FE, FI, FO, FOLIUM: A DISCOURSE ON DESCARTES' MATHEMATICAL CURIOSITY

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Abstract. A brief history and biography of Descartes and his scientific work is given followed by some of the mathematical details of a mathematical curiosity called the Folium of Descartes which he discovered in an attempt to challenge Fermat's extremum-finding techniques.

Keywords: Cartesian geometry, Descartes, Folium of Descartes, Parametric equations.

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1. Biographical Notes

René Descartes was born on 31 March, 1596 in the small French town of La Haye near the center of France, which has been renamed Descartes in his honor. He died prematurely of illness about which the circumstances are not totally clear on 11 February, 1650 in Stockholm. Descartes was in poor health during his early years at the Jesuit college of La Fleche where he began his studies at the age of eight; and was given permission to stay in bed until 11 AM. This he developed a lifelong habit; and only broke this custom in 1649 when Queen Christina of Sweden talked him into teaching her to draw tangents every day at 5AM. The most generally accepted account states that after walking to the palace for a few months in the cold Swedish climate Descartes died of pneumonia. But other stories say he was poisoned. Descartes is buried in the Pantheon in Paris.

Today Descartes is remembered most as a Mathematician and Philosopher; but he also had extensive careers as a soldier, teacher, and as a gentleman and world traveler. Descartes obtained a law degree in 1619 from the University of Poitiers in Paris and then enlisted in the military school at Breda. In 1618 he studied mathematics and mechanics under the Dutch scientist Isaac Beeckman after which he began to look for a unified natural science. Following this he joined the Bavarian army in 1619.

Descartes lived during one of the greatest intellectual times in human history; and was a contemporary of Galileo, Fermat, Pascal, Harvey, Huygens, Newton, Milton and Shakespeare to name a few. Perhaps this sufficiently fantastic to warrant using a giblet of giant-speak 'fe fi fo folium' from the fable *Jack and the Beanstalk* in the title of this piece to association with this lofty world of giants in the stalk of history.

Finally Descartes decided to settle down and shortly after he moved to Holland in 1628 he began a major treatise on Physics called *Le Monde, ou Traité de la Lumière*. The book was almost finished when Descartes heard of Galileo's house arrest by the Inquisition and became adamant about not publishing it and soon began writing *La Methode*.

Descartes is most famous for this work (called in full) *Discours De La Methode Pour Bien Conduire Sa Raison et Chercher La Verite Dans Les Sciences* published in Leiden in 1637 when Descartes was forty-one years old. The treatise had three appendices: *La Dioptrique* (on optics), *Les Meteores* (on meteorology), and *La Géométrie* (on Mathematics). Of this work Descartes wrote to his friend Mersenne:

> "I have tried in my *Dioptrique and my Meteores* to show that my *Methode* is better than the vulgar, and in my *Geometrie* to have demonstrated it."

Modern mathematics is founded on two fundamental advances:

1. The introduction of Calculus by Newton
2. The method for the integration of algebra with geometrical proof by Descartes which led to analytic geometry.

The general idea for developing the concept of analytic geometry came to Descartes in a dream on the 10th of November 1619 which is called by some scholars the birthday of modern mathematics.

Descartes was a perennial skeptic and questioned the basis for epistemology especially as it related to Aristotelian logic. Descartes felt that only mathematics was certain and provided the more satisfactory method of knowledge acquisition. This philosophy of Descartes is credited as giving birth to the scientific method.

2. Discussion Of The Folium And Some Of its History

Descartes was first to discuss the folium (leaf in Latin), which he discovered in an attempt to challenge Fermat's extremum-finding techniques, in 1638. Descartes challenged Fermat to find the tangent line at arbitrary points. Fermat achieved success immediately, much to the chagrin of Descartes. In French the Descartes Folium is sometimes called the *noeud de ruban*, or in German - *Kartesisches Blatt.*

Figure 1. This stamp with a graph of the folium was issued by Albania in 1966**.**

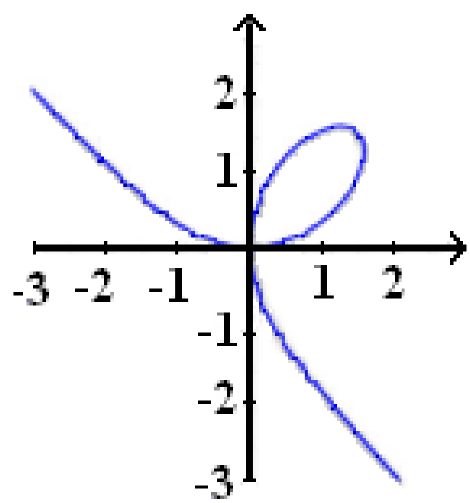

Figure 2. The folium of Descartes described by the implicit equation $x^3 - 3axy + y^3 = 0$.

The folium first appeared in Descartes writings in a section of *La Géométrie* called On the nature of curved lines which as already stated above was an appendix to his *Discours De La Methode*.

3. Mathematics Related To The Folium

3.1 IMPLICIT, EXPLICIT AND PARAMETRIC FUNCTIONS

Most simple functions take the form $y = f(x)$ where y is directly or *explicitly* expressed in terms of x. Typically y is defined as a function of x in terms of an equation of the form

$$F(x,y) = 0. \tag{1}$$

Equation (1) is not solved for *y*, because *x* and *y* are entangled with each other. If *x* is assigned a suitable numerical value, the result generally produces one or more corresponding values of *y*. In this kind of situation it is said that equation (1) determines *y* as an *implicit* function(s) of *x*. For example the trivial equation $xy = 1$ determines a single *implicit* function of x, which could be written as an *explicit* equation in the form

$$y = \frac{1}{x}. \tag{2}$$

The *implicit* equation $x^2 + y^2 = 144$ determines two functions of x, which can be written in *explicit* form as

$$y = \sqrt{144 - x^2} \qquad \text{And} \qquad y = -\sqrt{144 - x^2}.$$

As generally known, the graphic curves of these two functions are the upper and lower halves of a circle of radius 12. The *implicit* equation

$$x^3 + y^3 = 3axy \text{ with } (a > 0), \tag{3}$$

which is the general form for the Folium of Descartes also has a number of *implicit* functions. However the solving this equation for *y* is so challenging that it is usually not attempted.

By thinking of a curve as the plot of the path of a moving point, it is usually easier to study the curve by using two simple equations that describe *x* and *y* in terms of a third independent variable *t* as in

$$x = f(t) \qquad \text{and} \qquad y = g(t) \tag{4}$$

This makes studying the curve considerably more simple than using the implicit form as in equation (1). The variable *t* can be considered the time elements of the points motion during any interval $t_1 \leq t \leq t_2$. This third variable *t* is called a parameter of equation (1) and so the equations in (4) are called *parametric* equations of the curve. To revert back to the rectangular form (implicit and explicit) one simply removes the extra parameter from the equation.

A parametric equation is an equation of a curve expressed in the form of the parameters of the equation that locate points on the curve. For example the parametric equations of a straight line are $x = a + bt$, $y = c + dt$; and for a circle they are x = d cos θ, y = a sin θ.

3.2 MATHEMATICS OF THE FOLIUM - ASYMPTOTE AND TANGENT

Thus according to the insights above the parametric equations of the Folium of Descartes take the following form:

$$x(t) = \frac{3at}{1+t^3} \qquad \text{and} \qquad y(t) = \frac{3at^2}{1+t^3} \tag{5}$$

The folium, as can be readily observed from the graph in Fig. 1 has an **asymptote** as shown by the equation

$$y = -x - 1. \tag{6}$$

The equation of the **tangent** at the point where $t = p$ is

$$p(p^3 - 2)x + (1 - 2p^3)y + 3ap^2 = 0. \tag{7}$$

3.2 MATHEMATICS OF THE FOLIUM - AREA OF THE LOOP

The **area of the loop** of Descartes Folium is more readily found by using the polar equation for the folium and evaluating the area integral by substituting $u = \tan\theta$. The solution takes the general form $3a^2/2$. Using the parametric equations in (5) the area of the loop can be calculated using Green's theorem as given by the formula

$$A = \frac{1}{2}\oint_C -ydx + xdy \quad (8)$$

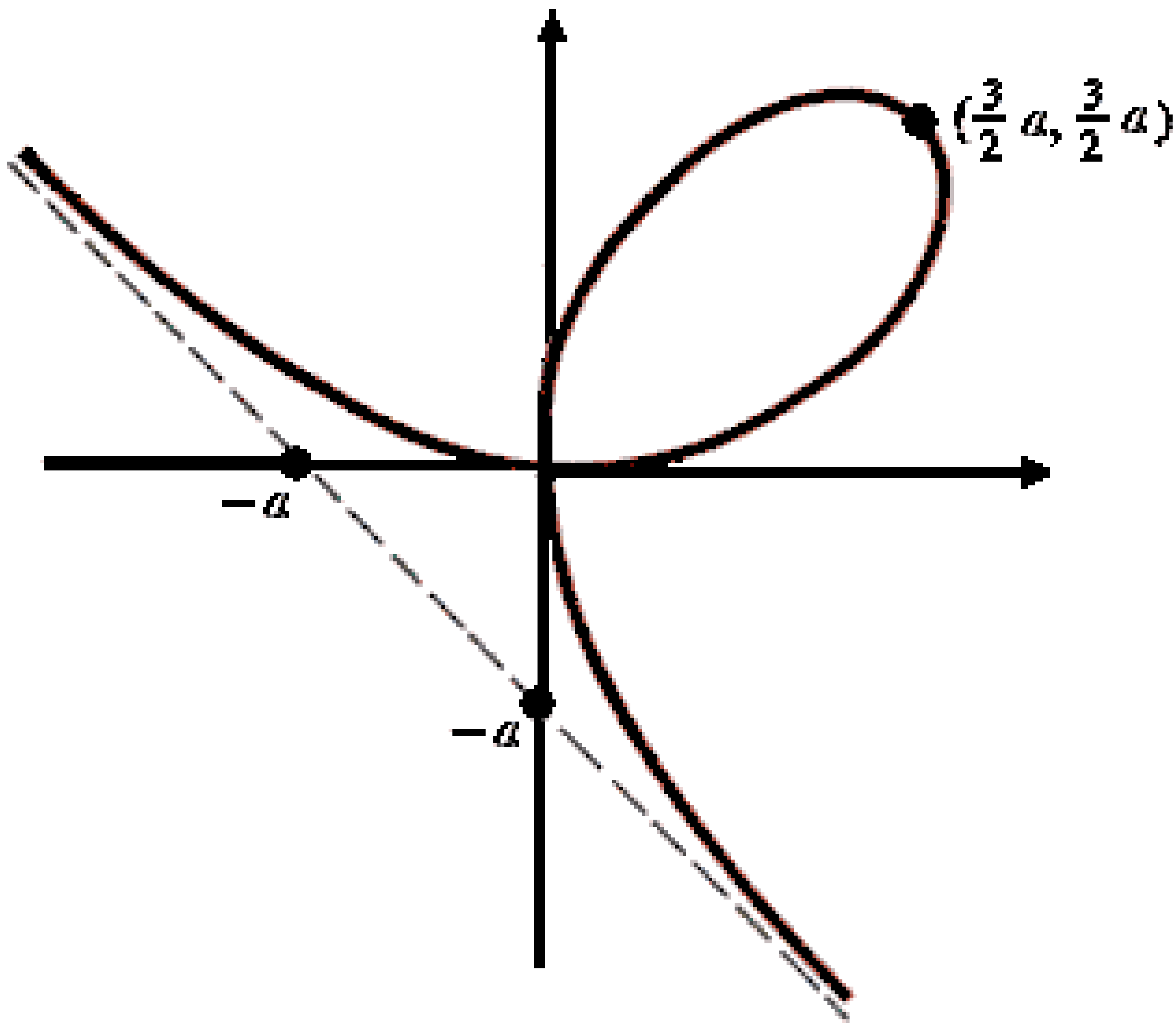

Figure 3. Graphic representation of the Folium of Descartes showing the relationship the constant *a* has to the curve and its asymptote.

For ease of working first we must convert the Cartesian (rectangular or implicit) form of the Folium as in equation (3) to it's parametric form as shown in equation (5)

$$x^3 + y^3 = 3axy \quad \Rightarrow \qquad x(t) = \frac{3at}{1+t^3} \quad \text{and} \quad y(t) = \frac{3at^2}{1+t^3} \quad .$$

Next the parametric form is converted to polar form

$$r^2 = \frac{(3at)^2(1+t^2)}{(1+t^3)^2} \tag{9}$$

With $\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1} t$ and $d\theta = \frac{dt}{1+t^2}$, the general equation for the area of a loop from equation (8) becomes

$$A = \frac{1}{2}\int_1^\infty \frac{(3at)^2(1+t^2)}{(1+t^3)^2}\frac{dt}{1+t^2} \quad = \frac{3}{2}a^2\int_1^\infty \frac{(3t)^2 dt}{(1+t^3)^2}. \tag{10}$$

To perform the integration we use $u = 1+t^3$ and therefore $du = 3t^2dt.$ Then the integral becomes

$$\frac{3}{2}a^2\int_1^\infty \frac{du}{u^2} \quad = \frac{3}{2}a^2\left[-\frac{1}{u}\right]_1^\infty \quad = \frac{3}{2}a^2(\theta+1) \quad = \frac{3}{2}a^2. \tag{11}$$

3.3 GRAPHING THE FOLIUM

It is more of a challenge to plot the graph of the Folium of Descartes without graphing software. It is easy to see that the graph of the folium is symmetric about the line $y = x$ because if one reverses y and x in the equation the curve stays exactly the same. All the graphs have a maximum when $t = 2^{1/3}$. If one tried to use the *implicit* equation (3) one can't simply create a table of x and y values from which to plot he curve since one cannot make either x or y the subject of the equation. To plot the equation in this manner one would have to try testing every point on the x - y plane to see if it fit the equation to some degree.

In the parametric form, equations (5) this difficulty is overcome because all one has to do is take several values of *t,* calculate the x and y values and then plot (x,y). In a program like Mathematica one utilizes the command:

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ParametricPlot [{3*t/(1+t^3),3*t^2(1+t^3)},{t,-100,100}, PlotRange ->{-1.5,2}]
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to obtain a preliminary graph.

The folium has a discontinuity at $t = -1$. The left wing is formed when t runs from -1 to 0, the loop as t runs from 0 to ∞ , and the right wing as t runs from ∞ to -1. How does the folium change for different values of a?

The **maximum** of the curve occurs when $dy/dx = 0$, this is when $ay = x^2$. Using $y = tx$ for the parametric form of the equation gives $atx = x^2$ and when $x \neq 0$ this is at $x = 3at/(1 + t^3)$ so $1 + t^3 = 3$ and thus $t = 2^{1/3}$ is at the stationary point. Looking at the parametric form, this occurs at the point $(2^{1/3}a, 2^{2/3}a)$. If one puts in an x-value just before and just after this confirms the point to be a maximum.

Other main graphical features of the folium in terms of t are:

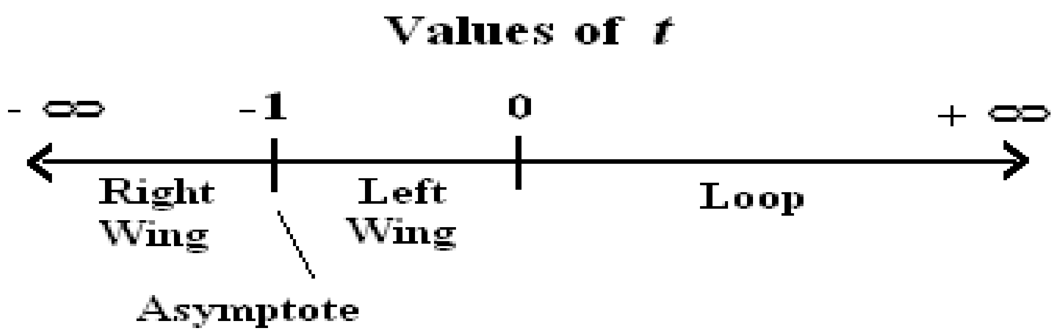
1. For $-\infty < t < -1$ from the parametric form of the equation the graphs lies in the fourth quadrant because x *is* positive and y is negative.

2. For $-1 < t < 0$ t the graphs lie in the second quadrant because x negative and y is positive.

3. For $0 < t < 1$ $3at > 0$ and $t^3 > 0$ so the graphs lie in the first quadrant because $0 < y < x$.

4. For $1 < t < +\infty$ Again when $3at > 0$ and $1 + t^3 > 0$ the graphs again lie in the first quadrant because x and y are positive $(0 < x < y)$ for as $y = tx$ then $x < y$.

What would happen to the graphs if $t = -1$? If $t = -1$ the graph is undefined; but as t approaches - 1 the absolute values of x and y move toward the asymptote of the curve at $y = -x$. This values of t can be seen in Fig. 4 below.



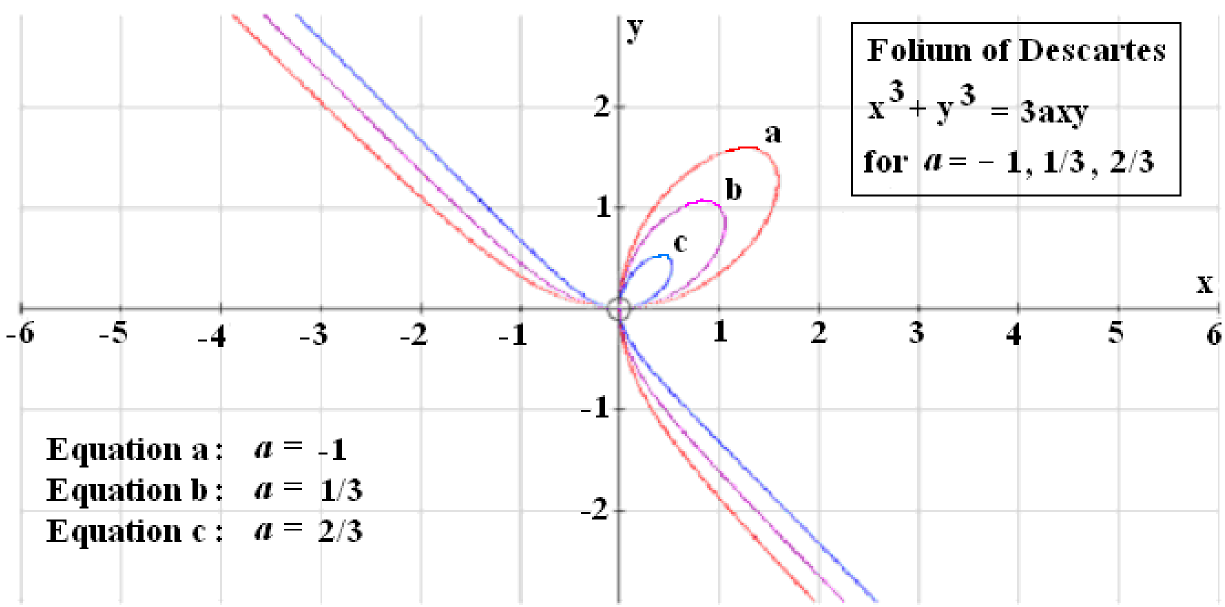


Figure 5. Variations in the graph of the Folium as *a* changes from positive to negative between -1 and 2/3.

A final consideration for graphing the curve of Descartes' Folium is the following. Is there a point such that $y = x$? Considering the equation, $2x^3=3ax^2$ at such a point where $y = x$; since we already know the point (0,0) is on the graph, for other points we can divide by x^2. We get $x = y = 3a/2$ which corresponds to $t = 1$. The important thing about this point is that considering our formula for dy/dx, if $x = y$ then $dy/dx = -1$ and so then we know that at this point the gradient is the same as that of the line $y = -x$. This tells us that (since the curve is symmetrical about $y = x$ and using parts [3] and [4] above concerning graphical features) there is a 'loop' at this point.

4. End Note

The Folium of Descartes, although seemingly little more than a curiosity today, played a role in the early days of the development of calculus; and is still (as perhaps you noticed) quite fascinating to *mathemagicians*! As one notes above in terms of the multiple forms the folium equation takes, especially relative to the challenges of plotting the curve in a simple manner, Descartes challenge to Fermat's method of finding tangents clearly demonstrates the elegance and beauty of calculus.

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