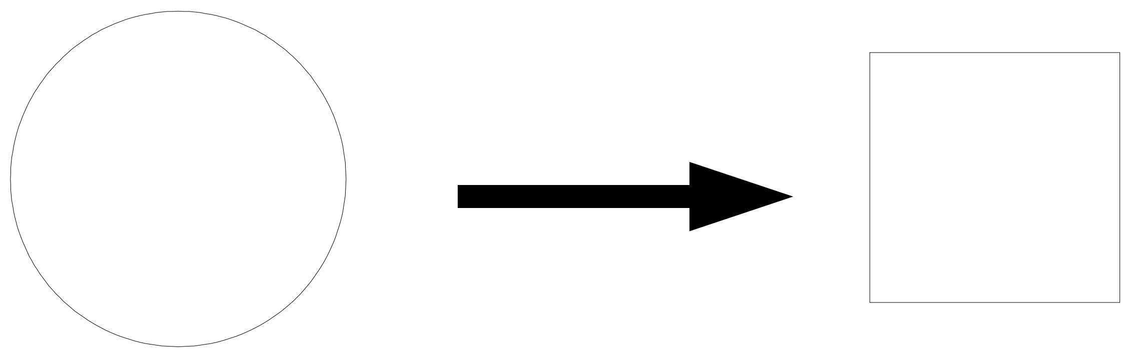
Squaring the Circle

A Case Study in the History of Mathematics

The Problem

Using only a compass and straightedge, construct for any given circle, a square with the same area as the circle.



The general problem of constructing a square with the same area as a given figure is known as the ***Quadrature*** of that figure. So, we seek a ***quadrature of the circle***.

The Answer

It has been known since 1822 that the quadrature of a circle with straightedge and compass is **impossible**.

Notes: First of all we are ***not*** saying that a square of equal area does not exist. If the circle has area A, then a square with side $\sqrt{A}$ clearly has the same area.

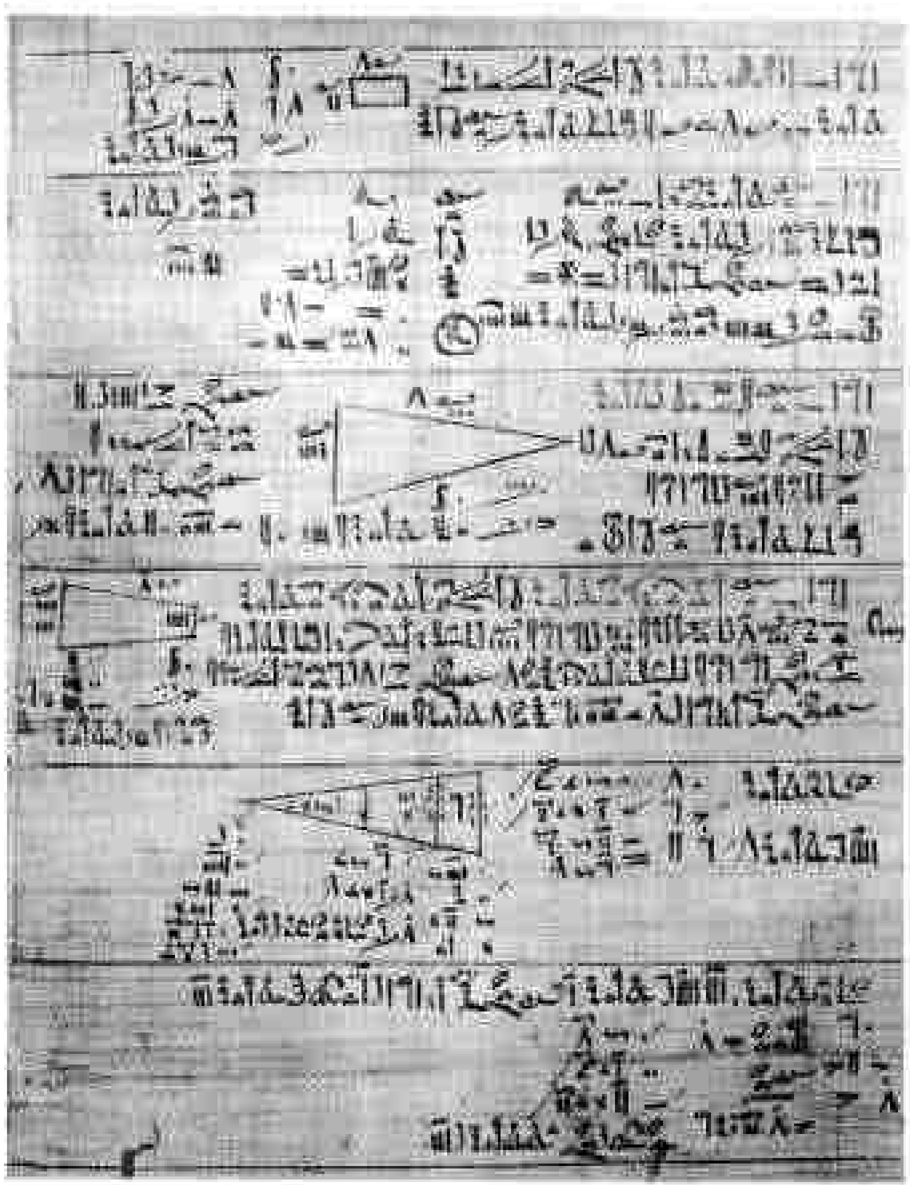
Secondly, we are ***not*** saying that a quadrature of a circle is impossible, since it is possible, **but** not under the restriction of using only a straightedge and compass.

Precursors

It has been written, in many places, that the quadrature problem appears in one of the earliest extant mathematical sources, the Rhind Papyrus (~ 1650 B.C.).

This is not really an accurate statement. If one means by the "quadrature of the circle" simply a quadrature *by any means*, then one is just asking for the determination of the area of a circle. This problem *does* appear in the Rhind Papyrus, but I consider it as just a ***precursor*** to the construction problem we are examining.

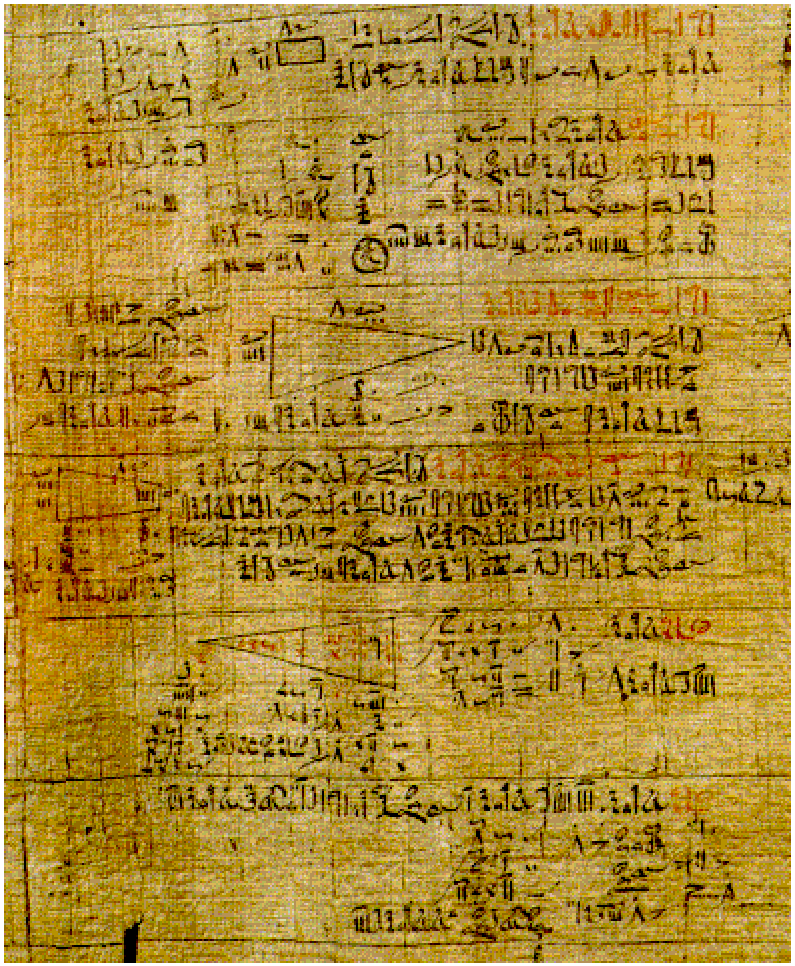
The Rhind Papyrus



The papyrus was found in Thebes (Luxor) in the ruins of a small building near the Ramesseum.[1] It was purchased in 1858 in Egypt by the Scottish Egyptologist A. Henry Rhind and acquired by the British Museum after his death.

The papyrus, written in *hieratic*, the cursive form of hieroglyphics, is a single roll which was originally about 5.4 meters long by 32 cms wide (~18 feet by 13 inches).

The Rhind Papyrus



However, when the British Museum acquired it, it was shorter and in two pieces, missing the central portion. About 4 years after Rhind made his purchase, the American Egyptologist Edwin Smith bought, in Egypt, what he thought was a medical papyrus. This was given to the New York Historical Society in 1932, where it was discovered that beneath a fraudulent covering lay the missing piece of the Rhind Paprus.

The Society then gave the scroll to the British Museum.[2]

The Ahmes (Rhind) Papyrus

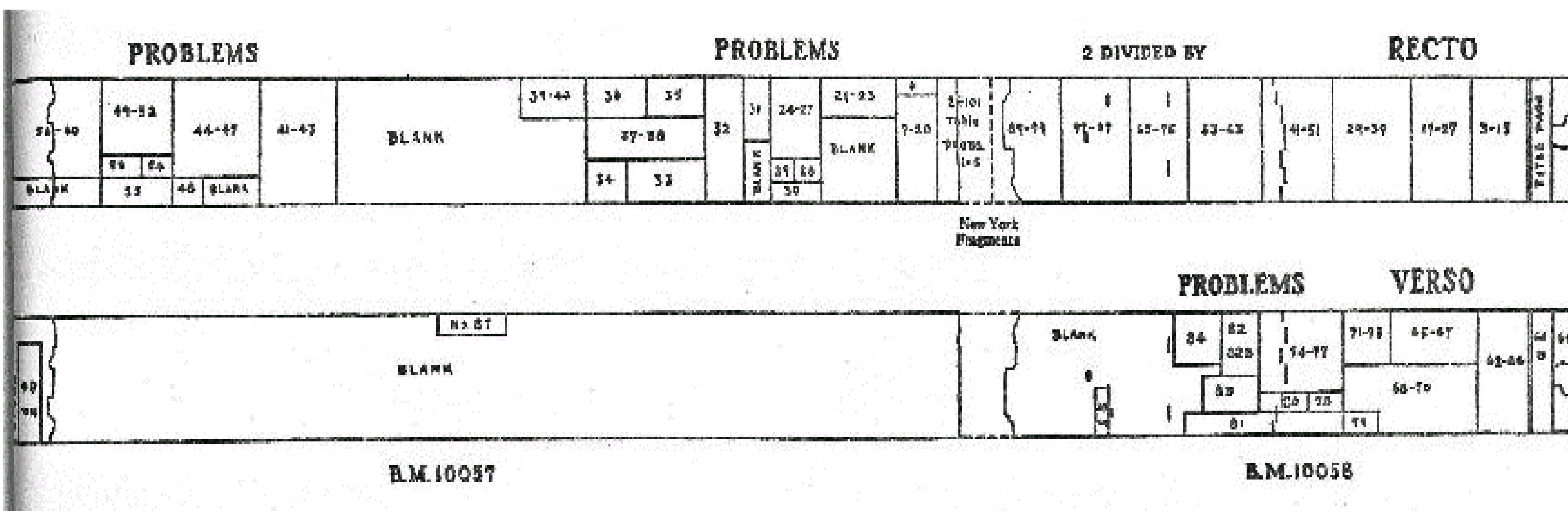
The Rhind Papyrus has been dated to about 1650 B.C. and there is only one older mathematical papyrus, the Moscow Papyrus dated 1850 B.C.

The papyrus was written by an Egyptian scribe A'h-mosè, commonly called Ahmes by modern writers. It appears to be a copy of an older work.

> *" Accurate reckoning of entering into things, knowledge of existing things all, mysteries, secrets all. Now was copied book this in year 33, month four of the inundation season, [under the majesty] of the king of [Upper and] Lower Egypt, 'A-user-Rê'* [15th Dynasty reign of the Hyksos Pharaoh, Apepi I]*, endowed with life, in likeness to writings of old made in the time of the king of Upper [and Lower] Egypt, Ne-ma'et-Rê'* [Amenemhet III (1842 - 1797 B.C.)]. *Lo the scribe A'h-mosè writes copy this."* [3]

The Ahmes (Rhind) Papyrus

The scroll consists of 87 problems (with solutions) and is a rich primary source of ancient Egyptian mathematics, describing the Egyptian methods of multiplying and dividing, the Egyptian use of unit fractions, their employment of false position, and many applications of mathematics to practical problems.



The Ahmes (Rhind) Papyrus

Our concern is with problem 50 which reads:

A circular field has diameter 9 khet. What is its area?

(A ***khet*** is a length measurement of about 50 meters.)

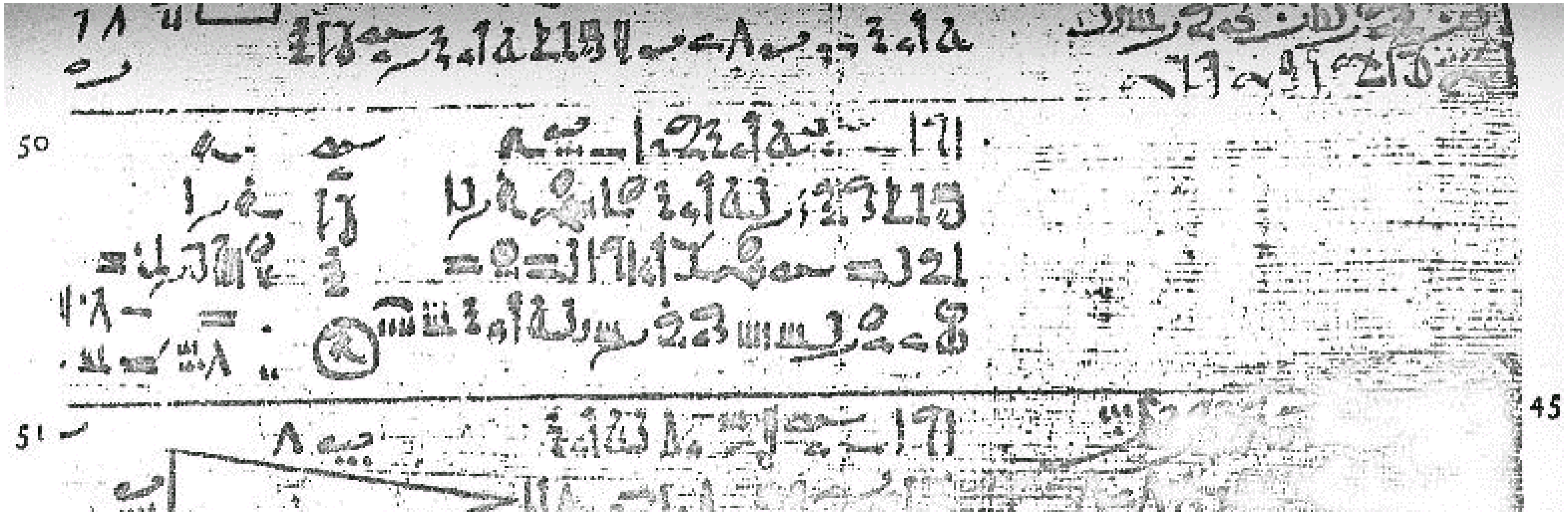


Photo of Problem 50

The Ahmes (Rhind) Papyrus

This problem (and solution) as well as all the other problems is phrased in terms of specific numbers. Does this mean that the problems are meant to be illustrative of general methods, or are all problems with different parameters considered different?

Examination of other problems indicates that a definite algorithm was being used since the same technique is applied to different parameters.

Ahmes Papyrus: Problem 50

Ahmes' solution is:

Take away thou 1/9 of it, namely 1; the remainder is 8. Make thou the multiplication 8 times 8; becomes it 64; the amount of it, this is, in area 64 setat.

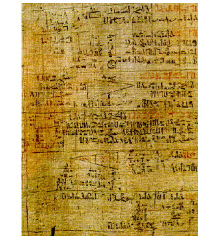
If we take this solution as a general formula, then in modern notation we obtain the “formula” for the area A of a circle of diameter d as:

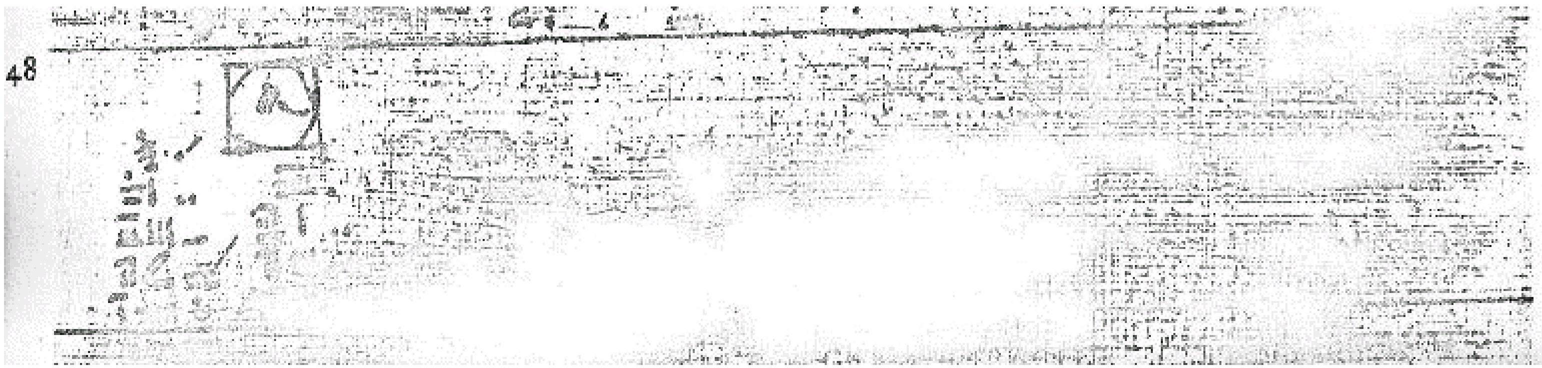
$$A = \left(d - \frac{d}{9}\right)^2 = \frac{64}{81} d^2 .$$

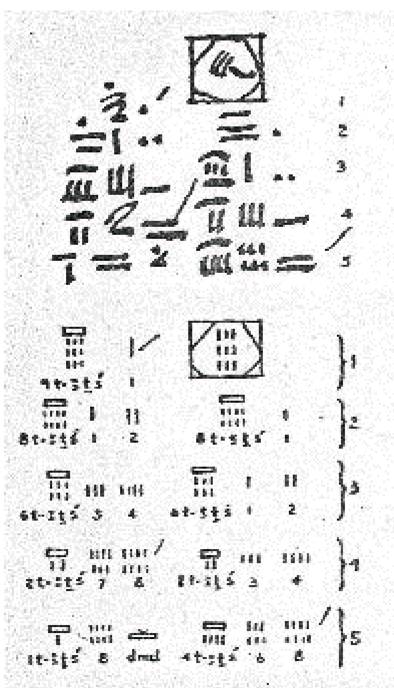
From this we deduce (since $A = \pi d^2/4$) that the ancient Egyptians implicitly used the value

$$\pi = 256/81 = (4/3)^4 = 3.\overline{160493827}\ldots$$

Ahmes Papyrus: Problem 48







Problem 48

1	šṯ3-t 8 setat	\1	šṯ3-t 9[2] setat
2	1 " 6[1]	2	1 " 8
4	3 " 2	4	3 " 6[1]
\8	6 " 4.	\8	7 " 2
		dmd *Total*	8 " 1.

This problem compares the areas of a circle of radius 9 and the circumscribing square.

The number before the word šṯ3-t denotes the number of times ten *setat*. Thus the second line of the first table represents 16 *setat*, the third line 32 *setat*, and so on. See volume 1, page 33. The writing of multiples of ten *setat* in this way is explained by the fact that ten *setat* is equivalent to the old Egyptian unit called a "thousand-of-land" equal to a thousand cubit-strips or cubits-of-land (see volume 1, page 33). Griffith and Peet consider these numbers as representing so many thousands-of-land (Griffith, volume 16, page 236; see also volume 14, pages 410–415. Peet, page 25 and under Problems 48–55).

[1] The numeral sign here which resembles the ordinary sign for 60 is probably a special sign used in writing both 6 *setat* and 6 *hekat*. See Introduction.

[2] The numeral 9 here is a special sign used in writing both 9 *setat* and 9 *hekat*. See Problems 53 and 84.

Ahmes Papyrus: Problem 48

Problem 48 is the only problem in the papyrus which does not have a statement. It consists only of a diagram and a calculation of 8^2 and 9^2.

Chace's interpretation is "This problem compares the areas of a circle of radius[sic] 9 and the circumscribing square."

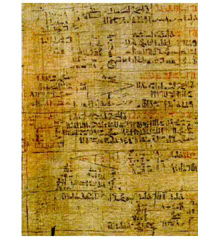
Gillings[4] challenges this interpretation, pointing out how well the circles of problems 50 and 41 are drawn.

Prob. 48

Prob. 50

Prob. 41

Ahmes Papyrus: Problem 48



Gillings' interpretation of this problem is that Ahmes is providing the justification of the rule for finding the area of a circle.

He proceeds to show how the rule could be obtained by examining the areas of an octagon (which would be “close” to the area of a circle) and the square which circumscribes it.

In his discussion he makes the “tongue in cheek” comment that Ahmes should be considered the first circle squarer![5] Others have seemingly missed the point that Gillings was joking, and have in all seriousness claimed this *honor* for Ahmes.

Earliest Greek Study

To talk about the quadrature problem with straightedge and compass, one must turn to the ancient Greeks; for these were the tools of the Greek geometers.

The first Greek known to be connected to the problem is **Anaxagoras** (c. 499 – c. 427 B.C.). Although his chief work was in philosophy, where his prime postulate was “reason rules the world,” he was interested in mathematics and wrote on the quadrature of the circle and perspective. Plutarch (c.46 - c.120) reports that he did this mathematical work while he was in jail (for being a Persian sympathizer)[6]. Only fragments of his work are extant, and it is unclear what his contribution actually is.

Lunes of Hippocrates

Hippocrates of Chios (c. 440 B.C.) was a contemporary of Anaxagoras and was described by Aristotle as being skilled in geometry but otherwise stupid and weak[7]. [Not to be confused with Hippocrates of Cos who also lived around this time on an island not far from the island of Chios and is considered to be the "father of medicine"; originator of the Hippocratic Oath.]

Hippocrates of Chios is mentioned by ancient writers as the first to arrange the propositions of geometry in a scientific fashion and as having published the secrets of Pythagoras in the field of geometry. Proclus (c. 460) credits him with the *method of reduction* ... reducing a problem to a simpler one, solving the simpler problem and then reversing the process.

Lunes of Hippocrates

Hippocrates enters our story because he provided the first example of a quadrature of a curvilinear figure (one whose sides are not line segments). He worked with certain ***lunes*** (crescents) formed by two circular arcs.

This work is also historically important, since it is the first *known* “proof”. We don't have Hippocrates' original words, rather Simplicius' summary (530 A.D.) of Eudemus' account (335 B.C.) in his now lost *History of Geometry,* of Hippocrates' proof (440 B.C.)[8].

Lunes of Hippocrates

Hippocrates' proof uses three preliminary results:

1. The Pythagorean Theorem ($a^2 + b^2 = c^2$)
2. An angle subtended by a semicircle is a right angle.
3. The ratio of the areas of two circles is the same as the ratio of the squares of their diameters. (Euclid XII.2)

The first two of these were well known to geometers of Hippocrates' time. Eudemos (again via Simplicus) credits Hippocrates with the third result, but Archimedes (c. 225 B.C.) implies that the result is due to Eudoxus (408 – c. 355 B.C.)[9]. This puts the credit for the proof in doubt and current thinking is that Hippocrates probably didn't have a rigorous proof.

Euclid XII 2

Before we examine Hippocrates' lunes, let's consider this result on the areas of circles.

We examine the result as given in Euclid's (c. 300 B.C.) masterpiece, *The Elements*. It is clear that this work is, at least in part, a compilation of earlier Greek work. The second proposition in book XII (out of 13) is:

Circles are to one another as the squares on the diameters.

Notice how the proposition is phrased in geometrical terms – not the "squares *of* the diameters" an algebraic operation, but the "squares *on* the diameters" referring to the geometric square with side equal to a diameter. The ancient Greeks had only a rudimentary algebraic notation and relied almost exclusively on geometric ideas in their writing and thinking.

Euclid XII 2

Today we would state the result as:

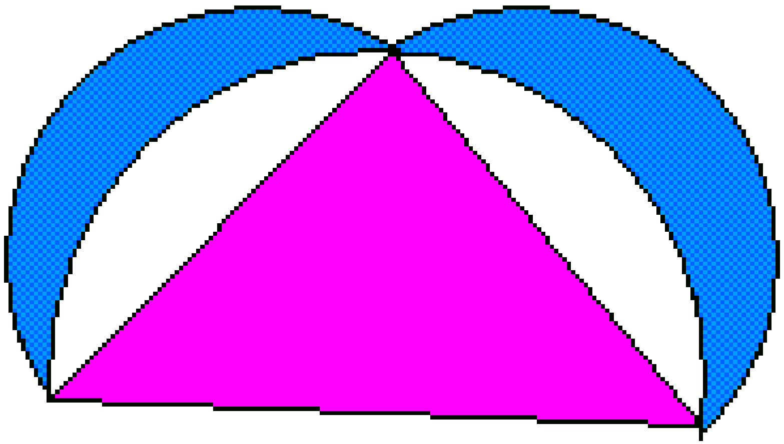
The ratio of the areas of two circles equals the ratio of the squares of their diameters.

This phrasing underlines the more algebraic way in which we view such problems. One needs to be careful in studying ancient mathematics not to dismiss the difficulties that were overcome by the ancients because they appear simple to us. This simplicity is a result of a viewpoint that took thousands of years to develop.

Euclid's proof is an example of the *method of exhaustion*, a technique used several times in Euclid and according to Archimedes, perfected by the mathematician Eudoxus. Simply put, the idea is to "exhaust" the area of a given circle by inscribing in it polygons of increasingly many sides. This is combined with a double *reductio ad absurdum,* that is, he proves $A = B$ by showing that $A < B$ and $B < A$ both lead to contradictions.

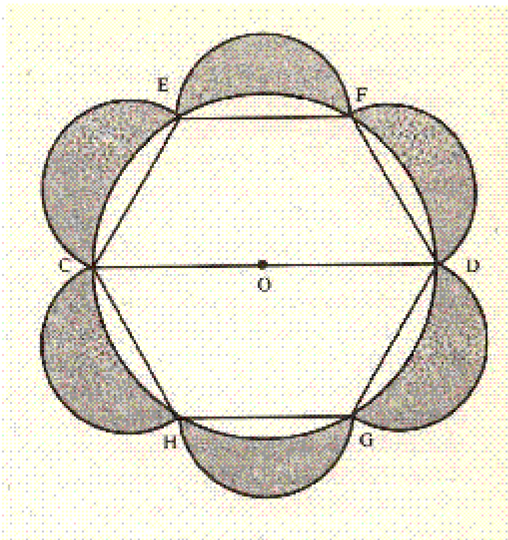
Quadrature of Hippocrate's Lunes

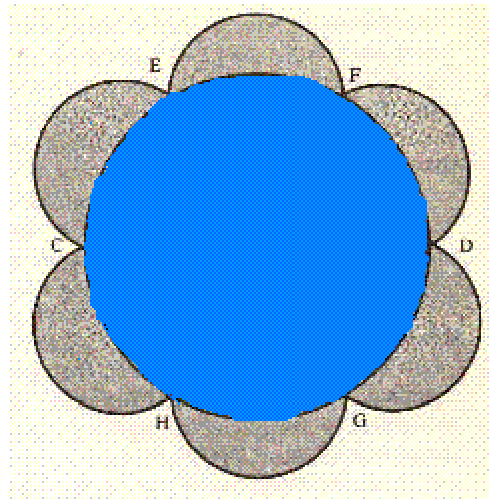
Given an isosceles right triangle, the area of the lunes determined by the semicircle on the hypotenuse and the semicircles on the sides of the triangle is equal to the area of the triangle.

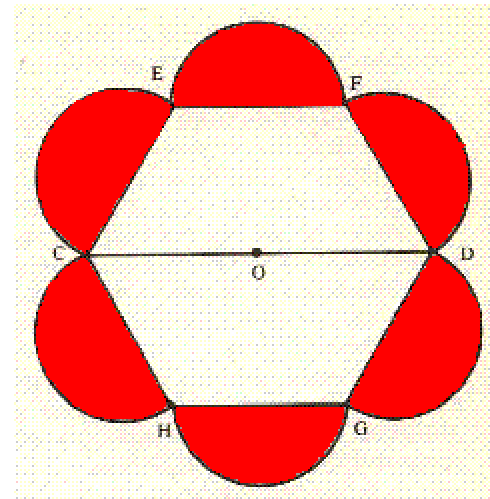


The theorem remains true for any right triangle, but Hippocrates does not seem to be aware of this.

Lunes of Hippocrates







Consider a regular hexagon inscribed in a circle whose side is ½ of the diameter of the circle. The figure above can be thought of in two ways: circle + 6 lunes = hexagon + 6 semicircles .
Since the diameter of the red semicircles is ½ the diameter of the blue circle, circle = 4 circles = 8 semicircles. So, we get

1 circle = 2 semicircles = hexagon – 6 lunes.

So if we can square these lunes, we can square a circle ....

... but these are **not** Hippocrates' lunes!

A Literary Aside

Arguments such as this may have engendered a hope that with enough work a quadrature of the circle could be accomplished.

The desire to find such a quadrature must have been well known to the general Greek populace, and not just the small set of mathematicians, for we find it referenced in one of Aristophanes famous comedies.

Not only is the problem known, but in order to achieve the comic effect, it must have been known as a fruitless endeavor.

Aristophanes

In the ***Birds***, performed in 414 BC, a new city has to be founded from scratch. The main character, Peisthetaerus, is visited by various people who offer their services.[11]

Enter

METON: I come amongst you ...

PEISTHETAERUS: Some new misery this! Come to do what? What's your scheme's form and outline? What's your design? What buskin's on your foot?

METON: I come to land-survey this Air of yours, and mete it out by acres.

PEISTHETAERUS: Heaven and earth! Whoever are you?

METON: Whoever am I? I'm Meton, known throughout Hellas and Colonnus.

PEISTHETAERUS: Aye, and what are these?

Aristophanes

METON: They're rods for Air-surveying. I'll just explain. The Air's, in outline, like one vast extinguisher; so then, observe, applying here my flexible rod, and fixing my compass there, - you understand?
PEISTHETAERUS: I don't.
METON: With the straight rod I measure out, that so the circle may be squared; and in the centre a market-place; and streets be leading to it straight to the very centre; just as from a star, though circular, straight rays flash out in all directions.
PEISTHETAERUS: Why, the man's a Thales!

π

It is lost in the mists of pre-history who first realized that the ratio of the circumference of a circle to its diameter is a constant. All the ancient civilizations knew this fact. Today we call this ratio π and express this relationship by saying that for ***any*** circle, the circumference C and the diameter d satisfy: $C = \pi d$.

The use of the symbol "π" for this ratio is of relatively recent origin; the Greeks did not use the symbol.

"π was first used by the English mathematicians Oughtred (1647), Isaac Barrow (1664) and David Gregory (1697) to represent the circumference of a circle. The first use of "π" to represent the ratio of circumference to diameter was the English writer William Jones (1706). However, it did not come into common use until Euler adopted the symbol in 1737.[1]

Archimedes

Euclid XII.2 says that the ratio of the area of any circle to the square of its diameter is also a constant, but does not determine the value of this constant.

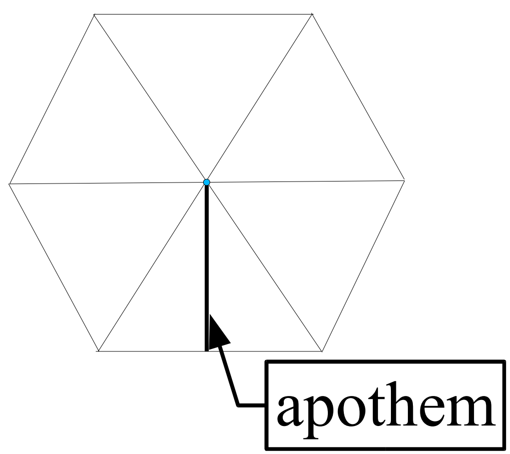
It was Archimedes (287 – 212 B.C.) who determined the constant in his remarkable treatise *Measurement of a Circle*. There are only three propositions in this short work (or at least, that is all of that work that has come down to us) and the second proposition is out of place – indicating that what we have is probably not the original version.[2]

We shall look at the first and third proposition.

Archimedes

A few preliminary ideas:

The area of a regular polygon is easily determined if you know that the area of a triangle = ½bh (b the length of a side, h the length of the altitude drawn to that side). In a regular polygon of n sides (all sides equal, all angles equal), draw the lines from the center to each of the vertices creating n congruent triangles.

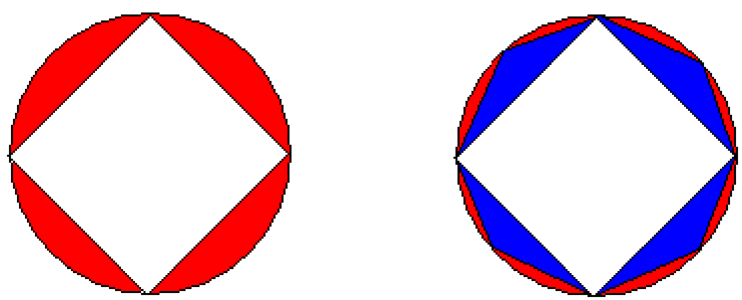


The area is thus n times the area of one of the triangles = n(½sa) where s is the length of a side and a the length of an ***apothem*** (line drawn from center perpendicular to a side). Since ns is the perimeter (Q) of the polygon we get :

$$A = \tfrac{1}{2}aQ.$$

Archimedes

Consider a regular polygon inscribed in a circle. Let K be the difference in the areas (area of circle – area of polygon). If you now double the number of sides of the polygon, the area you have added to the original polygon is more than ½K.



By repeating the procedure, you can make the area difference *as small as you like*, or in other words, for any positive number K, you can find a regular polygon (with enough sides) inscribed in a circle so that the area difference is less than K. This is Eudoxus' *method of exhaustion*. It also works for circumscribed polygons.

Archimedes

Archimedes has proved that for any circle, $A = \frac{1}{2}rC$, and since we know that $C = \pi d$, we get $A = \frac{1}{2}r(\pi d) = \frac{1}{2}r(\pi 2r) = \pi r^2$ our familiar high school formula.

Even though Archimedes showed the equivalence of a circle with a rectilinear figure, easily converted to a square, this is **not** a solution of the quadrature problem. The proof is indirect, it does not give a means for constructing the triangle with straightedge and compass.

Before looking at Proposition 3, let's consider one of several methods known to the Greeks of using curves to perform a quadrature of the circle – however, the curves used in this way can not themselves be constructed with straightedge and compass!

Archimedes

In proposition 3 Archimedes turns his attention to the circumference of a circle. Again using inscribed and circumscribed regular polygons, their perimeters provide upper and lower bounds for the circumference of the circle. This gives him a means of calculating bounds for the number π.

Proposition 3*: The ratio of the circumference of any circle to its diameter is less than 3 1/7 (*22/7*) but greater than 3 10/71 (*223/71*).*

$(3.140845... < \pi < 3.142857...)$

What is remarkable about this result is not the underlying idea, but rather the skill of Archimedes in carrying out the computations. He started with inscribed and circumscribed hexagons, then doubled the size, and again, and again and yet again, ending with 96-sided polygons. At each step he calculates the perimeters. This involves approximating radicals which is where he shows his genius.

The Computation of π Early Phase

There has always been an interest in the precise value of π. As we have seen, the ancient Egyptians used π = 3.1604938... . Other ancient civilizations were not as precise, generally using π = 3. This can be seen in Babylonian clay tablets and in the Bible (I Kings: 7:23)

*Then He made the molten sea (*circular*), ten cubits from brim to brim, while a line of 30 cubits measured it around.*

After Archimedes improvements were made by taking larger and larger polygons (except for the Romans – not very concerned with precision, they dropped back to the value 3 1/8)[6].

The Computation of π Early Phase[7]

ca. 150 A.D. - The first improvement over Archimedes values was given by Claudius Ptolemy of Alexandria in the *Almagest*, the most famous Greek work on astronomy. Ptolemy gives a value of $\pi = 377/120 = 3.1416$ ....

ca. 480 – The early Chinese worker in mechanics, Tsu Ch'ung-chih, gave the approximation $\pi = 355/113 = 3.1415929$ ... correct to 6 decimal places.

ca. 530 – The early Hindu mathematician Āryabhata gave $\pi = 62{,}832/20{,}000 = 3.1416$ ... as an approximation. It is not known how this was obtained, but it could have been calculated as the perimeter of a regular inscribed polygon of 384 sides.

The Computation of π Early Phase

ca. 1150 – The later Hindu mathematician Bhāskara gave several approximations. He gave 3927/1250 as an accurate value, 22/7 as an inaccurate value, and $\sqrt{10}$ for ordinary work.

1429 – Al-Kashi, astronomer royal to Ulugh Beg of Samarkand, computed π to sixteen decimal places using perimeters.

1579 – The eminent French mathematician François Viète found π correct to nine decimal places using polygons having 393,216 sides.

1593 – Adriaen van Roomen, more commonly known as Adrianus Romanus, of the Netherlands, found π correct to 15 places using polygons having 2^{30} sides.

The Computation of π Early Phase

1610 – Ludolph van Ceulen of the Netherlands computed π to 35 decimal places using polygons having 2^{62} sides. He spent a large part of his life on this task, and his achievement was considered so extraordinary that his widow had the number engraved on his tombstone (now lost). To this day, the number is sometimes referred to as "the Ludophine number."

1621 – The Dutch physicist Willebrord Snell, best known for his discovery of the law of refraction, devised a trigonometric improvement of the classical method so that from each pair of bounds given by the classical method he was able to obtain considerably closer bounds. By his method, he was able to get van Ceulen's 35 places using a polygon with only 2^{30} sides.

The Computation of π Early Phase

1630 – Grienberger, using Snell's refinement, computed π to 39 decimal places. This was the last major attempt to compute π by the method of Archimedes.