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# Compactness of certain integral operators

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Let $G = NA^+K$ be an Iwasawa decomposition of $G = GL(n, \mathbf{R})$, where $P = NA^+$ is the (connected component of the) standard minimal parabolic of upper-triangular matrices *with positive diagonal entries*, with unipotent radical $N$, and standard Levi component $M$ consisting of diagonal matrices with positive diagonal entries, and $K$ is the maximal compact subgroup $O(n)$. For $x \in G$, the decomposition $x = na_xk$ is unique. Let $\Gamma = GL(n, \mathbf{Z})$.

Let $\Sigma^o$ be the set of simple positive roots. Let $\delta$ be the sum of all the positive roots. Then for $a \in A^+$ and a measurable subset $X$ of $N$ we have

$$\text{meas}\,(aXa^{-1}) = \delta(a) \cdot \text{meas}\,(X)$$

where meas $(X)$ is Haar measure on $N$. For fixed $t > 0$, define a Siegel set $S_t$ by

$$S_t = \{nak : n \in N, a \in A^+, k \in K, \text{ with } a^\alpha \geq t \text{ for all } \alpha \in \Sigma^o\}$$

**Proposition:** Fix $\alpha \in \Sigma^o$. Fix a Siegel set $S_t$. Fix a character $\beta$ on $A^+$. Given $\eta \in C_c^\infty(G)$, there is $0 < t' < t$ and a constant $c$ so that, for any $x \in \mathbf{S}_t$ and for any $f \in L^2(\Gamma_N\backslash S_{t'})$

$$|\eta \cdot f(x)| \leq c \cdot a_x^{\delta-\beta} \cdot |a^\beta \cdot f|_{L^2(\Gamma_N\backslash S_t')}$$

Also, for $x \in S_t$ with $a_x^\delta$ sufficiently large,

$$|\eta \cdot f(x)| \leq c \cdot a_x^{\delta-\beta} \cdot |a^\beta \cdot f|_{L^2(\Gamma_N\backslash S_t)}$$

**Remark:** As is visible in the proof, the hypothesis and conclusion can be adjusted a little in various ways. The two different conclusions are the two most useful ones.

*Proof:* First, let $C$ be the compact support of $\eta$. Since $C \cdot K$ is compact, there are compact subsets $C_N \subset N$, $C_A \subset A^+$, so that

$$K \cdot C \subset C_N \cdot C_A \cdot K$$

For $x \in S_t$, writing $x \in \Omega_N \cdot a_x \cdot K$ with $\Omega_N$ a compact set in $N$ so that $\Gamma_N \cdot \Omega_N = N$. Then

$$x \cdot C \subset \Omega_N\, a_x\, K \cdot C \subset \Omega_N\, a_x\, C_N\, C_A\, K \subset a_x \cdot (a_x^{-1}\Omega_N a_x\, C_N)\, C_A\, K$$

Since $x \in S_t$, for every positive root $\beta$ we have $a_x^\beta \geq t$, so there is a compact subset $C_N'$ of $N$ so that for all $x \in S_t$ $a_x^{-1}\Omega_N a_x \subset C_N'$. Then $C_N' \cdot C_N$ is contained in a compact subset $C_N^*$ of $N$, independent of $x \in S_t$. Thus,

$$x \cdot C \subset a_x \cdot C_N^*\, C_A\, K \subset a_x \cdot C_N^* a_x^{-1}\, a_x C_A\, K$$

Let $\Omega_N$ be a compact subset of $N$ so that $\Gamma_N \cdot \Omega_N = N$. Then there is a constant $c_1$ depending only on $t$ and $C_N^*$ and $\Omega_N$ so that for all $x \in S_t$

$$a_x \cdot C_N^* a_x^{-1}$$

is contained in $c_1 \cdot a_x^\delta$ copies $\gamma\Omega_N$ of $\Omega_N$ for $\gamma \in \Gamma_N$. For $a_x^\delta$ large enough so that $x \cdot C_A \subset S_t$,

$$a_x \cdot C_N^* a_x^{-1} \cdot M_t C_A \cdot K$$

is contained in $c_1 \cdot a_x^\delta$ copies of $S_t$.

Let $C$ be the support of $\eta$. Then for any character $\beta$

$$|\eta f(x)| \leq \sup|\eta| \cdot \int_{xC} |f(y)|\, dy \leq \sup|\eta| \cdot \left(\int_{xC} |a_y^\beta f(y)|^2\, dy\right)^{1/2} \cdot \left(\int_{xC} |a_y^{-\beta}|^2\, dy\right)^{1/2}$$

by Cauchy-Schwarz-Bunyakowsky. Since the integrand $|a_y^\beta f(y)|^2$ is left $\Gamma_N$-invariant and since (from above) $xC$ is contained in at most $c_2 \cdot a_x^\delta$ copies of $S_t$, we have

$$|\eta f(x)| \leq\leq \sup|\eta| \cdot c_2 \cdot \Omega_N \cdot |a_y^\beta f(y)|_{L^2(\Gamma_N\backslash S_t)} \cdot \left(\int_{xC} |a_y^{-\beta}|^2\,dy\right)^{1/2}$$

Now we estimate the function $y \to a_y^{-\beta}$ on $xC$. Again, from above,

$$xC \subset \Omega_N\, a_x\, C_N\, C_M\, K = (\Omega_N\, a_x C_N a_x^{-1})\,(a_x\, C_M)\, K$$

Since $C_M$ is compact (depending only upon $\eta$), on $xC$ the function $y \to a_y^{-\beta}$ is bounded by a constant multiple of $a_x^{-\beta}$.

Altogether, we have (for some constant $c$ not depending upon $x$ nor upon $f$)

$$|\eta f(x)| \leq c \cdot |a_y^\beta f(y)|_{L^2(\Gamma_N\backslash S_t)} \cdot a_x^{\delta-\beta}$$

This proves the proposition. ///

**Corollary:** Given a Siegel set $S_t$, for a left $\Gamma_N$-invariant function $f$, if

$$|a^\delta f|^2_{L^2(\Gamma_N\backslash S_t)} = \int_{\Gamma_N\backslash S_t} |a_x^\delta f(x)|^2\,dx < \infty$$

then for any $\eta \in C_c^\infty(G)$ the function $\eta \cdot f$ is *bounded* on $S_t$. ///

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then for any $\eta \in C_c^\infty(G)$ and for any element $D$ of the universal enveloping algebra of $\mathbf{g}$, the function $D(\eta \cdot f)$ is *bounded* on $S_t$.

*Proof:* First, for $D$ in the Lie algebra itself (interpreted as left-invariant differential operators on $G$), we compute directly from the definition that

$$D(\eta f) = (-X^{\text{left}}\,\eta)\,f$$

where $X^{\text{left}}$ is the natural action as *right*-invariant differential operator on $G$. This reduces the question to the previous corollary. ///

Now the main theorem is just another corollary:

**Theorem:** Fix a Siegel set $S_t$ and define a Hilbert space

$$V = \{f : a^\delta \cdot f \in L^2(\Gamma_N\backslash S_t)\}$$

Then for any $\eta \in C_c^\infty(G)$ the map

$$f \to \eta \cdot f$$

is a compact operator $V \to V$.

*Proof:* First, note that the operator $f \to \eta \cdot f$ maps locally integrable functions to *continuous* functions, since $\eta$ is continuous. The previous two corollaries show that on this space $V$ the operator $f \to \eta \cdot f$ maps the unit ball to a set of continuous functions which are bounded and whose derivatives (by Lie algebra acting on the right) are bounded. The boundedness of derivatives implies that the functions in the image are *uniformly* continuous in the sense that given $\varepsilon > 0$ there is an open neighborhood $U$ of 1 in $G$ so that for $\varphi$ in the image and for $x \in G$ and for $h \in U$,

$$|\varphi(x \cdot h) - \varphi(x)| < \varepsilon$$

Cover $G$ by a countable collection of open sets $U_i$ with compact closures $\bar{U}_i$ sufficiently small so that the natural maps $\bar{U}_i \to \Gamma\backslash G$ are *inclusions*. (This is possible since $\Gamma$ is discrete.) Let $\{\psi_i\}$ be a locally finite partition of unity on $\Gamma\backslash G$ subordinate to the cover $\{U_i\}$. Then the Arzela-Ascoli theorem implies that the image in $C^o(\bar{U}_i)$ (with sup norm) under $f \to \psi_i \cdot \eta f$ of the unit ball in $V$ is pre-compact. Thus, by definition, for each $i$, the operator

$$f \to \psi_i \cdot \eta f$$

is compact as a map from $V$ to $C^o(\bar{U}_i)$. Thus, given a sequence $\{f_j\}$ in $V$, for each $i$ there is a subsequence $f_{j_i(1)}, f_{j_i(2)}, f_{j_i(3)}, \ldots$ which converges in sup norm on

$$\bigcup_{\ell \le i} \bar{U}_\ell$$

Then the diagonal sequence

$$f_{j_1(1)},\ f_{j_2(2)},\ f_{j_3(3)}, \ldots$$

converges in sup norm on $\Gamma\backslash G$. Since the total measure of $\Gamma\backslash G$ is finite, this gives convergence in $L^2(\Gamma\backslash G)$. ///