

On the Homotopy Perturbation Method for the Exact Solution of Fitzhugh–Nagumo Equation

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ABSTRACT

In this paper, the Homotopy Perturbation Method (HPM) is used to solve the Fitzhugh–Nagumo non-linear differential equations. In order to obtain the exact solution of Fitzhugh–Nagumo equation, two case study problems of the equation are solved by using the HPM. The trend of the rapid convergence of the sequences constructed by the method towards the exact solution is also numerically shown. As a result, the rapid convergence towards the exact solutions of HPM indicates that the method is powerful and efficient technique to solve the Fitzhugh–Nagumo non-linear differential equations. Also, the results present validity and great potential of the method as a powerful algorithm in order to obtain the exact solution of nonlinear differential equations.

Keywords: Fitzhugh–Nagumo equation, Homotopy Perturbation Method, Nonlinear Differential Equations.

Mathematics Subject Classification: 35D35, 35D05

1. INTRODUCTION

Many phenomena in various fields of physical science and engineering can be modeled by nonlinear partial differential equations. Most of the equations do not have an analytical solution and these equations have to be handled by the semi-analytical methods such as the Homotopy Perturbation Method (HPM).

The method was first introduced by He (1999). Then, He (2005) presented application of the method in solving the non-linear non-homogeneous partial differential equations. The HPM considers the solution of non-linear differential equation as a series expansion which can present a rapid convergence toward the exact solution. The HPM is used by Nourazar et al. (2015) in order to obtain the exact solution of Burgers-Huxley equation. Also, Nourazar et al. (2011) used the homotopy perturbation method to find exact solution of Newell-Whitehead-Segel equation. Barari et al. (2009) used the homotopy perturbation method for solving tenth order boundary value problems. In order to

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solve variational inequalities with general constraints on an unbounded set, Fan et al. (2014) used the homotopy perturbation method. Matinfar and Saeidy (2011) presented application of homotopy analysis method to functional integral equations. Kawahara and Tanaka (1983) obtained new exact solutions of the Fitzhugh–Nagumo equation using Hirota method. Nucci and Clarkson (1992) presented some new solutions of the Fitzhugh–Nagumo by using Jacobi elliptic function.

The Fitzhugh–Nagumo equation models the interaction of the effect of the diffusion term with the nonlinear effect of the reaction term. It is an important nonlinear reaction–diffusion equation which is applied to model the transmission of nerve impulses (Fitzhugh, 1961; Nagumo et al., 1962). It is also used in biology, the area of population genetics, transmission of heat mass transfer as well as circuit theory .The Fitzhugh–Nagumo equation is written as:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - u(1-u)(a-u) \tag{1.1}$$

Where a is arbitrary constant.

The first term on the right hand side, $\frac{\partial^2 u}{\partial x^2}$, expresses the variations of $u(x, t)$ with spatial variable x at a specific time. Also, the remaining terms on the right hand side, $u(1-u)(a-u)$, takes into account the effect of the source term. In Eq. (1.1), $u(x, t)$ is a function of the spatial variable x and the temporal variable t with $x \in R$ and $t \geq 0$.

In the present research work, the HPM is applied to obtain the closed form solution of the non-linear Fitzhugh–Nagumo equation. Two case study problems of non-linear Fitzhugh–Nagumo equations are solved by using the HPM. In terms of comparison to the exact solution, the trend of rapid convergence is shown.

The idea of homotopy perturbation method is presented in the section 2. Application of the homotopy perturbation method to the exact solution of Fitzhugh–Nagumo equation is presented in the section 3.

2. THE IDEA OF HOMOTOPY PERTURBATION METHOD

The homotopy perturbation method (HPM) is originally initiated by He (1999). This is a combination of the classical perturbation technique and homotopy technique. The basic idea of the HPM for solving nonlinear differential equations is as follow; consider the following differential equation:

$$E(u) = 0, \tag{2.1}$$

Where E is any differential operator. We construct a homotopy as follow:

$$H(u, p) = (1 - p)F(u) + p(E(u) - F(u)). \tag{2.2}$$

Where $F(u)$ is a functional operator with the known solution v_0 . It is clear that when p is equal to zero then $H(u,0) = F(u) = 0$, and when p is equal to 1, then $H(u,1) = E(u) = 0$. It is worth noting that as the embedding parameter p increases monotonically from zero to unity the zero order solution v_0 continuously deforms into the original problem $E(u) = 0$. The embedding parameter, $p \in [0,1]$ is considered as an expanding parameter (He, 2005). In the homotopy perturbation method the embedding parameter p is used to get series expansion for solution as:

$$u = \sum_{i=0}^{\infty} p^i v_i = v_0 + p v_1 + p^2 v_2 + p^3 v_3 + \dots \quad (2.3)$$

When, $p \rightarrow 1$ then Eq. (2.2) becomes the approximate solution to Eq. (2.1) as:

$$u = v_0 + v_1 + v_2 + v_3 + \dots \quad (2.4)$$

The series Eq. (2.4) is a convergent series and the rate of convergence depends on the nature of Eq. (2.1) (He, 1999; He, 2005). It is also assumed that Eq. (2.2) has a unique solution and by comparing the like powers of p the solution of various orders is obtained. These solutions are obtained using the Maple package.

3. THE FITZHUGH-NAGUMO EQUATION

To illustrate the capability and reliability of the method, two cases of nonlinear diffusion equations are presented.

Case I: In this case we will examine the Fitzhugh–Nagumo equation for $a = 2$, equation is written as:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - u(1-u)(2-u) \quad (3.1)$$

Subject to initial condition:

$$u(x,0) = \frac{e^{\frac{\sqrt{2}}{2}x} + 2e^{\sqrt{2}x}}{e^{\frac{\sqrt{2}}{2}x} + e^{\sqrt{2}x} + 1} \quad (3.2)$$

We construct a homotopy for Eq. (3.1) in the following form:

$$H(v, p) = (1-p) \left[\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t} \right] + p \left[\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} + v(1-v)(2-v) \right]. \quad (3.3)$$

The solution of Eq. (3.1) can be written as a power series in p as:

$$v = v_0 + p v_1 + p^2 v_2 + \dots \quad (3.4)$$

Substituting Eq. (3.4) and Eq. (3.2) into Eq. (3.3) and equating the terms with identical powers of p :

$$\begin{aligned} p^0 : \frac{\partial v_0}{\partial t} &= \frac{\partial u_0}{\partial t}, & v_0(x,0) &= \frac{e^{\frac{\sqrt{2}}{2}x} + 2e^{\sqrt{2}x}}{e^{\frac{\sqrt{2}}{2}x} + e^{\sqrt{2}x} + 1}, \\ p^1 : \frac{\partial v_1}{\partial t} + \frac{\partial u_0}{\partial t} &= \frac{\partial^2 v_0}{\partial x^2} - v_0(1-v_0)(2-v_0), & v_1(x,0) &= 0, \\ p^2 : \frac{\partial v_2}{\partial t} &= \frac{\partial^2 v_1}{\partial x^2} + 6v_0v_1 - 2v_1 - 3v_1v_0^2, & v_2(x,0) &= 0, \end{aligned} \quad (3.5)$$

$$p^3 : \frac{\partial v_3}{\partial t} = \frac{\partial^2 v_2}{\partial x^2} + 6v_0v_2 - 3v_2v_0^2 - 3v_0v_1^2 + 3v_1^2 - 2v_2, \quad v_3(x,0) = 0.$$

Using the Maple package to solve recursive sequences, Eq. (3.5), we obtain the followings:

$$v_0(x,t) = \frac{e^{\frac{\sqrt{2}}{2}x} + 2e^{\sqrt{2}x}}{e^{\frac{\sqrt{2}}{2}x} + e^{\sqrt{2}x} + 1},$$

$$v_1(x,t) = \frac{3}{2} \left(\frac{e^{\frac{\sqrt{2}}{2}x}(-1 + e^{\sqrt{2}x})}{\left(e^{\frac{\sqrt{2}}{2}x} + e^{\sqrt{2}x} + 1 \right)^2} \right) t,$$

$$v_2(x,t) = \frac{9}{4} \left(\frac{e^{\frac{\sqrt{2}}{2}x}(-1 + e^{\sqrt{2}x}) \left(e^{\frac{\sqrt{2}}{2}x} - e^{\sqrt{2}x} - 1 \right)}{\left(e^{\frac{\sqrt{2}}{2}x} + e^{\sqrt{2}x} + 1 \right)^3} \right) \frac{t^2}{2}, \tag{3.6}$$

$$v_3(x,t) = \frac{27}{8} \left(\frac{e^{\frac{\sqrt{2}}{2}x}(-1 + e^{\sqrt{2}x}) \left(\left(e^{\frac{\sqrt{2}}{2}x} \right)^2 + \left(e^{\sqrt{2}x} \right)^2 - 4e^{\frac{\sqrt{2}}{2}x} e^{\sqrt{2}x} + 2e^{\sqrt{2}x} + 1 - 4e^{\frac{\sqrt{2}}{2}x} \right)}{\left(e^{\frac{\sqrt{2}}{2}x} + e^{\sqrt{2}x} + 1 \right)^4} \right) \frac{t^3}{6},$$

By setting $p = 1$ in Eq. (3.4), the solution of Eq. (3.1) can be obtained as $v = v_0 + v_1 + v_2 + v_3 + \dots$.

Therefore the solution of Eq. (3.1) is written as:

$$v(x,t) = \frac{e^{\frac{\sqrt{2}}{2}x} + 2e^{\sqrt{2}x}}{e^{\frac{\sqrt{2}}{2}x} + e^{\sqrt{2}x} + 1} + \frac{3}{2} \left(\frac{e^{\frac{\sqrt{2}}{2}x}(-1 + e^{\sqrt{2}x})}{\left(e^{\frac{\sqrt{2}}{2}x} + e^{\sqrt{2}x} + 1 \right)^2} \right) + \frac{9}{4} \left(\frac{e^{\frac{\sqrt{2}}{2}x}(-1 + e^{\sqrt{2}x}) \left(e^{\frac{\sqrt{2}}{2}x} - e^{\sqrt{2}x} - 1 \right)}{\left(e^{\frac{\sqrt{2}}{2}x} + e^{\sqrt{2}x} + 1 \right)^3} \right) \frac{t^2}{2}$$

$$+ \frac{27}{8} \left(\frac{e^{\frac{\sqrt{2}}{2}x} (-1 + e^{\sqrt{2}x}) \left(\left(e^{\frac{\sqrt{2}}{2}x} \right)^2 + (e^{\sqrt{2}x})^2 - 4e^{\frac{\sqrt{2}}{2}x} e^{\sqrt{2}x} + 2e^{\sqrt{2}x} + 1 - 4e^{\frac{\sqrt{2}}{2}x} \right)}{\left(e^{\frac{\sqrt{2}}{2}x} + e^{\sqrt{2}x} + 1 \right)^4} \right) \frac{t^3}{6} + \dots \quad (3.7)$$

The Taylor series expansion for $\left(\frac{e^{\frac{\sqrt{2}}{2}x - \frac{3}{2}t} + 2e^{\sqrt{2}x}}{e^{\frac{\sqrt{2}}{2}x - \frac{3}{2}t} + e^{\sqrt{2}x} + 1} \right)$ is written as:

$$\left(\frac{e^{\frac{\sqrt{2}}{2}x - \frac{3}{2}t} + 2e^{\sqrt{2}x}}{e^{\frac{\sqrt{2}}{2}x - \frac{3}{2}t} + e^{\sqrt{2}x} + 1} \right) = \frac{e^{\frac{\sqrt{2}}{2}x} + 2e^{\sqrt{2}x}}{e^{\frac{\sqrt{2}}{2}x} + e^{\sqrt{2}x} + 1} + \frac{3}{2} \left(\frac{e^{\frac{\sqrt{2}}{2}x} (-1 + e^{\sqrt{2}x})}{\left(e^{\frac{\sqrt{2}}{2}x} + e^{\sqrt{2}x} + 1 \right)^2} \right) + \frac{9}{4} \left(\frac{e^{\frac{\sqrt{2}}{2}x} (-1 + e^{\sqrt{2}x}) \left(e^{\frac{\sqrt{2}}{2}x} - e^{\sqrt{2}x} - 1 \right)}{\left(e^{\frac{\sqrt{2}}{2}x} + e^{\sqrt{2}x} + 1 \right)^3} \right) \frac{t^2}{2} \quad (3.8)$$

$$+ \frac{27}{8} \left(\frac{e^{\frac{\sqrt{2}}{2}x} (-1 + e^{\sqrt{2}x}) \left(\left(e^{\frac{\sqrt{2}}{2}x} \right)^2 + (e^{\sqrt{2}x})^2 - 4e^{\frac{\sqrt{2}}{2}x} e^{\sqrt{2}x} + 2e^{\sqrt{2}x} + 1 - 4e^{\frac{\sqrt{2}}{2}x} \right)}{\left(e^{\frac{\sqrt{2}}{2}x} + e^{\sqrt{2}x} + 1 \right)^4} \right) \frac{t^3}{6} + \dots$$

By substituting Eq. (3.8) into Eq. (3.7), thus Eq. (3.7) can be rewritten as:

$$v(x, t) = \frac{e^{\frac{\sqrt{2}}{2}x - \frac{3}{2}t} + 2e^{\sqrt{2}x}}{e^{\frac{\sqrt{2}}{2}x - \frac{3}{2}t} + e^{\sqrt{2}x} + 1}, \quad (3.9)$$

This is the exact solution of the problem, Eq. (3.1). Table 1 shows the trend of rapid convergence of the results of $S_0(x, t) = v_0(x, t)$ to $S_5(x, t) = \sum_{i=0}^5 v_i(x, t)$ using the HPM. The rapid convergence of the solution toward the exact solution, the maximum relative error of less than 0.00005% is achieved as shown in table 1.

		Percentage of relative error (%RE)		
		$x = 1.7$	$x = 2.5$	$x = 5$
$t = 0.1$	$S_0(x, t)$	0.01212243657	0.008961933562	0.001944542565
	$S_1(x, t)$	0.0005389659700	0.0004976779710	0.0001412316169
	$S_2(x, t)$	0.000002301164956	0.000009908081124	0.000006349034810
	$S_3(x, t)$	0.000001008144018	4.086814607 e-7	1.823278692 e-7
	$S_4(x, t)$	3.598157652 e-8	3.332137355 e-8	3.090297025 e-9
	$S_5(x, t)$	8.027475176 e-11	9.968779601 e-10	7.506286372 e-10
$t = 0.3$	$S_0(x, t)$	0.03250166560	0.02373573301	0.005075070204
	$S_1(x, t)$	0.004698954162	0.004220046979	0.001162625454
	$S_2(x, t)$	0.00005370144463	0.0002801263997	0.0001614342617
	$S_3(x, t)$	0.00007427483338	0.00002746611476	0.00001435939656
	$S_4(x, t)$	0.000008554633162	0.000007802396706	6.123663113 e-7
	$S_5(x, t)$	4.384208065 e-7	5.464147246 e-7	4.561018885 e-8
$t = 0.5$	$S_0(x, t)$	0.04842166884	0.03502916290	0.007407255743
	$S_1(x, t)$	0.01255914876	0.01102481608	0.002964534234
	$S_2(x, t)$	0.0004254388900	0.001331059900	0.0007047880201
	$S_3(x, t)$	0.0005191187850	0.0001748677266	0.0001071637554
	$S_4(x, t)$	0.0001094809320	0.00009411780695	0.000008088311167
	$S_5(x, t)$	0.000006823246332	0.00001200658008	8.168657039 e-7

Table 1 shows: the percentage of relative errors of the results of $S_0(x, t) = v_0(x, t)$ to

$$S_5(x, t) = \sum_{i=0}^5 v_i(x, t) \text{ of the HPM solution of Eq. (3.1).}$$

Case II: In Eq. (1.1) for $a = 3$ the Fitzhugh–Nagumo equation is written as:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - u(1-u)(3-u) \tag{3.10}$$

Subject to initial condition:

$$u(x, 0) = \frac{e^{\frac{\sqrt{2}}{2}x} + 3e^{\frac{3\sqrt{2}}{2}x}}{e^{\frac{\sqrt{2}}{2}x} + e^{\frac{3\sqrt{2}}{2}x} + 1} \tag{3.11}$$

To solve Eq. (3.10), we construct a homotopy in the following form:

$$H(v, p) = (1-p) \left[\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t} \right] + p \left[\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} + v(1-v)(3-v) \right]. \tag{3.12}$$

The solution of Eq. (3.10) can be written as a power series in p as:

$$v = v_0 + pv_1 + p^2v_2 + \dots \tag{3.13}$$

Substituting Eq. (3.13) and Eq. (3.11) in to Eq. (3.12) and equating the term with identical powers of p , leads to:

$$p^0 : \frac{\partial v_0}{\partial t} = \frac{\partial u_0}{\partial t}, \quad v_0(x,0) = \frac{e^{\frac{\sqrt{2}}{2}x} + 3e^{\frac{3\sqrt{2}}{2}x}}{e^{\frac{\sqrt{2}}{2}x} + e^{\frac{3\sqrt{2}}{2}x} + 1},$$

$$p^1 : \frac{\partial v_1}{\partial t} + \frac{\partial u_0}{\partial t} = \frac{\partial^2 v_0}{\partial x^2} - v_0(1 - v_0)(3 - v_0), \quad v_1(x,0) = 0,$$

$$p^2 : \frac{\partial v_2}{\partial t} = \frac{\partial^2 v_1}{\partial x^2} + 8v_0v_1 - 3v_1 - 3v_1v_0^2, \quad v_2(x,0) = 0, \tag{3.14}$$

$$p^3 : \frac{\partial v_3}{\partial t} = \frac{\partial^2 v_2}{\partial x^2} + 8v_0v_2 - 3v_2v_0^2 - 3v_0v_1^2 + 4v_1^2 - 3v_2, \quad v_3(x,0) = 0.$$

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Using the Maple package to solve recursive sequences, Eq. (3.14), we obtain the followings:

$$v_0(x,t) = \frac{e^{\frac{\sqrt{2}}{2}x} + 3e^{\frac{3\sqrt{2}}{2}x}}{e^{\frac{\sqrt{2}}{2}x} + e^{\frac{3\sqrt{2}}{2}x} + 1},$$

$$v_1(x,t) = \frac{1}{2} \left(\frac{-5 \left(e^{\frac{\sqrt{2}}{2}x} \right) + 16 \left(e^{\frac{\sqrt{2}}{2}x} \right) \left(e^{\frac{3\sqrt{2}}{2}x} \right) + 9 \left(e^{\frac{3\sqrt{2}}{2}x} \right)}{\left(e^{\frac{\sqrt{2}}{2}x} + e^{\frac{3\sqrt{2}}{2}x} + 1 \right)^2} \right) t, \tag{3.15}$$

$$v_2(x, t) = \frac{1}{4} \left(\frac{1}{\left(e^{\frac{\sqrt{2}}{2}x} + e^{\frac{3\sqrt{2}}{2}x} + 1 \right)^3} \right) \left[-25 \left(e^{\frac{\sqrt{2}}{2}x} \right)^2 + 128 \left(e^{\frac{\sqrt{2}}{2}x} \right)^2 \left(e^{\frac{3\sqrt{2}}{2}x} \right) + 140 \left(e^{\frac{\sqrt{2}}{2}x} \right) \left(e^{\frac{3\sqrt{2}}{2}x} \right) \right. \\ \left. + 25 \left(e^{\frac{\sqrt{2}}{2}x} \right) - 128 \left(e^{\frac{\sqrt{2}}{2}x} \right) \left(e^{\frac{3\sqrt{2}}{2}x} \right)^2 - 27 \left(e^{\frac{3\sqrt{2}}{2}x} \right)^2 + 27 \left(e^{\frac{3\sqrt{2}}{2}x} \right) \right] \frac{t^2}{2},$$

$$v_3(x, t) = \frac{1}{8} \left(\frac{1}{\left(e^{\frac{\sqrt{2}}{2}x} + e^{\frac{3\sqrt{2}}{2}x} + 1 \right)^4} \right) \left[81 \left(e^{\frac{3\sqrt{2}}{2}x} \right)^3 - 125 \left(e^{\frac{\sqrt{2}}{2}x} \right)^3 - 125 \left(e^{\frac{\sqrt{2}}{2}x} \right) + 81 \left(e^{\frac{3\sqrt{2}}{2}x} \right) \right. \\ \left. + 1024 \left(e^{\frac{3\sqrt{2}}{2}x} \right) \left(e^{\frac{\sqrt{2}}{2}x} \right)^3 - 4096 \left(e^{\frac{3\sqrt{2}}{2}x} \right)^2 \left(e^{\frac{\sqrt{2}}{2}x} \right)^2 + 1024 \left(e^{\frac{3\sqrt{2}}{2}x} \right)^3 \left(e^{\frac{\sqrt{2}}{2}x} \right) \right. \\ \left. - 144 \left(e^{\frac{\sqrt{2}}{2}x} \right) \left(e^{\frac{3\sqrt{2}}{2}x} \right) - 324 \left(e^{\frac{3\sqrt{2}}{2}x} \right)^2 + 500 \left(e^{\frac{\sqrt{2}}{2}x} \right)^2 + 1399 \left(e^{\frac{\sqrt{2}}{2}x} \right)^2 \left(e^{\frac{3\sqrt{2}}{2}x} \right) \right. \\ \left. - 2235 \left(e^{\frac{\sqrt{2}}{2}x} \right) \left(e^{\frac{3\sqrt{2}}{2}x} \right)^2 \right] \frac{t^3}{6},$$

By setting $p=1$ in Eq. (3.13), the solution of Eq. (3.10) can be obtained as

$v = v_0 + v_1 + v_2 + v_3 + \dots$. Therefore the solution of Eq. (3.10) is written as:

$$v(x, t) = \frac{e^{\frac{\sqrt{2}}{2}x} + 3e^{\frac{3\sqrt{2}}{2}x}}{e^{\frac{\sqrt{2}}{2}x} + e^{\frac{3\sqrt{2}}{2}x} + 1} + \frac{1}{2} \left(\frac{-5 \left(e^{\frac{\sqrt{2}}{2}x} \right) + 16 \left(e^{\frac{\sqrt{2}}{2}x} \right) \left(e^{\frac{3\sqrt{2}}{2}x} \right) + 9 \left(e^{\frac{3\sqrt{2}}{2}x} \right)}{\left(e^{\frac{\sqrt{2}}{2}x} + e^{\frac{3\sqrt{2}}{2}x} + 1 \right)^2} \right) t$$

$$\begin{aligned}
 & + \frac{1}{4} \left(\frac{1}{\left(e^{\frac{\sqrt{2}}{2}x} + e^{\frac{3\sqrt{2}}{2}x} + 1 \right)^3} \right) \left[-25 \left(e^{\frac{\sqrt{2}}{2}x} \right)^2 + 128 \left(e^{\frac{\sqrt{2}}{2}x} \right) \left(e^{\frac{3\sqrt{2}}{2}x} \right) + 140 \left(e^{\frac{\sqrt{2}}{2}x} \right) \left(e^{\frac{3\sqrt{2}}{2}x} \right) \right. \\
 & \quad \left. + 25 \left(e^{\frac{\sqrt{2}}{2}x} \right) - 128 \left(e^{\frac{\sqrt{2}}{2}x} \right) \left(e^{\frac{3\sqrt{2}}{2}x} \right)^2 - 27 \left(e^{\frac{3\sqrt{2}}{2}x} \right)^2 + 27 \left(e^{\frac{3\sqrt{2}}{2}x} \right) \right] \frac{t^2}{2} \\
 & + \frac{1}{8} \left(\frac{1}{\left(e^{\frac{\sqrt{2}}{2}x} + e^{\frac{3\sqrt{2}}{2}x} + 1 \right)^4} \right) \left[81 \left(e^{\frac{3\sqrt{2}}{2}x} \right)^3 - 125 \left(e^{\frac{\sqrt{2}}{2}x} \right)^3 - 125 \left(e^{\frac{\sqrt{2}}{2}x} \right) + 81 \left(e^{\frac{3\sqrt{2}}{2}x} \right) \right. \\
 & \quad + 1024 \left(e^{\frac{3\sqrt{2}}{2}x} \right) \left(e^{\frac{\sqrt{2}}{2}x} \right)^3 - 4096 \left(e^{\frac{3\sqrt{2}}{2}x} \right)^2 \left(e^{\frac{\sqrt{2}}{2}x} \right)^2 + 1024 \left(e^{\frac{3\sqrt{2}}{2}x} \right)^3 \left(e^{\frac{\sqrt{2}}{2}x} \right) \\
 & \quad - 144 \left(e^{\frac{\sqrt{2}}{2}x} \right) \left(e^{\frac{3\sqrt{2}}{2}x} \right) - 324 \left(e^{\frac{3\sqrt{2}}{2}x} \right)^2 + 500 \left(e^{\frac{\sqrt{2}}{2}x} \right)^2 + 1399 \left(e^{\frac{\sqrt{2}}{2}x} \right)^2 \left(e^{\frac{3\sqrt{2}}{2}x} \right) \\
 & \quad \left. - 2235 \left(e^{\frac{\sqrt{2}}{2}x} \right) \left(e^{\frac{3\sqrt{2}}{2}x} \right)^2 \right] \frac{t^3}{6} + \dots
 \end{aligned} \tag{3.16}$$

The Taylor series expansion for $\frac{\left(e^{\frac{\sqrt{2}}{2}x - \frac{5}{2}t} + 3e^{\frac{3\sqrt{2}}{2}x + \frac{3}{2}t} \right)}{\left(e^{\frac{\sqrt{2}}{2}x - \frac{5}{2}t} + e^{\frac{3\sqrt{2}}{2}x + \frac{3}{2}t} + 1 \right)}$ is written as:

$$\frac{e^{\frac{\sqrt{2}}{2}x - \frac{5}{2}t} + 3e^{\frac{3\sqrt{2}}{2}x + \frac{3}{2}t}}{e^{\frac{\sqrt{2}}{2}x - \frac{5}{2}t} + e^{\frac{3\sqrt{2}}{2}x + \frac{3}{2}t} + 1}$$

$$\begin{aligned}
 &= \frac{e^{\frac{\sqrt{2}}{2}x} + 3e^{\frac{3\sqrt{2}}{2}x}}{e^{\frac{\sqrt{2}}{2}x} + e^{\frac{3\sqrt{2}}{2}x} + 1} + \frac{1}{2} \left(\frac{-5 \left(e^{\frac{\sqrt{2}}{2}x} \right) + 16 \left(e^{\frac{\sqrt{2}}{2}x} \right) \left(e^{\frac{3\sqrt{2}}{2}x} \right) + 9 \left(e^{\frac{3\sqrt{2}}{2}x} \right)}{\left(e^{\frac{\sqrt{2}}{2}x} + e^{\frac{3\sqrt{2}}{2}x} + 1 \right)^2} \right) t \\
 &+ \frac{1}{4} \left(\frac{1}{\left(e^{\frac{\sqrt{2}}{2}x} + e^{\frac{3\sqrt{2}}{2}x} + 1 \right)^3} \right) \left[-25 \left(e^{\frac{\sqrt{2}}{2}x} \right)^2 + 128 \left(e^{\frac{\sqrt{2}}{2}x} \right)^2 \left(e^{\frac{3\sqrt{2}}{2}x} \right) + 140 \left(e^{\frac{\sqrt{2}}{2}x} \right) \left(e^{\frac{3\sqrt{2}}{2}x} \right) \right. \\
 &\quad \left. + 25 \left(e^{\frac{\sqrt{2}}{2}x} \right) - 128 \left(e^{\frac{\sqrt{2}}{2}x} \right) \left(e^{\frac{3\sqrt{2}}{2}x} \right)^2 - 27 \left(e^{\frac{3\sqrt{2}}{2}x} \right)^2 + 27 \left(e^{\frac{3\sqrt{2}}{2}x} \right) \right] \frac{t^2}{2} \\
 &+ \frac{1}{8} \left(\frac{1}{\left(e^{\frac{\sqrt{2}}{2}x} + e^{\frac{3\sqrt{2}}{2}x} + 1 \right)^4} \right) \left[81 \left(e^{\frac{3\sqrt{2}}{2}x} \right)^3 - 125 \left(e^{\frac{\sqrt{2}}{2}x} \right)^3 - 125 \left(e^{\frac{\sqrt{2}}{2}x} \right) + 81 \left(e^{\frac{3\sqrt{2}}{2}x} \right) \right. \\
 &\quad \left. + 1024 \left(e^{\frac{3\sqrt{2}}{2}x} \right) \left(e^{\frac{\sqrt{2}}{2}x} \right)^3 - 4096 \left(e^{\frac{3\sqrt{2}}{2}x} \right)^2 \left(e^{\frac{\sqrt{2}}{2}x} \right)^2 + 1024 \left(e^{\frac{3\sqrt{2}}{2}x} \right)^3 \left(e^{\frac{\sqrt{2}}{2}x} \right) \right. \\
 &\quad \left. - 144 \left(e^{\frac{\sqrt{2}}{2}x} \right) \left(e^{\frac{3\sqrt{2}}{2}x} \right) - 324 \left(e^{\frac{3\sqrt{2}}{2}x} \right)^2 + 500 \left(e^{\frac{\sqrt{2}}{2}x} \right)^2 + 1399 \left(e^{\frac{\sqrt{2}}{2}x} \right)^2 \left(e^{\frac{3\sqrt{2}}{2}x} \right) \right. \\
 &\quad \left. - 2235 \left(e^{\frac{\sqrt{2}}{2}x} \right) \left(e^{\frac{3\sqrt{2}}{2}x} \right)^2 \right] \frac{t^3}{6} + \dots \tag{3.17}
 \end{aligned}$$

By substituting Eq. (3.17) into Eq. (3.16), the Eq. (3.16) can be reduced to:

$$v(x,t) = \frac{e^{\frac{\sqrt{2}}{2}x - \frac{5}{2}t} + 3e^{\frac{3\sqrt{2}}{2}x + \frac{3}{2}t}}{e^{\frac{\sqrt{2}}{2}x - \frac{5}{2}t} + e^{\frac{3\sqrt{2}}{2}x + \frac{3}{2}t} + 1} \tag{3.18}$$

This is the exact solution of the problem, Eq. (3.10). Table 2 shows the trend of rapid convergence of the results of $S_0(x, t) = v_0(x, t)$ to $S_5(x, t) = \sum_{i=0}^5 v_i(x, t)$ using the HPM solution toward the exact solution. The maximum relative error of less than 0.003% is achieved in comparison to the exact solution as shown in table 2.

		Percentage of relative error (%RE)		
		$x = 1.7$	$x = 2.5$	$x = 5$
$t = 0.1$	$S_0(x, t)$	0.02059762271	0.006810842481	0.0001899154162
	$S_1(x, t)$	0.003190548961	0.001279624879	0.00003997269697
	$S_2(x, t)$	0.0002649283847	0.0001518370273	0.000005462433330
	$S_3(x, t)$	0.000002931975582	0.00001163604905	5.516605232 e-7
	$S_4(x, t)$	0.000002282925629	4.085966308 e-7	4.476088152 e-8
	$S_5(x, t)$	3.081340559 e-7	3.258476457 e-8	2.271634917 e-9
$t = 0.3$	$S_0(x, t)$	0.04580565755	0.01481331978	0.0004042559687
	$S_1(x, t)$	0.02372206472	0.009262519008	0.0002852605197
	$S_2(x, t)$	0.006576792092	0.003516833984	0.0001235679891
	$S_3(x, t)$	0.0004692932012	0.0008613757397	0.00003877773369
	$S_4(x, t)$	0.0004180732891	0.0001063796743	0.000009522043300
	$S_5(x, t)$	0.0001953487480	0.00001574454885	0.000001904408054
$t = 0.5$	$S_0(x, t)$	0.05893509801	0.01874318914	0.0005021880149
	$S_1(x, t)$	0.05534996993	0.02122314627	0.0006468935438
	$S_2(x, t)$	0.02765545574	0.01413345558	0.0004886299426
	$S_3(x, t)$	0.004516457152	0.006055179716	0.0002628969901
	$S_4(x, t)$	0.004126679836	0.001382282303	0.0001097499620
	$S_5(x, t)$	0.003653432381	0.0002213565622	0.00003718074636

Table 2 shows: the percentage of relative errors of the results of $S_0(x, t) = v_0(x, t)$ to $S_5(x, t) = \sum_{i=0}^5 v_i(x, t)$ of the HPM solution of Eq. (3.10).

4. CONCLUSION

In the present research work, the exact solution of the Fitzhugh–Nagumo nonlinear diffusion equation is obtained by using the HPM. The validity and effectiveness of the HPM is shown by solving two non-homogenous non-linear differential equations and the very rapid convergence to the exact solutions is also numerically demonstrated. By using the HPM, the Taylor series expansions of the exact solutions

as the Eq. (3.9) and Eq. (3.18) in each problem are obtained which are convergent in the desired domains.

The trend of rapid and monotonic convergence of the solution towards the exact solution is clearly shown by tabulated numerical values. As a result, the maximum relative errors of less than 0.00005% and 0.003% are presented in comparison to the exact solution of two case study problems of the equation. The rapid convergence towards the exact solutions of the HPM indicates that, the HPM is a very powerful and efficient technique with a reasonable amount of computational work and acceptable accuracy in order to obtain the exact solution of the nonlinear differential equations.

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