11.10: Taylor Series

Wednesday, March 18

Recap

A certain power series centered at $x = 2$ converges at -2 and diverges at 7. Decide whether it converges or diverges at each of the following points, or whether you do not have enough information to tell.

1. $x = -4$: diverges
2. $x = -3$: unknown
3. $x = 2$: converges
4. $x = 5$: converges
5. $x = 6$: unknown
6. $x = 8$: diverges

Power Series

1. $e^x = 1 + x + x^2/2! + x^3/3! + \dots$
2. $\sin x = x - x^3/3! + x^5/5! - \dots$
3. $\cos x = 1 - x^2/2! + x^4/4! - \dots$
4. $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$
5. $\ln(1 + x) = x - x^2/2 + x^3/3 - x^4/4 + \dots$
6. $\arctan(x) = x - x^3/3 + x^5/5 - x^7/7 + \dots$

Power Series Arithmetic

1. $\frac{1}{1+2x} = 1 - 2x + 4x^2 - 8x^3 + \dots$
2. $e^{2x} + \sin(x) = 1 + 3x + 2x^2 + 7x^3/6 + 2x^4/3 + \dots$
3. $\sin x + 2\cos 2x = 2 + x - 4x^2 - x^3/6 + \dots$
4. $(\sin x)^2 = x^2 - x^4/3 + 2x^6/45 - \dots$
5. Show that $\frac{d}{dx} \sin x = \cos x$.
6. Derive the Taylor series for $\arctan x$ at $x = 0$ by integrating $\frac{1}{1+x^2}$.

$$\begin{aligned}
\arctan x &= \int \frac{1}{1+x^2} \\
&= \int \frac{1}{1-(-x^2)} \\
&= \int 1 - x^2 + x^4 - x^6 + \dots \\
&= x - x^3/3 + x^5/5 - x^7/7 = \dots
\end{aligned}$$

7. Show that $\sin 2x = 2 \sin x \cos x$, at least up to the x^3 term in their series expansions.

Taylor Series

Find the Taylor series expansions for the given functions around the given points.

1. $\sqrt{3 + x}$ around $x = 0$
   Answer: $\sqrt{3} + \frac{x}{2\sqrt{3}} - \frac{x^2}{24\sqrt{3}} + \frac{x^3}{144\sqrt{3}} - \dots$

2. $\sqrt{x}$ around $x = 3$

   Answer: $\sqrt{3} + \frac{(x-3)}{2\sqrt{3}} - \frac{(x-3)^2}{24\sqrt{3}} + \frac{(x-3)^3}{144\sqrt{3}} - \ldots$. Note the similarity to the previous answer, since one function is just a translation of the other.

3. $\sin(x)$ around $x = \pi/2$

   Answer: $1 - \frac{1}{2}(x - \pi/2)^2 + \frac{1}{4!}(x - \pi/2)^4 - \ldots$ This looks like the series for $\cos(x)$ because (letting $x = \pi/2 + y$) of the identity $\sin(\pi/2 + y) = \cos(y)$.

4. e^{2x} around $x = 1$

   Answer: $e^2 + 2e^2(x-1) + 2e^2(x-1)^2 + \frac{4}{3}e^2(x-1)^3 + \ldots$

Find the Taylor series expansions for the function $f(x) = x^3 - 3x$ at $x = 0$, $x = 1$, and $x = 2$. Sketch the linear and quadratic approximations at each of those points below:

1. Are the three full Taylor series expansions the same? If not, how do they differ?

   The three expansions are the same, as three derivatives is enough to fully describe the cubic polynomial.

2. What do the coefficients of the Taylor series expansions tell you about the behavior of the function (e.g. slope, concavity) at each of the three points?

   The constant term tells you about the value of the function at the given point, the x term tells you what the derivative is, and the x^2 term (giving the second derivative) tells you whether the function is concave up or down.

3. Compare the Taylor series expansions for $\sin x$ around $x = 0$ and $x = 2\pi$. How are these similar? How do they differ?

   The coefficients are the same, since $\sin x = \sin(x + 2\pi)$ in general. The two series both converge on $(-\infty, \infty)$, but if you cut off the series at any finite power of x the two polynomials will be different. The first is most accurate near $x = 0$ and the second is most accurate near $x = 2\pi$.