An introduction to using SDEs

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Outline

- Ito-Taylor expansion
- Basic numerical schemes & convergence
- Parameter estimation
- Filtering
- Local linearization

- Next meeting – Valdes Sosa (1999) paper

Resources

- Kloeden & Platen
- "An algorithmic introduction to numerical simulation of SDEs" by Higham
- "MAPLE and MATLAB for SDEs in finance" by Higham & Kloeden
- Jimenez. 1999 J. of Stat. Phys. Vol 94
- Ozaki. 1992 Statistica Sinica 2 pp113-135
- Ghahramani & Hinton. 1996 Technical report CRG-TR-96-2

Weiner process

- Properties

$$E(W_t) = 0, \qquad E(W_t^2) \sim t$$

Normally distributed

$$E((W_{t_4} - W_{t_3})(W_{t_2} - W_{t_1})) = 0$$

Independent increments

$$0 \le t_1 < t_2 < t_3 < t_4 \le T$$

- Update equation

$$W_{t_{n+1}} = W_{t_n} + dW_{t_{n+1}}, \quad dW_{t_{n+1}} \sim \sqrt{\delta t}\, N(0,1), \quad W_{t_0} = 0$$

```
T = 1; N = 500; dt = T/N;
dW = sqrt(dt)*randn(1,N);   % increments
W = cumsum(dW);             % cumulative sum
plot([0:dt:T],[0,W],'r-')   % plot W against t
```

Weiner process

- Properties

$$E(W_t) = 0, \qquad E(W_t^2) \sim t$$

Normally distributed

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plot([0:dt:T],[0,W],'r-')   % plot W against t
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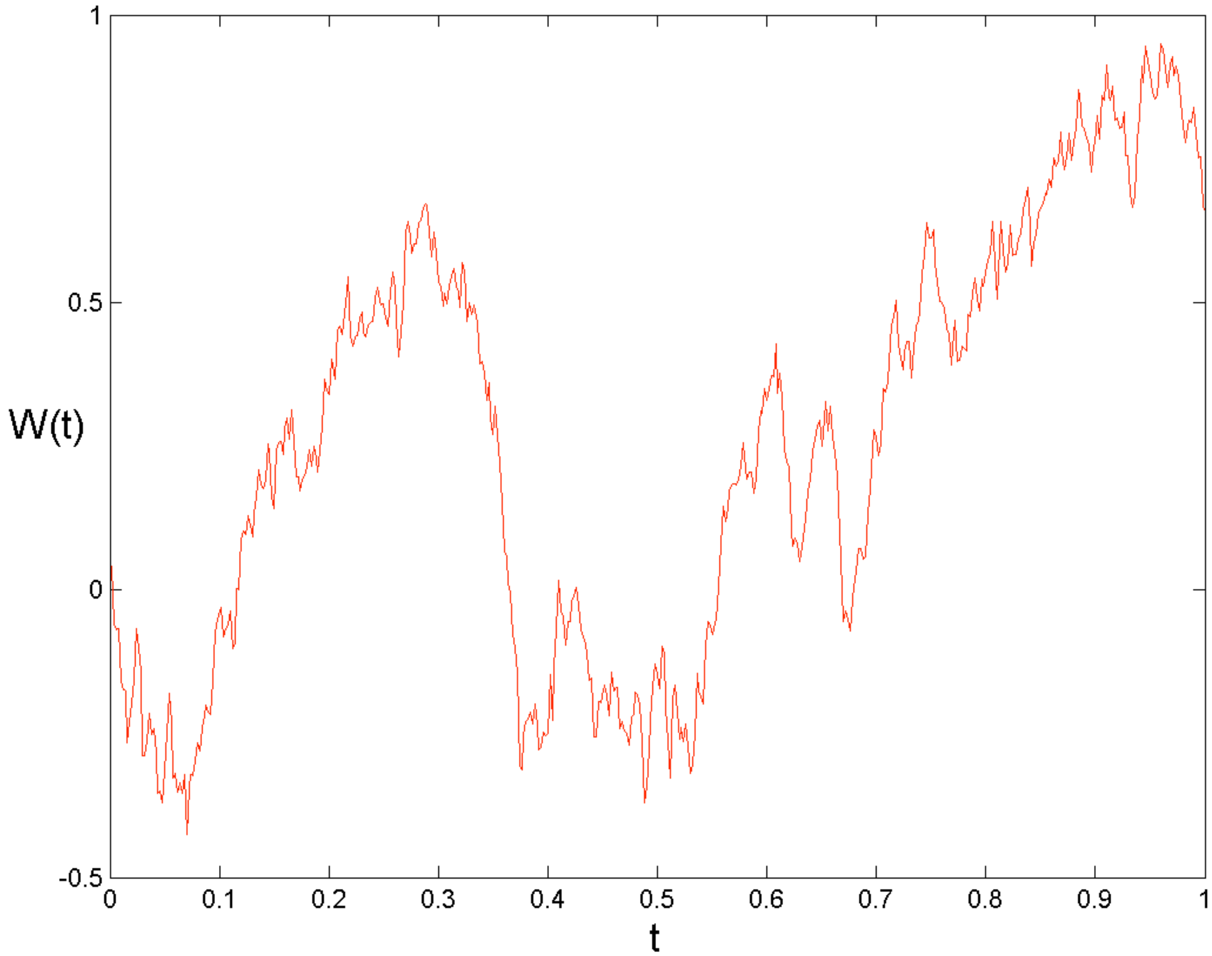
Sample path of a Weiner process



Scalar SDEs

- Integral form

Stochastic integral

$$X_t = X_{t_0} + \int_{t_0}^{t} a(X_s)ds + \int_{t_0}^{t} b(X_s)dW_s$$

- Differential form (symbolic)

$$dX_t = a(X_t)dt + b(X_t)dW_t$$

Stochastic integrals

- Ito

$$\int_0^T h(t)dW_s \approx \sum_{n=0}^{N-1} h(t_n)(W_{t_{n+1}} - W_{t_n})$$

- Stratonovich

$$\int_0^T h(t)dW_s \approx \sum_{n=0}^{N-1} h(\tfrac{t_n + t_{n+1}}{2})(W_{t_{n+1}} - W_{t_n})$$

Example of difference

- Ito

Additional term

$$\int_0^T W_s dW_s \approx \frac{1}{2}\left\{W_T^2 - T\right\}$$

- Stratonovich

$$\int_0^T W_s dW_s \approx \frac{1}{2} W_T^2$$

Same as classical calculus

Ito's formula

- Function of SDE solution

$$V_t = V(X_t), \qquad dX_t = a(X_t)dt + b(X_t)dW_t$$

- Ito's formula

$$dV_t = L^0 V(t, X_t)dt + L^1 V(t, X_t)dW_t$$

- Partial differential operators

$$L^0 = \frac{\partial}{\partial t} + a\frac{\partial}{\partial x} + \tfrac{1}{2}b^2\frac{\partial^2}{\partial x^2}$$

Additional term
c.f. classical calculus

$$L^1 = b\frac{\partial}{\partial x}$$

Ito's formula

- Ito stochastic integral equation

$$V_t = V_{t_0} + \int_{t_0}^{t} L^0 V(s, X_s)\,ds + \int_{t_0}^{t} L^1 V(s, X_s)\,dW_s$$

Ito's formula

- Ito stochastic integral equation

$$V_t = V_{t_0} + \int_{t_0}^{t} L^0 V(s, X_s)\,ds + \int_{t_0}^{t} L^1 V(s, X_s)\,dW_s$$

Stochastic integral

Ito's formula

- Ito stochastic integral equation

$$V_t = V_{t_0} + \int_{t_0}^{t} L^0 V(s, X_s)\,ds + \int_{t_0}^{t} L^1 V(s, X_s)\,dW_s$$

Coefficient functions

Stochastic integral

Ito's formula - example

- SDE

$$dX_t = (\alpha - X_t)dt + \beta\sqrt{X_t}dW_t$$

- Change of variable

$$V(X) = \sqrt{X}$$

Ito's formula - example

- SDE

$$dX_t = (\alpha - X_t)dt + \beta\sqrt{X_t}\,dW_t$$

- Change of variable

$$V(X) = \sqrt{X}$$

- Ito's formula

$$dV_t = L^0 V(X_t)dt + L^1 V(X_t)dW_t$$

$$\frac{\partial V}{\partial t} = 0, \quad \frac{\partial V}{\partial x} = \tfrac{1}{2} X^{-\frac{1}{2}}, \quad \frac{\partial^2 V}{\partial x^2} = -\tfrac{1}{4} X^{-\frac{3}{2}}$$

$$dV_t = \left(\frac{4\alpha - \beta^2}{8V} - \frac{1}{2}V\right)dt + \tfrac{1}{2}\beta dW_t$$

Higham's code - chain.m

```
alpha = 2; beta = 1; T = 1; N = 200; dt = T/N;  % Problem parameters
Xzero = 1; Xzero2 = sqrt(Xzero);                %

Dt = dt;                                        % EM steps of size Dt = dt
Xem1 = zeros(1,N); Xem2 = zeros(1,N);           % preallocate for
efficiency
Xtemp1 = Xzero;  Xtemp2 = Xzero2;
for j = 1:N
   Winc = sqrt(dt)*randn;
   f1 = (alpha-Xtemp1);
   g1 = beta*sqrt(abs(Xtemp1));
   Xtemp1 = Xtemp1 + Dt*f1 + Winc*g1;
   Xem1(j) = Xtemp1;
   f2 = (4*alpha-beta^2)/(8*Xtemp2) - Xtemp2/2;
   g2 = beta/2;
   Xtemp2 = Xtemp2 + Dt*f2 + Winc*g2;
   Xem2(j) = Xtemp2;
end

plot([0:Dt:T],[sqrt([Xzero,abs(Xem1)])],'b-',[0:Dt:T],[Xzero,Xem2],'ro')
```

$$dX_t = (\alpha - X_t)dt + \beta\sqrt{X_t}\,dW_t$$

Higham's code - chain.m

```
alpha = 2; beta = 1; T = 1; N = 200; dt = T/N;  % Problem parameters
Xzero = 1; Xzero2 = sqrt(Xzero);                %

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for j = 1:N
   Winc = sqrt(dt)*randn;
   f1 = (alpha-Xtemp1);
   g1 = beta*sqrt(abs(Xtemp1));
   Xtemp1 = Xtemp1 + Dt*f1 + Winc*g1;
   Xem1(j) = Xtemp1;
   f2 = (4*alpha-beta^2)/(8*Xtemp2) - Xtemp2/2;
   g2 = beta/2;
   Xtemp2 = Xtemp2 + Dt*f2 + Winc*g2;
   Xem2(j) = Xtemp2;
end

plot([0:Dt:T],[sqrt([Xzero,abs(Xem1)])],'b-',[0:Dt:T],[Xzero,Xem2],'ro')
```

$$dV_t = \left(\frac{4\alpha-\beta^2}{8V} - \frac{1}{2}V\right)dt + \frac{1}{2}\beta dW_t$$

Higham's code - chain.m

```
alpha = 2; beta = 1; T = 1; N = 200; dt = T/N;  % Problem parameters
Xzero = 1; Xzero2 = sqrt(Xzero);                %

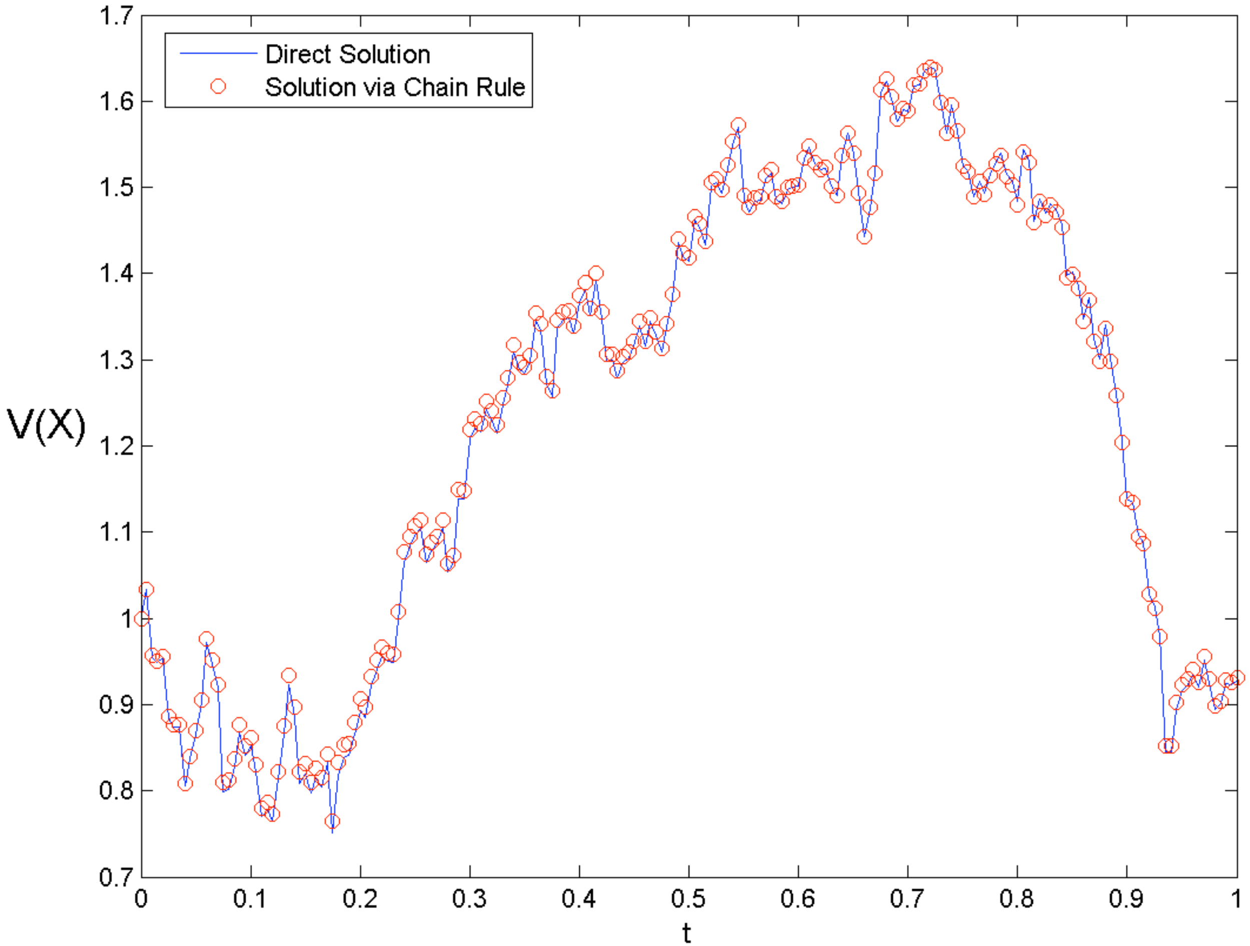
Dt = dt;                                        % EM steps of size Dt = dt
Xem1 = zeros(1,N); Xem2 = zeros(1,N);           % preallocate for
efficiency
Xtemp1 = Xzero;  Xtemp2 = Xzero2;
for j = 1:N
   Winc = sqrt(dt)*randn;
   f1 = (alpha-Xtemp1);
   g1 = beta*sqrt(abs(Xtemp1));
   Xtemp1 = Xtemp1 + Dt*f1 + Winc*g1;
   Xem1(j) = Xtemp1;
   f2 = (4*alpha-beta^2)/(8*Xtemp2) - Xtemp2/2;
   g2 = beta/2;
   Xtemp2 = Xtemp2 + Dt*f2 + Winc*g2;
   Xem2(j) = Xtemp2;
end

plot([0:Dt:T],[sqrt([Xzero,abs(Xem1)])],'b-',[0:Dt:T],[Xzero,Xem2],'ro')
```

$V(X_t) = \sqrt{X_t}$

V_t

Compare V(t) using dX & dV



Ito-Taylor expansion

- Iterated application of Ito's formula
- Generalization of Taylor expansion
- Convenient notation
  - Coefficient functions & stochastic integrals (Kloeden & Platen ch. 5)

Ito-Taylor expansion

- SDE

$$X_t = X_{t_0} + \int_{t_0}^{t} a(X_s)ds + \int_{t_0}^{t} b(X_s)dW_s$$

- Ito-Taylor expansion

$$X_k(t) = \sum_{\alpha \in \Lambda_k} I_\alpha [f_\alpha(t_0, X_{t_0})]_{t_0,t}$$

Series (hierarchy) of stochastic integrals

Ito-Taylor expansion

- SDE

$$X_t = X_{t_0} + \int_{t_0}^{t} a(X_s)ds + \int_{t_0}^{t} b(X_s)dW_s$$

- Ito-Taylor expansion

Coefficient functions

$$X_k(t) = \sum_{\alpha \in \Lambda_k} I_\alpha [f_\alpha (t_0, X_{t_0})]_{t_0,t}$$

Hierarchical set

Stochastic integrals

Truncated expansion

- SDE

$$X_t = X_{t_0} + \int_{t_0}^{t} a(X_s)ds + \int_{t_0}^{t} b(X_s)dW_s$$

- *e.g.*

$$X_t = X_{t_0} + f_{(0)}I_{(0)} + f_{(1)}I_{(1)} + \ldots$$

$$f_{(0,0)}I_{(0,0)} + f_{(0,1)}I_{(0,1)} + f_{(1,0)}I_{(1,0)} + f_{(1,1)}I_{(1,1)}$$

Coefficient functions

- Differential operators

$$L^0 = \frac{\partial}{\partial t} + a\frac{\partial}{\partial x} + \tfrac{1}{2}b^2\frac{\partial^2}{\partial x^2}, \qquad L^1 = b\frac{\partial}{\partial x}$$

- *e.g.*

$$f(t,x) = id_x = x$$

$$f_{(0)} = L^0 id_x = a, \quad f_{(1)} = L^1 id_x = b$$

$$f_{(0,0)} = L^0 L^0 id_x = aa' + \tfrac{1}{2}b^2 a'', \quad f_{(0,1)} = L^0 L^1 id_x = ab' + \tfrac{1}{2}b^2 b''$$

$$f_{(1,0)} = L^1 L^0 id_x = ba', \quad f_{(1,1)} = L^1 L^1 id_x = bb'$$

Coefficient functions

- Differential operators

$$L^0 = \frac{\partial}{\partial t} + a\frac{\partial}{\partial x} + \tfrac{1}{2}b^2\frac{\partial^2}{\partial x^2}, \qquad L^1 = b\frac{\partial}{\partial x}$$

- *e.g.*

$$f(t,x) = id_x = x$$

Euler-Maruyama

$$f_{(0)} = L^0 id_x = a, \quad f_{(1)} = L^1 id_x = b$$

$$f_{(0,0)} = L^0 L^0 id_x = aa' + \tfrac{1}{2}b^2 a'', \quad f_{(0,1)} = L^0 L^1 id_x = ab' + \tfrac{1}{2}b^2 b''$$

$$f_{(1,0)} = L^1 L^0 id_x = ba', \quad f_{(1,1)} = L^1 L^1 id_x = bb'$$

Milstein

Stochastic integrals

- Examples

$$I_{(0)} = \int_{t_n}^{t_{n+1}} ds = t_{n+1} - t_n = \Delta_n$$

$$I_{(1)} = \int_{t_n}^{t_{n+1}} dW_s = W_{t_{n+1}} - W_{t_n} = \Delta W_n$$

$$I_{(1,1)} = \int_{t_n}^{t_{n+1}} \left(\int_{t_n}^{t} dW_s \right) dW_t = \int_{t_n}^{t_{n+1}} \Delta W_n dW_t = \frac{1}{2} \left\{ (\Delta W_n)^2 - \Delta_n \right\}$$

Stochastic integrals

- Examples

$$I_{(0)} = \int_{t_n}^{t_{n+1}} ds = t_{n+1} - t_n = \Delta_n$$

$$I_{(1)} = \int_{t_n}^{t_{n+1}} dW_s = W_{t_{n+1}} - W_{t_n} = \Delta W_n$$

Euler-Maruyama

$$I_{(1,1)} = \int_{t_n}^{t_{n+1}} \left(\int_{t_n}^{t} dW_s \right) dW_t = \int_{t_n}^{t_{n+1}} \Delta W_n dW_t = \frac{1}{2} \left\{ (\Delta W_n)^2 - \Delta_n \right\}$$

Milstein

Basic numerical schemes & convergence

- Schemes based on truncated Ito-Taylor expansion
  - Euler-Maruyama
  - Milstein

SDE schemes – truncated Ito-Taylor expansion

Euler-Maruyama

Order 0.5, 1 strong, weak convergence resp.

$$X_t = X_{t_0} + f_{(0)} I_{(0)} + f_{(1)} I_{(1)} + f_{(1,1)} I_{(1,1)}$$

Order 1, 1 strong, weak convergence resp.

Milstein

Euler-Maruyama

- SDE

$$X_t = X_{t_0} + \int_{t_0}^{t} a(X_s)ds + \int_{t_0}^{t} b(X_s)dW_s, \quad 0 \le t \le T$$

- Truncated Ito-Taylor expansion

$$X_t = X_{t_0} + f_{(0)}I_{(0)} + f_{(1)}I_{(1)}, \quad \alpha = \{\upsilon, (0), (1)\}$$

- Update equation

$$X_{t_{n+1}} = X_{t_n} + a(X_{t_n})\Delta t + b(X_{t_n})(W(t_{n+1}) - W(t_n))$$

Euler-Maruyama

- SDE

$$X_t = X_{t_0} + \int_{t_0}^{t} a(X_s)ds + \int_{t_0}^{t} b(X_s)dW_s, \quad 0 \le t \le T$$

- Truncated Ito-Taylor expansion

Hierarchical set

$$X_t = X_{t_0} + f_{(0)}I_{(0)} + f_{(1)}I_{(1)}, \quad \alpha = \{v, (0), (1)\}$$

- Update equation

$$X_{t_{n+1}} = X_{t_n} + a(X_{t_n})\Delta t + b(X_{t_n})(W(t_{n+1}) - W(t_n))$$

Higham's code – em.m

$$dX_t = \lambda X_t dt + \mu X_t dW_t$$

$$X_{t_{n+1}} = X_{t_n} + \lambda X_{t_n} \Delta_n + \mu X_{t_n} \Delta W_n$$

```
lambda = 2; mu = 1; Xzero = 1;    % problem parameters
T = 1; N = 2^8; dt = T/N;
dW = sqrt(dt)*randn(1,N);         % Brownian increments
W = cumsum(dW);                   % discretized Brownian path

Xtrue = Xzero*exp((lambda-0.5*mu^2)*([dt:dt:T])+mu*W);

R = 4; Dt = R*dt; L = N/R;        % L EM steps of size Dt = R*dt
Xem = zeros(1,L);                 % preallocate for efficiency
Xtemp = Xzero;
for j = 1:L
  Winc = sum(dW(R*(j-1)+1:R*j));
  Xtemp = Xtemp + Dt*lambda*Xtemp + mu*Xtemp*Winc;
  Xem(j) = Xtemp;
end
```

Approximate $\int_{t_n}^{t_{n+1}} dW_s$

Higham's code – em.m

$$dX_t = \lambda X_t dt + \mu X_t dW_t$$

$$X_{t_{n+1}} = X_{t_n} + \lambda X_{t_n} \Delta_n + \mu X_{t_n} \Delta W_n$$

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T = 1; N = 2^8; dt = T/N;
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R = 4; Dt = R*dt; L = N/R;        % L EM steps of size Dt = R*dt
Xem = zeros(1,L);                 % preallocate for efficiency
Xtemp = Xzero;
for j = 1:L
   Winc = sum(dW(R*(j-1)+1:R*j));
   Xtemp = Xtemp + Dt*lambda*Xtemp + mu*Xtemp*Winc;
   Xem(j) = Xtemp;
end
```

Exact solution

$$X_t = X_{t_0} \exp((\lambda - \tfrac{1}{2}\mu^2)t + \mu W_t)$$

Exact solution

- SDE $$dX_t = \lambda X_t dt + \mu X_t dW_t$$
- Transformation $$V = \ln X$$
- Ito's formula $$dV_t = L^0 V(X_t)dt + L^1 V(X_t)dW_t$$
- Solution $$\frac{\partial V}{\partial t} = 0, \quad \frac{\partial V}{\partial x} = X^{-1}, \quad \frac{\partial^2 V}{\partial x^2} = -X^{-2}$$

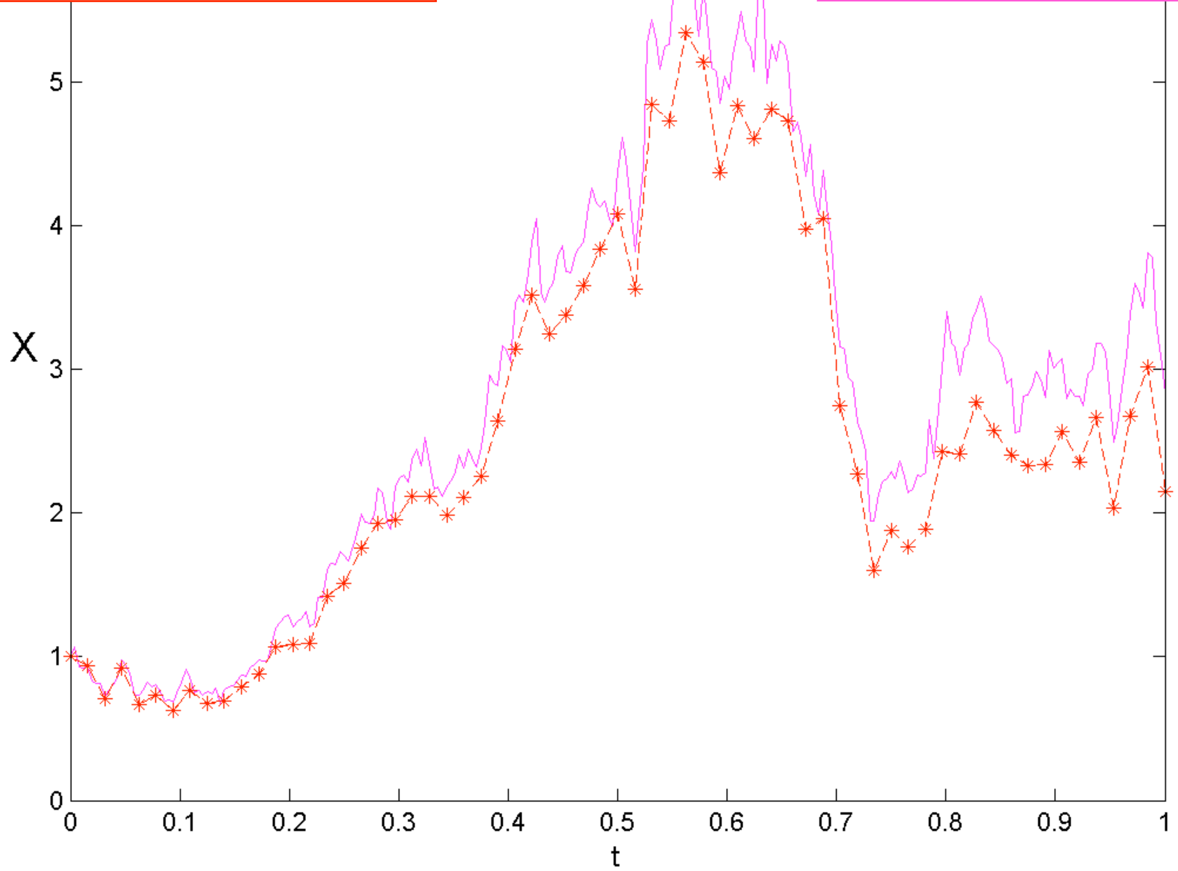
$$dV_t = (\lambda - \tfrac{1}{2}\mu^2)dt + \mu dW_t$$

$$X_t = X_0 \exp\left((\lambda - \tfrac{1}{2}\mu^2)t + \mu W_t\right)$$

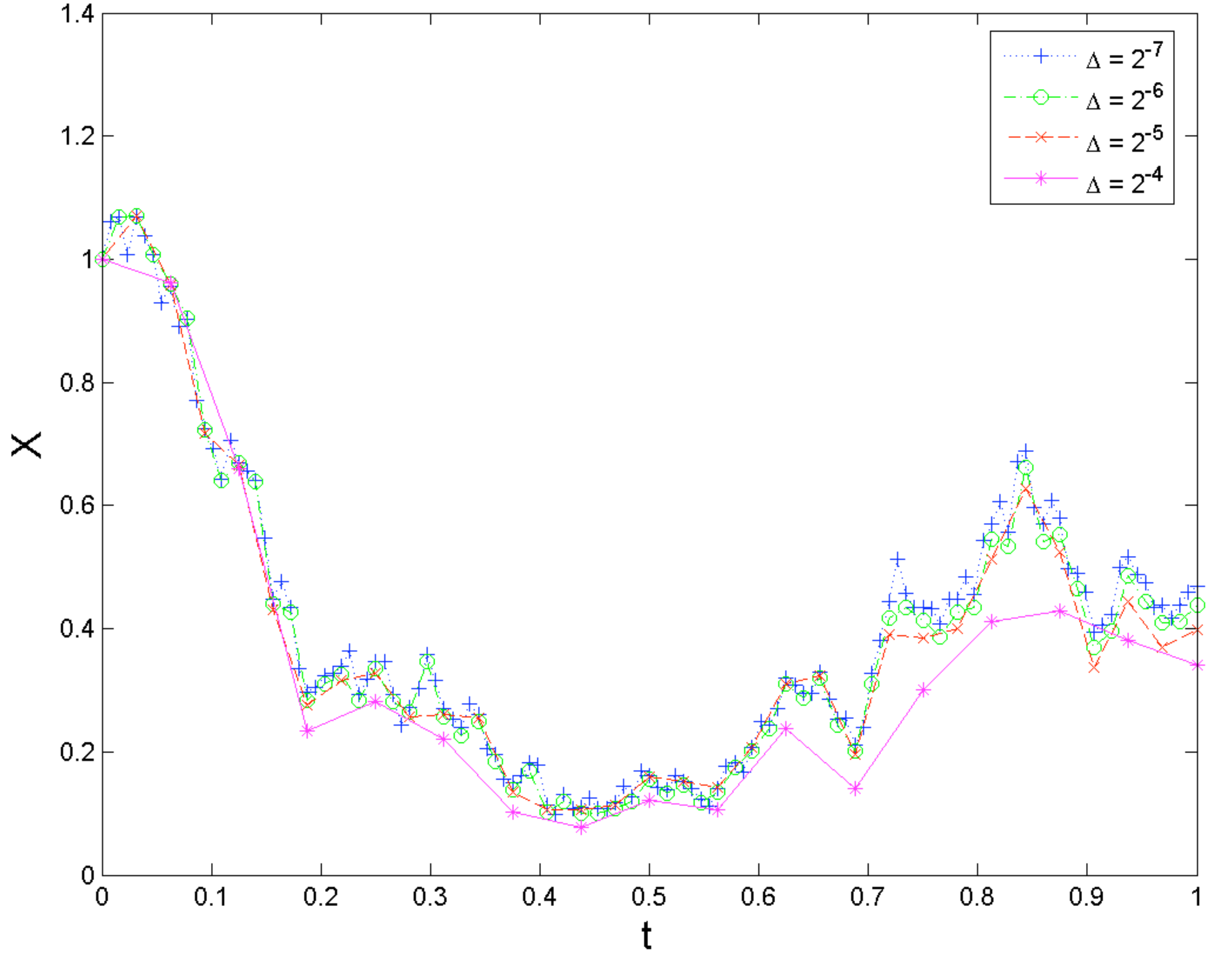
Compare exact with numerical

$$X_{t_{n+1}} = X_{t_n} + \lambda X_{t_n} \Delta_n + \mu X_{t_n} \Delta W_n$$

[redacted]



Convergence



Strong convergence of truncated Ito-Taylor expansion

Measures rate of decay of "mean of the error" as [redacted]

- Pathwise criteria

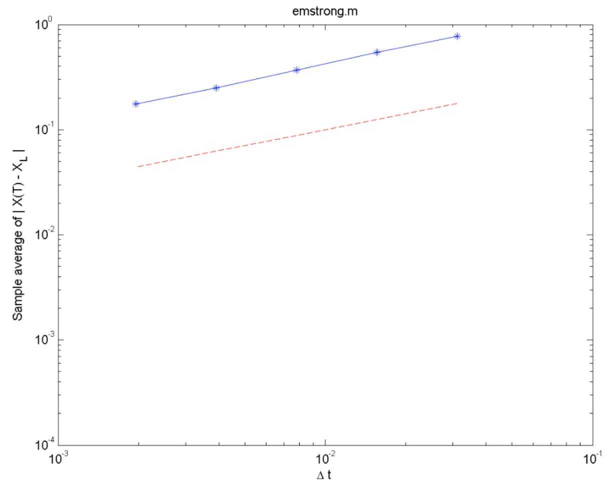
$$E\left(\left|X_n - X(\tau_n)\right|\right) \le C\Delta t^{\gamma}, \qquad \tau_n = n\Delta t \in [0,T]$$

Error @ end point

$$e_{\Delta t}^{strong} := E\left(\left|X_L - X(T)\right|\right) \quad L\Delta t = T$$

$$e_{\Delta t}^{strong} \le C\Delta t^{\gamma}$$

$$\log e_{\Delta t}^{strong} = \log C + q \log \Delta t$$



estimate $\quad q \approx \gamma$

Higham's code – emstrong.m

$$dX_t = \lambda X_t dt + \mu X_t dW_t$$

```
lambda = 2; mu = 1; Xzero = 1;    % problem parameters
T = 1; N = 2^9; dt = T/N;         %
M = 1000;                         % number of paths sampled

Xerr = zeros(M,5);                % preallocate array
for s = 1:M,                      % sample over discrete Brownian paths
    dW = sqrt(dt)*randn(1,N);     % Brownian increments
    W = cumsum(dW);               % discrete Brownian path
    Xtrue = Xzero*exp((lambda-0.5*mu^2)+mu*W(end));
    for p = 1:5
        R = 2^(p-1); Dt = R*dt; L = N/R;     % L Euler steps of size Dt = R*dt
        Xtemp = Xzero;
        for j = 1:L
            Winc = sum(dW(R*(j-1)+1:R*j));
            Xtemp = Xtemp + Dt*lambda*Xtemp + mu*Xtemp*Winc;
        end
        Xerr(s,p) = abs(Xtemp - Xtrue);      % store the error at t = 1
    end
end
```

Variable step size

Approximate

$$\int_{t_n}^{t_{n+1}} dW_s$$

Weak convergence of truncated Ito-Taylor expansion

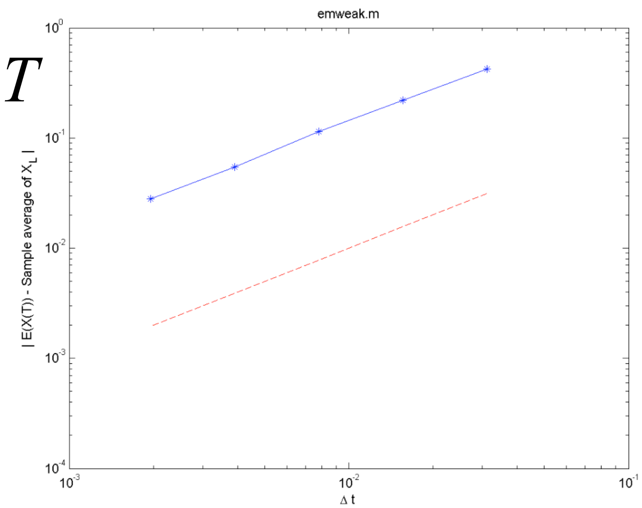
Measures rate of decay of "error of the means" as [redacted]

- statistical criteria

$$\left| E(p(X_n)) - E(p(X(\tau_n))) \right| \leq C \Delta t^{\gamma}, \qquad \tau_n = n\Delta t \in [0,T]$$

Error @ end point

$$e_{\Delta t}^{weak} := \left| E(X_L) - E(X(T)) \right|, \quad L\Delta t = T$$



$$e_{\Delta t}^{weak} \leq C^{\gamma} \Delta t$$

$$\log e_{\Delta t}^{weak} = \log C + q \log \Delta t$$

estimate $q \approx \gamma$

Higham's code – emweak.m

$$dX_t = \lambda X_t dt + \mu X_t dW_t$$

```
lambda = 2; mu = 0.1; Xzero = 1; T = 1;  % problem parameters
M = 50000;                               % number of paths sampled

Xem = zeros(5,1);                        % preallocate arrays
for p = 1:5                              % take various Euler timesteps
    Dt = 2^(p-10); L = T/Dt;             % L Euler steps of size Dt
    Xtemp = Xzero*ones(M,1);
    for j = 1:L
      Winc = sqrt(Dt)*randn(M,1);
      Xtemp = Xtemp + Dt*lambda*Xtemp + mu*Xtemp.*Winc;
    end
    Xem(p) = mean(Xtemp);
end
Xerr = abs(Xem - exp(lambda));
```

Note: can use two point random variable to generate random numbers for **weak** convergence (see Higham p.13)

Milstein

- Integral form

$$X_t = X_{t_0} + \int_{t_0}^{t} a(X_s)ds + \int_{t_0}^{t} b(X_s)dW_s, \quad 0 \le t \le T$$

- Truncated Ito-Taylor expansion

$$X_t = X_{t_0} + f_{(0)}I_{(0)} + f_{(1)}I_{(1)} + f_{(1,1)}I_{(1,1)}, \quad \alpha = \{\upsilon, (0), (1), (1,1)\}$$

- Update equation

$$X_{t_{n+1}} = X_{t_n} + a(X_{t_n})\Delta t + b(X_{t_n})(W(t_{n+1}) - W(t_n)) + \ldots$$

$$\tfrac{1}{2} b(X_{t_n}) \tfrac{\partial}{\partial x} b(X_{t_n})((W(t_{n+1}) - W(t_n))^2 - \Delta t)$$

Milstein

- Integral form

$$X_t = X_{t_0} + \int_{t_0}^{t} a(X_s)ds + \int_{t_0}^{t} b(X_s)dW_s, \quad 0 \le t \le T$$

- Truncated Ito-Taylor expansion

Hierarchical set

$$X_t = X_{t_0} + f_{(0)}I_{(0)} + f_{(1)}I_{(1)} + f_{(1,1)}I_{(1,1)}, \quad \alpha = \{\upsilon, (0), (1), (1,1)\}$$

- Update equation

$$X_{t_{n+1}} = X_{t_n} + a(X_{t_n})\Delta t + b(X_{t_n})(W(t_{n+1}) - W(t_n)) + \ldots$$

$$\tfrac{1}{2} b(X_{t_n}) \tfrac{\partial}{\partial x} b(X_{t_n})\left(\left(W(t_{n+1}) - W(t_n)\right)^2 - \Delta t\right)$$

double stochastic integral

Higham's code – milstrong.m

$$dX_t = rX_t(K - X_t)dt + \beta X_t dW_t$$

$$X_{t_{n+1}} = X_{t_n} + rX_{t_n}(K - X_{t_n})\Delta_n + \beta X_{t_n}\Delta W_n + \frac{1}{2}\beta^2 X_{t_n}\left\{(\Delta W_n)^2 - \Delta_n\right\}$$

```
r = 2; K = 1; beta = 0.25; Xzero = 0.5;   % problem parameters
T = 1; N = 2^(11); dt = T/N;              %
M = 500;                                  % number of paths sampled
R = [1; 16; 32; 64; 128];                 % Milstein stepsizes are R*dt

dW = sqrt(dt)*randn(M,N);                 % Brownian increments
Xmil = zeros(M,5);                        % preallocate array
for p = 1:5
    Dt = R(p)*dt; L = N/R(p);             % L timesteps of size Dt = R dt
    Xtemp = Xzero*ones(M,1);
    for j = 1:L
        Winc = sum(dW(:,R(p)*(j-1)+1:R(p)*j),2);
        Xtemp = Xtemp + Dt*r*Xtemp.*(K-Xtemp) + beta*Xtemp.*Winc ...
          + 0.5*beta^2*Xtemp.*(Winc.^2 - Dt);
    end
    Xmil(:,p) = Xtemp;  % store Milstein solution at t =1
end
```

Linear SDEs

- Moment equations
- Parameter estimation
- Filtering

Mean, variance & auto-covariance

- Scalar SDE

$$dX_t = -aX_t dt + b dW_t$$

- Exact solution (using integrating factor)

$$\frac{d}{dt}\left(\exp(at)X_t\right) = b\exp(at)dW_t$$

$$\exp(at)X_t - X_0 = b\int_0^t \exp(as)dW_s$$

$$X_t = \exp(-at)X_0 + b\int_0^t \exp(-a(t-s)dW_s$$

Mean, variance & auto-covariance

- Mean

$$E(X_t) = \exp(-at)X_0 = m_t$$

- Auto-covariance

$$E((X_t - m_t)(X_{t+s} - m_{t+s})) = b^2 \int_0^t \exp(-a(2t + s - 2t'))dt'$$

$$= \frac{b^2}{2a}\left(\exp(-as) - \exp(-a(2t + s))\right)$$

- Variance

$$E((X_t - m_t)^2) = \frac{b^2}{2a}\left(1 - \exp(-2at)\right)$$

- Stationary variance

$$E((X_t)^2) = \frac{b^2}{2a}$$

Alternatively …

- SDE

$$dX_t = -aX_t dt + b dW_t$$

- 1st and 2nd moments

$$\frac{dm}{dt} = -am \Rightarrow m_t = \exp(-at) m_0$$

$$\frac{dP}{dt} = -2aP + b^2 \Rightarrow P_t = \frac{b^2}{2a}\left(1 - \exp(-2at)\right)$$

- Using result from next slide

… 2nd moment

- Transformation

$$U(t,x) = X^2$$

$$dU_t = L^0 U dt + L^1 U dW$$

$$\frac{\partial U}{\partial t} = 0, \quad \frac{\partial U}{\partial x} = 2x, \quad \frac{\partial^2 U}{\partial x^2} = 2$$

$$L^0 U = -2aX^2 + b^2, \qquad L^1 U = 2bX$$

$$dU_t = (-2aX^2 + b^2)dt + 2bXdW$$

Expectation is zero

- Expectation

$$P = \langle U \rangle$$

$$\frac{dP}{dt} = -2aP + b^2$$

Solve using integrating factor

Parameter estimation

- With or without observation noise *i.e.* no filtering
- Use transition probability density to compute likelihood of data points
- Maximize likelihood

Transition probability density

- Density

$$p(X_2, t+s \mid X_1, t) = (2\pi V_s)^{-1/2} \exp\left(-\frac{(X_2 - m_s)^2}{2V_s}\right)$$

- Mean

$$m_s = \exp(-as)X_1$$

- Variance

$$V_s = \frac{b^2}{2a}\left(1 - \exp(-2as)\right)$$

Log-likelihood

$$p(\{X\}) = p(X_0)\prod_{i=1}^{N} p(X_{t_i} \mid X_{t_{i-1}})$$

Transition probability density

Initial distribution

$$\log p(\{X\}) = \sum_{i=1}^{N} \log p(X_{t_i} \mid X_{t_{i-1}}) + \log p(X_{t_0})$$

$$= -\frac{1}{2}\sum_{i=1}^{N}\left[\frac{(X_{t_i} - \exp(-a\Delta t)X_{t_{i-1}})^2}{V_{t_{i-1}}} + \log(2\pi V_{t_{i-1}})\right] + \log p(X_{t_0})$$

Filtering

- Observation noise
- 2 level model
- Update equations for conditional moments
- Kalman gain

Filtering

- Discretized state and observation equations

Hidden states $x_{t+1} = Ax_t + w_t, \quad w_t \sim N(0,Q) \quad x_t \in \Re^k$

Observed data $y_t = Cx_t + v_t, \quad v_t \sim N(0,R) \quad y_t \in \Re^p$

- Can include inputs (see Ghahramani & Hinton 1996)

$$x_{t+1} = Ax_t + Bu_t + w_t$$

Filtering: joint likelihood

$$p(x_{t+1} \mid x_t) = (2\pi)^{-k/2} \left| Q \right|^{-1/2} \exp\left(-\frac{1}{2}(x_t - Ax_{t-1})^T Q^{-1} (x_t - Ax_{t-1}) \right)$$

$$p(y_t \mid x_t) = (2\pi)^{-p/2} \left| R \right|^{-1/2} \exp\left(-\frac{1}{2}(y_t - Cx_{t-1})^T R^{-1} (y_t - Cx_{t-1}) \right)$$

$$p(\{x\},\{y\}) = p(x_0) \prod_{t=0}^{N} p(x_{t+1} \mid x_t) \prod_{t=0}^{N} p(y_t \mid x_t)$$

Conditional moments

- Update equations

$$x_t^{t-1} = A x_{t-1}^{t-1}$$

$$V_t^{t-1} = A V_{t-1}^{t-1} A^T + Q$$

$$K_t = V_t^{t-1} C^T (C V_t^{t-1} C^T + R)^{-1}$$

$$x_t^t = x_t^{t-1} + K_t \left(y_t - C x_t^{t-1} \right)$$

$$V_t^t = V_t^{t-1} - K_t C V_t^{t-1}$$

- where

$$x_t^\tau = E(x_t \mid \{y\}_1^\tau), \quad V_t^\tau = Var(x_t \mid \{y\}_1^\tau)$$

Nonlinear SDEs

- Local linearization (Ozaki 1992)
  - Ito-Taylor expansion
  - Analytic solution
  - Ito-integral
  - Discretized equation
    - Uses
      - Simulation
      - Parameter estimation
      - Filtering

Nonlinear SDEs

- Local linearization (Ozaki 1992)
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Local linearization for deterministic dynamics – basic intuition

$$\dot{x} = f(x(t))$$

$$J_t = \left.\frac{\partial f(x)}{\partial x}\right|_{x=x(t)}$$

$$\ddot{x} = J_t \dot{x}$$

$$\dot{x}(t+\tau) = \exp(J_t \tau)\dot{x}(t) = \exp(J_t \tau) f(x(t))$$

$$\int_0^{\Delta t} \dot{x}(t+\tau)d\tau = x(t+\Delta t) - x(t) = J_t^{-1}(\exp(J_t \Delta t) - 1) f(x(t))$$

$$x(t+\Delta t) = x(t) + J_t^{-1}(\exp(J_t \Delta t) - 1) f(x(t))$$

Value of x @ $t+dt$ as function of x @ t

Discretization

- Nonlinear SDE

$$dx(t) = f(t, x(t))dt + gdw$$

- Discretized update equation

$$X_{t_{n+1}} = X_{t_n} + \Phi(t_n, X_{t_n}; h) + \xi(t_n, X_{t_n}; h)$$

Increment term

Noise term

$$c.f. \qquad x_{t_{n+1}} = Ax_{t_n} + Bu_{t_n} + w_{t_n}$$

Ito-Taylor expansion – *f(s,x(s))* and *x(s)*

$$dx(t) = f(t, x(t))dt + gdw$$

$$f(s, x(s)) \approx f(t, x(t)) + L^0 f(t, x(t)) \int_t^s du + L^1 f(t, x(t)) \int_t^s dw(u)$$

$$x(s) \approx x(t) + f \int_t^s du + g \int_t^s dw(u)$$

- where

$$L^0 f = f_t + ff_x + \tfrac{1}{2} g^2 f_{xx}, \qquad L^1 f = gf_x$$

Re-arrange

$$f(s,x(s)) \approx f(t,x(t)) + \left(f_t + \tfrac{1}{2} g^2 f_{xx}\right)\int_t^s du + f_x\left(f\int_t^s du + g\int_t^s dw(u)\right)$$

$$\approx f(t,x(t)) + \left(f_t + \tfrac{1}{2} g^2 f_{xx}\right)(s-t) + f_x(x(s) - x(t))$$

$$dX(s) \approx \left(A(t)X(s) + a(t,s)\right)ds + g\,dw(s)$$

$$A(t) = f_x$$

$$a(t,s) = f + \left(f_t + \tfrac{1}{2} g^2 f_{xx}\right)(s-t) - f_x x$$

Evaluated @ $t, X(t)$

Exact solution

- Linear SDE

$$dX_t = \left(A(t)X_t + a(t)\right)dt + g(t)dW_t$$

- Exact solution

$$X(t+h) = \exp(A(t)h)\cdot$$

$$\left(X(t) + \int_t^{t+h} \exp(-A(t)(u-t))a(u,t)du + \int_t^{t+h} \exp(-A(t)(u-t))g(u)dW_u \right)$$

Re-write

$$X_{t_{n+1}} = X_{t_n} + \Phi(t_n, X_{t_n}; h) + \xi(t_n, X_{t_n}; h)$$

- Using (see Jimenez 1999)

$$\Phi(t, X(t); h) = r_0\big(A(t), h\big) f(t, X(t)) + \ldots$$

$$\big(h r_0\big(A(t), h\big) - r_1\big(A(t), h\big)\big)\left(f_t + \frac{1}{2} g^2 f_{xx} \right)$$

$$r_n\big(M, a\big) = \int_0^a \exp(Mu) u^n du$$

Approximate noise

- Substitute Gaussian noise term

$$\eta_t \approx \xi(t, X(t); h)$$

$$\eta_t \sim N(0, \Sigma_\eta)$$

$$\Sigma_\eta = \frac{g^2}{2A(t)}\left(1 - \exp(2A(t)h)\right)$$

- Substitute into discretized update equation

$$X_{t_{n+1}} = X_{t_n} + \Phi_{t_n} + \eta_{t_n}$$

Summary

- Ito-Taylor expansion
- Basic numerical schemes & convergence
- Parameter estimation
- Filtering
- Local linearization

- Next meeting – Valdes Sosa (1999) paper

If you notice any typos (or mistakes!) please email L.Harrison@fil.ion.ucl.ac.uk