Complex Analysis, the low down

I've once heard this class described as locally easy but globally hard. Which I think means that each step is small, but builds on the step before so that the overall distance traveled is large. Complex analysis is different from Real analysis. (We will list some of the ways it's different shortly.) One should embrace the differences rather than only finding ways to related it back to the Reals.

What you lose going from $\mathbb{R}$ to $\mathbb{C}$:

- **Total Ordering**: For reals a, b, one has the trichotomy $a < b$ or $a = b$ or $a > b$. But it is impossible to say which is larger $2 + i$ or $1 + 2i$, in a way that preserves the usual ideas of order and arithmetic.
- A corollary of total ordering. There are no left and right limits, and in particular there is only one point at infinitely, instead of the dual $\pm\infty$ on the real line. This difference is similar to what happens with functions, $f(x, y)$, of two real variables and the way they are handled in calculus 3.
- Also a corollary of total ordering. Well known functions become multi-valued. For example, for real $x \geq 0$, $\sqrt{x}$ is the positive number y so that $y^2 = x$. What does positive mean for a complex number? Now there are two $w = \sqrt{z}$ and you can't just divide the cases into $\pm\sqrt{z}$. In fact you can't tell the difference between $i = \sqrt{-1}$ and $-i = -\sqrt{-1}$ mathematically. One is picked arbitrarily and called i.
- **The Complex Part**: The algebra becomes a little messier, the simplification tricks are more varied, but it is not that different.
- **Pathology disappears**: It is "harder" to be differentiable as a complex function and so most of the pathological real variable examples are just not complex differentiable. (Real function $f(x) = \int_0^x |t|\, dt$ is differentiable once but not twice.) This means theorems starting with "for all" are easier than Real analysis and theorems starting with "there exists" are harder than for Real analysis.

All in all it is not a big lost. The next jump from $\mathbb{C}$ to the quaternions $\mathbb{H}$ loses commutativity of multiplication. The jump after that is to the Cayley numbers or the Octonions $\mathbb{O}$ where multiplication is not even associative. Lots of things are true only for dimension two.

What you gain going from $\mathbb{R}$ to $\mathbb{C}$:

- **Multiple Views**. More than one way of representing $\mathbb{C}$. You already know several of these. It is the Euclidean plane $\mathbb{R}^2$ which we know from geometry, from analytic geometry as rectangular coordinates $z = \Re z + i\Im z = x + iy = (x, y) = (\Re z, \Im z)$ and from polar coordinates as (r, θ) and eventually $z = r\exp(i\theta) = re^{i\theta}$. Having several ways to look at something provides multiple simplification routes. (The math symbols $\Re$ and $\Im$ are often written Re and Im for Real and Imaginary)
- **Algebraically Complete**. The complex numbers are algebraically complete. Every real or complex polynomial has a complex root. Actually an n-th degree polynomial has exactly n roots if you count multiplicities. While one can handle real polynomials that arise from differential equations with sines and cosines, the expressions are cleaner in the language of complex numbers. One can often appear to tricks like if the differential equation and the initial conditions are Real, then this complex solution has to be Real for all time.
- **Geometry**. The geometric connection is very strong, adding the point at infinity, for example, makes the plane a sphere. This makes circles and lines the same thing as lines become circles with the point at infinity attached. Complex analysis provides many fast methods of obtaining trigonometric identities. The geometric operations of translation, dilation, rotation, and reflection are all simple to describe as complex functions.
- **Analytic completeness**. The analysis is very complete. Differentiable functions are infinity differentiable, they always have power series, they are always 'conservative' or exact, they solve strong partial differential equations, and line integrals (called contour integrals) are a fast computational tool.
- **Awesome Theorems**. The theorems are absolutely astounding. One we will prove is that a function like the real function $\sin x$ which varies, is bounded and has a derivative can't happen for a complex

function. (Note there is a complex function $w = \sin z$ but it is not bounded.) This is Louiville's theorem and we will use it to prove the fundamental theorem of algebra, namely, every complex polynomial has a complex root. In fact, one could say these theorems are the result of considering the smallest class of functions that are "infinite polynomials". And there is a larger interesting class which are the "infinite rational functions" where a rational function is the quotient of two polynomials.

The gains are so large that one course cannot come close to making a real dent in the material. Another limitation is that many of the truly wonderful theorems do not generalize to functions of several complex variables. Some of the theorems are so strong that they cloak Real analysis theorems which are not quit as easy to prove, but whose proofs provide additional insight.

Complex analysis is also interesting from a historical point of view. The imaginary numbers and the complexes in general were hard to shallow initially. The historic reasons for their eventual acceptance is generally not presented. Our mathematical fore fathers didn't introduce complex numbers so that all quadratics had roots or so the that quadratic formula worked. It was because complex numbers gave "real answers" that were not available otherwise. Generally speaking some cubics have 3 real roots, but the formula's to acquire all three roots travels through the complex universe. Indeed even today the area of "Analytic Number Theory" is based on the using the power of applying complex analysis to the plane to say something about much smaller subset of the natural numbers.

The most famous unsolved problem in mathematics is the Riemann Hypothesis and it is a statement about the zeros of a complex function (the zeta function, $\zeta(s)$) which has important consequences about the distributions of prime numbers. Analytic number theory's current biggest success is called the Prime Number Theorem, which was proved at the end of the 19-th century (and is also about the zeros of the zeta function).

But complex analysis is even more. The multi-valued functions lead to Riemannian manifolds which in turn is the foundation for Einstein's theory of general relativity. One way of thinking about this is that this physical theory of relativity is more a climax of the 19-th century which has shape contrast with the probabilistic based quantum mechanics of the 20-th century.