LESSON 6:
TRIGONOMETRIC IDENTITIES

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1. Introduction

An **identity** is an equality relationship between two mathematical expressions. For example, in basic algebra students are expected to master various **algbriac factoring identities** such as

$$a^2 - b^2 = (a-b)(a+b) \quad \text{or}$$
$$a^3 + b^3 = (a+b)(a^2 - ab + b^2).$$

Identities such as these are used to simplifly algebriac expressions and to solve algebriac equations. For example, using the third identity above, the expression $\dfrac{a^3+b^3}{a+b}$ simplifies to $a^2 - ab + b^2$. The first identiy verifies that the equation $(a^2 - b^2) = 0$ is true precisely when $a = \pm b$. The formulas or trigonometric identities introduced in this lesson constitute an integral part of the study and applications of trigonometry. Such identities can be used to simplifly complicated trigonometric expressions. This lesson contains several examples and exercises to demonstrate this type of procedure. Trigonometric identities can also used solve trigonometric equations. Equations of this type are introduced in this lesson and examined in more detail in Lesson 7. For student's convenience, the identities presented in this lesson are sumarized in Appendix A

2. The Elementary Identities

Let (x, y) be the point on the unit circle centered at $(0,0)$ that determines the angle t rad. Recall that the definitions of the trigonometric functions for this angle are

$$\begin{array}{lll} \sin t = y & \tan t = \frac{y}{x} & \sec t = \frac{1}{y} \\ \cos t = x & \cot t = \frac{x}{y} & \csc t = \frac{1}{x} \end{array}.$$

These definitions readily establish the first of the **elementary** or **fundamental identities** given in the table below. For obvious reasons these are often referred to as the **reciprocal** and **quotient** identities. These and other identities presented in this section were introduced in Lesson 2 Sections 2 and 3.

$\sin t = \frac{1}{\csc t}$	$\cos t = \frac{1}{\sec t}$	$\tan t = \frac{1}{\cot t} = \frac{\sin t}{\cos t}$
$\csc t = \frac{1}{\sin t}$	$\sec t = \frac{1}{\cos t}$	$\cot t = \frac{1}{\tan t} = \frac{\cos t}{\sin t}$

Table 6.1: Reciprocal and Quotient Identities.

Example 1 *Use the reciprocal and quotient formulas to verify*

$$\sec t \cot t = \csc t.$$

Solution: Since $\sec t = \frac{1}{\cos t}$ *and* $\cot t = \frac{\cos t}{\sin t}$ *we have*

$$\sec t \cot t = \frac{1}{\cos t}\frac{\cos t}{\sin t} = \frac{1}{\sin t} = \csc t.$$

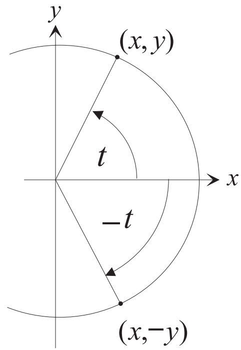
Example 2 *Use the reciprocal and quotient formulas to verify*

$$\sin t \cot t = \cos t.$$

Solution: We have

$$\sin t \cot t = \sin t \frac{\cos t}{\sin t} = \cos t.$$

Several fundamental identities follow from the symmetry of the unit circle centered at $(0, 0)$. As indicated in the figure, if (x, y) is the point on this circle that determines the angle t rad, then $(x, -y)$ is the point that determines the angle $(-t)$ rad. This suggests that $\sin(-t) = -y = -\sin t$ and $\cos(-t) = x = \cos t$. Such functions are called **odd** and **even** respectively[1]. Similar reasoning verifies that the tangent, cotangent, and secant functions are odd while the cosecant function is even. For example, $\tan(-t) = \dfrac{-y}{x} = -\dfrac{y}{x} = \tan t$. Identities of this type, often called the **symmetry identities,** are listed in the following table.



[1]A function f is **odd** if $f(-x) = -f(x)$ and **even if** $f(-x) = f(x)$ for all x in its domain. (See section 2in section 5for more information about these two properties of functions.

$\sin(-t) = -\sin t$	$\cos(-t) = \cos t$	$\tan(-t) = -\tan t$
$\csc(-t) = -\csc t$	$\sec(-t) = \sec t$	$\cot(-t) = -\cot t$

Table 6.2: The Symmetry Identities.

The next example illustrates an alternate method of proving that the tangent function is odd.

Example 3 *Using the symmetry identities for the sine and cosine functions verify the symmetry identity* $\tan(-t) = -\tan t$.
Solution: Armed with theTable 6.1 we have

$$\tan(-t) = \frac{\sin(-t)}{\cos(-t)} = \frac{-\sin t}{\cos t} = -\tan t.$$

This strategy can be used to establish other symmetry identities as illustrated in the following example and in Exercise 1.)

Example 4 *The symmetry identity for the tangent function provides an easy method for verifying the symmetry identity for the cotnagent function. Indeed,*

$$\cot(-t) = \frac{1}{\tan(-t)} = \frac{1}{-\tan t} = -\frac{1}{\tan t} = -\cot t.$$

The last of the elementary identities covered in this lesson are the **Pythagorean identities**[2] given in Table 6.3**.** Again let (x, y) be the point on the unit circle with center $(0, 0)$ that determines the angle t rad. Replacing x and y by $\cos t$ and $\sin t$ respectively in the equation $x^2 + y^2 = 1$ of the unit circle yields the identity[3] $\sin^2 t + \cos^2 t = 1$. This is the first of the Pythagorean identities. Dividing this last equality through by $\cos^2 t$ gives

$$\frac{\sin^2 t}{\cos^2 t} + \frac{\cos^2 t}{\cos^2 t} = \frac{1}{\cos^2 t}$$

which suggest the second Pythagorean identity $\tan^2 t + 1 = \sec^2 t$. The proof of the last identity is left to the reader. (See Exercise 2.)

$\sin^2 t + \cos^2 t = 1$	$\tan^2 t + 1 = \sec^2 t$	$1 + \cot^2 t = \csc^2 t$

Table 6.3: Pythagorean Identities.

[2]These identities are so named because angles formed using the unit circle also describe a right triangle with hypotenuse 1 and sides of length x and y. These identities are an immediate consequence of the Pythagorean Theorem.

[3]The expression $\sin^2 t$ is used to represent $(\sin t)^2$ and should not be confused with the quantity $\sin t^2$.

The successful use of trigonometry often requires the simplification of complicated trigonometric expressions. As illustrated in the next example, this is frequently done by applying trigonometric identities and algebraic techniques.

Example 5 *Verify the following identity and indicate where the equality is valid:*

$$\frac{\cos^2 t}{1-\sin t} = 1+\sin t.$$

Solution: *By first using the Pythagorean identity* $\sin^2 t + \cos^2 t = 1$ *and then the factorization* $1-\sin^2 t = (1+\sin t)(1-\sin t)$, *the following sequence of equalities can be established:*

$$\frac{\cos^2 t}{1-\sin t} = \frac{1-\sin^2 t}{1-\sin t} = \frac{(1+\sin t)(1-\sin t)}{1-\sin t} = 1+\sin t, \quad 1-\sin t \neq 0.$$

As indicated, the formula is valid as long as $1-\sin t \neq 0$ *or* $\sin t \neq 1$. *Since* $\sin t = 1$ *only when* $t = \frac{\pi}{2} + 2k\pi$ *where* k *denotes any integer, the identity is valid on the set*

$$\Re - \{t : t = \frac{\pi}{2} + 2k\pi \textit{ where } k \textit{ is an integer}\}.$$

The process of using trigonometric identities to convert a complex expression to a simpler one is an intuitive mathematical strategy for most people. Sometimes, however, problems are solved by initially replacing a simple expression with a more complicated one. For example, in some applications the expression $1+\sin t$ is replaced by the more complex quantity $\dfrac{\cos^2 t}{1-\sin t}$. This essentially involves redoing the steps in Example 5 in reverse order as indicated in the following calculations:

$$1+\sin t = \frac{(1+\sin t)(1-\sin t)}{1-\sin t} = \frac{1-\sin^2 t}{1-\sin t} = \frac{\cos^2 t}{1-\sin t}.$$

In particular, the first step would be to multiply $1+\sin t$ by the fraction $\dfrac{1-\sin t}{1-\sin t}$ (which has value one as long as $1-\sin t \neq 0$) to obtain the quantity $\dfrac{(1+\sin t)(1-\sin t)}{1-\sin t}$. The reader is advised to review the calculatons above while keeping in mind the insights required to perform the steps. The strategy of replacing seemingly simple expressions by more sophisticated ones is a rather unnatural and confusing process. However, with practice the strategy can be mastered and understood. The next example further illustrates this type of problem.

Example 6 *Determine the values of t such that $2\sin t + \cos^2 t = 2$.*
Solution: Equations such as these are usually solved by rewriting the expression in terms of one trigonometric function. In this case it is reasonable to use the first identity in Table 6.3 to change $\cos^2 t$ to the more complicated expression $1 - \sin^2 t$. This will produce the following equation involving only the sine function:

$$2\sin t + 1 - \sin^2 t = 2.$$

This last equation should remind the reader of the corresponding quadratic equation $2x + 1 - x^2 = 2$ which can be solved by factoring. That is what we will do here. First, subtract 2 from both sides of the above equation and then multiply through by (-1) to obtain $\sin^2 t - 2\sin t + 1 = 0$. Factoring this expression yields

$$(\sin t - 1)^2 = 0.$$

The only solution to this last expression is given by $\sin t = 1$ or $t = \frac{\pi}{2} + 2k\pi$ where k is any integer.

3. The sum and difference formulas

This section begins with the verification of the **difference formula** for the cosine function:

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

where $\alpha - \beta$ denotes the measure of the difference of the two angles α and β. Once this identity is established it can be used to easily derive other important identities. The verification of this formula is somewhat complicated. Perhaps the most difficult part of the proof is the complexity of the notation. A drawing (Figure 6.1)should provide insight and assist the reader overcome this obstacle. Before presenting the argument, two points should be reviewed. First, recall the formula for the distance between two points in the plane. Specifically, if (a, b) and (c, d) are planer points, then the distance between them is given by

$$\sqrt{(a-c)^2 + (b-d)^2}. \tag{1}$$

Second, the argument given below conveniently assumes that $0 < \alpha - \beta < 2\pi$. The assumption that $\alpha - \beta < 2\pi$ is justified because complete wrappings of angles (integer multiples of 2π) can be ignored since the cosine function has period 2π. The assumption that the angle $\alpha - \beta$ is positive is justified because the Symmetry Identities guarantee that $\cos(\alpha - \beta) = \cos(\beta - \alpha)$. We can now derive the formula.

First, observe that the angle $\alpha - \beta$ appears in Figure 6.1(a) and (b) and is in standard position in Figure 6.1b. This angle determines a **chord** or line segment in each drawing (not shown), one connecting (x_1, y_1) to (x_2, y_2) (in Figure 6.1a) and one connecting (w, z) to $(1, 0)$ (in Figure 6.1b). These chords have the same length since they subtend angles of equal measure on circles of equal radii. (See Lesson 3 Section 6 .) This observation and the distance formula (Equation 1) permit the equality

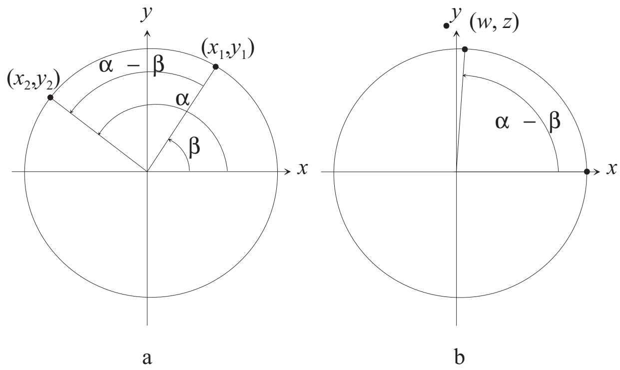


Figure 6.1: Differences of angles.

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(w - 1)^2 + z^2}. \tag{2}$$

Squaring both sides of Equation 2 removes the radicals resulting in

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = (w - 1)^2 + z^2.$$

Expanding the squares of the binomials suggests that

$$x_2^2 - 2x_1x_2 + x_1^2 + y_2^2 - 2y_1y_2 + y_1^2 = w^2 - 2w + 1 + z^2. \tag{3}$$

Since (x_1, y_1) is a point on the unit circle so that the equality $x_1^2 + y_1^2 = 1$ holds, the sum of x_1^2 and y_1^2 in Equation 3 can be replaced by 1. Similar statements hold for the points (x_2, y_2), and (w, z). These replacements yield

$$2 - 2x_1x_2 - 2y_1y_2 = 2 - 2w,$$

or, after dividing by 2 and solving for w,

$$w = x_1x_2 + y_1y_2.$$

The desired formula

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \tag{4}$$

follows from the observations that (Refer to Figure 6.1.) $w = \cos(\alpha - \beta)$, $x_2 = \cos\alpha$, $x_1 = \cos\beta$, $y_2 = \sin\alpha$, and $y_1 = \sin\beta$. This completes the proof.

Example 7 *Without the use of a calculator determine the value of* $\cos(\pi/12)$*.* (Computing the value of $\cos(\pi/12)$ is not the instructional goal of this example. The purpose is to provide the reader with some experience using the cosine formula for the difference of two angles. Being able to derive a correct answer using a computing device will never serve as a substitute for analytical thinking and understanding mathematical concepts.)
Solution: Note that

$$\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}.$$

Equation 4 and Figure 6.1 yield

$$\begin{aligned}\cos\frac{\pi}{12} &= \cos(\frac{\pi}{3} - \frac{\pi}{4}) \\ &= \cos\frac{\pi}{3}\cos\frac{\pi}{4} + \sin\frac{\pi}{3}\sin\frac{\pi}{4} \\ &= \frac{1}{2}\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} \\ &= \frac{\sqrt{2}}{4}(1+\sqrt{3}).\end{aligned}$$

Example 8 *Without the use of a calculator determine the value of* $\cos(7\pi/12)$.
Solution: This problem can easily be done using the formula for the cosine of the sum of two angles which is covered in the sequel (Equation 6). Presently, however, we must use Equation 4. First, write

$$\frac{7\pi}{12} = \frac{\pi}{3} + \frac{\pi}{4} = \frac{\pi}{3} - \left(-\frac{\pi}{4}\right).$$

Then

$$\begin{aligned}
\cos\frac{7\pi}{12} &= \cos\left[\frac{\pi}{3} - \left(-\frac{\pi}{4}\right)\right] \\
&= \cos\frac{\pi}{3}\cos\left(-\frac{\pi}{4}\right) + \sin\frac{\pi}{3}\sin\left(-\frac{\pi}{4}\right) \\
&= \cos\frac{\pi}{3}\cos\frac{\pi}{4} + \sin\frac{\pi}{3}\left(-\sin\frac{\pi}{4}\right) \qquad \textit{Table 6.2} \\
&= \cos\frac{\pi}{3}\cos\frac{\pi}{4} - \sin\frac{\pi}{3}\sin\frac{\pi}{4} \\
&= \frac{1}{2}\frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{4}\left(1 - \sqrt{3}\right).
\end{aligned}$$

The **cofunction identities** are immediate consequences of Equation 4. These express the values of the trigonometric functions at α in terms of their cofunctions at the complementary angle $\frac{\pi}{2} - \alpha$. For example, by Equation 4

$$\cos\left(\frac{\pi}{2} - \alpha\right) = \cos\frac{\pi}{2}\cos\alpha + \sin\frac{\pi}{2}\sin\alpha = 0 \cdot \cos\alpha + 1 \cdot \sin\alpha = \sin\alpha,$$

so

$$\cos\left(\frac{\pi}{2} - \alpha\right) = \sin\alpha. \tag{5}$$

Replacing α with $\frac{\pi}{2} - \alpha$ in Equation 5 validates the following cofunction identity for the sine function:

$$\sin\left(\frac{\pi}{2} - \alpha\right) = \cos\left[\frac{\pi}{2} - \left(\frac{\pi}{2} - \alpha\right)\right] = \cos\alpha.$$

Example 9 *Verify that* $\sin\left(\frac{\pi}{4}+t\right)=\cos\left(\frac{\pi}{4}-t\right)$.
Solution: Since

$$\sin\left(\frac{\pi}{4}+t\right)=\sin\left[\frac{\pi}{2}-(\frac{\pi}{4}-t)\right]=\cos(\frac{\pi}{4}-t)$$

the desired equality follows.

The cofunction identity for the tangent function is easily established since

$$\tan\left(\frac{\pi}{2}-t\right)=\frac{\sin\left(\frac{\pi}{2}-t\right)}{\cos\left(\frac{\pi}{2}-t\right)}=\frac{\cos t}{\sin t}=\cot t.$$

The reader should verify the remaining cofunction identities. (See Exercise 3.) Table 6.4 summarizes these identities.

$\sin\left(\frac{\pi}{2}-t\right)=\cos t$	$\cos\left(\frac{\pi}{2}-t\right)=\sin t$	$\tan\left(\frac{\pi}{2}-t\right)=\cot t$
$\cot\left(\frac{\pi}{2}-t\right)=\tan t$	$\sec\left(\frac{\pi}{2}-t\right)=\csc t$	$\csc\left(\frac{\pi}{2}-t\right)=\sec t$

Table 6.4: The cofunction identities.

Now we return to the general discussion of sum and difference formulas for the trigonometric functions. Their derivation uses the formula for the cosine of the difference of two angles (Equation 4) in conjunction with the symmetry indentities (Table 6.2). The following sequence of equalities demonstrates this strategy:

$$\begin{aligned}\cos(\alpha+\beta) &= \cos[\alpha-(-\beta)]\\ &= \cos\alpha\cos(-\beta)+\sin\alpha\sin(-\beta)\\ &= \cos\alpha\cos\beta+\sin\alpha(-\sin\beta)\\ &= \cos\alpha\cos\beta-\sin\alpha\sin\beta.\end{aligned}$$

This establishes the cosine formula for the sum of two angles:

$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta. \tag{6}$$

The sum and difference formulas for the sine and cosine functions are

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha \tag{7}$$

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha. \tag{8}$$

The proof of the first of these is given below while that for the second is left as an exercise (Exercise 4). The identity $\cos\left(\frac{\pi}{2}-t\right) = \sin t$ in Table 6.4 allows the equality

$$\begin{aligned}\sin(\alpha+\beta) &= \cos\left[\frac{\pi}{2}-(\alpha+\beta)\right]\\ &= \cos\left[\left(\frac{\pi}{2}-\alpha\right)-\beta)\right].\end{aligned}$$

Appealing to Equation 4 using the two angles $\left(\frac{\pi}{2}-\alpha\right)$ and β yields

$$\begin{aligned}\cos\left[\left(\frac{\pi}{2}-\alpha\right)-\beta)\right] &= \cos\left(\frac{\pi}{2}-\alpha\right)\cos(-\beta) - \sin\left(\frac{\pi}{2}-\alpha\right)\sin(-\beta)\\ &= \sin\alpha\cos\beta - \cos\alpha(-\sin\beta)\\ &= \sin\alpha\cos\beta + \cos\alpha\sin\beta.\end{aligned}$$

The two intermediate steps made use of identities in Table 6.2 and Table 6.4. This completes the proof.

The following example uses the sum formula for the sine function (Equation 7).

Example 10 *Without the use of a calculator determine the value of* $\sin(7\pi/12)$*.*
Solution: This problem requires a strategy similar to that used in Example 7. Consider

$$\begin{aligned}\sin\frac{7\pi}{12} &= \sin(\frac{\pi}{3}+\frac{\pi}{4}) \\ &= \sin\frac{\pi}{3}\cos\frac{\pi}{4}+\sin\frac{\pi}{4}\cos\frac{\pi}{3} \\ &= \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\frac{1}{2} \\ &= \frac{\sqrt{2}}{4}(1+\sqrt{3}).\end{aligned}$$

Example 11 *Note that the answers for both Example 7 and Example 10 are the same. Verify this result using a cofucntion identity. That is, prove that* $\cos\frac{\pi}{12}=\sin\frac{7\pi}{12}$*.*
Solution: Consider $\cos\frac{\pi}{12}=\cos\left(\frac{7\pi}{12}-\frac{\pi}{2}\right)=\cos\left(\frac{\pi}{2}-\frac{7\pi}{12}\right)=\sin\frac{7\pi}{12}$*. The second step in the sequence used the fact that* $\cos(-t)=\cos t$*.*

The sum formulas can be used to establish some of the properties of the trigonometric functions discussed in Lesson 2 Section 4 as illustrated in the following example.

Example 12 *Use the sum formula for the cosine function (Equation 6) to establish the identities*

$$\cos(t+\pi) = -\cos t \quad \textit{and} \quad \cos(t+2\pi) = \cos t.$$

Solution: The sum formula for the cosine function and the values $\cos\pi = -1$ *and* $\sin\pi = 0$ *give*

$$\cos(t+\pi) = \cos t\cos\pi - \sin t\sin\pi = \cos t(-1) - \sin t(0) = -\cos t.$$

Similarly, because $\cos(2\pi) = 1$ *and* $\sin(2\pi) = 0$,

$$\cos(t+2\pi) = \cos t\cos(2\pi) - \sin t\sin(2\pi) = \cos t.$$

The first identity in the previous example can be proven using the Cofunction Identities as illustrated by the following calculations:

$$\cos\left(t+\pi\right) = \cos\left(\frac{\pi}{2} - \left(t - \frac{\pi}{2}\right)\right) = \sin\left(t - \frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{2} - t\right) = -\cos t.$$

The last sum and difference formulas to be treated in this tutorial are those for the tangent function. Consider

$$\tan(\alpha+\beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \frac{\sin\alpha\cos\beta+\sin\beta\cos a}{\cos\alpha\cos\beta-\sin\alpha\sin\beta}.$$

Dividing the numerator and denominator of this last fraction by $\sin\alpha\cos\beta$ yields

$$\begin{aligned}\frac{\sin\alpha\cos\beta+\sin\beta\cos a}{\cos\alpha\cos\beta-\sin\alpha\sin\beta} &= \frac{\frac{\sin\alpha\cos\beta}{\sin\alpha\cos\beta}+\frac{\sin\beta\cos a}{\sin\alpha\cos\beta}}{\frac{\cos\alpha\cos\beta}{\sin\alpha\cos\beta}-\frac{\sin\alpha\sin\beta}{\sin\alpha\cos\beta}} = \frac{1+\frac{\tan\beta}{\tan\alpha}}{\cot\alpha-\tan\beta}\\ &= \frac{\frac{\tan\alpha+\tan\beta}{\tan\alpha}}{\frac{1}{\tan\alpha}-\tan\beta} = \frac{\frac{\tan\alpha+\tan\beta}{\tan\alpha}}{\frac{1-\tan^2\beta}{\tan\alpha}} = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}.\end{aligned}$$

Hence, the sum formula for the tangent function is

$$\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}. \tag{9}$$

Example 13 *Determine the value of* $\tan\left(\frac{11\pi}{12}\right)$.
Solution: First observe that $\frac{11\pi}{12} = \frac{2\pi}{3} + \frac{\pi}{4}$ *so that*

$$\tan\left(\frac{11\pi}{12}\right) = \tan\left(\frac{2\pi}{3} + \frac{\pi}{4}\right) = \frac{\tan\frac{2\pi}{3} + \tan\frac{\pi}{4}}{1 - \tan\frac{2\pi}{3}\tan\frac{\pi}{4}} = \frac{1-\sqrt{3}}{1+\sqrt{3}} = \frac{1-\sqrt{3}}{1+\sqrt{3}}\frac{1-\sqrt{3}}{1-\sqrt{3}} = -2+\sqrt{3}.$$

The difference formula for the tangent function is easily derived using the formula for the tangent of the sum of two angles (Equation 9) and the symmetry identity $\tan(-\beta) = \tan\beta$ (Table 6.2). Consider

$$\begin{aligned} \tan(\alpha - \beta) &= \tan[\alpha + (-\beta)] \\ &= \frac{\tan\alpha + \tan(-\beta)}{1 - \tan\alpha\tan(-\beta)} \\ &= \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}. \end{aligned} \tag{10}$$

Example 14 *Determine the value of* $\tan\left(\frac{5\pi}{12}\right)$.
Solution: Write $\frac{5\pi}{12} = \frac{2\pi}{3} - \frac{\pi}{4}$ *so that*

$$\tan\left(\frac{5\pi}{12}\right) = \tan\left(\frac{2\pi}{3} - \frac{\pi}{4}\right) = \frac{\tan\frac{2\pi}{3} - \tan\frac{\pi}{4}}{1 + \tan\frac{2\pi}{3}\tan\frac{\pi}{4}} = \frac{-1-\sqrt{3}}{1-\sqrt{3}} = \frac{-1-\sqrt{3}}{1-\sqrt{3}}\frac{1+\sqrt{3}}{1+\sqrt{3}} = 2+\sqrt{3}.$$

The following table lists the sum and difference formulas presented in this section.

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$
$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha$
$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$
$\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$
$\tan(\alpha-\beta) = \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$

Table 6.5: Sum and Difference Formulas.

4. The double and half angle formulas

Some rather simple applications of the sum formulas result in additional useful identities. The **double angle formulas** fall into this category. By Equation 7 $\sin(2\alpha) = \sin(\alpha + \alpha) = \sin\alpha\cos\alpha + \sin\alpha\cos\alpha = 2\sin\alpha\cos\alpha$, resulting in the double angle formula for the sine function:

$$\sin(2\alpha) = 2\sin\alpha\cos\alpha \tag{11}$$

There are three double angle identities for the cosine function. The first of these is obtained by using the sum formula for the cosine function: $\cos(2\alpha) = \cos\alpha\cos\alpha - \sin\alpha\sin\alpha = \cos^2\alpha - \sin^2\alpha$. Replacing $\cos^2\alpha$ by $1 - \sin^2\alpha$ in this last formula yields the double angle formula $\cos(2\alpha) = 1 - 2\sin^2\alpha$. The proof of the last double angle formula for the cosine function, $\cos(2\alpha) = 2\cos^2\alpha - 1$, is left to the reader. (See Exercise 9.) The three double angle formulas for the cosine function are

$$\begin{aligned} \cos(2\alpha) &= \cos^2\alpha - \sin^2\alpha && (12)\\ &= 1 - 2\sin^2\alpha && (13)\\ &= 2\cos^2\alpha - 1. && (14) \end{aligned}$$

Example 15 *Show that* $\cos(3\alpha) = 4\cos^3\alpha - 3\cos\alpha$.
Solution: The proof of this identity involves the formula the sum of two angles for the cosine functions, the double angle formulas for both the sine and cosine functions,

and Equation 6 Consider

$$\begin{aligned}
\cos(3\alpha) &= \cos(2\alpha + \alpha) \\
&= \cos(2\alpha)\cos\alpha - \sin(2\alpha)\sin\alpha \\
&= (2\cos^2\alpha - 1)\cos\alpha - (2\sin\alpha\cos\alpha)\sin\alpha \\
&= 2\cos^3\alpha - \cos\alpha - 2\sin^2\alpha\cos\alpha \\
&= 2\cos^3\alpha - \cos\alpha - 2(1-\cos^2\alpha)\cos\alpha \\
&= 2\cos^3\alpha - 3\cos\alpha + 2\cos^3\alpha \\
&= 4\cos^3\alpha - 3\cos\alpha.
\end{aligned}$$

The double angle formula for the tangent function is an immediate consequence of the sum formula for that function. Consider

$$\begin{aligned} \tan(2\alpha) &= \tan(\alpha + \alpha) \\ &= \frac{\tan\alpha + \tan\alpha}{1 - \tan\alpha\tan\alpha} \\ &= \frac{2\tan\alpha}{1 - \tan^2\alpha}. \end{aligned} \tag{15}$$

The following table summarizes the double angle formulas

$\sin(2\alpha) = 2\sin\alpha\cos\alpha$
$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha = 1 - 2\sin^2\alpha = 2\cos^2\alpha - 1$
$\tan(2\alpha) = \frac{2\tan\alpha}{1-\tan^2\alpha}$.

Table 6.6: The Double Angle Formulas

The **half angle formulas** for the sine and cosine functions can be derived from two of the double angle formulas for the cosine function. Consider the double angle formula $\cos(2t) = 2\cos^2 t - 1$. Solving this for $\cos^2 t$ gives

$$\cos^2 t = \frac{1 + \cos(2t)}{2}.$$

Taking square roots of both sides of this last equation and replacing t with $\frac{\alpha}{2}$ results in the half-angle formula for the cosine function:

$$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1 + \cos\alpha}{2}}. \tag{16}$$

The $\pm$ in front of the radical is determined by the quadrant in which the angle $\frac{\alpha}{2}$ resides. It is left to the reader (See Exercise 10) to verify that the half angle formula for the sine function is

$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1 - \cos\alpha}{2}}. \tag{17}$$

Example 16 *Without the use of a calculator, determine the value of* $\cos(\pi/24)$*.*
Solution: Recall that $\cos(\pi/12) = \frac{\sqrt{2}}{4}(1+\sqrt{3})$ *was calculated in Example 7. Using this quantity in Equation 16 yields*

$$\begin{aligned}
\cos(\pi/24) &= \cos\left(\frac{\pi/12}{2}\right) \\
&= \sqrt{\frac{1+\cos\left[2\left(\frac{\pi}{24}\right)\right]}{2}} \\
&= \sqrt{\frac{1+\cos\frac{\pi}{12}}{2}} \\
&= \sqrt{\frac{1+\frac{\sqrt{2}}{4}(1+\sqrt{3})}{2}}.
\end{aligned}$$

Similar calculations would provide the values of $\cos(\pi/48)$, $\cos(\pi/64)$, and so on. This technique also works with the other trigonometric functions as demonstrated in Exercise 11.

The half-angle formulas for the sine and cosine functions provide a means for establishing a similar identity for the tangent function. To see this construct the quotient of Equation 17 and Equation 16 to obtain

$$\begin{aligned} \tan\left(\frac{\alpha}{2}\right) = \frac{\sin\left(\frac{\alpha}{2}\right)}{\cos\left(\frac{\alpha}{2}\right)} &= \frac{\pm\sqrt{\frac{1-\cos\alpha}{2}}}{\pm\sqrt{\frac{1+\cos\alpha}{2}}} \\ &= \pm\sqrt{\frac{\frac{1-\cos\alpha}{2}}{\frac{1+\cos\alpha}{2}}} \\ &= \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}. \end{aligned} \tag{18}$$

Once again, the sign of the last expression above is determined by the location of the angle $\frac{\alpha}{2}$. There are two additional half-angle formulas for the tangent function. The derivation of these is left to the reader. (See Exercise 12).

Example 17 *By Equation 18*

$$\tan 15^\circ = \tan\left(\frac{30^\circ}{2}\right) = \sqrt{\frac{1-\cos 30^\circ}{1+\cos 30^\circ}} = \sqrt{\frac{1-\sqrt{3}/2}{1+\sqrt{3}/2}} = \sqrt{\frac{2-\sqrt{3}}{2+\sqrt{3}}}.$$

Example 18 *A half-angle formula for the cotangent function follows from Equation 18 since*

$$\cot\frac{\alpha}{2} = \frac{1}{\tan\frac{\alpha}{2}} = \frac{1}{\pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}} = \pm\sqrt{\frac{1+\cos\alpha}{1-\cos\alpha}}.$$

Hence,

$$\cot 15^\circ = \sqrt{\frac{2+\sqrt{3}}{2-\sqrt{3}}}.$$

Table 6.7 lists the half angle formulas covered in this lesson.

$\sin \frac{\alpha}{2} = \pm\sqrt{\dfrac{1-\cos\alpha}{2}}$
$\cos \frac{\alpha}{2} = \pm\sqrt{\dfrac{1+\cos\alpha}{2}}$
$\tan\left(\frac{\alpha}{2}\right) = \pm\sqrt{\dfrac{1-\cos\alpha}{1+\cos\alpha}} = \dfrac{\sin\alpha}{1+\cos\alpha} = \dfrac{1-\cos\alpha}{\sin\alpha}.$

Table 6.7: The Half Angle Formulas

5. Product Identities and Factor formulas

There are at least three useful trigonometric identities that arise from the sum formulas. For example, adding Equation 4 and Equation 6 yields $\cos(\alpha+\beta)+\cos(\alpha-\beta) = 2\cos\alpha\cos\beta$. Dividing by 2 results in the product formula for the cosine function:

$$\cos\alpha\cos\beta = \frac{1}{2}\left(\cos(\alpha+\beta)+\cos(\alpha-\beta)\right). \tag{19}$$

Two additional product formulas are

$$\sin\alpha\sin\beta = \frac{1}{2}\left(\cos(\alpha-\beta)-\cos(\alpha+\beta)\right) \tag{20}$$

and

$$\cos\alpha\sin\beta = \frac{1}{2}\left(\sin(\alpha+\beta)-\sin(\alpha-\beta)\right). \tag{21}$$

The reader should derive the last two product formulas. (See Exercise 14.) Table 6.8 contains a list of the Product Identities.

$\cos\alpha\cos\beta = \frac{1}{2}\left(\cos(\alpha+\beta)+\cos(\alpha-\beta)\right)$
$\sin\alpha\sin\beta = \frac{1}{2}\left(\cos(\alpha-\beta)-\cos(\alpha+\beta)\right)$
$\cos\alpha\sin\beta = \frac{1}{2}\left(\sin(\alpha+\beta)-\sin(\alpha-\beta)\right)$

Table 6.8: The Product Identities

The last collection of identities are called the **factor formulas** (sometimes called the sum formulas). These are listed in the table below. The development of a strategy for verifying these formulas is left to the reader (See Exercise 1515).

$\sin s + \sin t = 2\sin\left(\frac{s+t}{2}\right)\cos\left(\frac{s-t}{2}\right)$	$\cos s + \cos t = 2\cos\left(\frac{s+t}{2}\right)\cos\left(\frac{s-t}{2}\right)$
$\sin s - \sin t = 2\cos\left(\frac{s+t}{2}\right)\sin\left(\frac{s-t}{2}\right)$	$\cos s - \cos t = 2\sin\left(\frac{s+t}{2}\right)\sin\left(\frac{s-t}{2}\right)$

Table 6.9: The Factor Formulas

6. Exercises

EXERCISE 1. Using the strategy presented in Example 3 in show that the cotangent and cosecant functions are odd. Also show that the secant function is even.

EXERCISE 2. Verify that $1 + \cot^2 t = \csc^2 t$.

EXERCISE 3. Show that $\cot(\frac{\pi}{2} - \alpha) = \tan \alpha$, $\sec(\frac{\pi}{2} - \alpha) = \csc \alpha$, and $\csc(\frac{\pi}{2} - \alpha) = \sec \alpha$.

EXERCISE 4. Verify the sine formula for the difference of two angles. That is, establish the identity

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \sin \beta \cos \alpha.$$

EXERCISE 5. Without a calculator determine the value of $\sin(\pi/12)$.

EXERCISE 6. Trigonometric identities are independent of the dimension used to measure angles. With this in mind determine $\tan 105\,^\circ$

EXERCISE 7. Prove that $\csc(\frac{\pi}{2} - t) = \sec t$.

EXERCISE 8. Prove that $\tan\left(t - \frac{\pi}{4}\right) = \dfrac{\tan t - 1}{\tan t + 1}$.

EXERCISE 9. Verify the double-angle formula $\cos(2\alpha) = 2\cos^2 \alpha - 1$.

EXERCISE 10. Verify the half-angle formula for the sine function:

$$\sin \frac{\alpha}{2} = \pm\sqrt{\frac{1 - \cos \alpha}{2}}.$$

(See Equation 17.)

EXERCISE 11. Use the formula derived in Exercise 10 and the value computed in Example 7 to evaluate $\sin(\pi/24)$.

EXERCISE 12. Verify the half-angle identities

$$\tan\left(\frac{\alpha}{2}\right) = \frac{\sin\alpha}{1+\cos\alpha} = \frac{1-\cos\alpha}{\sin\alpha}.$$

EXERCISE 13. Use Example 15 to establish the identity

$$\cos(4\alpha) = 8\cos^4\alpha - 4\cos^2\alpha + 2.$$

EXERCISE 14. Derive Equation 20 and Equation 21

EXERCISE 15. Verify the identity

$$\sin s + \sin t = 2\sin\left(\frac{s+t}{2}\right)\cos\left(\frac{s-t}{2}\right).$$

EXERCISE 16. Verify the identity

$$\frac{\sin 4t - \sin 2t}{\cos 4t + \cos 2t} = \tan t.$$

Exercise 17. Verify the identity

$$\frac{\sin t - \sin 3t}{\sin^2 t - \cos^2 t} = 2\sin t.$$