Representation of functions as power series

Dr. Philippe B. Laval
Kennesaw State University

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Abstract

This document is a summary of the theory and techniques used to represent functions as power series.

1 Representation of Functions as Power Series (8.6)

1.1 Theory

In this section, we develop several techniques to help us represent a function as a power series. More precisely, given a function $f(x)$, we will try to find a power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ such that $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$. Part of the work will involve finding the values of x for which this is valid. Part of the reason for doing this is that a power series looks like a polynomial (except that it has infinitely many terms). Polynomials are among the easiest functions to work with. They are easy to differentiate, integrate, ... So, if f is a complicated function, replacing it with a power series amounts to replacing it with a polynomial. Therefore, working with f becomes easier. There are some technical difficulties to resolve, we will address those as we develop the technique.

First, we will look at techniques which will allow us to obtain a series representation by using known series representations. The techniques involved are substitution, differentiation and integration. Then, we will learn a technique which will allow us to find a series representation for a given function f directly, without having to use known series.

We now look at each technique (substitution, differentiation and integration) in details, using examples. At this point, we only know the following series representation:

$$
\begin{aligned}
\frac{1}{1-x} &= 1 + x + x^2 + x^3 + \dots \text{ in } (-1,1) \\
&= \sum_{n=0}^{\infty} x^n
\end{aligned}
$$

When finding the power series of a function, you must find both the series representation and when this representation is valid (its domain).

1.2 Substitution

We derive the series for a given function using another function for which we already have a power series representation. Then, we do the following:

1. Figure out which substitution can be applied to transform the function for which we know the series representation to the function for which we want a series representation.

2. Apply the same substitution to the known series. This will give us the series representation we wanted.

3. The domain of the new function is obtained by applying the same substitution to the domain of the known series.

Example 1 *Find a power series representation for* $f(x) = \dfrac{1}{1+x}$ *and its domain.*

We use the representation of $\dfrac{1}{1-x}$, *replacing* x *by* $-x$. *We obtain:*

$$
\begin{aligned}
\frac{1}{1+x} &= \frac{1}{1-(-x)} \\
&= 1 + (-x) + (-x)^2 + (-x)^3 + \dots \\
&= 1 - x + x^2 - x^3 + \dots \\
&= \sum_{n=0}^{\infty} (-1)^n x^n
\end{aligned}
$$

The series representation for $\dfrac{1}{1-x}$ *was valid if* $|x| < 1$. *If we apply the same substitution, we see that this representation is valid if* $|-x| < 1$ *or* $|x| < 1$. *In other words, the interval of convergence is also* $(-1, 1)$.

Example 2 *Find a power series representation for* $f(x) = \dfrac{1}{1-x^2}$.

Proceeding as above and remembering that $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, *we have:*

$$
\begin{aligned}
\frac{1}{1-x^2} &= \sum_{n=0}^{\infty} \left(x^2\right)^n \\
&= \sum_{n=0}^{\infty} x^{2n} \\
&= 1 + x^2 + x^4 + \dots
\end{aligned}
$$

2

This is a geometric series which converges when $\left|x^2\right| < 1$ that is when $x^2 < 1$ or $-1 < x < 1$.

Example 3 *Find a power series representation for $\dfrac{1}{2+x}$ and find its domain.*

First, we rewrite $\dfrac{1}{2+x} = \dfrac{1}{2}\dfrac{1}{1+\dfrac{x}{2}}$. Now, we find a power series representation

for $\dfrac{1}{1+\dfrac{x}{2}}$, then we will multiply it by $\dfrac{1}{2}$. $\dfrac{1}{1+\dfrac{x}{2}}$ can be obtained from $\dfrac{1}{1-x}$ by

replacing x by $-\dfrac{x}{2}$. Since $\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n$, we have:

$$\frac{1}{1+\dfrac{x}{2}} = \sum_{n=0}^{\infty} \left(-\frac{x}{2}\right)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n \left(\frac{x}{2}\right)^n$$

Therefore,

$$\frac{1}{2+x} = \frac{1}{2}\sum_{n=0}^{\infty} (-1)^n \left(\frac{x}{2}\right)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{2^{n+1}}$$

$\dfrac{1}{1-x}$ *converges when $|x| < 1$, so this series converges when $\left|-\dfrac{x}{2}\right| < 1 \implies |x| <$ 2. So, the interval of convergence is $(-2, 2)$, the radius of convergence is 2.*

Remark 4 *In the first of these two examples, we applied the substitution to the expanded form of $\dfrac{1}{1-x}$. In the second, we applied it to the compact form. You can do it either way, it is simply a matter of choice.*

Example 5 *Suppose that you are given that a series repersentation for e^x is $e^x = \displaystyle\sum_{n=0}^{\infty} \dfrac{x^n}{n!}$ in $(-\infty, \infty)$. What is a series representation for e^{x^2} and what is its domain?*

We go from e^x to e^{x^2} using the substitution $x \to x^2$. Thus

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{\left(x^2\right)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$$

also valid in $(-\infty, \infty)$

1.3   Differentiation and Integration

The driving force behind the integration and differentiation techniques is the theorem below which we state without proof.

Theorem 6   *If the power series $\sum_{n=0}^{\infty} c_n (x - a)^n$ has a radius of convergence $R > 0$ then the function defined by $f(x) = \sum_{n=0}^{\infty} c_n (x - a)^n = c_0 + c_1 (x - a) + c_2 (x - a)^2 + ...$ is differentiable (hence) continuous on $(a - R, a + R)$ and*

1. *$f'(x) = c_1 + 2c_2 (x - a) + 3c_3 (x - a)^2 + ...$ In other words, the series can be differentiated term by term.*

2. *$\int f(x)\, dx = C + c_0 (x - a) + c_1 \dfrac{(x - a)^2}{2} + c_2 \dfrac{(x - a)^3}{3} + ...$ In other words, the series can be integrated term by term.*

3. *Note that whether we differentiate or integrate, the radius of convergence is preserved. However, convergence at the endpoints must be investigated every time.*

Remark 7   *This theorem simply says that the sum rule for derivatives and integrals also applies to power series. Remember that a power series is a sum, but it is an infinite sums. So, in general, the results we know for finite sums do not apply to infinite sums. The theorem above says that it does in the case of infinite series.*

Remark 8   *The formula in part 1 of the theorem is obtained simply by differentiating the series term by term. Since*

$$f(x) = c_0 + c_1 (x - a) + c_2 (x - a)^2 + ...$$

then

$$
\begin{aligned}
f'(x) &= \left(c_0 + c_1 (x - a) + c_2 (x - a)^2 + ... \right)' \\
&= (c_0)' + (c_1 (x - a))' + \left(c_2 (x - a)^2 \right)' + ... \\
&= 0 + c_1 + 2c_2 (x - a) + 3c_3 (x - a)^2 + ... \quad\quad (1)
\end{aligned}
$$

Alternatively, one can also differentiate the general formula. Since

$$f(x) = \sum_{n=0}^{\infty} c_n (x - a)^n$$

then

$$f'(x) = \left(\sum_{n=0}^{\infty} c_n (x-a)^n \right)'$$

$$= \sum_{n=0}^{\infty} \left(c_n (x-a)^n \right)' \text{ by the theorem}$$

$$= \sum_{n=0}^{\infty} n c_n (x-a)^{n-1}$$

The first term of this series (when $n = 0$) is 0, thus we can start summation at $n = 1$. Hence, we have

$$f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1} \qquad (2)$$

The reader should check that formulas 1 and 2 are identical.

Remark 9 *The formula in part 2 of the theorem is obtained by integrating term by term. It can also be obtained by integrating the general formula. Since*

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

then

$$\int f(x)\, dx = \int \left(\sum_{n=0}^{\infty} c_n (x-a)^n \right) dx$$

$$= \sum_{n=0}^{\infty} \int \left(c_n (x-a)^n \right) dx \text{ by the theorem}$$

Since an antiderivative of $c_n (x-a)^n$ is $C + \dfrac{c_n (x-a)^{n+1}}{n+1}$, we have

$$\int f(x)\, dx = C + \sum_{n=0}^{\infty} \frac{c_n (x-a)^{n+1}}{n+1}$$

If we expand this, we get

$$\int f(x)\, dx = C + c_0 (x-a) + c_1 \frac{(x-a)^2}{2} + c_2 \frac{(x-a)^3}{3} + \dots$$

Which is the formula which appears in the theorem.

Example 10 *Given that a power series representation for $f(x)$ is*

$$\begin{aligned} f(x) &= 1 + x + x^2 + x^3 + \dots \\ &= \sum_{n=0}^{\infty} x^n \end{aligned}$$

find a power series representation for $f'(x)$ and $\int f(x)\, dx$.

- *First, we find $f'(x)$. From the theorem, we know it is enough to differentiate term by term. Thus,*

$$\begin{aligned} f'(x) &= 0 + 1 + 2x + 3x^2 + \dots \\ &= 1 + 2x + 3x^2 + \dots \end{aligned}$$

Note that we can also obtain the same result by using the general formula. In this case,

$$f(x) = \sum_{n=0}^{\infty} x^n$$

Thus

$$\begin{aligned} f'(x) &= \sum_{n=0}^{\infty} (x^n)' \\ &= \sum_{n=0}^{\infty} n x^{n-1} \\ &= \sum_{n=1}^{\infty} n x^{n-1} \end{aligned}$$

Which gives us the same answer.

- *Next, we find $\int f(x)\, dx$. From the theorem, it is enough to integrate term by term. Thus since $f(x) = 1 + x + x^2 + x^3 + \dots$, we have:*

$$\int f(x)\, dx = C + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \dots$$

Alternatively, we can work from the general formula. Since $f(x) = \sum_{n=0}^{\infty} x^n$, we have

$$\begin{aligned} \int f(x)\, dx &= \sum_{n=0}^{\infty} \int (x^n)\, dx \\ &= C + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \end{aligned}$$

Which is the same formula.

6

1.3.1 Differentiation

This time, we find the series representation of a given series by differentiating the power series of a known function. More precisely, if $f'(x) = g(x)$, and if we have a series representation for f and need one for g, we simply differentiate the series representation of f. The theorem above tells us that the radius of convergence will be the same. However, we will have to check the endpoints.

Example 11 *Find a series representation for* $\dfrac{1}{(1-x)^2}$, *find the interval of convergence.*

We begin by noting that $\left(\dfrac{1}{1-x}\right)' = \dfrac{1}{(1-x)^2}$. *Since* $\dfrac{1}{1-x} = 1+x+x^2+x^3+....,$ *we have:*

$$
\begin{aligned}
\frac{1}{(1-x)^2} &= \left(1 + x + x^2 + x^3 + ...\right)' \\
&= 1 + 2x + 3x^2 + 4x^3 + ... \\
&= \sum_{n=1}^{\infty} n x^{n-1}
\end{aligned}
$$

The radius of convergence is still 1. *Since* $\dfrac{1}{1-x}$ *converges for* x *in* $(-1,1)$, *we know that* $\dfrac{1}{(1-x)^2}$ *will also converge in* $(-1,1)$. *It might also converge at the endpoints, so we need to check them. Do it as an exercise.*

Remark 12 *If you prefer to work from the compact form of the series, it can be done.*

$$
\begin{aligned}
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n \\
\frac{1}{(1-x)^2} &= \left(\sum_{n=0}^{\infty} x^n\right)' \\
&= \sum_{n=0}^{\infty} (x^n)' \\
&= \sum_{n=1}^{\infty} n x^{n-1}
\end{aligned}
$$

However, you have to be careful with the starting value of n.

Example 13 *Suppose you know that a series representation for* $\sin x$ *is*

$$
\begin{aligned}
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ... \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}
\end{aligned}
$$

7

Find a power series representation for $\cos x$.

$$
\begin{aligned}
\cos x &= (\sin x)' \\
&= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ... \right)' \\
&= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ... \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}
\end{aligned}
$$

Example 14 *Find a power series representation for* $2xe^{x^2}$ *given that a series repersentation for* e^x *is* $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ *in* $(-\infty, \infty)$.
We can do this problem two ways.

Method 1 *Using substitution, we can find a power series representation for* e^{x^2}, *then multiply what we find by* $2x$. *The first part was done earlier, and we found that* $e^{x^2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$ *on* $(-\infty, \infty)$. *Thus,*

$$
\begin{aligned}
2xe^{x^2} &= \sum_{n=0}^{\infty} \frac{2x x^{2n}}{n!} \\
&= \sum_{n=0}^{\infty} \frac{2x^{2n+1}}{n!}
\end{aligned}
$$

also in $(-\infty, \infty)$.

Method 2 *We note that* $2xe^{x^2} = \left(e^{x^2} \right)'$. *So, using substitution, we can find a power series representation for* e^{x^2}, *then differentiate it to get a series*

representation for $2xe^{x^2}$.

$$
\begin{aligned}
2xe^{x^2} &= \left(e^{x^2}\right)' \\
&= \left(\sum_{n=0}^{\infty} \frac{x^{2n}}{n!}\right)' \\
&= \sum_{n=0}^{\infty} \frac{2nx^{2n-1}}{n!} \\
&= \sum_{n=1}^{\infty} \frac{2nx^{2n-1}}{n!} \quad \textit{when } n = 0, \ \frac{2nx^{2n-1}}{n!} = 0 \\
&= \sum_{n=1}^{\infty} \frac{2x^{2n-1}}{(n-1)!} \\
&= \sum_{n=0}^{\infty} \frac{2x^{2n+1}}{n!}
\end{aligned}
$$

So, either way, we find the same answer.

1.3.2   Integration

This time we find the power series representation of a function by integrating the power series representation of a known function. If $g(x) = \int f(x)\,dx$ and we know a power series representation for $f(x)$, we can get a series representation for $g(x)$ by integrating the series representation of f.

Example 15 *Find a power series representation for* $\ln(1-x)$.

We know that $\ln(1-x) = -\int \dfrac{dx}{1-x}$. *By our earlier work, we found that* $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + ...$ *in* $(-1, 1)$, *so*

$$
-\ln(1-x) = C + x + \frac{x^2}{2} + \frac{x^3}{3} + ...
$$

We also know that $\ln 1 = 0$, *From the above equation, we get that* $C = 0$ *by letting* $x = 0$. *Therefore*

$$
\begin{aligned}
-\ln(1-x) &= x + \frac{x^2}{2} + \frac{x^3}{3} + ... \\
&= \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}
\end{aligned}
$$

Therefore,

$$
\ln(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}
$$

Since the original series converges in $(-1, 1)$, this one will also converge there. The end points should be checked, we leave it as an exercise.

Example 16 *Find a power series rpresentation for $\tan^{-1} x$.*

We use the fact that $\tan^{-1} x = \int \dfrac{dx}{1 + x^2}$. First, since $\dfrac{1}{1-x} = 1+x+x^2+x^3+...$, se see that $\dfrac{1}{1 + x^2} = 1 - x^2 + x^4 - x^6 + ...$ and therefore

$$\tan^{-1} x = C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ...$$

Using the fact that $\tan^{-1} 0 = 0$ gives us $C = 0$. Thus

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ...$$

1.4 Things to Know

- Be able to find the series representation of a function by substitution, integration or differentiation.

- Problems assigned: # 1, 3, 5, 7, 11, 13, 21, 35 on pages 604, 605

2 Taylor and Maclaurin's Series (8.7)

2.1 Introduction

The previous section showed us how to find the series representation of some functions by using the series representation of known functions. The methods we studied are limited since they require us to relate the function to which we want a series representation with one for which we already know a series representation. In this section, we develop a more direct approach. When dealing with functions and their power series representation, there are three fundamental questions one has to answer:

1. Does a given function have a power series representation?

2. If it does, how do we find it?

3. What is the domain in other words, for which values of x is the representation valid?

Question 1 is more theoretical, we won't address it. We will concentrate on questions 2 and 3.

Assuming f has a power series representation that is $f(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + ...$, we want to find what the power series representation is, that is we need to find the coefficients $c_0, c_1, c_2, ...$ It turns out that it is not very difficult. The technique used is worth remembering. We first find c_0. Having found c_0 we next find c_1. Then, we find c_2 and so on.

- **Finding c_0.** Since

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \dots \quad (3)$$

  it follows that

$$
\begin{aligned}
f(a) &= c_0 + c_1(0) + c_2(0)^2 + \dots \\
&= c_0
\end{aligned}
$$

- **Finding c_1.** Differentiating equation 3 gives us

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \dots$$

  Therefore,

$$f'(a) = c_1$$

- **Finding c_2.** We proceed the same way. First, we compute $f''(x)$, then $f''(a)$.

$$f''(x) = 2c_2 + (2)(3)c_3(x-a) + (3)(4)c_4(x-a)^2 + \dots$$

  Therefore

$$f''(a) = 2c_2$$

  or

$$c_2 = \frac{f''(a)}{2}$$

- **In general.** Continuing this way, we can see that

$$c_n = \frac{f^{(n)}(a)}{n!}$$

2.2 Definitions and Theorems

Theorem 17 *If the function f has a power series representation, that is if*
$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots \text{ for } |x-a| < R \text{ then}$$
its coefficients are given by:
$$c_n = \frac{f^{(n)}(a)}{n!}$$
In other words

$$f(x) = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2} + f'''(a)\frac{(x-a)^3}{3!} + \dots$$

Definition 18   *1. The series in the previous theorem* $\left(\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \right)$
   is called the Taylor series of the function f at a.

2. *The nth partial sum is called the nth order Taylor polynomial It is denoted T_n. So,*

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

3. *In the special case $a = 0$, the series is called a Maclaurin's Series. So, a Maclaurin's series is of the form $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$ and the Maclaurin's series for a function f is given by*

$$
\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \\
&= f(0) + f'(0)x + f''(0)\frac{x^2}{2} + f'''(0)\frac{x^3}{3!} + \ldots
\end{aligned}
$$

2.3 Examples

We now look how to find the Taylor and Maclaurin's series of some functions.

Example 19 *Find the Maclaurin's series for $f(x) = e^x$, find its domain. The series will be of the form $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$, we simply need to find the coefficients $f^{(n)}(0)$. This is easy. Since all the derivatives of e^x are e^x, it follows that $f^{(n}(x) = e^x$, thus $f^{(n)}(0) = e^0 = 1$, hence*

$$
\begin{aligned}
e^x &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \\
&= \sum_{n=0}^{\infty} \frac{x^n}{n!}
\end{aligned}
$$

To find where this series converges, we use the ration test and compute:

$$
\begin{aligned}
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \to \infty} \left| \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} \right| \\
&= \lim_{n \to \infty} \frac{n! \, |x|^{n+1}}{(n+1)! \, |x|^n} \\
&= \lim_{n \to \infty} \frac{|x|}{n+1} = 0
\end{aligned}
$$

Thus the domain is all real numbers.

Example 20 *Find a Taylor series for $f(x) = e^x$ centered at 2.*
The series will be of the form $\sum_{n=0}^{\infty} \dfrac{f^{(n)}(2)}{n!}(x-2)^n$, we simply need to find the coefficients $f^{(n)}(2)$. This is easy. Since all the derivatives of e^x are e^x, it follows that $f^{(n}(x) = e^x$, thus $f^{(n)}(2) = e^2$, hence

$$
\begin{aligned}
e^x &= \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n \\
&= \sum_{n=0}^{\infty} \frac{e^2}{n!}(x-2)^n
\end{aligned}
$$

Example 21 *Find the nth order Taylor polynomial for e^x centered at 0 and centered at 2. Plot these polynomials for $n = 1, 2, 3, 4, 5$. What do you notice? We already computed the power series corresponding to these two situations. We found that*

$$
e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \dots
$$

and

$$
\begin{aligned}
e^x &= e^2 + e^2(x-2) + e^2\frac{(x-2)^2}{2!} + e^2\frac{(x-2)^3}{3!} + \dots \\
&= e^2\left(1 + (x-2) + \frac{(x-2)^2}{2!} + \frac{(x-2)^3}{3!} + \frac{(x-2)^4}{4!} + \frac{(x-2)^5}{5!} + \dots\right)
\end{aligned}
$$

If we denote T_n the nth order Taylor polynomial for e^x centered at 0 and Q_n the nth order Taylor polynomial for e^x centered at 2, we have:
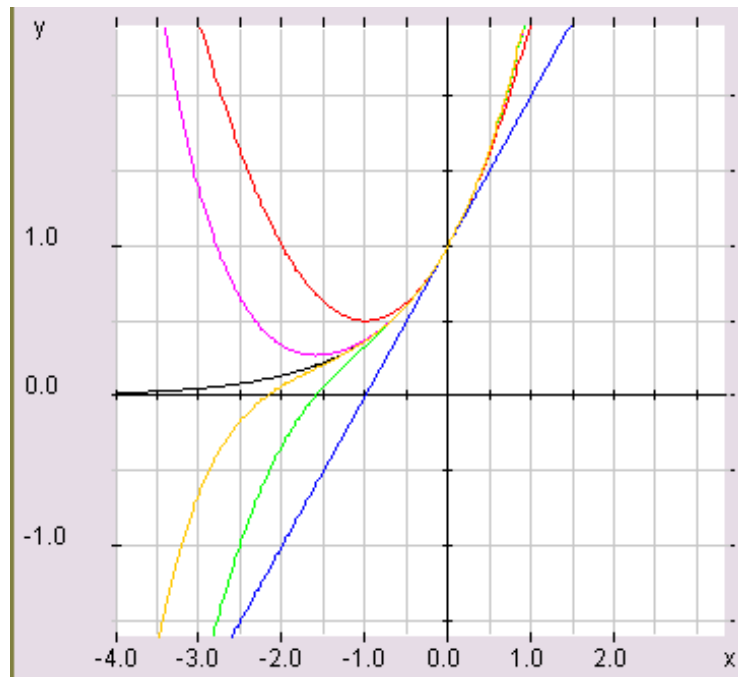
- n^{th} *order Taylor polynomials for e^x centered at 0.*

$$
\begin{aligned}
T_1(x) &= 1 + x \\
T_2(x) &= 1 + x + \frac{x^2}{2!} \\
T_3(x) &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \\
T_4(x) &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \\
T_5(x) &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}
\end{aligned}
$$

The graphs is shown on the next page.

- *Graphs of e^x and the first 5 Taylor polynomials centered at 0. The functions have the following colors:*

$$
\begin{array}{rcl}
e^x & : & black \\
T_1 & : & blue \\
T_2 & : & red \\
T_3 & : & green \\
T_4 & : & purple \\
T_5 & : & yellow
\end{array}
$$

- n^{th} order Taylor polynomial for e^x centered at 2:

$$Q_1(x) = e^2(1 + (x-2))$$

$$Q_2(x) = e^2\left(1 + (x-2) + \frac{(x-2)^2}{2!}\right)$$

$$Q_3(x) = e^2\left(1 + (x-2) + \frac{(x-2)^2}{2!} + \frac{(x-2)^3}{3!}\right)$$
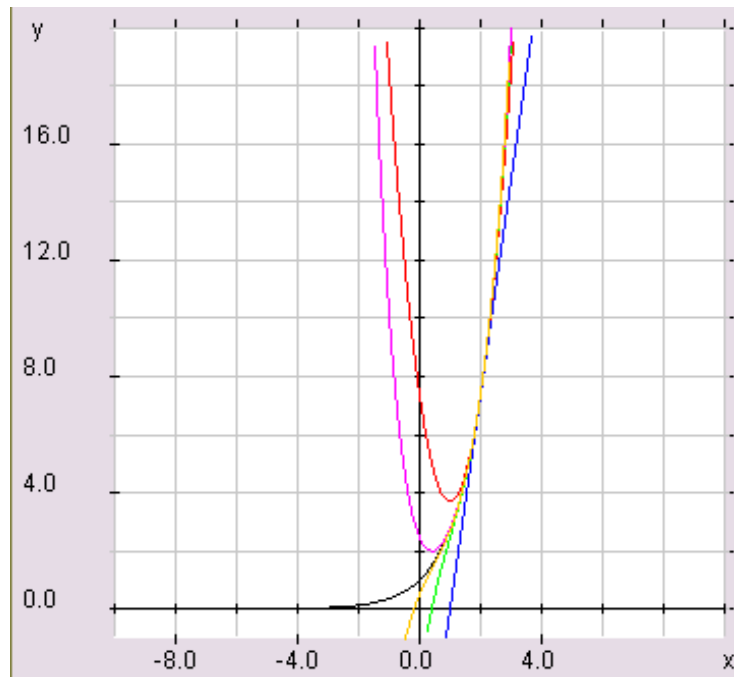
$$Q_4(x) = e^2\left(1 + (x-2) + \frac{(x-2)^2}{2!} + \frac{(x-2)^3}{3!} + \frac{(x-2)^4}{4!}\right)$$

$$Q_5(x) = e^2\left(1 + (x-2) + \frac{(x-2)^2}{2!} + \frac{(x-2)^3}{3!} + \frac{(x-2)^4}{4!} + \frac{(x-2)^5}{5!}\right)$$

The graphs are shown on the next page..

- *Graphs of e^x and the first 5 Taylor polynomials centered at 2. The functions have the following colors:*

$$
\begin{array}{ccl}
e^x & : & black \\
Q_1 & : & blue \\
Q_2 & : & red \\
Q_3 & : & green \\
Q_4 & : & purple \\
Q_5 & : & yellow
\end{array}
$$

We can see the following:

- *The Taylor polynomial approximates the functions well near the point at which the series is centered. As we move away from this point, the approximation deteriorates very quickly.*

- *The approximation is better the higher the degree of the Taylor polynomial. More specifically, the Taylor polynomial stay closer to the function over a larger interval, the higher its degree.*

Example 22 *Find a Maclaurin's series for $f(x) = \sin x$ and find its domain*

The series will be of the form $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$, we simply need to find the coeffi-cients $f^{(n)}(0)$. That is we need to find all the derivatives of $\sin x$ and evaluate them at $x = 0$. We summarize our findings in the table below:

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\sin x$	0
1	$\cos x$	1
2	$-\sin x$	0
3	$-\cos x$	-1
4	$\sin x$	0

So, we see that we will have coefficients only for odd values of n. In addition, the sign of the coefficients will alternate. Thus, since

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2} + f'''(0)\frac{x^3}{3!} + f^{(4)}(0)\frac{x^4}{4!} + \ldots$$

we have

$$\begin{aligned}
\sin x &= 0 + x + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} - \ldots \\
&= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}
\end{aligned}$$

To find where this series converges, we use the ration test and compute $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$.

Since $a_n = \dfrac{x^{2n+1}}{(2n+1)!}$, $a_{n+1} = \dfrac{x^{2n+3}}{(2n+3)!}$. Therefore,

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{\frac{x^{2n+3}}{(2n+3)!}}{\frac{x^{2n+1}}{(2n+1)!}}\right| \\
&= \lim_{n\to\infty}\frac{|x|^{2n+3}(2n+1)!}{|x|^{2n+1}(2n+3)!} \\
&= |x|^2 \lim_{n\to\infty}\frac{(2n+1)!}{(2n+3)!} \\
&= |x|^2 \lim_{n\to\infty}\frac{1}{(2n+2)(2n+3)} \\
&= 0
\end{aligned}$$

Thus the series representation is valid for all x.

Here are some important functions and their corresponding Maclaurin's series

$$
\begin{aligned}
\frac{1}{1-x} &= 1 + x + x^2 + x^3 + x^4 + \dots \quad (-1,1) \\
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \quad (-\infty, \infty) \\
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \quad (-\infty, \infty) \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \quad (-\infty, \infty) \\
\tan^{-1} x &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \quad [-1,1]
\end{aligned}
$$

Remark 23 *The nth order Taylor polynomial (T_n) associated with the series expansion of a function f is the nth degree polynomial obtained by truncating the series expansion and keeping only the terms of degree less than or equal to n. In the case of $\sin x$, since*

$$
\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots
$$

It follows that

$$
T_1 = x
$$
$$
T_2 = x
$$
$$
T_3 = x - \frac{x^3}{3!}
$$
$$
T_4 = x - \frac{x^3}{3!}
$$
$$
T_5 = x - \frac{x^3}{3!} + \frac{x^5}{5!}
$$
$$
T_6 = x - \frac{x^3}{3!} + \frac{x^5}{5!}
$$

and so on.

2.4   Summary

We now have four different techniques to find a series representation for a function. These techniques are:

1. Substitution

2. Differentiation

3. Integration

4. Taylor/Maclaurin's series.

It may appear that the last technique is much more powerful, as it gives us a direct way to derive the series representation. In contrast, the first three techniques require we start from a known series representation. However, the first three techniques should not be ignored. In many cases, they make the work easier. We illustrate this with a few examples.

Example 24 *Find a Maclaurin's series for $f(x) = e^{-x^2}$.*
Of course, this can be done directly. But consider the problem of finding all the derivative of e^{-x^2}. One can also start from e^x and substitute $-x^2$ for x. Since

$$
\begin{aligned}
e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\
&= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!}
\end{aligned}
$$

it follows that

$$
\begin{aligned}
e^{-x^2} &= \sum_{n=0}^{\infty} \frac{\left(-x^2\right)^n}{n!} \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}
\end{aligned}
$$

*This was pretty painless. To convince yourself that this is **the** way to do it, try the direct approach instead!*

Example 25 *Find a Maclaurin's series for $\cos x$.*
Again, one can try the direct approach as in example 22. It is not too difficult. One can also realize that $(\sin x)' = \cos x$. Thus, one can start with the series for $\sin x$ (which we already derived) and find the series for $\cos x$ by differentiating it. We get

$$
\begin{aligned}
\cos x &= (\sin x)' \\
&= \left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \right)' \\
&= \sum_{n=0}^{\infty} \left((-1)^n \frac{x^{2n+1}}{(2n+1)!} \right)' \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \\
&= 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!}
\end{aligned}
$$

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2.5   Things to Remember

- Given a function, be able to find its Taylor or Maclaurin's series.

- Be able to find the radius and interval of convergence of Taylor or Maclaurin's series.

- Section 8.7: # 3, 5, 7, 9, 11, 13, 19, 21, 23, 25, 31 on pages 615, 616.

3   Applications

The purpose of this section is to show the reader how Taylor series can be used to approximate functions. The approximation can then be used to either evaluate a function at specific values of x, to integrate or to differentiate the function. Of course, we would only use this technique with functions for which the traditional calculus methods do not work. For example, we may need to compute $\int_0^1 e^{-x^2} dx$. This cannot be done using the integration techniques learned in a traditional calculus class since e^{-x^2} does not have an antiderivative which can be expressed in terms of elementary functions. Another application might be to approximate $\sin(0.01)$, without a calculator. The difficulty does not lie in the series representation of a given function, we now know how to represent functions as power series. However, series have infinitely many terms, for practical purposes, we can only use a finite number of them. Thus, we replace the infinite series by the corresponding Taylor polynomial (see definition 18) of order n (T_n), for some n. We then use the Taylor polynomial instead of the function. This can be used to evaluate a function, integrate a function or differentiate a function. However, when we perform the following approximation

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

there are several questions to answer before we can carry it out:

1. How do we pick a, the number around which the series is centered?

2. How do we pick n so the Taylor polynomial T_n approximates the given function f within the desired accuracy?

Answer to question 1   There are several factors to take under consideration. First, since we have to evaluate $f^{(n)}(a)$, a must be picked so we can do this evaluation easily. Second, you will recall that the accuracy of the approximation decreases as x gets further away from a. Therefore, we need to pick a so that the values at which $f(x)$ will be approximated are in the domain of the series, and not too far from a. For example, if we had to approximate $\sin(.001)$, then $a = 0$ would be a good choice because it satisfies both criteria.

Answer to question 2 Once we have selected a and have a series representation for f, we use the techniques studied to approximate a series and find the error.

The examples below illustrate these applications.

Example 26 *Find the nth order Taylor polynomial for $f(x) = \cos x$ when $n = 2, 3, 4, 5, 6$. Sketch the graph of $\cos x$ as well as the Taylor polynomials found.*
We already know the power series for $\cos x$.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!}....$$

So,

$$
\begin{aligned}
P_2(x) &= 1 - \frac{x^2}{2!} \\
P_3(x) &= 1 - \frac{x^2}{2!} \\
P_4(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \\
P_5(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \\
P_6(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}
\end{aligned}
$$

You will note that because every other coefficient in the series expansion of $\cos x$ is 0, $P_3 = P_2$, $P_5 = P_4$. The graph of these polynomials is shown on figure 1.

Remark 27 *You will notice that as n increases, P_n gets closer to the graph of $\cos x$. In other words, the accuracy of our approximation increases with n. However, P_n is a good approximation for $\cos x$ in a neighborhood of 0, as we move away from 0, P_n gets further and further away from $\cos x$. This is important. When one approximates a function with a Taylor polynomial about a, the approximation is good only for values of x close to a.*

Example 28 *Approximate $\cos 0.01$ with an error less than 10^{-20}.*
First, we note that since .01 is close to 0, we can use a Taylor polynomial centered at 0 to approximate $\cos(.01)$. Therefore, using the Taylor polynomial for $\cos x$ centered at 0, and replacing x by 0.01, we get:

$$\cos 0.01 = \sum_{i=0}^{n} (-1)^i \frac{0.01^{2i}}{(2i)!}$$

We need to find n so that if we approximate $\sum_{i=0}^{\infty} (-1)^i \frac{0.01^{2i}}{(2i)!}$ by $\sum_{i=0}^{n} (-1)^i \frac{0.01^{2i}}{(2i)!}$, the error is less than 10^{-20}. We notice that $\sum_{i=0}^{\infty} (-1)^i \frac{0.01^{2i}}{(2i)!}$ is an alternating
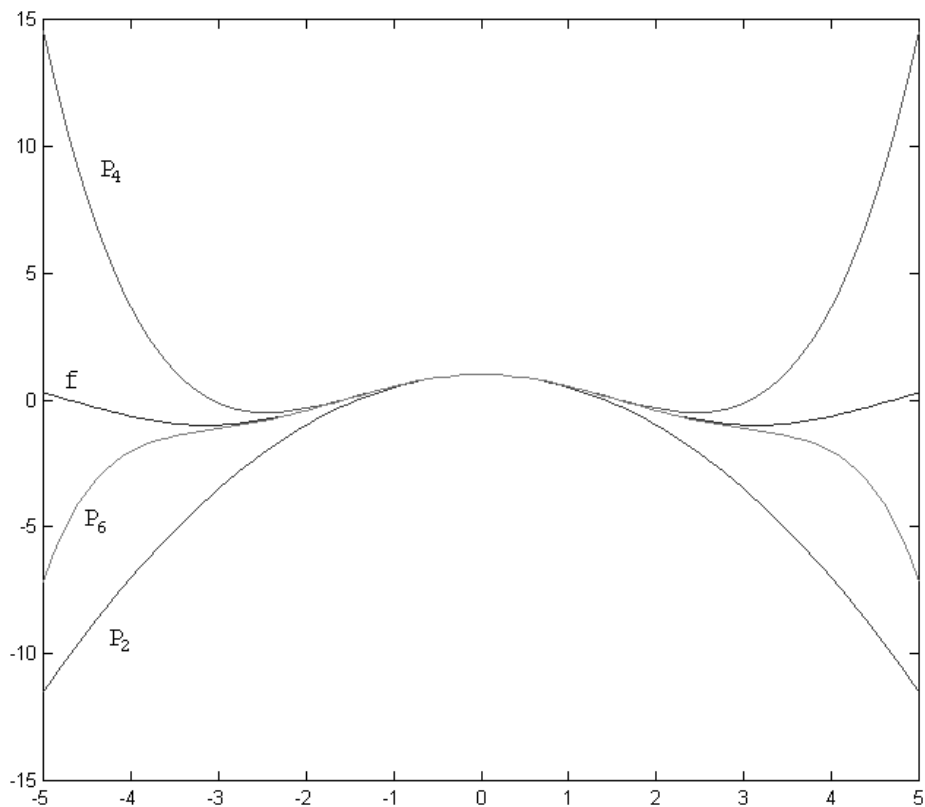
Figure 1: Graph of $\cos x$ and some Taylor polynomials

series with $b_n = \dfrac{0.01^{2n}}{(2n)!}$, so we know how to estimate its error. The error is always less than b_{n+1}. So, if we want the error to be less than 10^{-20}, it is enough to solve:

$$
\begin{aligned}
b_{n+1} &< 10^{-20} \\
\frac{0.01^{2(n+1)}}{(2n+2)!} &< 10^{-20}
\end{aligned}
$$

We solve this by trying various values of n . The table below shows this procedure:

n	$\dfrac{0.01^{2(n+1)}}{(2n+2)!}$
1	4×10^{-10}
2	1×10^{-15}
3	2×10^{-21}

We see that $n = 3$ is enough. Therefore:

$$\cos 0.01 \approx \sum_{n=0}^{3} (-1)^n \frac{0.01^{2n}}{(2n)!}$$

$$\approx 0.999950000416665$$

Example 29 *Approximate $\int_0^1 e^{-x^2}\,dx$ with an error less than 0.001.*
First, we find a series representation for e^{-x^2}. Since

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

it follows that

$$e^{-x^2} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}$$

Therefore

$$\int e^{-x^2}\,dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)\,n!}$$

and therefore

$$\int_0^1 e^{-x^2}\,dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)\,n!}\bigg|_0^1$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)\,n!}$$

This is an alternating series $\displaystyle\sum_{n=0}^{\infty} (-1)^n b_n$ with $b_n = \dfrac{1}{(2n+1)\,n!}$. From our

knowledge of alternating series, we know that if we approximate $\displaystyle\sum_{i=0}^{\infty} (-1)^i \frac{1}{(2i+1)\,i!}$

by $\displaystyle\sum_{i=0}^{n} (-1)^i \frac{1}{(2i+1)\,i!}$, the error will be less than $b_{n+1} = \dfrac{1}{(2n+3)\,(n+1)!}$. So,
we find n such that

$$\frac{1}{(2n+3)\,(n+1)!} < .001$$

We try several values of n, we find that when $n = 3$, $\dfrac{1}{(2n+3)(n+1)!} = .00463$ and when $n = 4$, $\dfrac{1}{(2n+3)(n+1)!} = .0007576$. So,

$$\sum_{n=0}^{4} (-1)^n \frac{1}{(2n+1)\,n!} = 0.747\,49$$

gives us the desired approximation.

3.1 Problems

The problems assigned for power series, Taylor series and representation of functions as power series were:

- Section 8.6: # 1, 3, 5, 7, 11, 13, 21, 35 on pages 610, 611.

- Section 8.7: # 3, 5, 7, 9, 11, 13, 19, 21, 23, 25, 31 on pages 615, 616.

- In addition, for the applications discussed in this section, do # 3, 5, 7 on page 628.