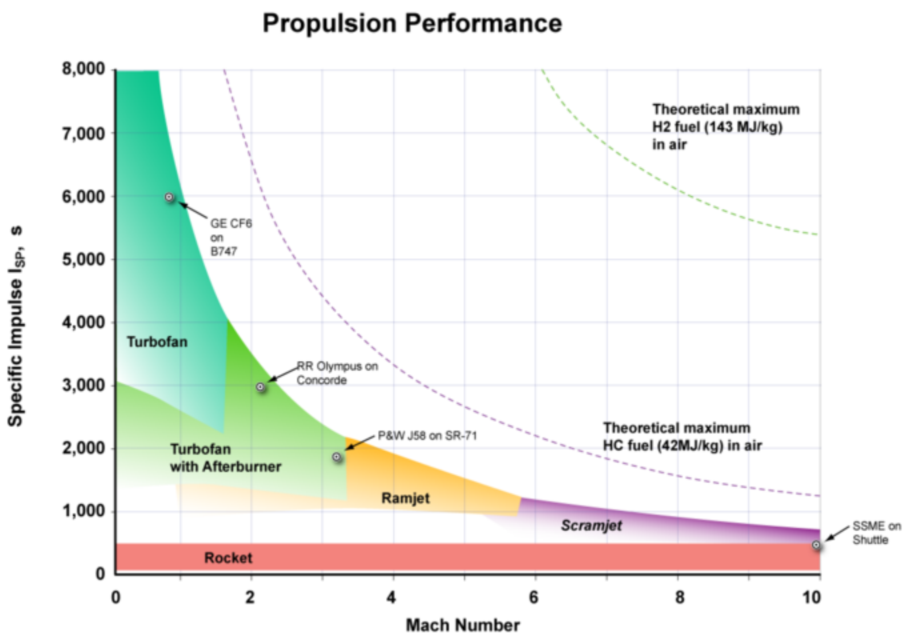
1.3 Specific Impulse

The specific impulse is simply a measure of the efficiency of a rocket engine. Phyically, it is the impulse delivered per unit of propellant consumed, and has the same dimensions as thrust produced per propellant flow rate. If mass is used as the unit of propellant, then I_{sp} has the unit m/s. However if propellant is expressed as weight in $Newtons$, the I_{sp} has the unit *seconds*. This latter definition of specific impulese is used, because *seconds* are common to both measurement systems, imperial and metric.

Put together, the thrust produced by a rocket is:

$$F_{thrust} = I_{sp} \cdot \dot{m}$$

The product of the specific impulse and the mass of the craft. We can conclude that a higher specific impulse generates more thrust per unit of propellant, and hence a more efficient rocket.



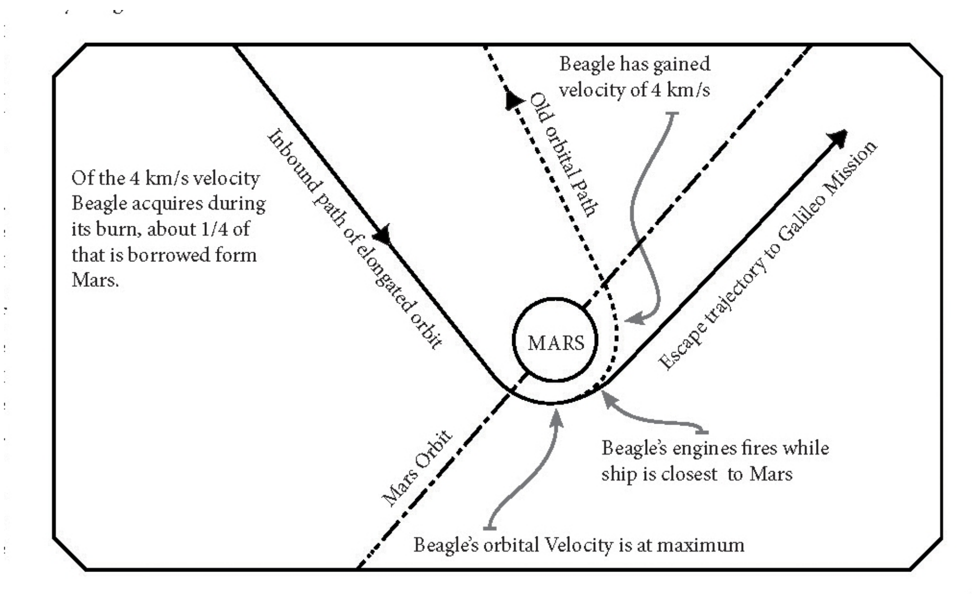
1.4 Thrust-to-Weight Ratio

The thrust-to-weight ratio (TWR) of a rocket is defined as the dimensionless ratio of the thrust of the craft to it's weight on the surface of the Earth. Mathematically:

$$TWR = \frac{T}{W}$$

1.5 Oberth Effect

When a spacecraft travels close to a large body, at a very high speed due to the body's gravitational well, an impulse by the craft's engine produces more thrust than in normal situations. This is attributed to the Oberth effect. Simply described, the faster a spacecraft travels, the more kinetic energy it has in it's propellant. Hence the propellant uses this additional energy in addition to it's own chemical energy, to propel a rocket further and faster. Hence, engine burns are far more efficient closer to a planetary body, and the Oberth effect can be exploited to gain more Δv from a burn.



Example of the Oberth Effect. The Beagle spacecraft performed it's burn at it's closest, fastest point relative to Mars.

2 Orbital Mechanics

2.1 Orbital Velocity

An 'orbit' is the path of an object in space around another body. In order for a spacecraft to be 'orbital' or 'in orbit', it must travel fast enough laterally for the surface of the Earth to literally fall away from underneath it.

Depending on the velocity of the craft, it may be on a parabolic/hyperbolic escape trajectory, or a stable elliptical orbit around a planet. To determine the velocity required at a certain altitude to maintain a circular orbit, we use the equation:

$$v = \sqrt{\frac{GM}{r}}$$

where G is the gravitational constant, M is the mass of the planet, and r is the radius from the centre of the planet. This equation is of course an approximation that ignores the spacecrafts own mass, as it is negligible compared to that of the planet.

Now in order for a spacecraft to escape the planet's gravity, and hence no longer be on a closed elliptical orbit, the following inequality must be satisfied:

$$v \geq \sqrt{\frac{2GM}{r}}$$

The majority of orbits in the real world are ellipses, not perfect circles. Hence the lowest point in an orbit is called the 'periapsis', and the highest point is called the 'apoapsis'. In a perfectly circular orbit, the periapse and apoapse are the same.

In order to better understand these equations, let's see their application in the real world. The International Space Station orbits at a mean altitude of approximately $400Km$ above the surface of the Earth. The equatorial radius of the Earth is $6371Km$, hence the radius from the centre of the Earth to the ISS is $6,771,000m$. The mass of the Earth is $6.972 \times 10^{24}kg$ and the gravitational constant is approximately 6.67384×10^{-11}. Hence using the circular orbit equation, we get:

$$v = \sqrt{\frac{6.67384 \times 10^{-11} \times 6.972 \times 10^{24}}{6771000}} = 7672m/s$$

The true velocity of the ISS is reported as $7677m/s$, within $\pm 10m/s$ of our calculation.

To calculate the escape velocity of the Earth at $400km$ above the surface, we use the second equation and get:

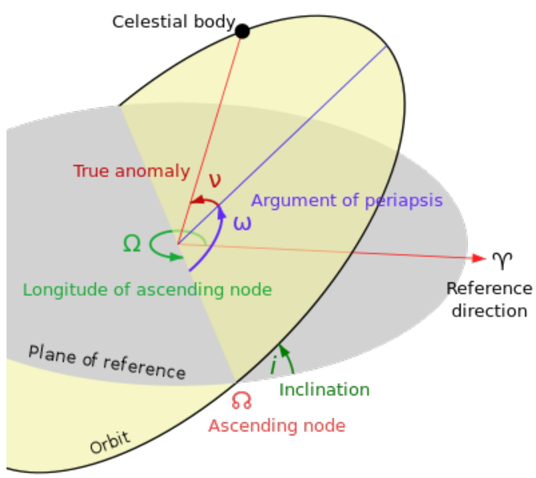
$$v = \sqrt{\frac{2 \times 6.67384 \times 10^{-11} \times 6.972 \times 10^{24}}{6771000}} = 10850m/s$$

We can thus use simple mathematics to calculate the velocity of a spacecraft in different orbits.

2.2 Orbital Characteristics

In order to accurately describe an orbital trajectory, far more information is required in addition to velocity. The parameters required to identify a unique orbit are known as the 'orbitale elements', and the six elements are:

1. Eccentricity - Elongation of the orbit compared to a circle. Has the value 1 for a perfectly circular orbit.
2. Semi-major Axis - In circular orbits, the distance between the centre of the spacecraft and the centre of the body it orbits. In eccentric orbits, the sum of the apoapsis and periapsis (relative to the centre of the Earth) divided by two; simply, the average radius of the orbit relative to the centre of the planet.
3. Inclination - Vertical tilt of the orbit, relative to the reference plane. The point of intersection between the reference plane and the orbital plane is called the Ascending Node (AN). The inclination is numerically the angle between the two planes at the AN.
4. Longtitude of the Ascending Node - The horizontal location of the AN with respect to the reference plane.
5. Argument of Periapsis - Angle between AN and periapsis, defining the orientation of the ellipse.
6. Mean anomoly at epoch - The position of the spacecraft on the orbital ellipse at a given time.



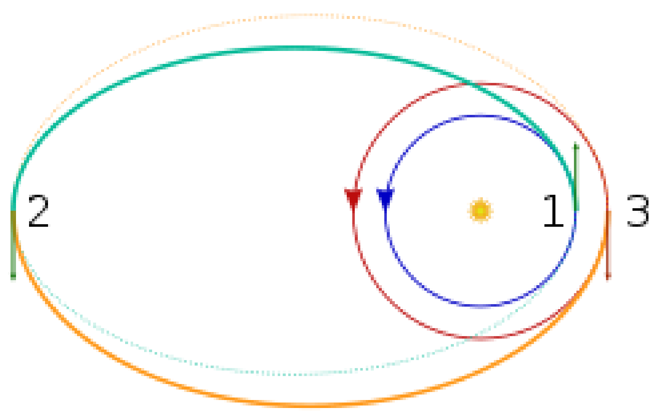
3 Orbital Manouvers

3.1 Hohmann Transfer

A Hohmann transfer is the simplest propulsive manouver, used to transfer between two different circular orbits around the same body. At it's most basic, it involves two impulsive thrusts. The first burn raises/lowers the apoapsis/periapsis of the orbit to the desired altitude. The second burn occurs at the desired altitude, and circularises the orbit by bringing the other periapsis/apoapsis to the desired alititude. Hence, a circular orbit at a different altitude is obtained.

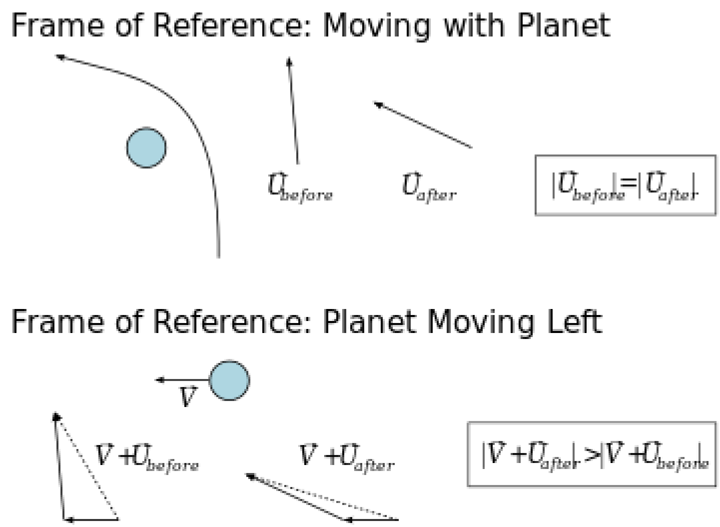
3.2 Bi-elliptic Transfer

A bi-elliptic transfer serves the same purpose as a Hohmann transfer, but can in some cases reduce the fuel requirement to change orbits. An initial burn is applied to raise the apoapsis of the orbit and create a highly elliptical trajectory. At the tip of the ellipse, another burn is performed to raise/lower the periapsis to the desired altitude. Finally, at the desired altitude, a third and final burn circularises the orbit. Whilst these transfers take longer, they are more efficient for large changes in the orbit.



3.3 Gravity Assist

A gravity assist is the simplest form of manouver that spacecraft perform. The principle behind a gravity assist is the usage of the angular momentum and gravity of a planet to change the velocity of a spacecraft, without using propellant or thrust. This may at first seem like a violation of conservation of energy. However momentum is in fact conserved because whatever momentum the spacecraft gains, the planet loses. The effect on the planet's velocity is, however, negligible to the point of non-existence.

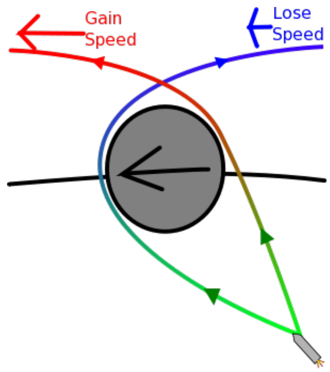


3.3.1 Acceleration

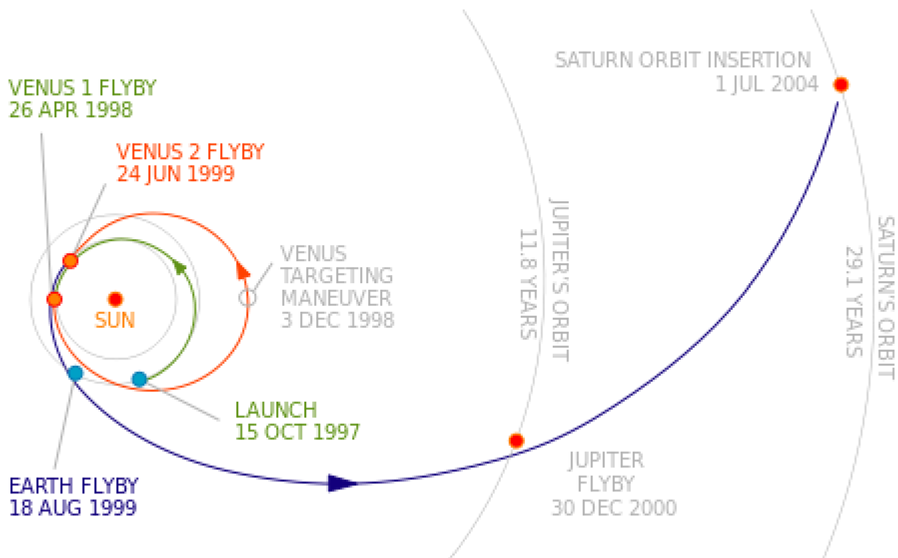
A gravity assist to increase the velocity of a spacecraft uses the gravity of a planet to pull the craft towards itself. The spacecraft hence travels on a hyperbolic escape trajectory relative to the planet. However, the planet iteself is moving at a very high speed around the Sun, to the order of tens of kilometres per second. Thus, the gravity of the planet 'pulls' the spacecraft towards it with a force proportional to the orbital velocity of the planet around the Sun. The craft is hence 'sling-shotted' on a hyperbolic trajectory away from the planet, with a net velocity greater than before the assist.

3.3.2 Deceleration

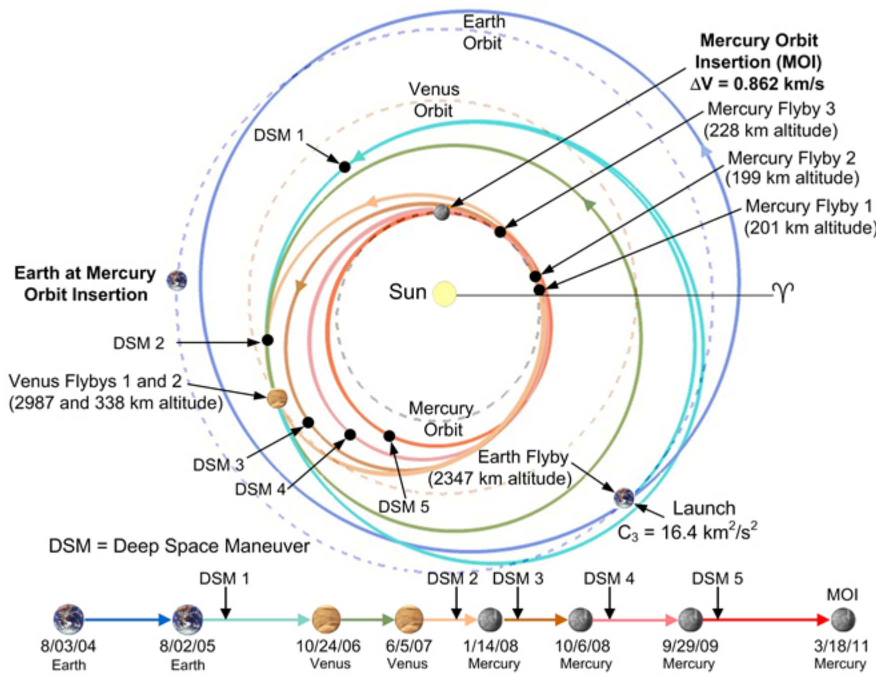
A decelerating gravity assist is essentially an inverted accelerating gravity assist. Here, the angular momentum of the planet is subtracted from that of the spacecraft, and the net velocity is less than before the assist.



A practical example of accelerating gravity assists is the Cassini Saturn Orbiter. Cassini used several flybys around Venus, Earth and Jupiter to increase it's speed and reach Saturn far quicker than by a normal trajectory:



Decelerating gravity assists were heavily used by the MESSENGER probe, to continually reduce it's velocity relative to the Sun and lower it to the orbit of Mercury:



References

[1] Feynman, R. The Feynman lectures on physics. (Pearson Addison Wesley, 2006).

[2] Susskind, L. & Hrabovsky, G. Classical mechanics.

[3] Braeunig.us,. Basics of Space Flight: Orbital Mechanics. (2015). at <http://www.braeunig.us/space/orbmech.htm>

[4] Misner, C., Thorne, K. & Wheeler, J. Gravitation.

[5] Susskind, L. & Hrabovsky, G. The theoretical minimum. (Basic Books, 2013).

[6] A. Konstantin Tsiolkovsky, 1857-1935. (General Editorial Board for Foreign Publications, Nauka Publishers, 1985).

[7] Turner, M. Rocket and spacecraft propulsion. (Springer, 2000).

[8] Labunsky, A., Papkov, O. & Sukhanov, K. Multiple gravity assist interplanetary trajectories. (Gordon and Breach Science Publishers, 1998).

[9] Prussing, J. & Conway, B. Orbital mechanics. (Oxford University Press, 1993).

[10] Practical Astrodynamics. Nature 238, 471-472 (1972).

[11] OpenCourseWare, M. Lecture Notes | Physics I | Physics | MIT OpenCourseWare. Ocw.mit.edu (2015). at <http://ocw.mit.edu/courses/physics/8-01-physics-i-fall-2003/lecture-notes/>