Section 1.5. Taylor Series Expansions

In the previous section, we learned that any power series represents a function and that it is very easy to differentiate or integrate a power series function. In this section, we are going to use power series to represent and then to approximate general functions. Let us start with the formula

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad \text{for } |x| < 1. \tag{1}$$

We call the power series the power series representation (or expansion) for the function

$$f(x) = \frac{1}{1-x} \quad \text{about } x = 0.$$

It is very important to recognize that though the function $f(x) = (1-x)^{-1}$ is defined for all $x \neq 1$, the representation holds only for $|x| < 1$. In general, if a function $f(x)$ can be represented by a power series as

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

then we call this power series

power series representation (or expansion) of $f(x)$ about $x = a$.

We often refer to the power series as

Taylor series expansion of $f(x)$ about $x = a$.

Note that for the same function $f(x)$, its Taylor series expansion about $x = b$,

$$f(x) = \sum_{n=0}^{\infty} d_n (x-b)^n$$

if $a \neq b$, is completely different from the Taylor series expansion about $x = a$. Generally speaking, the interval of convergence for the representing Taylor series may be different from the domain of the function.

Example 5.1. Find Taylor series expansion at given point $x = a$:

(a) $f(x) = \dfrac{1}{1+x^2}, \ a = 0;$

(b) $g(x) = \dfrac{x}{x+2}$, $a = 0$;

(c) $h(x) = \dfrac{1}{2x+3}$, $a = 1$.

Solution: (a) We shall use (1) by first rewriting the function as follows:

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} \overset{y=-x^2}{=} \frac{1}{1-y} = \sum_{n=0}^{\infty} y^n, \quad \text{for } |y| < 1.$$

Formula (1) leads to

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} y^n = \sum_{n=0}^{\infty} \left(-x^2\right)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}, \quad \text{for } |y| < 1.$$

Note that, since $y = -x^2$,and

$$|y| < 1 \iff \left|-x^2\right| < 1 \iff |x| < 1 \ ,$$

we know conclude

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}, \quad \text{for } |x| < 1.$$

(b) Write

$$\begin{aligned} g(x) &= \frac{x}{x+2} = x\left(\frac{1}{2+x}\right) = x\left(\frac{1}{2(1+x/2)}\right) \\ &= \frac{x}{2} \cdot \frac{1}{1+x/2} = \frac{x}{2} \cdot \frac{1}{1-(-x/2)}. \end{aligned} \tag{2}$$

We now use (1) to derive

$$\begin{aligned} \frac{1}{1-(-x/2)} \overset{y=-x/2}{=} \frac{1}{1-y} &= \sum_{n=0}^{\infty} y^n \\ &= \sum_{n=0}^{\infty} \left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} x^n, \text{ for } |y| < 1. \end{aligned}$$

Substituting this into (2), we obtain

$$\begin{aligned} \frac{x}{x+2} &= \frac{x}{2} \cdot \frac{1}{1-(-x/2)} = \frac{x}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} x^n \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^{n+1}, \text{ for } |y| < 1. \end{aligned}$$

Now since

$$|y| = \left|\frac{x}{2}\right| < 1 \quad \Longleftrightarrow \quad |x| < 2,$$

we conclude

$$\frac{x}{x+2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^{n+1}, \text{ for } |x| < 2.$$

(c) For

$$h(x) = \frac{1}{2x+3}, \; a = 1,$$

we need to rewrite the denominator in terms of $(x-1)$as follows:

$$\begin{aligned}
\frac{1}{2x+3} &= \frac{1}{2[(x-1)+1]+3} = \frac{1}{2(x-1)+5} \\
&= \frac{1}{5[1+2(x-1)/5]} = \frac{1}{5}\frac{1}{1+2(x-1)/5} \\
&\overset{y=-2(x-1)/5}{=} \frac{1}{5}\frac{1}{1-y} = \frac{1}{5}\sum_{n=0}^{\infty} y^n \qquad (\text{for } |y| < 1).
\end{aligned}$$

We then substitute $y = -\dfrac{2(x-1)}{5}$ back to obtain

$$\begin{aligned}
h(x) = \frac{1}{2x+3} &= \frac{1}{5}\sum_{n=0}^{\infty} y^n \\
&= \frac{1}{5}\sum_{n=0}^{\infty} \left(-\frac{2(x-1)}{5}\right)^n \qquad \left(\text{for } |y| = \left|\frac{2(x-1)}{5}\right| < 1\right) \\
&= \frac{1}{5}\sum_{n=0}^{\infty} (-1)^n \left(\frac{2}{5}\right)^n (x-1)^n \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{5^{n+1}} (x-1)^n, \quad \text{for } |x-1| < \frac{5}{2}.
\end{aligned}$$

Example 5.2. Find Taylor series about $a = 0$ for
(a) $f(x) = \dfrac{1}{(1-x)^2}$;
(b) $g(x) = \ln(1-x)$;
(c) $h(x) = \arctan x$.

Solution: (a) Differentiate

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad \text{for } |x| < 1,$$

we obtain

$$f = \frac{1}{(1-x)^2} = \left(\frac{1}{1-x}\right)' = \sum_{n=0}^{\infty} (x^n)' = \sum_{n=1}^{\infty} n x^{n-1}.$$

(b) Take anti-derivative on both sides of

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad \text{for } |x| < 1,$$

we obtain

$$\int \frac{1}{1-x} dx = \sum_{n=0}^{\infty} \left(\int x^n dx\right) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} + C.$$

So

$$\ln(1-x) = -\int \frac{1}{1-x} dx = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} - C.$$

To determine the constant, we insert $x = 0$ into both sides:

$$0 = \ln(1-0) = -\left[\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}\right]_{x=0} - C = -C.$$

We have to choose $C = 0$ and

$$\ln(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = -\sum_{n=1}^{\infty} \frac{x^n}{n}. \qquad \text{(Memorize it)}$$

(c) Note

$$h' = \frac{d}{dx} \arctan x = \frac{1}{1+x^2}.$$

So

$$h(x) = \arctan x = \int \frac{1}{1+x^2} dx.$$

From **Example 5.1 (a),** we know

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}.$$

Thus

$$\arctan x = \int \frac{1}{1+x^2} dx = \sum_{n=0}^{\infty} (-1)^n \int x^{2n} dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} + C.$$

By setting $x = 0$ above, we find $C = 0$. So

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}.$$

Taylor Series for General Functions.

Consider power series expansion

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + c_3 (x-a)^3 + ... \quad (3)$$

for general function $f(x)$ about $x = a$. Setting $x = a$, we obtain

$$f(a) = c_0.$$

Next, we take derivative on (3) so that

$$f'(x) = \sum_{n=1}^{\infty} c_n n (x-a)^{n-1} = c_1 + c_2 \cdot 2 (x-a) + c_3 \cdot 3 (x-a)^2 + c_4 \cdot 4 (x-a)^3 + ... \quad (4)$$

Setting $x = a$, we have

$$f'(a) = c_1.$$

We repeat the same process again and again: take derivative again on (4)

$$f''(x) = \sum_{n=2}^{\infty} c_n n (n-1) (x-a)^{n-2} = c_2 \cdot 2 \cdot 1 + c_3 \cdot 3 \cdot 2 (x-a) + c_4 \cdot 4 \cdot 3 (x-a)^2 + ... \quad (5)$$

and set $x = a$ to obtain

$$f''(a) = c_2 \cdot 2 \cdot 1 \Longrightarrow c_2 = \frac{f''(a)}{2!};$$

take derivative again on (5)

$$f^{(3)}(x) = \sum_{n=3}^{\infty} c_n n(n-1)(n-2)(x-a)^{n-3} = c_3 3 \cdot 2 \cdot 1 + c_4 4 \cdot 3 \cdot 2(x-a) + c_5 5 \cdot 4 \cdot 3(x-a)^2 + ...$$

and insert $x = a$ to obtain

$$f^{(3)}(a) = c_3 3 \cdot 2 \cdot 1 \Longrightarrow c_3 = \frac{f^{(3)}(a)}{3!}.$$

In general, we have

$$c_n = \frac{f^{(n)}(a)}{n!}, \quad n = 0, 1, 2, ...$$

here we adopt the convention that $0! = 1$. All above process can be carried on as long as any number of order of derivative at $x = a$ exists, i.e., $f(x)$ must be a smooth function near a. Then, we have Taylor series expansion formula

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n. \qquad \text{(Taylor Series)}$$

When $a = 0$, it becomes

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n, \qquad \text{(Maclaurin Series)}$$

we call it Maclaurin Series of $f(x)$.

Example 5.3. Find Maclaurin series for

(a) $f(x) = e^x$;

(b) $g(x) = b^x \quad (b > 0)$

Solution: (a) For $f = e^x$, we know

$$f' = e^x, \ f'' = e^x, ..., f^{(n)} = e^x.$$

Thus

$$c_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{n!},$$

$$e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^n. \qquad \text{(Maclaurin Series For } e^x\text{)}$$

This is one of the most useful Taylor series, and must be memorized.

(b) We offer two methods to solve this problem. First is the direct method by using formula for Maclaurin Series. To this end, we compute derivatives

$$\begin{aligned} g' &= b^x \ln b \\ g'' &= (b^x)' \ln b = (b^x \ln b) \ln b = b^x (\ln b)^2, \\ &\dots \\ g^{(n)} &= b^x (\ln b)^n. \end{aligned}$$

So

$$b^x = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{(\ln b)^n}{n!} x^n.$$

Another method is to use Taylor series for e^x above. Write

$$b^x = e^{\ln(b^x)} = e^{(x \ln b)} \overset{y = x \ln b}{=} e^y = \sum_{n=0}^{\infty} \frac{1}{n!} y^n = \sum_{n=0}^{\infty} \frac{1}{n!} (x \ln b)^n = \sum_{n=0}^{\infty} \frac{(\ln b)^n}{n!} x^n.$$

Example 5.4. Find Maclaurin series for

(a) $f(x) = \sin x$;

(b) $g(x) = \cos x$.

Solution: (a) We observe that

$$\begin{aligned} f &= \sin x \implies f(0) = 0, \\ f' &= \cos x \implies f'(0) = 1, \\ f\text{''} &= -\sin x \implies f\text{''}(0) = 0, \\ f^{(3)} &= -\cos x \implies f^{(3)}(0) = -1 \\ f^{(4)} &= \sin x \implies f^{(4)}(0) = 0, \end{aligned}$$

and that this cyclic pattern repeats every 4 times differentiations. In particular, we see that when n is even, i.e., $n = 2m$, $f^{(n)}(0) = 0$. When n is odd, i.e., $n = 2m+1$, $f^{(n)}(0)$ equals 1 and -1 alternating. Thus,

$$\begin{aligned}
\sin x &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \\
&= \sum_{\substack{n=0 \\ n=odd}}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \\
&= \sum_{m=0}^{\infty} \frac{f^{(2m+1)}(0)}{(2m+1)!} x^{2m+1} \\
&= \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m+1} \qquad \text{(Maclaurin Series for } \cos x) \\
&= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...
\end{aligned}$$

(b) Maclaurin series for $\cos x$ may be derived analogously. Another simple way to find Maclaurin series for $\cos x$ is to use the above Maclaurin series for $\sin x$. We know that $\cos x = (\sin x)'$. So

$$\begin{aligned}
\cos x &= \left(\sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m+1} \right)' \\
&= \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} (2m+1) x^{2m} \\
&= \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} x^{2m} \qquad \text{(Maclaruin Series for } \cos x) \\
&= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...
\end{aligned}$$

Example 5.5. Some applications.
(a) Find Maclaurin series for $x \sin(2x)$;
(b) Find Maclaurin series for $\int e^{-x^2} dx$;
(c) Find the limit $\lim_{x \to 0} \frac{e^x - 1 - x - x^2/2}{x^3}$.

Solution: (a) Set $y = 2x$, and use Maclaurin Series for $\sin x$ to get

$$\begin{aligned}
\sin(2x) &= \sin y \\
&= \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} y^{2m+1} \\
&= \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} (2x)^{2m+1} \\
&= (2x) - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \ldots.
\end{aligned}$$

So

$$\begin{aligned}
x\sin(2x) &= x \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} (2x)^{2m+1} \\
&= \sum_{m=0}^{\infty} \frac{(-1)^m 2^{2m+1}}{(2m+1)!} x^{2m+2}.
\end{aligned}$$

(b) Letting $y = -x^2$ and using Maclaurin Series for e^x, we find

$$\begin{aligned}
e^{-x^2} &= e^y \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} y^n \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \left(-x^2\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int e^{-x^2} dx &= \sum_{n=0}^{\infty} \int \frac{(-1)^n}{n!} x^{2n} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} x^{2n+1} + C.
\end{aligned}$$

(c) Again, we use Maclaurin Series for e^x

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + O\left(x^4\right)$$

to get

$$e^x - 1 - x - \frac{x^2}{2} = \frac{x^3}{3!} + O\left(x^4\right)$$

and consequently

$$\frac{e^x - 1 - x - \dfrac{x^2}{2}}{x^3} = \frac{1}{3!} + \frac{O\left(x^4\right)}{x^3} = \frac{1}{6} + O\left(x\right).$$

So

$$\lim_{x \to 0} \frac{e^x - 1 - x - \dfrac{x^2}{2}}{x^3} = \frac{1}{6}.$$

Taylor Polynomials.

Consider Taylor series expansion

$$f\left(x\right) = \sum_{n=0}^{\infty} \frac{f^{(n)}\left(a\right)}{n!} \left(x - a\right)^n$$

for a function $f\left(x\right)$. The partial sum of order n,

$$T_n\left(x\right) = \sum_{k=0}^{n} \frac{f^{(k)}\left(a\right)}{k!} \left(x - a\right)^k,$$

is called Taylor Polynomial of order n. This is often used for approximation. When $n = 1$,

$$T_1\left(x\right) = \sum_{k=0}^{1} \frac{f^{(k)}\left(a\right)}{k!} \left(x - a\right)^k = f\left(a\right) + f'\left(a\right)\left(x - a\right)$$

is exactly the linear approximation of $f\left(x\right)$ about $x = a$, as we learned in Calculus I. When $n = 2$,

$$T_2\left(x\right) = \sum_{k=0}^{2} \frac{f^{(k)}\left(a\right)}{k!} \left(x - a\right)^k = f\left(a\right) + f'\left(a\right)\left(x - a\right) + \frac{f"\left(a\right)}{2!} \left(x - a\right)^2$$

is called Quadratic approximation. In general, $T_n\left(x\right)$ is considered as the approximation to $f\left(x\right)$ of order n, and this approximation is valid only when x is very close to a.

Example 5.6. Find $T_3(x)$ about $x = 0$ for
(a) $e^x \sin x$;
(b) $\tan x$.
Solution. (a) Using Maclaurin expansions, we have

$$\begin{aligned} e^x \sin x &= \left(\sum_{n=0}^{\infty} \frac{x^n}{n!} \right) \left(\sum_{n=0} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \right) \\ &= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + ... \right) \left(x - \frac{x^3}{3!} + ... \right) \\ &= \left(x - \frac{x^3}{3!} \right) + x(x) + \frac{x^2}{2}(x) + O\left(x^4\right) \\ &= x + x^2 + \frac{x^3}{3} + O\left(x^4\right), \end{aligned}$$

where

$O\left(x^4\right)$ indicates the sum of all terms being equal or higher order than x^4.

So

$$T_3 = x + x^2 + \frac{x^3}{3}.$$

(b) Solution 1.

$$\begin{aligned}
\tan x = \frac{\sin x}{\cos x} &= \frac{\sum_{n=0} \dfrac{(-1)^n x^{2n+1}}{(2n+1)!}}{\sum_{n=0} \dfrac{(-1)^n x^{2n}}{(2n)!}} \\
&= \frac{\left(x - \dfrac{x^3}{3!} + O\left(x^5\right)\right)}{\left(1 - \dfrac{x^2}{2!} + O\left(x^4\right)\right)} \\
&= \frac{\left(x - \dfrac{x^3}{3!} + O\left(x^5\right)\right)}{1-y} \qquad\qquad \left(y = \frac{x^2}{2!} + O\left(x^4\right)\right) \\
&= \left(x - \frac{x^3}{3!} + O\left(x^5\right)\right)\left(1 + y + y^2 + ...\right) \\
&= \left(x - \frac{x^3}{3!} + O\left(x^5\right)\right)\left(1 + \frac{x^2}{2!} + O\left(x^4\right)\right) \\
&= \left(x - \frac{x^3}{3!}\right) + x\left(\frac{x^2}{2!}\right) + O\left(x^4\right) \\
&= x + \frac{x^3}{3} + O\left(x^4\right).
\end{aligned}$$

So

$$T_3 = x + \frac{x^3}{3}.$$

Solution 2: We use formula

$$T_3(x) = \sum_{k=0}^{3} \frac{f^{(k)}(0)}{k!} x^k.$$

Now

$$\begin{aligned}
\tan 0 &= 0, \\
(\tan x)' &= \sec^2 x, \quad (\tan x)'|_{x=0} = 1 \\
(\tan x)'' &= 2\tan x \sec^2 x, \quad (\tan x)''|_{x=0} = 0 \\
(\tan x)''' &= 2(\tan x)' \sec^2 x + 2\tan x \left(\sec^2 x\right)' \\
(\tan x)'''|_{x=0} &= 2(\tan x)'|_{x=0} = 2.
\end{aligned}$$

So

$$T_3(x) = \sum_{k=0}^{3} \frac{f^{(k)}(0)}{k!} x^k = x + \frac{2}{3!} x^3 = x + \frac{x^3}{3}.$$

Homework (problems marked with * are optional):

1. Find Taylor series and determine the interval of convergence.

   (a) $f(x) = \dfrac{3}{1-x^4}, \quad a = 0$

   (b) $f(x) = \dfrac{x}{4x+1}, \quad a = 0$

   (c) $f(x) = \ln(x+1), \quad a = 0$

   (d) $*f(x) = \ln x, \quad a = 2$ ($hint$: write $\ln x = \ln(1 - (2 - x))$)

   (e) $*f(x) = x^{-1}, \quad a = 1$ ($hint$: write $x^{-1} = \dfrac{1}{1-(1-x)}$)

   (f) $*f(x) = 2^x, \quad a = 1$

2. Using a Maclaurin series derived in this section to obtain the Maclaurin series for the given function.

   (a) $f(x) = e^{-x/2}$

   (b) $f(x) = x\cos(2x)$

   (c) $f(x) = \sin(x^4)$

3. Find the Taylor polynomial of degree 3 about the given point a.

   (a) $f(x) = \cos(2x)\, e^{-x/2}, \quad a = 0.$

   (b) $f(x) = \dfrac{2^x}{1-x}, \quad a = 0.$

4. *Evaluate the indefinite integral as a power series, and determine the radius of convergence.

   (a) $*\int \dfrac{3}{1-x^4} dx$

   (b) $*\int \dfrac{\ln(1-t)}{t} dt$

5. *Use the first 3 terms of a power series to approximate the definite integral:

$$\int_0^{0.4} \ln\left(1 + x^2\right) dx.$$