Chapter 2

Asymptotic Expansions

In this chapter, we define the order notation and asymptotic expansions. For additional discussion, see [3], and [12].

2.1 Order notation

The O and o order notation provides a precise mathematical formulation of ideas that correspond — roughly — to the 'same order of magnitude' and 'smaller order of magnitude.' We state the definitions for the asymptotic behavior of a real-valued function $f(x)$ as $x \to 0$, where x is a real parameter. With obvious modifications, similar definitions apply to asymptotic behavior in the limits $x \to 0^+$, $x \to x_0$, $x \to \infty$, to complex or integer parameters, and other cases. Also, by replacing $|\cdot|$ with a norm, we can define similiar concepts for functions taking values in a normed linear space.

Definition 2.1 Let $f, g : \mathbb{R} \setminus 0 \to \mathbb{R}$ be real functions. We say $f = O(g)$ as $x \to 0$ if there are constants C and $r > 0$ such that

$$|f(x)| \leq C|g(x)| \text{ whenever } 0 < |x| < r.$$

We say $f = o(g)$ as $x \to 0$ if for every $\delta > 0$ there is an $r > 0$ such that

$$|f(x)| \leq \delta|g(x)| \text{ whenever } 0 < |x| < r.$$

If $g \neq 0$, then $f = O(g)$ as $x \to 0$ if and only if f/g is bounded in a (punctured) neighborhood of 0, and $f = o(g)$ if and only if $f/g \to 0$ as $x \to 0$.

We also write $f \ll g$, or f is 'much less than' g, if $f = o(g)$, and $f \sim g$, or f is asymptotic to g, if $f/g \to 1$.

Example 2.2 A few simple examples are:

(a) $\sin 1/x = O(1)$ as $x \to 0$

(b) it is not true that $1 = O(\sin 1/x)$ as $x \to 0$, because $\sin 1/x$ vanishes is every neighborhood of $x = 0$;
(c) $x^3 = o(x^2)$ as $x \to 0$, and $x^2 = o(x^3)$ as $x \to \infty$;
(d) $x = o(\log x)$ as $x \to 0^+$, and $\log x = o(x)$ as $x \to \infty$;
(e) $\sin x \sim x$ as $x \to 0$;
(f) $e^{-1/x} = o(x^n)$ as $x \to 0^+$ for any $n \in \mathbb{N}$.

The o and O notations are not quantitative without estimates for the constants C, δ, and r appearing in the definitions.

2.2 Asymptotic expansions

An asymptotic expansion describes the asymptotic behavior of a function in terms of a sequence of gauge functions. The definition was introduced by Poincaré (1886), and it provides a solid mathematical foundation for the use of many divergent series.

Definition 2.3 A sequence of functions $\varphi_n : \mathbb{R} \setminus 0 \to \mathbb{R}$, where $n = 0, 1, 2, \ldots$, is an *asymptotic sequence* as $x \to 0$ if for each $n = 0, 1, 2, \ldots$ we have

$$\varphi_{n+1} = o\left(\varphi_n\right) \qquad \text{as } x \to 0.$$

We call φ_n a *gauge function.* If $\{\varphi_n\}$ is an asymptotic sequence and $f : \mathbb{R} \setminus 0 \to \mathbb{R}$ is a function, we write

$$f(x) \sim \sum_{n=0}^{\infty} a_n \varphi_n(x) \qquad \text{as } x \to 0 \tag{2.1}$$

if for each $N = 0, 1, 2, \ldots$ we have

$$f(x) - \sum_{n=0}^{N} a_n \varphi_n(x) = o(\varphi_N) \qquad \text{as } x \to 0.$$

We call (2.1) the asymptotic expansion of f with respect to $\{\varphi_n\}$ as $x \to 0$.

Example 2.4 The functions $\varphi_n(x) = x^n$ form an asymptotic sequence as $x \to 0^+$. Asymptotic expansions with respect to this sequence are called asymptotic power series, and they are discussed further below. The functions $\varphi_n(x) = x^{-n}$ form an asymptotic sequence as $x \to \infty$.

Example 2.5 The function $\log \sin x$ has an asymptotic expansion as $x \to 0^+$ with respect to the asymptotic sequence $\{\log x, x^2, x^4, \ldots\}$:

$$\log \sin x \sim \log x + \frac{1}{6} x^2 + \ldots \qquad \text{as } x \to 0^+.$$

If, as is usually the case, the gauge functions φ_n do not vanish in a punctured neighborhood of 0, then it follows from Definition 2.1 that

$$a_{N+1} = \lim_{x \to 0} \frac{f(x) - \sum_{n=0}^{N} a_n \varphi_n(x)}{\varphi_{N+1}}.$$

Thus, if a function has an expansion with respect to a given sequence of gauge functions, the expansion is unique. Different functions may have the same asymptotic expansion.

Example 2.6 For any constant $c \in \mathbb{R}$, we have

$$\frac{1}{1-x} + ce^{-1/x} \sim 1 + x + x^2 + \ldots + x^n + \ldots \qquad \text{as } x \to 0^+,$$

since $e^{-1/x} = o(x^n)$ as $x \to 0^+$ for every $n \in \mathbb{N}$.

Asymptotic expansions can be added, and — under natural conditions on the gauge functions — multiplied. The term-by-term integration of asymptotic expansions is valid, but differentiation may not be, because small, highly-oscillatory terms can become large when they are differentiated.

Example 2.7 Let

$$f(x) = \frac{1}{1-x} + e^{-1/x} \sin e^{1/x}.$$

Then

$$f(x) \sim 1 + x + x^2 + x^3 + \ldots \qquad \text{as } x \to 0^+,$$

but

$$f'(x) \sim -\frac{\cos e^{1/x}}{x^2} + 1 + 2x + 3x^2 + \ldots \qquad \text{as } x \to 0^+.$$

Term-by-term differentiation is valid under suitable assumptions that rule out the presence of small, highly oscillatory terms. For example, a convergent power series expansion of an analytic function can be differentiated term-by-term

2.2.1 *Asymptotic power series*

Asymptotic power series,

$$f(x) \sim \sum_{n=0}^{\infty} a_n x^n \qquad \text{as } x \to 0,$$

are among the most common and useful asymptotic expansions.

If f is a smooth (C^∞) function in a neighborhood of the origin, then Taylor's theorem implies that

$$\left| f(x) - \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!} x^n \right| \le C_{N+1} x^{N+1} \qquad \text{when } |x| \le r,$$

where

$$C_{N+1} = \sup_{|x| \le r} \frac{\left| f^{(N+1)}(x) \right|}{(N+1)!}.$$

It follows that f has the asymptotic power series expansion

$$f(x) \sim \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \qquad \text{as } x \to 0. \tag{2.2}$$

The asymptotic power series in (2.2) converges to f in a neighborhood of the origin if and only if f is analytic at $x = 0$. If f is C^∞ but not analytic, the series may converge to a function different from f (see Example 2.6 with $c \neq 0$) or it may diverge (see (2.4) or (3.3) below).

The Taylor series of $f(x)$ at $x = x_0$,

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n,$$

does not provide an asymptotic expansion of $f(x)$ as $x \to x_1$ when $x_1 \neq x_0$ even if it converges. The partial sums therefore do not generally provide a good approximation of $f(x)$ as $x \to x_1$. (See Example 2.9, where a partial sum of the Taylor series of the error function at $x = 0$ provides a poor approximation of the function when x is large.)

The following (rather surprising) theorem shows that there are no restrictions on the growth rate of the coefficients in an asymptotic power series, unlike the case of convergent power series.

Theorem 2.8 (Borel-Ritt) Given any sequence $\{a_n\}$ of real (or complex) coefficients, there exists a C^∞-function $f : \mathbb{R} \to \mathbb{R}$ (or $f : \mathbb{R} \to \mathbb{C}$) such that

$$f(x) \sim \sum a_n x^n \qquad \text{as } x \to 0.$$

Proof. Let $\eta : \mathbb{R} \to \mathbb{R}$ be a C^∞-function such that

$$\eta(x) = \begin{cases} 1 & \text{if } |x| \le 1, \\ 0 & \text{if } |x| \ge 2. \end{cases}$$

We choose a sequence of positive numbers $\{\delta_n\}$ such that $\delta_n \to 0$ as $n \to \infty$ and

$$|a_n| \left\| x^n \eta\left(\frac{x}{\delta_n}\right) \right\|_{C^n} \le \frac{1}{2^n}, \tag{2.3}$$

where

$$\|f\|_{C^n} = \sup_{x\in\mathbb{R}} \sum_{k=0}^{n} \left| f^{(k)}(x) \right|$$

denotes the C^n-norm. We define

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \eta\left(\frac{x}{\delta_n}\right).$$

This series converges pointwise, since when $x = 0$ it is equal to a_0, and when $x \neq 0$ it consists of only finitely many terms. The condition in (2.3) implies that the sequence converges in C^n for every $n \in \mathbb{N}$. Hence, the function f has continuous derivatives of all orders. □

2.2.2 *Asymptotic versus convergent series*

We have already observed that an asymptotic series need not be convergent, and a convergent series need not be asymptotic. To explain the difference in more detail, we consider a formal series

$$\sum_{n=0}^{\infty} a_n \varphi_n(x),$$

where $\{a_n\}$ is a sequence of coefficients and $\{\varphi_n(x)\}$ is an asymptotic sequence as $x \to 0$. We denote the partial sums by

$$S_N(x) = \sum_{n=0}^{N} a_n \varphi_n(x).$$

Then convergence is concerned with the behavior of $S_N(x)$ as $N \to \infty$ with x fixed, whereas asymptoticity (at $x = 0$) is concerned with the behavior of $S_N(x)$ as $x \to 0$ with N fixed.

A convergent series define a unique limiting sum, but convergence does not give any indication of how rapidly the series it converges, nor of how well the sum of a fixed number of terms approximates the limit. An asymptotic series does not define a unique sum, nor does it provide an arbitrarily accurate approximation of the value of a function it represents at any $x \neq 0$, but its partial sums provide good approximations of these functions that when x is sufficiently small.

The following example illustrates the contrast between convergent and asymptotic series. We will examine another example of a divergent asymptotic series in Section 3.1.

Example 2.9 The error function erf $: \mathbb{R} \to \mathbb{R}$ is defined by

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt.$$

Integrating the power series expansion of e^{-t^2} term by term, we obtain the power series expansion of $\operatorname{erf} x$,

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}}\left\{x - \frac{1}{3}x^3 + \ldots + \frac{(-1)^n}{(2n+1)n!}x^{2n+1} + \ldots\right\},$$

which is convergent for every $x \in \mathbb{R}$. For large values of x, however, the convergence is very slow. the Taylor series of the error function at $x = 0$. Instead, we can use the following divergent asymptotic expansion, proved below, to obtain accurate approximations of $\operatorname{erf} x$ for large x:

$$\operatorname{erf} x \sim 1 - \frac{e^{-x^2}}{\sqrt{\pi}}\sum_{n=0}^{\infty}(-1)^{n+1}\frac{(2n-1)!!}{2^n}\frac{1}{x^{n+1}} \qquad \text{as } x \to \infty, \tag{2.4}$$

where $(2n-1)!! = 1 \cdot 3 \cdot \ldots \cdot (2n-1)$. For example, when $x = 3$, we need 31 terms in the Taylor series at $x = 0$ to approximate $\operatorname{erf} 3$ to an accuracy of 10^{-5}, whereas we only need 2 terms in the asymptotic expansion.

Proposition 2.10 The expansion (2.4) is an asymptotic expansion of $\operatorname{erf} x$.

Proof. We write

$$\operatorname{erf} x = 1 - \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}\,dt,$$

and make the change of variables $s = t^2$,

$$\operatorname{erf} x = 1 - \frac{1}{\sqrt{\pi}}\int_{x^2}^\infty s^{-1/2}e^{-s}\,ds.$$

For $n = 0, 1, 2, \ldots$, we define

$$F_n(x) = \int_{x^2}^\infty s^{-n-1/2}e^{-s}\,ds.$$

Then an integration by parts implies that

$$F_n(x) = \frac{e^{-x^2}}{x^{2n+1}} - \left(n + \frac{1}{2}\right)F_{n+1}(x).$$

By repeated use of this recursion relation, we find that

$$\begin{aligned}
\operatorname{erf} x &= 1 - \frac{1}{\sqrt{\pi}}F_0(x) \\
&= 1 - \frac{1}{\sqrt{\pi}}\left[\frac{e^{-x^2}}{x} - \frac{1}{2}F_1(x)\right] \\
&= 1 - \frac{1}{\sqrt{\pi}}\left[e^{-x^2}\left(\frac{1}{x} - \frac{1}{2x^3}\right) + \frac{1\cdot 3}{2^2}F_2(x)\right]
\end{aligned}$$

$$
\begin{aligned}
&= 1 - \frac{1}{\sqrt{\pi}}\left[e^{-x^2}\left(\frac{1}{x} - \frac{1}{2x^3} + \ldots (-1)^N \frac{1\cdot 3\cdot \ldots \cdot (2N-1)}{2^N x^{2N+1}}\right)\right. \\
&\qquad \left. + (-1)^{N+1}\frac{1\cdot 3\cdot \ldots \cdot (2N+1)}{2^{N+1}} F_{N+1}(x)\right].
\end{aligned}
$$

It follows that

$$
\operatorname{erf} x = 1 - \frac{e^{-x^2}}{\sqrt{\pi}} \sum_{n=0}^{N} (-1)^n \frac{1\cdot 3\cdot \ldots \cdot (2n-1)}{2^n x^{2n+1}} + R_{N+1}(x)
$$

where

$$
R_{N+1}(x) = (-1)^{N+1}\frac{1}{\sqrt{\pi}}\frac{1\cdot 3\cdot \ldots \cdot (2N+1)}{2^{N+1}} F_{N+1}(x).
$$

Since

$$
\begin{aligned}
|F_n(x)| &= \left|\int_{x^2}^{\infty} s^{-n-1/2} e^{-s}\, ds\right| \\
&\le \frac{1}{x^{2n+1}} \int_{x^2}^{\infty} e^{-s}\, ds \\
&\le \frac{e^{-x^2}}{x^{2n+1}},
\end{aligned}
$$

we have

$$
|R_{N+1}(x)| \le C_{N+1} \frac{e^{-x^2}}{x^{2N+3}},
$$

where

$$
C_N = \frac{1\cdot 3\cdot \ldots \cdot (2N+1)}{2^{N+1}\sqrt{\pi}}.
$$

This proves the result. □

2.2.3 *Generalized asymptotic expansions*

Sometimes it is useful to consider more general asymptotic expansions with respect to a sequence of gauge functions $\{\varphi_n\}$ of the form

$$
f(x) \sim \sum_{n=0}^{\infty} f_n(x),
$$

where for each $N = 0, 1, 2, \ldots$

$$
f(x) - \sum_{n=0}^{N} f_n(x) = o(\varphi_{N+1}).
$$

For example, an expansion of the form

$$f(x) \sim \sum_{n=0}^{\infty} a_n(x)\varphi_n(x)$$

in which the coefficients a_n are bounded functions of x is a generalized asymptotic expansion. Such expansions provide additional flexibility, but they are not unique and have to be used with care in many cases.

Example 2.11 We have

$$\frac{1}{1-x} \sim \sum_{n=0}^{\infty} x^n \qquad \text{as } x \to 0^+.$$

We also have

$$\frac{1}{1-x} \sim \sum_{n=0}^{\infty} (1+x)x^{2n} \qquad \text{as } x \to 0^+.$$

This is a generalized asymptotic expansion with respect to $\{x^n \mid n = 0, 1, 2, \ldots\}$ that differs from the first one.

Example 2.12 According to [12], the following generalized asymptotic expansion with respect to the asymptotic sequence $\{(\log x)^{-n}\}$

$$\frac{\sin x}{x} \sim \sum_{n=1}^{\infty} \frac{n!e^{-(n+1)x/(2n)}}{(\log x)^n} \qquad \text{as } x \to \infty$$

is an example showing that "the definition admits expansions that have no conceivable value."

Example 2.13 A physical example of a generalized asymptotic expansion arises in the derivation of the Navier-Stokes equations of fluid mechanics from the Boltzmann equations of kinetic theory by means of the Chapman-Enskog expansion. If λ is the mean free path of the fluid molecules and L is a macroscopic length-scale of the fluid flow, then the relevant small parameter is

$$\varepsilon = \frac{\lambda}{L} \ll 1.$$

The leading-order term in the Chapman-Enskog expansion satisfies the Navier-Stokes equations in which the fluid viscosity is of the order ε when nondimensionalized by the length and time scales characteristic of the fluid flow. Thus the leading order solution depends on the perturbation parameter ε, and this expansion is a generalized asymptotic expansion.

2.2.4 *Nonuniform asymptotic expansions*

In many problems, we seek an asymptotic expansion as $\varepsilon \to 0$ of a function $u(x, \varepsilon)$, where x is an independent variable.* The asymptotic behavior of the function with respect to ε may depend upon x, in which case we say that the expansion is nonuniform.

Example 2.14 Consider the function $u : [0, \infty) \times (0, \infty) \to \mathbb{R}$ defined by

$$u(x, \varepsilon) = \frac{1}{x + \varepsilon}.$$

If $x > 0$, then

$$u(x, \varepsilon) \sim \frac{1}{x}\left[1 - \frac{\varepsilon}{x} + \frac{\varepsilon^2}{x^2} + \dots\right] \qquad \text{as } \varepsilon \to 0^+.$$

On the other hand, if $x = 0$, then

$$u(0, \varepsilon) \sim \frac{1}{\varepsilon} \qquad \text{as } \varepsilon \to 0^+.$$

The transition between these two different expansions occurs when $x = O(\varepsilon)$. In the limit $\varepsilon \to 0^+$ with $y = x/\varepsilon$ fixed, we have

$$u(\varepsilon y, \varepsilon) \sim \frac{1}{\varepsilon}\left(\frac{1}{y + 1}\right) \qquad \text{as } \varepsilon \to 0^+.$$

This expansion matches with the other two expansions in the limits $y \to \infty$ and $y \to 0^+$.

Nonuniform asymptotic expansions are not a mathematical pathology, and they are often the crucial feature of singular perturbation problems. We will encounter many such problems below; for example, in boundary layer problems the solution has different asymptotic expansions inside and outside the boundary layer, and in various problems involving oscillators nonuniformities aries for long times.

2.3 Stokes phenomenon

An asymptotic expansion as $z \to \infty$ of a complex function $f : \mathbb{C} \to \mathbb{C}$ with an essential singularity at $z = \infty$ is typically valid only in a wedge-shaped region $\alpha < \arg z < \beta$, and the function has different asymptotic expansions in different wedges.† The change in the form of the asymptotic expansion across the boundaries of the wedges is called the Stokes phenomenon.

*We consider asymptotic expansions with respect to ε, not x.

†We consider a function with an essential singularity at $z = \infty$ for definiteness; the same phenomenon occurs for functions with an essential singularity at any $z_0 \in \mathbb{C}$.

Example 2.15 Consider the function $f : \mathbb{C} \to \mathbb{C}$ defined by

$$f(z) = \sinh z^2 = \frac{e^{z^2} - e^{-z^2}}{2}.$$

Let $|z| \to \infty$ with $\arg z$ fixed, and define

$$\Omega_1 = \{z \in \mathbb{C} \mid -\pi/4 < \arg z < \pi/4\},$$
$$\Omega_2 = \{z \in \mathbb{C} \mid \pi/4 < \arg z < 3\pi/4 \text{ or } -3\pi/4 < \arg z < -\pi/4\}.$$

If $z \in \Omega_1$, then $\operatorname{Re} z^2 > 0$ and $e^{z^2} \gg e^{-z^2}$, whereas if $z \in \Omega_2$, then $\operatorname{Re} z^2 < 0$ and $e^{z^2} \ll e^{-z^2}$. Hence

$$f(z) \sim \begin{cases} \frac{1}{2} e^{z^2} & \text{as } |z| \to \infty \text{ in } \Omega_1, \\ \frac{1}{2} e^{-z^2} & \text{as } |z| \to \infty \text{ in } \Omega_2. \end{cases}$$

The lines $\arg z = \pm\pi/4, \pm 3\pi/4$ where the asymptotic expansion changes form are called anti-Stokes lines. The terms e^{z^2} and e^{-z^2} switch from being dominant to subdominant as z crosses an anti-Stokes lines. The lines $\arg z = 0, \pi, \pm\pi/2$ where the subdominant term is as small as possible relative to the dominant term are called Stokes lines.

This example concerns a simple explicit function, but a similar behavior occurs for solutions of ODEs with essential singularities in the complex plane, such as error functions, Airy functions, and Bessel functions.

Example 2.16 The error function can be extended to an entire function $\operatorname{erf} : \mathbb{C} \to \mathbb{C}$ with an essential singularity at $z = \infty$. It has the following asymptotic expansions in different wedges:

$$\operatorname{erf} z \sim \begin{cases} 1 - \exp(-z^2)/(z\sqrt{\pi}) & \text{as } z \to \infty \text{ with } z \in \Omega_1, \\ -1 - \exp(-z^2)/(z\sqrt{\pi}) & \text{as } z \to \infty \text{ with } z \in \Omega_2, \\ -\exp(-z^2)/(z\sqrt{\pi}) & \text{as } z \to \infty \text{ with } z \in \Omega_3. \end{cases}$$

where

$$\Omega_1 = \{z \in \mathbb{C} \mid -\pi/4 < \arg z < \pi/4\},$$
$$\Omega_2 = \{z \in \mathbb{C} \mid 3\pi/4 < \arg z < 5\pi/4\},$$
$$\Omega_3 = \{z \in \mathbb{C} \mid \pi/4 < \arg z < 3\pi/4 \text{ or } 5\pi/4 < \arg z < 7\pi/4\}.$$

Often one wants to determine the asymptotic behavior of such a function in one wedge given its behavior in another wedge. This is called a connection problem (see Section 3.5.1 for the case of the Airy function). The apparently discontinuous change in the form of the asymptotic expansion of the solutions of an ODE across an anti-Stokes line can be understood using exponential asymptotics as the result of a continuous, but rapid, change in the coefficient of the subdominant terms across the Stokes line [2].