Filter design

- FIR filters
- Chebychev design
- linear phase filter design
- equalizer design
- filter magnitude specifications

FIR filters

finite impulse response (FIR) filter:

$$y(t) = \sum_{\tau=0}^{n-1} h_{\tau} u(t-\tau), \quad t \in \mathbf{Z}$$

- (sequence) $u: \mathbf{Z} \to \mathbf{R}$ is input signal
- (sequence) $y : \mathbf{Z} \to \mathbf{R}$ is output signal
- \bullet h_i are called *filter coefficients*
- *n* is filter *order* or *length*

filter frequency response: $H: \mathbb{R} \to \mathbb{C}$

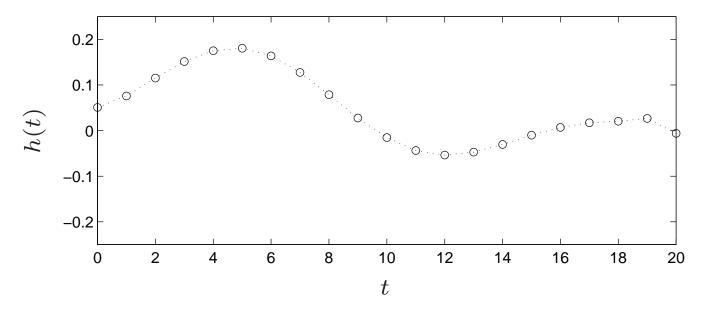
$$H(\omega) = h_0 + h_1 e^{-i\omega} + \dots + h_{n-1} e^{-i(n-1)\omega}$$
$$= \sum_{t=0}^{n-1} h_t \cos t\omega + i \sum_{t=0}^{n-1} h_t \sin t\omega$$

- (EE tradition uses $j = \sqrt{-1}$ instead of i)
- \bullet H is periodic and conjugate symmetric, so only need to know/specify for $0 \leq \omega \leq \pi$

FIR filter design problem: choose h so it and H satisfy/optimize specs

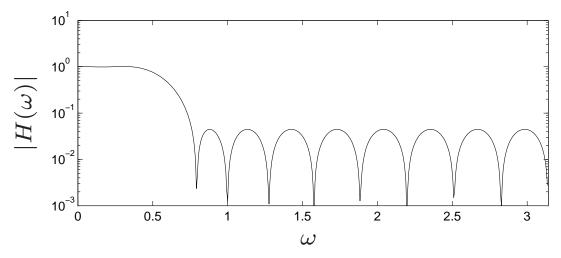
example: (lowpass) FIR filter, order n=21

impulse response h:

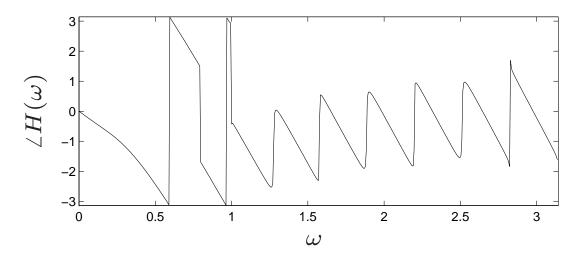


Filter design

frequency response magnitude (i.e., $|H(\omega)|$):



frequency response phase (i.e., $\angle H(\omega)$):



Chebychev design

minimize
$$\max_{\omega \in [0,\pi]} |H(\omega) - H_{\mathrm{des}}(\omega)|$$

- h is optimization variable
- $H_{\mathrm{des}}: \mathbf{R} \to \mathbf{C}$ is (given) desired transfer function
- convex problem
- can add constraints, e.g., $|h_i| \leq 1$

sample (discretize) frequency:

minimize
$$\max_{k=1,...,m} |H(\omega_k) - H_{\mathrm{des}}(\omega_k)|$$

- sample points $0 \le \omega_1 < \cdots < \omega_m \le \pi$ are fixed (e.g., $\omega_k = k\pi/m$)
- $m\gg n$ (common rule-of-thumb: m=15n)
- yields approximation (relaxation) of problem above

Chebychev design via SOCP:

minimize
$$t$$
 subject to $\left\|A^{(k)}h - b^{(k)}\right\| \leq t, \quad k = 1, \dots, m$

where

$$A^{(k)} = \begin{bmatrix} 1 & \cos \omega_k & \cdots & \cos(n-1)\omega_k \\ 0 & -\sin \omega_k & \cdots & -\sin(n-1)\omega_k \end{bmatrix}$$

$$b^{(k)} = \begin{bmatrix} \Re H_{\text{des}}(\omega_k) \\ \Im H_{\text{des}}(\omega_k) \end{bmatrix}$$

$$h = \begin{bmatrix} h_0 \\ \vdots \\ h_{n-1} \end{bmatrix}$$

Linear phase filters

suppose

- n=2N+1 is odd
- impulse response is symmetric about midpoint:

$$h_t = h_{n-1-t}, \quad t = 0, \dots, n-1$$

then

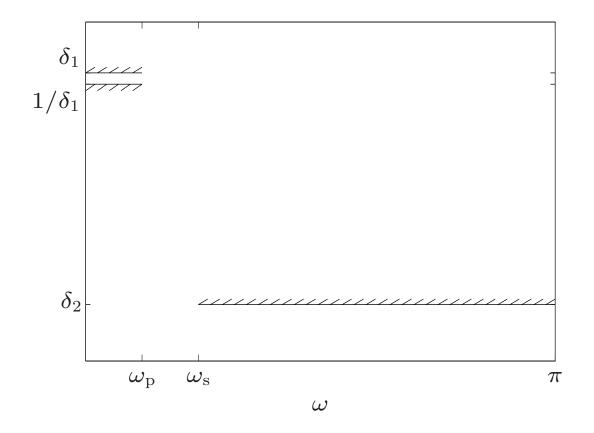
$$H(\omega) = h_0 + h_1 e^{-i\omega} + \dots + h_{n-1} e^{-i(n-1)\omega}$$

$$= e^{-iN\omega} \left(2h_0 \cos N\omega + 2h_1 \cos(N-1)\omega + \dots + h_N \right)$$

$$\stackrel{\Delta}{=} e^{-iN\omega} \widetilde{H}(\omega)$$

- ullet term $e^{-iN\omega}$ represents N-sample delay
- $\widetilde{H}(\omega)$ is real
- $|H(\omega)| = |\widetilde{H}(\omega)|$
- called **linear phase** filter $(\angle H(\omega))$ is linear except for jumps of $\pm \pi$)

Lowpass filter specifications



idea:

- ullet pass frequencies in passband $[0,\omega_{
 m p}]$
- ullet block frequencies in $stopband \ [\omega_{
 m s},\pi]$

specifications:

• maximum passband ripple ($\pm 20 \log_{10} \delta_1$ in dB):

$$1/\delta_1 \le |H(\omega)| \le \delta_1, \quad 0 \le \omega \le \omega_p$$

• minimum stopband attenuation ($-20 \log_{10} \delta_2$ in dB):

$$|H(\omega)| \leq \delta_2, \quad \omega_s \leq \omega \leq \pi$$

Linear phase lowpass filter design

- sample frequency
- ullet can assume wlog $\widetilde{H}(0)>0$, so ripple spec is

$$1/\delta_1 \le \widetilde{H}(\omega_k) \le \delta_1$$

design for maximum stopband attenuation:

- passband ripple δ_1 is given
- an LP in variables h, δ_2
- known (and used) since 1960's
- can add other constraints, e.g., $|h_i| \leq \alpha$

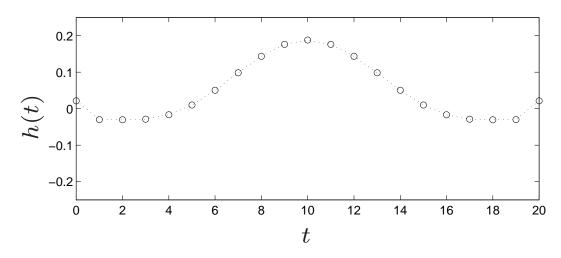
variations and extensions:

- fix δ_2 , minimize δ_1 (convex, but not LP)
- fix δ_1 and δ_2 , minimize ω_s (quasiconvex)
- fix δ_1 and δ_2 , minimize order n (quasiconvex)

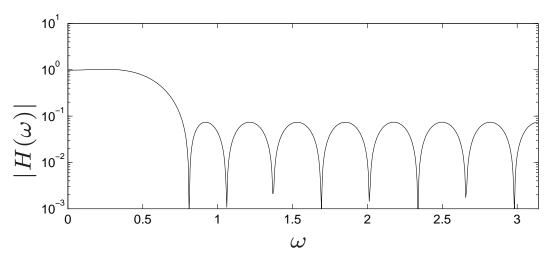
example

- linear phase filter, n = 21
- passband $[0, 0.12\pi]$; stopband $[0.24\pi, \pi]$
- max ripple $\delta_1 = 1.012 \; (\pm 0.1 dB)$
- design for maximum stopband attenuation

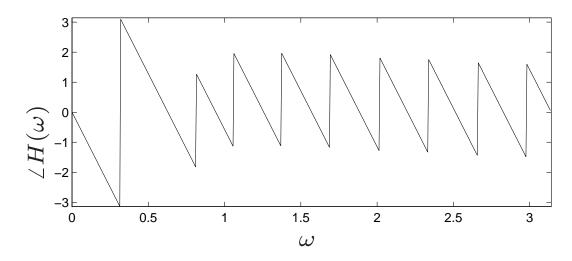
impulse response h:



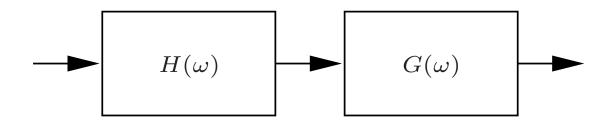
frequency response magnitude (i.e., $|H(\omega)|$):



frequency response phase (i.e., $\angle H(\omega)$):



Equalizer design



equalization: given

- \bullet G (unequalized frequency response)
- $G_{\rm des}$ (desired frequency response)

design (FIR equalizer) H so that $\widetilde{G} \stackrel{\Delta}{=} GH \approx G_{\mathrm{des}}$

- common choice: $G_{\text{des}}(\omega) = e^{-iD\omega}$ (delay) i.e., equalization is deconvolution (up to delay)
- ullet can add constraints on H, e.g., limits on $|h_i|$ or $\max_{\omega} |H(\omega)|$

Chebychev equalizer design:

$$\operatorname{minimize} \max_{\omega \in [0,\pi]} \left| \widetilde{G}(\omega) - G_{\operatorname{des}}(\omega) \right|$$

convex; SOCP after sampling frequency

time-domain equalization: optimize impulse response \tilde{g} of equalized system

e.g., with
$$G_{\rm des}(\omega)=e^{-iD\omega}$$
,

$$g_{\text{des}}(t) = \begin{cases} 1 & t = D \\ 0 & t \neq D \end{cases}$$

sample design:

minimize
$$\max_{t \neq D} |\tilde{g}(t)|$$

subject to $\tilde{g}(D) = 1$

- an LP
- \bullet can use $\sum_{t\neq D} \tilde{g}(t)^2$ or $\sum_{t\neq D} |\tilde{g}(t)|$

extensions:

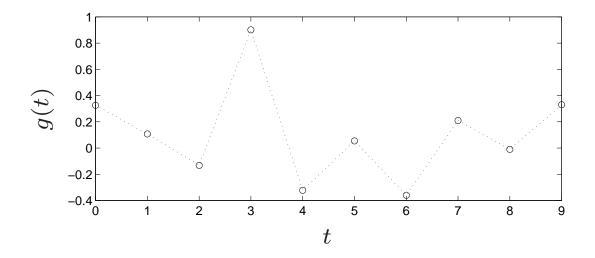
- can impose (convex) constraints
- can mix time- and frequency-domain specifications
- ullet can equalize multiple systems, i.e., choose H so

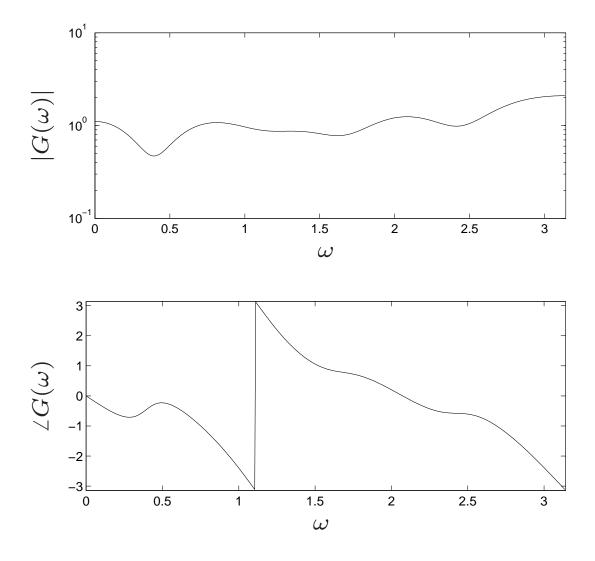
$$G^{(k)}H \approx G_{\text{des}}, \quad k = 1, \dots, K$$

- can equalize multi-input multi-output systems (i.e., G and H are matrices)
- \bullet extends to multidimensional systems, e.g., image processing

Equalizer design example

unequalized system G is 10th order FIR:



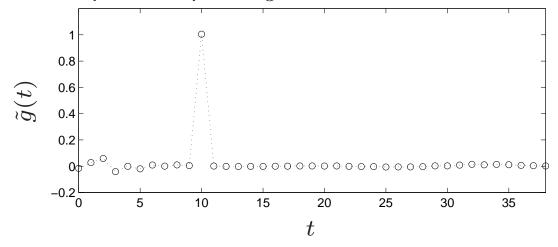


design $30{\rm th}$ order FIR equalizer with $\widetilde{G}(\omega)\approx e^{-i10\omega}$

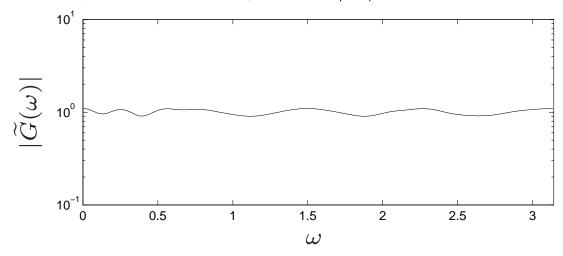
Chebychev equalizer design:

minimize
$$\max_{\omega} \left| \tilde{G}(\omega) - e^{-i10\omega} \right|$$

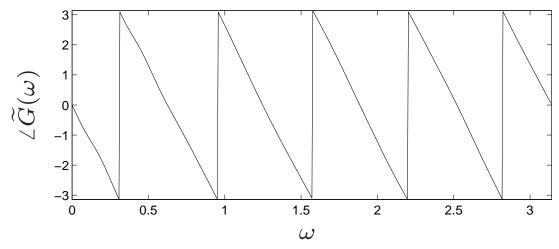
equalized system impulse response $ilde{g}$



equalized frequency response magnitude $|\widetilde{G}|$



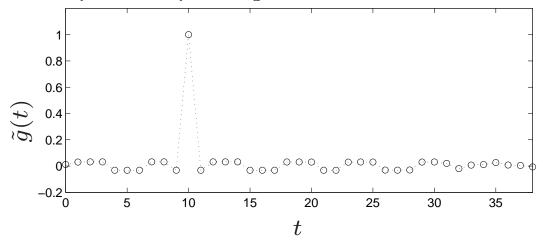
equalized frequency response phase $\angle \widetilde{G}$



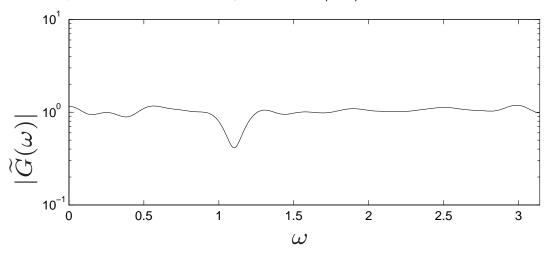
time-domain equalizer design:

minimize
$$\max_{t \neq 10} |\tilde{g}(t)|$$

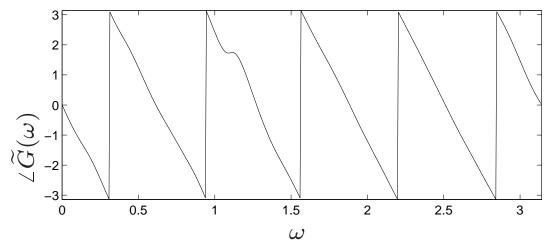
equalized system impulse response \tilde{g}



equalized frequency response magnitude $|\widetilde{G}|$



equalized frequency response phase $\angle \widetilde{G}$



Filter magnitude specifications

transfer function magnitude spec has form

$$L(\omega) \le |H(\omega)| \le U(\omega), \quad \omega \in [0, \pi]$$

where $L, U : \mathbf{R} \to \mathbf{R}_+$ are given

- lower bound is **not** convex in filter coefficients h
- \bullet arises in many applications, e.g., audio, spectrum shaping
- can change variables to solve via convex optimization

Autocorrelation coefficients

autocorrelation coefficients associated with impulse response $h = (h_0, \dots, h_{n-1}) \in \mathbf{R}^n$ are

$$r_t = \sum_{\tau} h_{\tau} h_{\tau+t}$$

(we take $h_k = 0$ for k < 0 or $k \ge n$)

- $r_t = r_{-t}$; $r_t = 0$ for $|t| \ge n$
- hence suffices to specify $r = (r_0, \dots, r_{n-1}) \in \mathbf{R}^n$

Fourier transform of autocorrelation coefficients is

$$R(\omega) = \sum_{\tau} e^{-i\omega\tau} r_{\tau} = r_0 + \sum_{t=1}^{n-1} 2r_t \cos \omega t = |H(\omega)|^2$$

- always have $R(\omega) \geq 0$ for all ω
- can express magnitude specification as

$$L(\omega)^2 \le R(\omega) \le U(\omega)^2, \quad \omega \in [0, \pi]$$

 \dots convex in r

Spectral factorization

question: when is $r \in \mathbb{R}^n$ the autocorrelation coefficients of some $h \in \mathbb{R}^n$? **answer:** (spectral factorization theorem) if and only if $R(\omega) \geq 0$ for all ω

- ullet spectral factorization condition is convex in r
- many algorithms for spectral factorization, *i.e.*, finding an h s.t. $R(\omega) = |H(\omega)|^2$

magnitude design via autocorrelation coefficients:

- use r as variable (instead of h)
- ullet add spectral factorization condition $R(\omega) \geq 0$ for all ω
- ullet optimize over r
- use spectral factorization to recover h

log-Chebychev magnitude design

choose h to minimize

$$\max_{\omega} |20 \log_{10} |H(\omega)| - 20 \log_{10} D(\omega)|$$

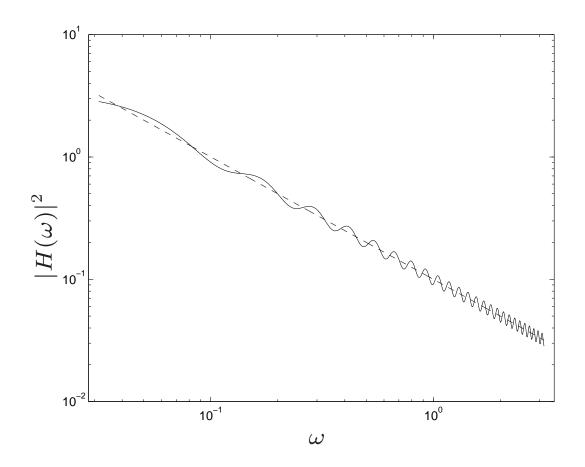
- D is desired transfer function magnitude $(D(\omega) > 0 \text{ for all } \omega)$
- find minimax logarithmic (dB) fit

reformulate as

minimize
$$t$$
 subject to $D(\omega)^2/t \leq R(\omega) \leq tD(\omega)^2, \quad 0 \leq \omega \leq \pi$

- \bullet convex in variables r, t
- constraint includes spectral factorization condition

example: 1/f (pink noise) filter (i.e., $D(\omega)=1/\sqrt{\omega}$), n=50, log-Chebychev design over $0.01\pi \le \omega \le \pi$



optimal fit: $\pm 0.5 dB$