Complex Analysis and Riemann Surfaces

Note Title 9/2/2008

A graduate course taught by
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§1. Local Theory of Holomorphic Functions

Notation: $f: \Omega \longrightarrow \mathbb{C}$, Ω : (connected) open domain in $\mathbb{C}$

$z = x + iy$, $i^2 = -1$

Def. f: holomorphic on $\Omega \Leftrightarrow \forall z_0 \in \Omega$. $f'(z_0) \triangleq \lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h}$ exists.

Example: $f(z) = x$.

Although this function is linear, it's not holomorphic:

$z_0 = x_0 + iy_0$, $h = h_1 + ih_2$. Then $\frac{f(z_0+h)-f(z_0)}{h} = \frac{x_0+h_1-x_0}{h_1+ih_2} = \frac{h_1}{h_1+ih_2}$ has no limit as $h \to 0$.

- Consequences of holomorphicity

Assume f holomorphic, i.e. $f'(z) = \lim_{h\to 0}\frac{f(z+h)-f(z)}{h}$ exists, $\forall z \in \Omega$.

Take $h = h_1 \to 0$, $\in \mathbb{R}$, and view $f(z) = f(x,y)$

$$\left.\begin{array}{l} \Rightarrow \lim_{h_1\to 0}\frac{f(x+h_1,y)-f(x,y)}{h_1} = \frac{\partial f}{\partial x}(x,y) \\ \text{On the other hand, take } h = ih_2 \to 0,\ h_2 \in \mathbb{R}. \\ \Rightarrow \lim_{h_2\to 0}\frac{f(x,y+h_2)-f(x,y)}{ih_2} = -i\frac{\partial f}{\partial y}(x,y) \end{array}\right\} \Rightarrow \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y} \quad (= f'(z))$$

Moreover, if we write $f = f(x,y) = u(x,y) + iv(x,y)$, $u, v: \Omega \longrightarrow \mathbb{R}$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Notation: $\frac{\partial f}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$, then f holomorphic $\Rightarrow \frac{\partial f}{\partial \bar z} = 0$.

Thm. The following conditions are equivalent (in standard notations)

(i) f holomorphic in Ω.

(ii) f is C^1 in Ω & $\frac{\partial f}{\partial \bar z} \equiv 0$ in Ω

(iii) f is C^1 in Ω & $\forall$ region $\bar D \subseteq \Omega$ with piecewise C^1 boundary the line integral $\oint_{\partial D} f(z)dz = 0$

(iv) f is C^1 in Ω & $\forall$ disc $\bar D(z_0,r) \subseteq \Omega$, $z \in D(z_0,r)$, we have

$f(z) = \frac{1}{2\pi i}\oint_{\partial D}\frac{f(w)}{w-z}\,dw$ (Cauchy integral formula)

(v) $\forall z_0 \in \Omega$, $\exists$ disc $\bar D(z_0,r) \subseteq \Omega$ s.t. $f(z) = \sum_{n=0}^{\infty} c_n(z-z_0)^n$, $\forall z \in D(z_0,r)$ (uniform convergence). In particular $f \in C^\omega(\Omega) \subseteq C^\infty(\Omega)$.

Observation:

Green's thm. in the plane: Let D be a region in $\mathbb{R}^2$ with piecewise C^1 boundary ∂D, then
$\oint_{\partial D} P(x,y)dx + Q(x,y)dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$

Apply this thm to the case of functions of a complex variable z, $f(z) = u + iv$

$$\oint_{\partial D} f(z)dz = \oint_{\partial D} (u+iv)(dx+idy) = \oint_{\partial D} udx - vdy + i\oint_{\partial D} udy + vdx$$
$$\overset{\text{Green}}{=} \iint_D \left(-\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) dxdy + i\iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dxdy$$
$$= i\iint_D \left[\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + i\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] dxdy$$
$$= 2i\iint_D \frac{\partial f}{\partial \bar{z}} dxdy$$

$\Rightarrow \oint_{\partial D} f(z)dz = 0$ if f holomorphic $\left(\frac{\partial f}{\partial \bar z} \equiv 0\right)$

Proof of thm.

(ii) $\Leftrightarrow$ (iii):

"$\Rightarrow$" is easy, since $f \in C^1(\Omega)$, we may apply Green's formula

"$\Leftarrow$" We argue by contradiction. Assume $\frac{\partial f}{\partial \bar z}(z_0) \neq 0$ for some $z_0 \in \Omega$.

Take an arbitrary disc D around z_0. By (iii)

$$0 = \left|\oint_{\partial D} f(z)dz\right| = \left|2i\iint_D \frac{\partial f}{\partial \bar z} dxdy\right| = \left|2i\iint_D \left(\frac{\partial f}{\partial \bar z}(z) - \frac{\partial f}{\partial \bar z}(z_0)\right)dxdy + 2i\frac{\partial f}{\partial \bar z}(z_0)\iint_D dxdy\right|$$
$$\geq 2\left|\frac{\partial f}{\partial \bar z}(z_0)\iint_D dxdy\right| - 2\left|\iint_D \left(\frac{\partial f}{\partial \bar z}(z) - \frac{\partial f}{\partial \bar z}(z_0)\right) dxdy\right|$$
$$\geq 2\left|\frac{\partial f}{\partial \bar z}(z_0)\right| \operatorname{Area}(D) - 2\iint_D \left|\frac{\partial f}{\partial \bar z}(z) - \frac{\partial f}{\partial \bar z}(z_0)\right| dxdy$$

Since f is C^1, $\exists\, \delta > 0$ s.t. $|z - z_0| < \delta \Rightarrow \left|\frac{\partial f}{\partial \bar z}(z) - \frac{\partial f}{\partial \bar z}(z_0)\right| < \frac{1}{2}\left|\frac{\partial f}{\partial \bar z}(z_0)\right|$

$\Rightarrow 0 \geq \left|\frac{\partial f}{\partial \bar z}(z_0)\right| \pi\delta^2 > 0$, contradiction.

(iii) $\Rightarrow$ (iv)

We fix $z \in \Omega$, and set $g(w) = \dfrac{f(w) - f(z)}{w - z}$, $w \in \Omega \setminus \{z\}$

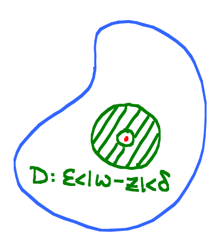
Claim: $\frac{\partial}{\partial \bar w} g(w) \equiv 0$ on $\Omega \setminus \{z\}$.

This is because $\frac{\partial}{\partial \bar w} g(w) = \dfrac{\frac{\partial}{\partial \bar w}(f(w) - f(z))(w-z) - (f(w) - f(z))\frac{\partial}{\partial \bar w}(w-z)}{(w-z)^2} = 0$ since $f(w) - f(z)$, $w - z$ are holomorphic.

Moreover, g is C^1 on $\Omega \setminus \{z\}$ since $f(w) - f(z)$ and $w - z$ are.

$\Rightarrow g$ satisfies (ii) on $\Omega \setminus \{z\}$

$\Rightarrow g$ satisfies (iii) on $\Omega \setminus \{z\}$



Apply (iii) to $\varepsilon < |w - z| < \delta$:

$$\Rightarrow 0 = \oint_{\partial D} g(w)\,dw = \oint_{|w-z|=\delta} g(w)\,dw - \oint_{|w-z|=\varepsilon} g(w)\,dw \quad (*)$$

Claim: $\oint_{|w-z|=\varepsilon} g(w)\,dw \to 0$ as $\varepsilon \to 0$

This is because by Taylor's formula: (for C^1)

$$f(w) = f(z) + \frac{\partial f}{\partial w}(w-z) - \cancel{\frac{\partial f}{\partial \bar w}}\,\overline{(w-z)} + o(|w-z|)$$

$$\Rightarrow g(w) = \frac{\partial f}{\partial w}(w-z) + o(|w-z|)$$

$\Rightarrow g$ can be extended as a continuous function on Ω, thus bounded on $|w-z| \le \varepsilon$

$$\Rightarrow \left|\oint_{|w-z|=\varepsilon} g(w)\,dw\right| \le \oint_{|w-z|=\varepsilon} \sup_{|w-z|\le\varepsilon} |g(w)| = 2\pi\varepsilon \cdot \sup_{|w-z|\le\varepsilon} |g(w)| \to 0 \text{ as } \varepsilon \to 0,$$

and we know that it's independent of ε by $(*)$.

$$\text{Thus}\quad 0 = \oint_{|w-z|=\delta} \frac{f(w)-f(z)}{w-z}\,dw = \oint_{|w-z|=\delta} \frac{f(w)}{w-z}\,dw - f(z)\oint_{|w-z|=\delta} \frac{dw}{w-z}$$

$$= \oint_{|w-z|=\delta} \frac{f(w)}{w-z}\,dw - f(z)\int_{|\zeta|=\delta} \frac{d\zeta}{\zeta}, \quad \text{let } \zeta = \delta e^{i\theta}$$

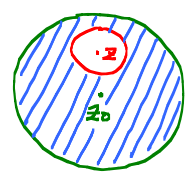
$$= \oint_{|w-z|=\delta} \frac{f(w)}{w-z}\,dw - f(z)\int_0^{2\pi} \frac{i\delta e^{i\theta}}{\delta e^{i\theta}}\,d\theta$$

$$= \oint_{|w-z|=\delta} \frac{f(w)}{w-z}\,dw - 2\pi i f(z)$$

$$\Rightarrow f(z) = \frac{1}{2\pi i}\oint_{|w-z|=\delta} \frac{f(w)}{w-z}\,dw$$

This is the case that $z = z_0$ in (iv). More generally, for any point z in $|z-z_0|<\delta$, consider the integral of $\frac{f(w)}{w-z}$ on the boundary of the region bounded by $|w-z_0|<\delta$ and $|w-z|<\varepsilon$, ε small enough so that it's contained in $|w-z_0|<\delta$

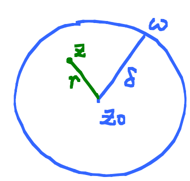
we have:



$$\frac{1}{2\pi i}\oint_{|w-z_0|=\delta} \frac{f(w)}{w-z}\,dw - \frac{1}{2\pi i}\oint_{|w-z|=\varepsilon} \frac{f(w)}{w-z}\,dw = 0$$

$$\Rightarrow f(z) = \frac{1}{2\pi i}\oint_{|w-z|=\varepsilon} \frac{f(w)}{w-z}\,dw = \frac{1}{2\pi i}\oint_{|w-z_0|=\delta} \frac{f(w)}{w-z}\,dw$$

(iv) $\Rightarrow$ (v)



$$\left|\frac{z-z_0}{w-z_0}\right| = \frac{r}{\delta} < 1$$

$$\text{Now } f(z) = \frac{1}{2\pi i}\oint_{|w-z|=\delta} f(w)\frac{1}{(w-z_0)-(z-z_0)}\,dw$$

$$= \frac{1}{2\pi i}\oint_{|w-z|=\delta} \frac{f(w)}{w-z_0}\,\frac{1}{1-\frac{z-z_0}{w-w_0}}\,dw$$

$$= \frac{1}{2\pi i}\oint_{|w-z|=\delta} \frac{f(w)}{w-z_0}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{w-z_0}\right)^n dw$$

$$= \frac{1}{2\pi i}\sum_{n=0}^{\infty}\left(\oint_{|w-z|=\delta} \frac{f(w)}{(w-z_0)^{n+1}}\,dw\right)(z-z_0)^n$$

Here the equalities are valid since $\left|\frac{z-z_0}{w-z_0}\right| < 1$ and the convergence is uniform.

(v) $\Rightarrow$ (i) trivial.

(i) $\Rightarrow$ (ii) (Harder!)

We already know that $\frac{\partial f}{\partial \bar{z}} = 0$ by analysis before the theorem. Thus our primary step would be to show that f holo. $\Rightarrow f \in C^1$ (Since we now don't know apriori that f is C^1, we can NOT apply Green's thm. to prove (i) $\Rightarrow$ (iii))

It suffices to show that for any disc $D \subseteq \Omega$, there exists a C^1 function F s.t. $F'(z) = f(z)$ and $\frac{\partial F}{\partial \bar{z}} = 0$. Then F satisfies (ii) thus (v) by the previous proof. Then F would be C^ω and so is $f(z) = F'(z)$ C^ω, in particular $f(z)$ is C^1.

In the disc D, let $F(z) = \int_{L_z} f(w)\,dw$, where L_z is the line segment joining z_0 and z

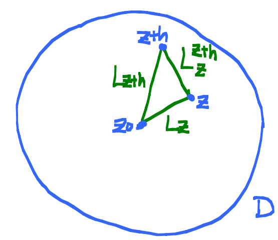
We compute $F'(z) = \lim_{h\to 0} \frac{F(z+h) - F(z)}{h}$

$$= \lim_{h\to 0} \frac{1}{h}\left(\int_{L_{z+h}}(\cdot) - \int_{L_z}(\cdot) \right)$$

$$\overset{\text{Claim}}{=} \lim_{h\to 0} \frac{1}{h}\left(\int_{L_z^{z+h}}(\cdot) \right)$$

$$= \lim_{h\to 0} \frac{1}{h} \int_0^1 f(z+th)\,h\,dt$$

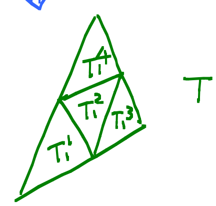
$$= f(z).$$



Proof of the Claim:

The claim $\Longleftrightarrow \oint_{\text{Triangle}} f(z)\,dz = 0$

Let $I = \oint_T f(z)\,dz$ and subdivide the triangle into the four equal pieces as in the picture on the right.

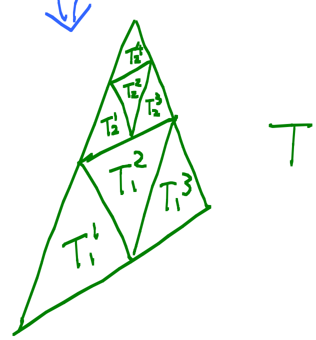


$$I = \sum_{i=1}^{4} \oint_{T_1^i} f(z)\,dz$$

We can now pick one T_1^i $(i=1,\cdots,4)$ s.t. (say, T_1^i)

$$\left| \oint_{T_1^i} f(z)\,dz \right| \geq \frac{I}{4}$$

We further subdivide T_1^i into 4 equal pieces $T_2^1, \cdots, T_2^4$, and pick T_2^i s.t.



$$\left| \oint_{T_2^i} f(z)\,dz \right| \geq \frac{1}{4} \left| \oint_{T_1^i} f(z)\,dz \right| \geq \frac{I}{4^2}$$

$\cdots\cdots$ (iterate this process)

We obtain a sequence of "nested triangles":

$$T \supseteq T_1^i \supseteq T_2^i \supseteq \cdots$$

with $\left| \oint_{T_k^i} f(z)\,dz \right| \geq \frac{I}{4^k}$

Moreover $\bigcap_{k=1}^{\infty} T_k^i = \{z_0\}$. Since f is holomorphic at z_0, $f(z) = f(z_0) + f'(z_0)(z-z_0) + o(|z-z_0|)$ in a sufficiently small neighborhood of z_0. When k is large enough, T_k^i will be contained in this neighborhood. Thus

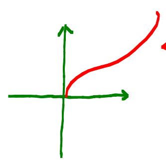
$$\left| \oint_{\partial T_k^i} f(z)\,dz \right| = \left| \oint_{\partial T_k^i} \left(f(z_0) + f'(z_0)(z-z_0) + o(|z-z_0|) \right) dz \right|$$

$$\leq \left| \oint_{\partial T_k^i} (f(z_0) + f'(z_0)(z-z_0))dz \right| + \left| \oint_{\partial T_k^i} o(|z-z_0|)dz \right|$$
$$\leq \oint_{\partial T_k^i} |o(|z-z_0|)|\,|dz|$$
$$\leq 3\cdot \operatorname{diam} T_k^i \cdot \ \varepsilon \operatorname{diam} T_k^i$$
$$= 3\varepsilon \cdot \frac{\operatorname{diam} T}{2^k} \cdot \frac{\operatorname{diam} T}{2^k}$$
$$= 3 \cdot \frac{(\operatorname{diam} T)^2}{4^k} \varepsilon \qquad \text{where } \varepsilon \text{ is arbitrarily small, since } \lim_{z\to z_0} \frac{o(|z-z_0|)}{|z-z_0|} \to 0$$
$$\Rightarrow \frac{I}{4^k} \leq 3\varepsilon \frac{(\operatorname{diam} T)^2}{4^k}$$
$$\Rightarrow I \leq 3(\operatorname{diam} T)^2 \cdot \varepsilon$$

Thus $I = 0$ since ε is arbitrarily small. □

Examples of holomorphic functions

- $e^z \triangleq \sum_{n=1}^{\infty} \frac{z^n}{n!}$ converges for all z. $\Rightarrow e^z$ is holomorphic on $\mathbb{C}$ and $\frac{d}{dz} e^z = e^z$. and $e^{z+w} = e^z e^w$
- We want to define $\log z$ as the inverse of e^z, i.e. $e^{\log z} = z$. Set $\log z = u + iv$ $\quad z = e^{\log z} = e^u e^{iv} \Rightarrow u = |z|,\ v = \operatorname{Arg} z$, but until we introduce a cut on $\mathbb{C}$, $\operatorname{Arg} z$ can't be a well-defined function on $\mathbb{C}$.



- Open mapping and maximal modulous theorems

Take f holomorphic in Ω

1). $f \not\equiv 0 \Rightarrow$ the zero's of f are isolated, i.e. $f(z_0) = 0 \Rightarrow \exists V$, neighborhood of z_0 s.t. $\forall z \in V \setminus \{z_0\}$, $f(z) \neq 0$

This is because, we can expand f as a power series near z_0 $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ in a nhd of z_0. $f \not\equiv 0 \Rightarrow \exists n.\ a_n \neq 0$. Take the smallest such n, say, $a_N \neq 0$. $\Rightarrow f(z) = (z-z_0)^N (a_N + a_{N+1}(z-z_0) + \cdots) = (z-z_0)^N g(z)$, where $g(z_0) = a_N \neq 0$. Thus in a small enough neighborhood $g(z) \neq 0$.

Def: The order of vanishing of f at $z_0 \triangleq N$.

2). The class of meromorphic functions in Ω.

Def: g meromorphic in $\Omega \Leftrightarrow \forall z_0 \in \Omega$, $g(z_0) = \sum_{n=N}^{\infty} a_n (z-z_0)^n$, $N \in \mathbb{Z}$.

Meromorphic functions satisfy the following version of Cauchy integral formula.

$$\frac{1}{2\pi i} \oint_{\partial D} g(z)dz = \sum_{z_i} \mathrm{Res}(g)(z_i),$$

where z_i are poles of g inside D and there are no poles on ∂D. (The poles inside D are isolated thus finite, since $\bar{D}$ compact). Here if z_i is a pole $g(z_i) = \sum_{n=N}^{\infty} a_n (z-z_i)^n \quad \mathrm{Res}(g)(z_i) \triangleq a_{-1}$

Pf: Consider $g(z)$ as a holomorphic function on $V \setminus \{z_1, \cdots, z_N\}$, where V is a neighborhood of $\bar{D}$

$\Rightarrow 0 = \oint_{\partial(D \setminus \bigcup_{i=1}^{n} \{|z-z_i|<\varepsilon\})} g(z)dz = \oint_{\partial D} g(z)dz - \sum_{i=1}^{N} \oint_{|z-z_i|=\varepsilon} g(z)dz$

$\Rightarrow \oint_{\partial D} g(z)dz = \sum_{i=1}^{N} \oint_{|z-z_i|=\varepsilon} g(z)dz$

In each $|z-z_i| \leq \varepsilon$, $g(z) = \sum_{n=N_i}^{\infty} a_n (z-z_i)^n$

$\Rightarrow \oint_{|z-z_i|=\varepsilon} g(z)dz = \sum_{n=N_i}^{\infty} \oint_{|z-z_i|=\varepsilon} a_n (z-z_i)^n dz$

Now $\oint_{|z-z_i|=\varepsilon} (z-z_i)^n dz = \begin{cases} 0 & n \neq -1 \\ 2\pi i & n = -1 \end{cases}$

$\Rightarrow \oint_{\partial D} g(z)dz = \sum_{i=1}^{N} 2\pi i \, \mathrm{Res}(g)(z_i)$ $\square$

3). Simple consequences

$g(z)$ meromorphic in $\Omega \Rightarrow g'(z)$ is meromorphic with poles at the poles of $g(z)$

$\Rightarrow \frac{g'(z)}{g(z)}$ is meromorphic with poles at the poles and zeroes of $g(z)$; and the residue of $\frac{g'(z)}{g(z)}$ at a zero of $g(z)$ is $+1$ (to be counted with multiplicity), while is -1 (to be counted with multiplicity) at a pole of $g(z)$.

Pf: Near a zero or pole of $g(z)$, $g(z) = \sum_{n=N_0}^{\infty} a_n (z-z_0)^n \quad (a_{N_0} \neq 0)$

$\Rightarrow g(z) = (z-z_0)^{N_0} \sum_{n=N_0}^{\infty} a_n (z-z_0)^{n-N_0} = (z-z_0)^{N_0} u(z)$, $u(z_0) \neq 0$.

$\Rightarrow \frac{g'(z)}{g(z)} = \frac{N_0 (z-z_0)^{N_0-1} u(z) + (z-z_0)^{N_0} u'(z)}{(z-z_0)^{N_0} u(z)} = \frac{N_0}{z-z_0} + \frac{u'(z)}{u(z)}$

We have a pole of $\frac{g'(z)}{g(z)}$ if $N_0 \neq 0$:

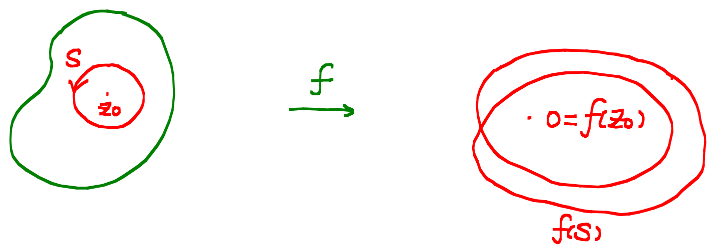
$\begin{cases} N_0 > 0, & g(z) \text{ has a zero of order } N_0 \text{ at } z_0 \\ N_0 < 0, & g(z) \text{ has a pole of order } -N_0 \text{ at } z_0 \end{cases}$ $\square$

In conclusion:

$$\frac{1}{2\pi i} \oint_{\partial D} \frac{g'(z)}{g(z)} dz = \sum_{\text{poles}} \mathrm{Res}\left(\frac{g'(z)}{g(z)}\right)(z_i) = \sum_{\substack{\text{zero's} \\ \text{of } g}} 1 - \sum_{\substack{\text{poles of} \\ g}} 1$$

(Open Mapping Thm.) $f: \Omega \longrightarrow \mathbb{C}$ holomorphic, not constant. Assume f has order $n>0$ at z_0 (i.e. $f(z)=(z-z_0)^n u(z)$ $u(z_0)\neq 0$) $\Rightarrow \exists\, U$ neighborhood of z_0 and V neighborhood of 0 s.t. $\forall\, v\in V$, $\exists$ exactly n points of $z_1, \cdots, z_n \in U$ s.t. $f(z_1)=\cdots=f(z_n)=v$

Pf: Since the zero's of f and f' are isolated, we may assume that $S=\partial D$ is a small circle around z_0 such that $f(z)\neq 0$, $f'(z)\neq 0$ for $z\in \bar{D}\setminus\{z_0\}$.



Consider the following function of ω: $\omega \mapsto \frac{1}{2\pi i}\oint_S \frac{f'(z)}{f(z)-\omega}dz$

The R.H.S. = integer valued continuous function on $\mathbb{C}\setminus f(S)$, thus locally constant.

Let V be the connected component of $\mathbb{C}\setminus f(S)$ containing 0.

$\Rightarrow \forall\, \omega\in V$ $\frac{1}{2\pi i}\oint_S \frac{f'(z)}{f(z)-\omega}dz = n = \#\{\text{zeros of } f(z)-\omega \text{ in } D\}$

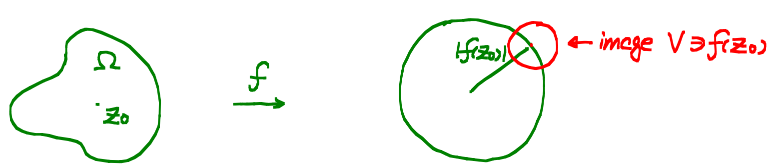
Thus $f(z)=\omega$ has exactly n-zeros for $\omega\in V$ in $f^{-1}(V)\cap D \triangleq U$. The zero's are distinct since we have assumed that $\forall\, z\neq z_0$, $f'(z)\neq 0$ $\square$

Cor. If f is holomorphic and not a constant $\Rightarrow$ the image of f is open.

Pf: $\forall\, \omega\in \mathrm{Im} f$. Take V for $f(z)-\omega$ as in the thm, $V\subseteq \mathrm{Im} f$. $\square$

Cor. (Maximum Modulous Principle) f holomorphic on Ω. If $\exists\, z_0\in\Omega$ s.t. $|f(z_0)|\geq |f(z)|$, $\forall\, z\in\Omega$, then f is a constant.

Pf:

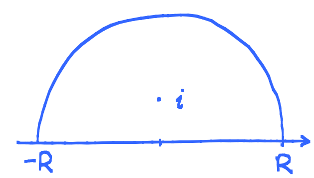


If f were not constant, then $f(z_0)$ would be in an open set of images of f $\Rightarrow$ there would be $z'\in\Omega$ with $|f(z')| > |f(z_0)|$, contradiction. $\square$

- Applications : method of residues

1). Calculate $\int_0^{+\infty} \frac{1}{1+x^2}dx$ using residue formula.

$I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{1+x^2}dx = \frac{1}{2}\lim_{R\to\infty}\int_{-R}^{R}\frac{1}{1+z^2}dz$. Construct an integral contour as follows :



$I = \frac{1}{2}\lim_{R\to\infty}\left(\int_{\cap\!\!\!-} \frac{dz}{1+z^2} - \int_{\cap}\frac{dz}{1+z^2}\right)$

By the residue thm $\int_{\cap\!\!\!-} \frac{dz}{1+z^2} = 2\pi i\, Res(\frac{1}{1+z^2})(i)$,

and $\frac{1}{1+z^2} = \frac{1}{z+i}\,\frac{1}{z-i} \Rightarrow Res(\frac{1}{1+z^2})(i) = \frac{1}{2i}$

$\Rightarrow \int_{\cap\!\!\!-}\frac{dz}{1+z^2} = \pi$

Moreover $\left|\int_{\cap}\frac{dz}{1+z^2}\right| \leq \int_0^{\pi}\frac{|dz|}{|z^2|-1} = \int_0^{\pi}\frac{R\,d\theta}{R^2-1} = \frac{\pi R}{R^2-1} \longrightarrow 0$ as $R \longrightarrow 0$.

$\Rightarrow I = \frac{\pi}{2}$.

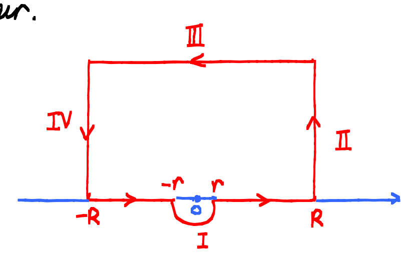
2). $\int_0^{\infty}\frac{\sin x}{x}dx$

First of all, the integral converges : $x\to 0$, $\frac{\sin x}{x} \longrightarrow 1$, and the function is smooth (analytic) thus integrable near 0 ; $\left|\int_1^{\infty}\frac{\sin x}{x}dx\right| = \left|\frac{-\cos x}{x}\Big|_1^{\infty} - \int_1^{\infty}\frac{\cos x}{x^2}dx\right| \leq 1 + \int_0^{\infty}\frac{1}{x^2}dx$, thus is integrable near ∞.

We want to apply the method of residue. Consider $e^{iz} = \cos z + i\sin z$

$\int_{-\infty}^{\infty}\frac{\sin x}{x}dx = Im\left(\int_{-\infty}^{\infty}\frac{e^{iz}}{z}dz\right)$

To calculate $\int_{-\infty}^{\infty}\frac{e^{iz}}{z}dz$ we take the following contour.



$\oint \frac{e^{iz}}{z}dz = Res(\frac{e^{iz}}{z})(0) = 2\pi i$

On I : $\frac{e^{iz}}{z} = \frac{1}{z} + u(z)$ where $u(z)$ is holomorphic

$\int_{I}\frac{e^{iz}}{z}dz = \int_{\pi}^{2\pi}\frac{1}{z}dz + \int_{\pi}^{2\pi}u(r,\theta)\,ire^{i\theta}d\theta$

$= \pi i + O(r) \longrightarrow \pi i \quad (r\longrightarrow 0)$

Ⅱ and Ⅳ : $\left|\int_{Ⅱ}\frac{e^{iz}}{z}dz\right| = \left|\int_0^{R}\frac{e^{i(R+iy)}}{R+iy}dy\right| = \left|\int_0^{R}\frac{e^{iR}e^{-y}}{R+iy}dy\right| \leq \int_0^{R}\left|\frac{e^{iR}e^{-y}}{R+iy}\right|dy$

$\leq \int_0^{R}\frac{e^{-y}}{R}dy \leq \frac{1}{R}\int_0^{\infty}e^{-y}dy \longrightarrow 0 \quad (R\longrightarrow\infty)$

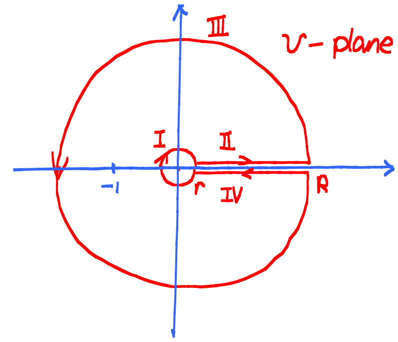
Ⅲ : $\left|\int_{Ⅲ}\frac{e^{iz}}{z}dz\right| = \left|\int_R^{-R}\frac{e^{ix-R}}{x+iR}dx\right| \leq \int_R^{-R}\left|\frac{e^{ix-R}}{x+iR}\right|dx \leq e^{-R}\int_R^{-R}\frac{1}{R}dx = 2e^{-R} \longrightarrow 0 \quad (R\to\infty)$

Thus it follows that $2\pi i = \int_{-\infty}^{\infty}\frac{e^{iz}}{z}dz + \pi i + 0 + 0$

$\Rightarrow \int_0^{\infty}\frac{\sin x}{x}dx = \frac{1}{2}\int_{-\infty}^{\infty}\frac{\sin x}{x}dx = \frac{\pi}{2}$.

3). Evaluate $\int_0^{\infty}\frac{u^{z-1}}{1+u}du$ $(0<z<1$, which ensures the convergence$)$

$u^{z-1} = e^{(z-1)\ln u}$, but since "ln" function can only be well-defined after we introduce a cut on the plane $\mathbb{C}$, we make the following contour on $\mathbb{C}\setminus\{x\geq 0,\ y=0\}$



We pick the ordinary "ln" function on the upper half of the positive axis.

Now the function $f(v)=\frac{v^{z-1}}{1+v}$ is holomorphic on $\mathbb{C}\setminus\{-1\}$ and in particular has a simple pole inside our contour. Thus by the residue thm.

$$\oint f(v)dv = 2\pi i \, \mathrm{Res}(f)(-1) = 2\pi i (e^{i\pi})^{z-1} = 2\pi i \, e^{i\pi(z-1)}$$

Moreover, $\left|\int_{I} f(v)dv\right| = \left|\int_{2\pi}^{0} \frac{(re^{i\theta})^{z-1}}{1+re^{i\theta}} r i e^{i\theta} d\theta\right| \le \int_0^{2\pi} \frac{r^z}{1-r} dr = \frac{2\pi r^z}{1-r} \to 0 \quad (r\to 0)$

$\left|\int_{III} f(v)dv\right| = \left|\int_0^{2\pi} \frac{R^{z-1}e^{(z-1)i\theta}}{1+Re^{i\theta}} R i e^{i\theta} d\theta\right| \le \int_0^{2\pi} \frac{R^z}{R-1} d\theta = \frac{2\pi R^z}{R-1} \to 0 \quad (R\to\infty)$

Now $\int_{II+IV} f(v)dv = \int_0^R \frac{v^{z-1}}{1+v} dv - \int_0^R \frac{v^{z-1}e^{2\pi i(z-1)}}{1+v} dv$

$$= (1-e^{2\pi i(z-1)}) \int_0^R \frac{v^{z-1}}{1+v} dv$$

$\Rightarrow (1-e^{2\pi i(z-1)}) \int_0^\infty \frac{v^{z-1}}{1+v} + 0 + 0 = 2\pi i \, e^{i\pi(z-1)}$

$\Rightarrow \int_0^\infty \frac{v^{z-1}}{1+v} dv = \frac{2\pi i \, e^{i\pi(z-1)}}{1-e^{2\pi i(z-1)}} = \frac{\pi}{\sin \pi z}$ □

• Analytic Continuation

Basic Observation:

Let Ω be a domain (connected), f a holomorphic function. If $f=0$ on a non-empty open set $\Omega' \subseteq \Omega$, then $f\equiv 0$ on Ω.

Pf: Let $\tilde{\Omega} = \{z_0\in\Omega \mid f=0 \text{ in a neighborhood of } z_0\}$

(i). $\tilde{\Omega} \supseteq \Omega'$ thus $\neq \phi$.

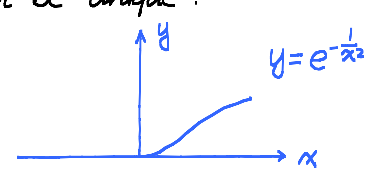
(ii). $\tilde{\Omega}$ is open from definition

(iii). $\tilde{\Omega}$ is closed •

$\Rightarrow \tilde{\Omega} = \Omega$ since Ω is connected.

• $\forall z_1 \in \tilde{\Omega}^c$, i.e. f is not identically 0 around z_1. Write $f(z)=\sum_{n=0}^{\infty} C_n(z-z_1)^n$, then some C_N must be non-zero. Take minimal such $\Rightarrow f(z)=(z-z_1)^N \hat{f}(z)$ with $\hat{f}(z_1)\neq 0$. Thus $f(z)\neq 0$ on an open neighborhood where $\hat{f}(z)\neq 0$. □

A Basic Question: Given $\Omega\subseteq\mathbb{C}$ and f holomorphic on Ω. What is the largest Ω' s.t. $\exists\, g(z)$ holomorphic on Ω' s.t. $g|_{\Omega} = f$. (Note that for smooth functions smooth extensions need not be unique:



We may extend y by 0 or $e^{-\frac{1}{x^2}}$ itself!

Model cases:

- Let $\varphi \in C^\infty[0,1]$ and consider $f(z) = \int_0^1 x^z \varphi(x)\,dx$.

The integral converges for $\mathrm{Re}\,z > -1$ and defines a holomorphic function on $\{\mathrm{Re}\,z > -1\}$

Claim: $f(z)$ admits a meromorphic extension with possibly poles at the negative integes.

Pf: Let N be any positive integer, we will write $\varphi(x)$ in terms of its Taylor expansion at 0. $\varphi(x) = \sum_{n=0}^{N} \frac{1}{n!} \varphi^{n}(0)\, x^n + E_N(x)$ where $|E_N(x)| \leq C x^{n+1}$

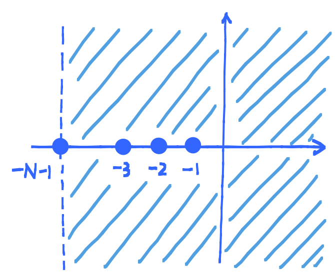
Let $\mathrm{Re}\,z > -1$, then:

$$f(z) = \int_0^1 x^z \varphi(x)\,dx = \int_0^1 \sum_{n=0}^{N} \frac{1}{n!} \varphi^{n}(0)\, x^{n+z}\,dx + \int_0^1 E_N(x)\, x^z\,dx$$

Now, since $|E_N(x)| \leq C x^{n+1}$ $\int_0^1 E_n(x)\, x^z\,dx$ converges on $\mathrm{Re}(z+n+1) \geq -1$ or equivalently $\mathrm{Re}\,z > -n-2$ and defines a holomorphic function there. Moreover

$$\int_0^1 \sum_{n=0}^{N} \frac{1}{n!} \varphi^{n}(0)\, x^{n+z}\,dx = \sum_{n=0}^{N} \frac{\varphi^{n}(0)}{n!} \frac{1}{n+1+z},$$

which is a meromorphic function with possibly poles at $\{-1,-2,\cdots,-N-1\}$.



$f(z) = \underbrace{\text{mero. fun}}_{\text{globally defined}} + \underbrace{\text{holo}}_{\text{defined only on } \mathrm{Re}\,z \geq -N-1}$ on $\mathrm{Re}\,z > -N-1$

This works for arbitrary $N \in \mathbb{N}$ $\Rightarrow$ f extends to a meromorphic function on $\mathbb{C}$, with residue at $-N-1$ equal to $\frac{\varphi^{N}(0)}{N!}$. □

- The $\Gamma(z)$-function

$\Gamma(z) = \int_0^\infty e^{-x} x^{z-1}\,dx$ $(\mathrm{Re}\,z > -1)$

Claim: $\Gamma(z)$ extends to a meromorphic function on $\mathbb{C}$, with simple poles in $\{-n \mid n \in \mathbb{N}\}$.

Pf: $\Gamma(z) = \int_0^1 e^{-x} x^{z-1}\,dx + \int_1^\infty e^{-x} x^{z-1}\,dx$, and $\int_1^\infty e^{-x} x^{z-1}\,dx$ is well-defined and holomorphic in the whole plane $\mathbb{C}$.

Thus we may apply the previous result. □

- The $\zeta(z)$ (Riemann zeta) function.

$\zeta(s) \triangleq \sum_{n=1}^{\infty} \frac{1}{n^s}$, well-defined for $Re(s) > 1$

Observe that $\frac{1}{n^s}$ can be written as $\int_0^\infty e^{-nt} t^{s-1} dt = \Gamma(s) n^{-s}$

$\Rightarrow \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{\Gamma(s)} \int_0^1 \sum_{n=0}^{\infty} e^{-nt} t^{s-1} dt = \frac{1}{\Gamma(s)} \int_0^\infty \frac{1}{e^t - 1} t^{s-1} dt$

Similar as above, $\int_0^\infty \frac{1}{e^t-1} t^{s-1} dt$ admits an extension to a meromorphic function on $\mathbb{C}$. (In fact holomorphic on $\mathbb{C} \setminus \{1\}$, with a simple pole at 1., since $\Gamma(s)$ also has simple poles at $-n$.)

Ex. Compute to show that $\zeta(0) = -\frac{1}{12}$, $\zeta'(0) = -\frac{1}{2} \log 2\pi$

Now, take X to be a smooth compact manifold. Assume that $\Delta: C^\infty(X) \longrightarrow C^\infty(X)$ is an operator with eigenvalues λ_n ($\lambda_n \geq 0$) $\Delta \varphi_n = \lambda_n \varphi_n$.

We want to define $\det \Delta = \prod_{n=1}^{\infty} \lambda_n$, is this possible?

In interesting cases, $\lambda_n \longrightarrow \infty$, there is no chance that it will converge.

E.g. $X = S^1$, $\Delta = -\frac{d^2}{d\theta^2}$ $\varphi = \sum_{n=-\infty}^{\infty} \varphi_n e^{in\theta}$ and $\{e^{in\theta}\}$ is a basis

$\Delta e^{in\theta} = -(in)^2 e^{in\theta} = n^2 e^{in\theta}$

$\prod n^2 = ?$

In interesting cases, $\det \Delta = \prod_{\lambda_n > 0} \lambda_n$. We apply a zeta-function definition

Formally, define $\zeta_\Delta(s) = \sum_{\lambda_n > 0} \frac{1}{\lambda_n^s}$.

$\Rightarrow \zeta_\Delta'(s) = \sum_{\lambda_n > 0} (e^{-s \ln \lambda_n})' = \sum_{\lambda_n > 0} -\ln \lambda_n \cdot \lambda_n^{-s}$

$\Rightarrow \zeta_\Delta'(0) = \sum_{\lambda_n > 0} -\ln \lambda_n = -\ln \prod_{\lambda_n > 0} \lambda_n$.

$\Rightarrow \prod_{n > 0} \lambda_n = e^{-\zeta_\Delta'(0)}$

E.g. $X = S^1$, $\zeta_\Delta(s) = 2 \sum_{n=1}^{\infty} \frac{1}{n^{2s}} = 2\zeta(2s)$.

Note that $\lambda_n^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-\lambda_n t} t^{s-1} dt$

$\Rightarrow \sum_{\lambda_n > 0} \lambda_n^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty \sum_{\lambda_n > 0} e^{-\lambda_n t} t^{s-1} dt$. Define $K(t) = \sum_{\lambda_n > 0} e^{-\lambda_n t}$

Using the same analysis as we did for $\zeta(s)$, we obtain a sufficient condition for $\zeta_\Delta(s)$ to extend holomorphically for s near $s=0$

Prop. Assume λ_n grows polynomially in n and $K(t)$ is a smooth function for $t>0$, furthermore:

(a). $K(t) \leq Ce^{-\mu t}$ for $t >> 0$

(b). $K(t) \sim \sum_{\ell=1}^{n} C_\ell t^{-\ell}$ + smooth function for $t \in [0,1)$

Then $\zeta_\Delta(s)$ admits a meromorphic extension for $s \in \mathbb{C}$, which is holomorphic at 0.

Pf: Since $|\int_0^\infty K(t)t^{s-1}dt| \leq \int_0^\infty |K(t)||t^{s-1}|dt < \int_0^\infty C\cdot e^{-\mu t}t^{s-1}dt$ converges for $Re s > -1$, thus defines a holomorphic function there.

Now $\int_0^\infty K(t)t^{s-1}dt = \int_0^1 K(t)t^{s-1} + \int_1^\infty K(t)t^{s-1}dt$.

Since $\int_1^\infty K(t)t^{s-1}dt$ is well-defined and holomorphic for all $s \in \mathbb{C}$, it suffices to show that $\int_0^1 K(t)t^{s-1}dt$ can be so extended.

Indeed, $\int_0^1 K(t)t^{s-1}dt = \sum_{m=0}^{N} C_m \int_0^1 t^{-m-1+s}ds + \int_0^1 E_N(t)t^{s-1}dt$

$$= \sum_{m=1}^{N} \frac{C_m}{s-m} + \int_0^1 E_N(t)t^{s-1}dt \quad \text{for } Re(s) > N+1$$

Since $\sum_{m=1}^{N}\frac{C_m}{s-m}$ is already meromorphic on $\mathbb{C}$, it suffices to show that $\int_0^1 E_N(t)t^{s-1}dt$ extends to be a meromorphic function on $\mathbb{C}$

Take the Taylor expansion of $E_N(t) = \sum_{m=0}^{M} a_m t^m + F_M(t)$, $a_m = \frac{E_N^{(m)}(0)}{m!}$. Then $|F_M(t)| \leq D\cdot t^{M+1}$ for $t\in(0,1)$, and we have:

$$\int_0^1 \left(\sum_{m=0}^{M} a_m t^m + F_M(t)\right)t^{s-1}dt$$
$$= \sum_{m=0}^{M} a_m \frac{1}{m+s} + \int_0^1 F_M(t)t^{s-1}dt$$

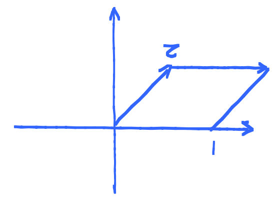
But $|\int_0^1 F_M(t)t^{s-1}dt| < \int_0^1 |F_M(t)t^{s-1}|dt < \int_0^1 D\cdot t^{M+s}dt$, converges for $Re s > -M-1$ and defines a meromorphic function there. The result follows.

Finally $\zeta_K(t)$ has at most simple poles at $\{N, N-1, \cdots 1, 0, -1, \cdots\}$ and $\Gamma(s)^{-1}$ vanishes at $s=0$, the last statement follows. $\square$

Observations: we shall later prove that if Δ is an elliptic PDE of positive order, then $\lambda_n \longrightarrow \infty$ polynomially and $Tr e^{-t\Delta}$ admits an expansion as in (b). Where $Tr e^{-t\Delta} = \dim\ker\Delta + \sum_{\lambda_n>0} e^{-\lambda_n t}$. We will show that $Tr e^{-t\Delta}$ satisfies (b) by solving $(\frac{\partial}{\partial t} + \Delta)H(t) = 0$ $H(t)|_{t=0} = I$ and set $H(t) = e^{-t\Delta}$.

Basic example:

$X = \mathbb{C}/\mathbb{Z}\oplus\mathbb{Z}\tau$, $\quad \tau = \tau_1 + \tau_2 i$, $\quad \tau_2 > 0$



$\Delta = -4\dfrac{\partial^2}{\partial \bar{z}\,\partial z}$ acting on periodic functions

$$\begin{cases} \varphi(z+1) = \varphi(z) \\ \varphi(z+\tau) = \varphi(z) \end{cases}$$

Ex. (1) Show that the eigenvalues of Δ are given by $\lambda_{mn} = \dfrac{4\pi^2}{\tau_2^2}|m+n\tau|^2$, $m, n \in \mathbb{Z}$.

(2). $(\det\Delta)' = \dfrac{1}{(2\pi)^4}\tau_2^2\,|\eta(q)|^4$ where $\eta(q) = q^{\frac{1}{12}}\prod_{n=1}^{\infty}(1-q^n)$ $\quad q = e^{2\pi\tau}$.

§2. Riemann Surfaces

- A Simple Model

Problem: We would like to define the function $\omega=\sqrt{z}$ on a maximal domain, where it is holomorphic.

$\sqrt{z}=e^{\frac{1}{2}\ln z}$, since there is $\ln$ involved, we start by picking a branch of $\ln z$ on $\mathbb{C}\setminus\mathbb{R}_{>0}$: if we choose the usualy definition of $\ln z$, then $z=re^{i\theta}\Rightarrow\sqrt{z}=\sqrt{r}\,e^{i\frac{\theta}{2}}$ and on upper and lower $\mathbb{R}_{>0}$ axis, $\sqrt{z}$ differ by $e^{\frac{2\pi i}{2}}=e^{\pi i}=-1$. Take 2 copies of $\mathbb{C}\setminus\mathbb{R}_{>0}$, then we can glue them together:

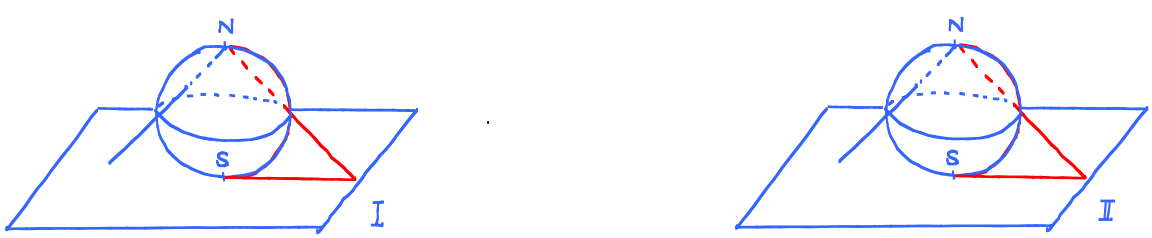
Glue I_+ with II_- and glue I_- with II_+, we obtain $X=(I)\cup(II)$ and define a function W on X as follows $W=\sqrt{z}$ if $z\in I$; $W=-\sqrt{z}$ if $z\in II$.

Observastions:

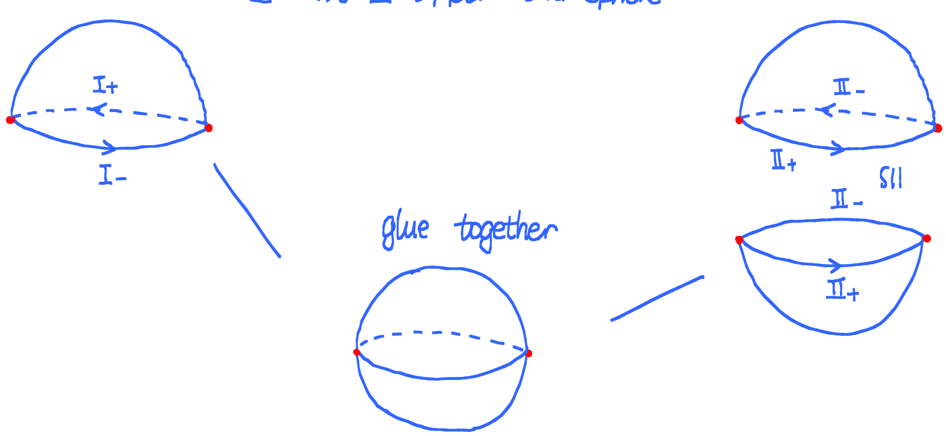
(1). W is continuous along the cut

(2). $X\cong S^2\setminus\{N,S\}$

For (2):



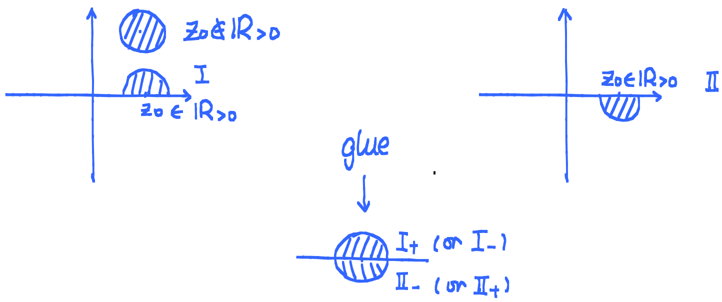
$\mathbb{C}\setminus\mathbb{R}_{>0}\cong$ upper hemi sphere



Claim: each point z_0 of X admits a neighborhood W_{z_0} which is in 1-1 correspondence with a disc D in $\mathbb{C}$.

This is obvious for $z_0 \notin \mathbb{R}_{>0}$ (the cut) $\quad W_{z_0} = \{ z \mid |z - z_0| < \delta \}$

If z_0 is on the cut, we can just take 2 half discs in I and II to glue together.



Thus we can give $X = S^2 \setminus \{N, S\}$ coordinate charts by these W_{z_0}'s, $W_{z_0} \longrightarrow D$

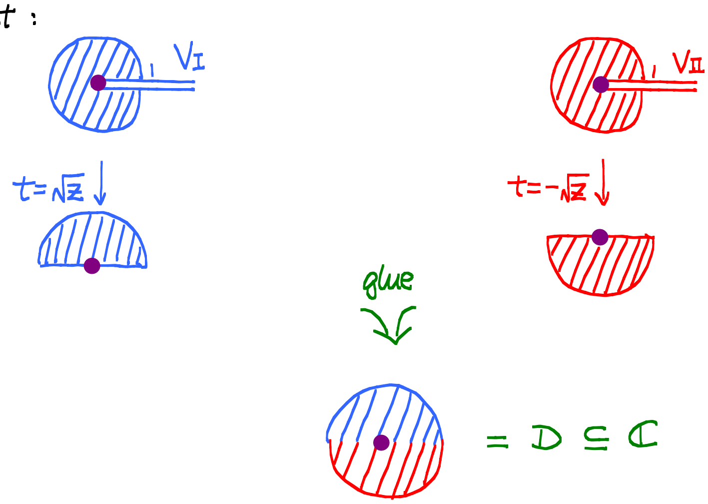
$z \mapsto t = z - z_0$

Def. Let f be a function on S. We say f is holomorphic if $\forall z_0 \in S$, $f|_{W_{z_0}}(z(t))$ is holomorphic as a function of t.

In this sense, $w(z) = \begin{cases} \sqrt{z}, & z \in \mathrm{I} \\ -\sqrt{z}, & z \in \mathrm{II} \end{cases}$ is holomorphic on X.

Claim: The point 0 (South pole) also admits a neighborhood V_0 in 1-1 correspondence with a disk D in $\mathbb{C}$.

In fact:



and plug in 0's of V_{I} and V_{II} to map to 0 of D

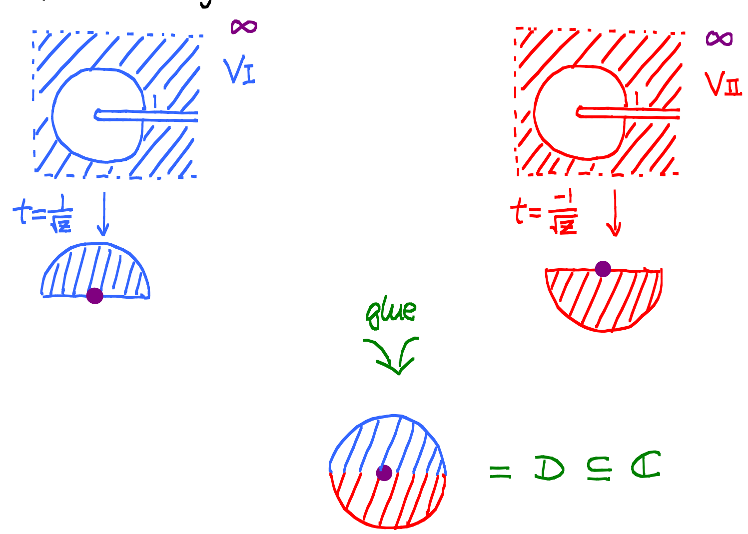
Now, let $\hat{X} = X \cup \{S, N\}$. A function f is said to be holomorphic near S if

$f|_{V_{\mathrm{I}} \cup V_{\mathrm{II}}}(z(t))$ is holomorphic in D.

Two natural examples are $\omega(z(t)) = t$ and $z(t) = t^2$, both holomorphic near S, and have zero's at S of order 1 and 2 respectively.

Near infinity of I and II, we introduce the coordinate $u = \frac{1}{z}$ and $u(\infty) = 0$.

Claim: ∞ (north pole) also admits a neighborhood V_∞ in 1-1 correspondence with the unit disk $D \subseteq \mathbb{C}$, and the local coordinate is given by $z \mapsto \frac{1}{\sqrt{z}} = t$, $z \in \mathrm{I}$ and $z \mapsto \frac{-1}{\sqrt{z}} = t$, $z \in \mathrm{II}$, and ∞ (of both I and II) $\mapsto 0$.



$\omega|_{V_\infty}(z(t)) = \frac{1}{t}$, simple pole at N, and $z|_{V_\infty}(t) = \frac{1}{t^2}$, double pole at $t=0$.

- In summary, the function $\omega = \sqrt{z}$ defined on $\mathbb{C} \setminus \mathbb{R}_+$ is extended to a meromorphic function on the space $\hat{X} = \mathrm{I} \sqcup \mathrm{II} \sqcup \{S, N\} \cong S^2$.

  The function ω has a simple 0 at 0, and a simple pole at ∞.

  The function z has a double 0 at 0, and a double pole at ∞.

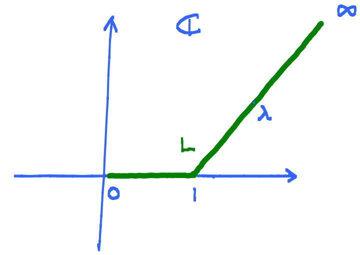
  Under the natural involution (switching) $\mathrm{I} \leftrightarrow \mathrm{II}$, the function z is even and the function ω is odd.

- Another Basic Example

$w^2 = z(z-1)(z-\lambda) \quad (\lambda \neq 0, 1)$

Problem: analytically continue $w = \sqrt{z(z-1)(z-\lambda)}$ defined on some domain of $\mathbb{C}$.

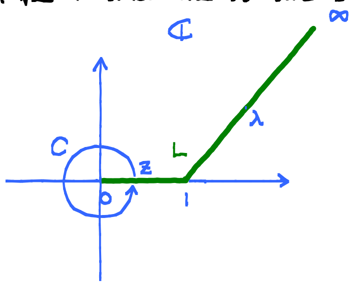
For w to be defined, it suffices $\sqrt{z}$, $\sqrt{z-1}$, $\sqrt{z-\lambda}$ be well-defined. Thus take the domain $\mathbb{C}\setminus L$, where L passes through $0, 1, \lambda$.



Clearly, $\sqrt{z}$, $\sqrt{z-1}$ and $\sqrt{z-\lambda}$ are all well defined and holomorphic on $\mathbb{C}\setminus L$.

Consider a point $z \in [0,1]$: $z \to e^{2\pi i}z$ on the circle C.

but $\sqrt{z-1}$ and $\sqrt{z-\lambda}$ are both well-defined, while $\sqrt{z} \xrightarrow{C} -\sqrt{z}$



$\Rightarrow$ $w \to -w = -\sqrt{z(z-1)(z-\lambda)}$ under C.

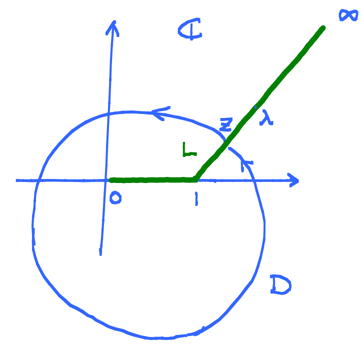
Next, Consider $z \in L_{1,\lambda}$

Similarly, we have

$\sqrt{z} \xrightarrow{D} -\sqrt{z}$

$\sqrt{z-1} \xrightarrow{D} -\sqrt{z-1}$

$\sqrt{z-\lambda} \xrightarrow{D} \sqrt{z-\lambda}$



$\Rightarrow$ $w \xrightarrow{D} w$ is invariant, continuous (holomorphic)

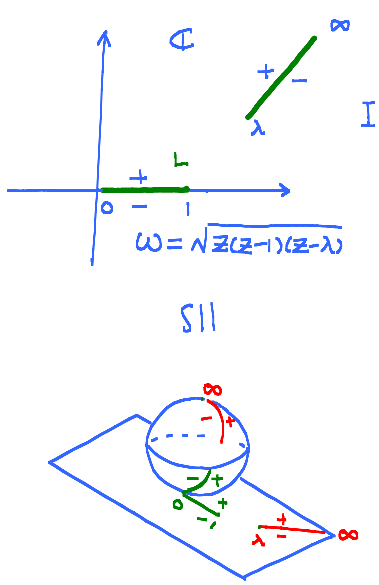
Finally, consider $z \in L_{\lambda\infty}$, and a similar loop.

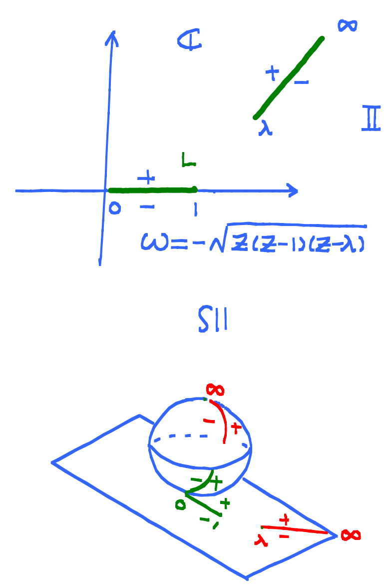
$\Rightarrow$ $\sqrt{z} \longrightarrow -\sqrt{z}$, $\sqrt{z-1} \longrightarrow -\sqrt{z-1}$, $\sqrt{z-\lambda} \longrightarrow -\sqrt{z-\lambda}$

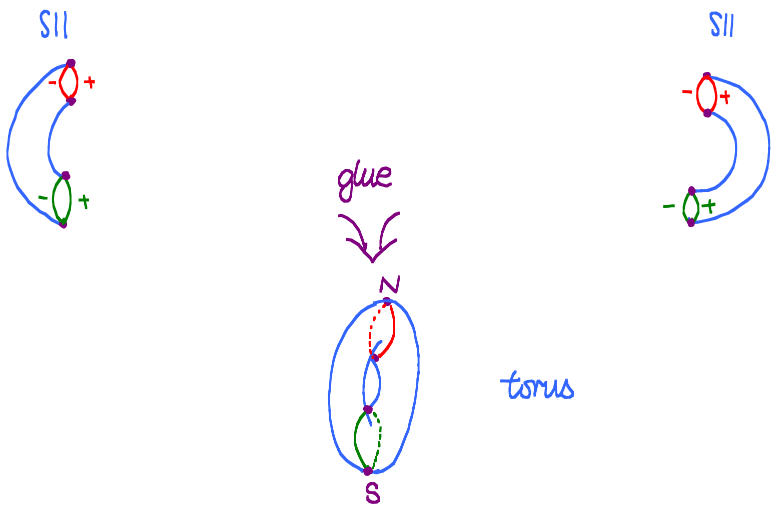
and $w \longrightarrow -w$ under the loop.

In summary, singularities occur only on $[0,1] \amalg L_{\lambda\infty}$.

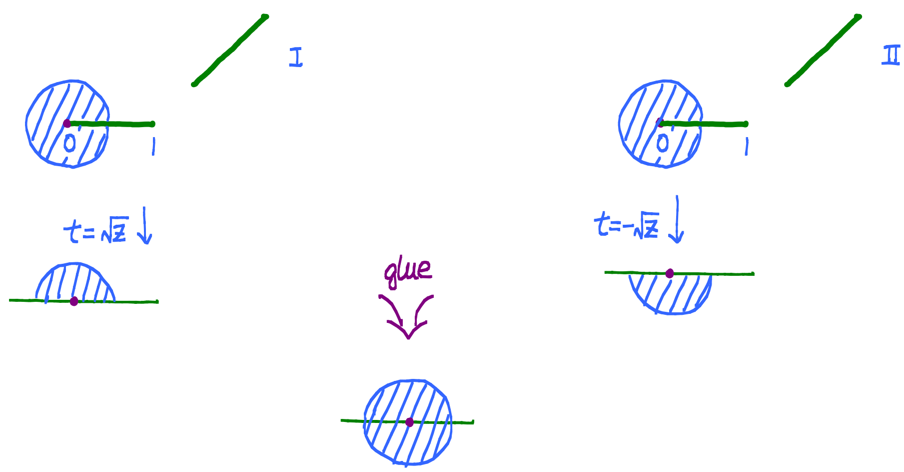
Thus $w = \pm\sqrt{z(z-1)(z-\lambda)}$ are holomorphic on $\mathbb{C}\setminus([0,1]\amalg L_{\lambda\infty})$. Take two copies and glue similarly as in the previous case.







We can construct explicitly local coordinates near the south pole S.



Claim: w is holomorphic near 0 (S). (similarly for 1 and λ)

$w = \sqrt{t^2(t^2-1)(t^2-\lambda)} = t\sqrt{(t^2-1)(t^2-\lambda)}$; for $|t-0| << 0$, $\sqrt{(t^2-1)(t^2-\lambda)}$ is holomorphic

$\Rightarrow$ w has a simple 0 at $t=0$

Similarly, near ∞ (N), introduce the holomorphic coordinate $t=\frac{1}{\sqrt{z}}$ on I and $t=-\frac{1}{\sqrt{z}}$ on II. $w = \sqrt{\frac{1}{t^2}(\frac{1}{t^2}-1)(\frac{1}{t^2}-\lambda)} = \frac{1}{t^3}\sqrt{(1-t^2)(1-\lambda t^2)}$; for $|t-0| << 0$ $\sqrt{(1-t^2)(1-\lambda t^2)}$ is holomorphic

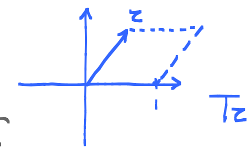
$\Rightarrow$ w is meromorphic with a pole of order 3.

In conclusion, w has been extended to a meromorphic function on T and w has 3 simple 0's and a pole of order 3.

The function z is also extended meromorphically, with an order 2 0 at S and an order 2 pole at N.

• Question: There are many tori with a complex structure (i.e. a notion of holomorphic or meromorphic function applies), for instance:

$T_\tau \triangleq \mathbb{C}/\mathbb{Z}\oplus\mathbb{Z}\tau$, $\tau\in\mathbb{C}$, $\operatorname{Im}\tau>0$



A function φ on T_τ can be identified with a function on $\mathbb{C}$ satisfying $\varphi(z+m+n\tau)=\varphi(z)$, $\forall m,n\in\mathbb{Z}$, and φ on T_τ is holomorphic (meromorphic) if the corresponding φ on $\mathbb{C}$ is holomorphic (meromorphic).

Thus we need to see if X can be identified with T_τ for some τ, and if yes how do we determine such a τ?

• Key strategy: There are very few (in fact, only constants, by maximal modulus theorem) holomorphic functions on $\hat{X}$. Rather we work instead with

(1). holomorphic forms

and/or (2). meromorphic functions

And then proceed in the following 3 steps.

Step 1: Construct these objects (← explicit forms)

Step 2: Develop techniques for manipulating them (← Riemann bilinear relations and abelian integrals)

Step 3: Use them to construct a holomorphic map $\hat{X}\longrightarrow\mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau$ (← The Abel map: Jacobi inversion thm, based on Abel's thm.)

I. Construction of holomorphic differentials.

Recall that $\hat{X}$ has two meromorphic functions on $\hat{X}$, namely $z:\hat{X}\longrightarrow\mathbb{C}\cup\{\infty\}$, $p\longmapsto z(p)$ and
$$w(p)=\begin{cases}\sqrt{z(z-1)(z-\lambda)} & \text{on I}\\ -\sqrt{z(z-1)(z-\lambda)} & \text{on II}\end{cases}$$

Consider the differential "$\frac{dz}{w}$" on $\hat{X}$

Key observation: "$\frac{dz}{w}$" is actually a holomorphic differential form on all of $\hat{X}$

To see this, express $\frac{dz}{w}$ in a local coordinate system.

$\frac{dz}{w}=\frac{dz}{\sqrt{z(z-1)(z-\lambda)}}$. Near $z=0$, recall that the holomorphic coordinate is given by $t=\sqrt{z}\Rightarrow z=t^2$, thus $dz=2t\,dt$

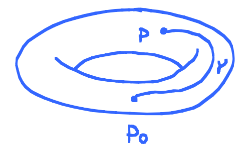
$\Rightarrow\frac{dz}{w}=\frac{2t\,dt}{\sqrt{t^2(t-1)(t-\lambda)}}=\frac{2dt}{\sqrt{(t-1)(t-\lambda)}}$, which is holomorphic for $|t|\ll1$.

The same reasoning applies to $z_0 = 1, \lambda$. Near $z_0 = \infty$, the holomorphic coordinate is given by $t = \frac{1}{\sqrt{z}} \Rightarrow z = \frac{1}{t^2}$, $dz = -2\frac{dt}{t^3}$

Thus $\frac{dz}{w} = \frac{-2dt/t^2}{\sqrt{\frac{1}{t^2}(\frac{1}{t^2}-1)(\frac{1}{t^2}-\lambda)}} = \frac{-2dt/t^2}{\frac{1}{t^2}\sqrt{(1-t^2)(1-\lambda t^2)}} = \frac{-2dt}{\sqrt{(1-t^2)(1-\lambda t^2)}}$, which is again holomorphic for $|t| \ll 1$.

Using this form $\omega = \frac{dz}{w}$, we can construct a map from $\hat{X}$ to $\mathbb{C}/\text{lattice}$ in the following way:

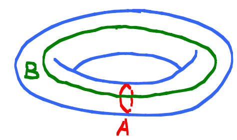
(The Abel map) Fix $P_0 \in \hat{X}$, $\forall P \in \hat{X}$.



Consider $\int_{P_0}^{P} \omega$, integration along γ. If γ can be continuously deformed to γ', then $\int_\gamma \omega = \int_{\gamma'} \omega$.

Pick A, B two cycles on $\hat{X}$

(A topological fact): Any two γ, γ' connecting P_0 and P have $\gamma - \gamma' \simeq nA + mB$, $n, m \in \mathbb{Z}$



$\Rightarrow \int_\gamma \omega = \int_{\gamma'} \omega + n\oint_A \omega + m\oint_B \omega$, $n, m \in \mathbb{Z}$.

Thus we may view $\int_\gamma \omega$ as an equivalence class in $\mathbb{C}/\mathbb{Z}\alpha + \mathbb{Z}\beta$ rather than a complex number, where $\alpha = \oint_A \omega$, $\beta = \oint_B \omega$

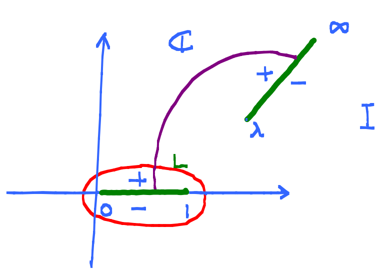
We shall later show that $\alpha \neq 0$, $\beta \neq 0$, and then we can normalize ω and define $\hat{\omega} = \frac{\omega}{\oint_A \omega} \Rightarrow \oint_A \hat{\omega} = 1$, and $\tau \triangleq \oint_B \hat{\omega}$. The Abel map is then defined as:

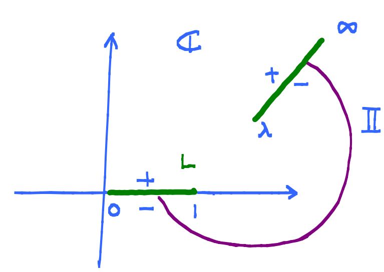
$$\hat{X} \longrightarrow \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau. \quad P \longmapsto A(P) \triangleq [\int_{P_0}^{P} \hat{\omega}]$$

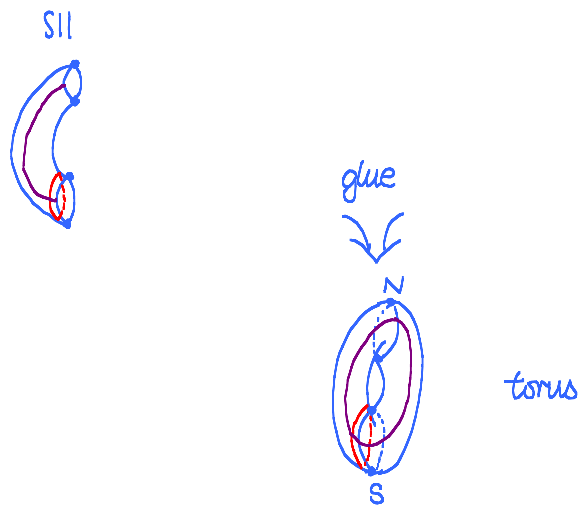
Furthermore, we will also show that $\operatorname{Im}\tau > 0$, so that $\mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau \cong T^2$

Thm. (Jacobi Inversion Thm.) The Abel map is holomorphic from $\hat{X}$ to $\mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$, and is 1-1 and onto.

Pf: We may choose the cycles A and B as follows:







$\Rightarrow \alpha = 2\int_0^1 \frac{dx}{\sqrt{x(x-1)(x-\lambda)}}$. We will show later that $\tau \notin \mathbb{R}$.

The Abel map is holomorphic since $dA(P) = \omega(P)$. Furthermore ω is never 0 $\Rightarrow$ the Abel map is locally an isomorphism. The open mapping thm $\Rightarrow$ the image of A is open ; while $\hat{X}$ compact $\Rightarrow$ the image is closed. Thus we conclude that A maps $\hat{X}$ holomorphically onto $\mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau$.

To show that it is 1-1, we shall use Abel's thm, to be proved below, which states that $\forall$ $P, Q \in \hat{X}$. $A(P) = A(Q)$ iff there exists a meromorphic function f, with a simple zero at P and a simple pole at Q.

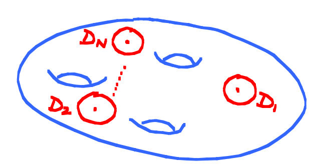
Thus if $P \neq Q$ but $A(P) = A(Q)$ we would obtain a meromorphic function f with a simple pole at Q. Consider the meromorphic differential $f\frac{dz}{\omega}$, which has a simple pole at Q, since $\frac{dz}{\omega}$ is globally holomorphic and non-vanishing. Thus the residue of $f\frac{dz}{\omega}$ is then non-zero at Q. which violates the following lemma.

$\square$

Lemma. A meromorphic differential ω on a compact Riemann surface has

$$\sum_{P:\text{poles}} (\operatorname{Res}\omega)(P) = 0$$

Pf: Let $P_1, \cdots, P_k$ be all the poles of ω. Then



$$\begin{aligned}\sum_{i=1}^{k} (\operatorname{Res}\omega)(P_i) &= \sum_{i=1}^{k} \oint_{\partial D_i} \omega = -\int_{\partial(X\setminus \bigcup_{i=1}^{k} D_i)} \omega \\ &= -\int_{X\setminus \bigcup_{i=1}^{k} D_i} d\omega \\ &= 0\end{aligned}$$

The last equality holds since ω is a holomorphic differential form on $X\setminus \bigcup_{i=1}^{k} D_i$ $\square$

Thm. (Abel) Let $P_1, \cdots, P_M$, $Q_1, \cdots, Q_N$ be points on $\hat{X}$, counted with multiplicity. Then $\exists f$ on $\hat{X}$, f meromorphic with 0's at P_i and poles at Q_j $\Leftrightarrow$ $M = N$ and $\sum_{i=1}^{M} A(P_i) = \sum_{j=1}^{N} A(Q_j)$, the addition being induced from the group addition of $\mathbb{C}$.

Sketch of proof of Abel's theorem.

- Idea: try to construct / identify f from $\frac{df}{f}$, which is easier to deal with since its residues are always ± 1: $+1$ for a pole of f, -1 for a zero of f, both to be counted with multiplicity.

The basic building block is the so called Abelian differential of the 3rd kind.

Lemma: $\forall\, Q_1 \neq Q_2$ on $\hat{X}$, $\exists$ a meromorphic form $\omega_{Q_1Q_2}(P)$ with simple poles at exactly Q_1 and Q_2 and residues $-1, +1$ respectively.

Assuming this lemma, a candidate of $\frac{df}{f}$ would be

$$\frac{df}{f} \overset{?}{=} \underbrace{\sum_{i=1}^{N} \omega_{P_0P_i}}_{\substack{\text{with a simple}\\ \text{pole at each } P_i\\ \text{whose residue is } 1}} - \underbrace{\sum_{i=1}^{N} \omega_{P_0Q_i}}_{\substack{\text{with a simple}\\ \text{pole at each } Q_i,\\ \text{whose residue is } -1}} \quad (\text{Fixing } P_0)$$

no poles at P_0 since they all got canceled

Then $\frac{df}{f} - \sum_{i=1}^{N} \omega_{P_0P_i} - \sum_{i=1}^{N} \omega_{P_0Q_i} = c \cdot \frac{dz}{w}$ for some constant c.

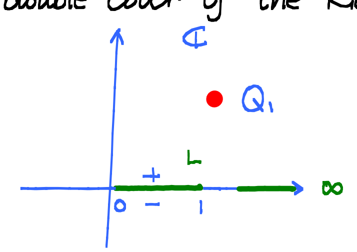
In case we know $\frac{df}{f}$, we may reconstruct f by formally:

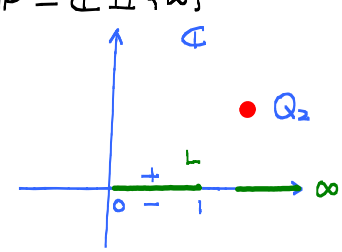
$$f \text{ "="} \exp\left(\int^{z} \frac{df}{f}\right)$$

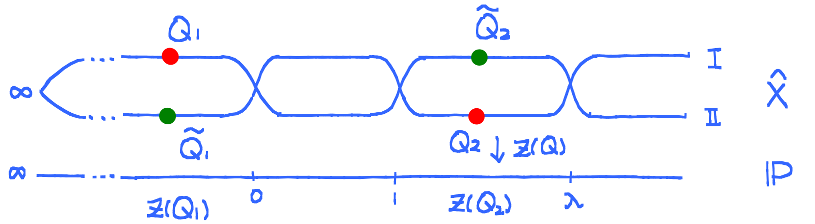
where we have to make sure that the right hand side is well-defined. This will be the part where $\sum_{i=1}^{N} A(P_i) = \sum_{j=1}^{N} A(Q_j)$ becomes a necessary and sufficient condition.

Existence of $\omega_{Q_1Q_2}$ (proof of lemma)

Now we come back to the realization of $\hat{X}$ as $w^2 = z(z-1)(z-\lambda)$, i.e. a ramified double cover of the Riemann sphere $S^2 \cong \mathbb{P}^1 = \mathbb{C} \sqcup \{\infty\}$







Now given Q_1, Q_2 on $\hat{X}$, we try to construct $\omega_{Q_1Q_2}$. Firstly we assume that $z(Q_1) \neq z(Q_2)$. Consider:

$$\omega_{Q_1Q_2} \overset{?}{=} \frac{1}{(z-z(Q_1))(z-z(Q_2))} \frac{dz}{\omega}$$

But this actually doesn't work since $z(\widetilde{Q}_1) = z(Q_1)$, $z(\widetilde{Q}_2) = z(Q_2)$, there will be 2 more poles than we actually need. Hence we can try to multiply a function to kill these poles off. Take:

$$\omega_{Q_1Q_2} = \frac{\alpha z + \beta + w}{(z-z(Q_1))(z-z(Q_2))} \frac{dz}{w}$$

(w must be involved since z is an "even" function, wherever Q_1 occurs, $\widetilde{Q}_1$ would be of the same value for z).

Thus we want: $\alpha z + \beta + w = 0$, $z = z(Q_1)$, $w = w(\widetilde{Q}_1)$

$\alpha z + \beta + w = 0$, $z = z(Q_2)$, $w = w(\widetilde{Q}_2)$

For this equation to have a solution, it suffices that $\begin{vmatrix} z(Q_1) & 1 \\ z(Q_2) & 1 \end{vmatrix} \neq 0$, which is automatic since we assumed $z(Q_1) \neq z(Q_2)$ from the outset. Moreover, this function doesn't create new poles at ∞ since $|z| = O(|z|)$ $|w| = O(|z|^{\frac{3}{2}})$ as $|z| \to \infty$ and the denominator is $O(|z|^2)$ as $z \longrightarrow \infty$

If $z(Q_1) = z(Q_2)$, it suffices to consider $\frac{1}{z - z(Q_1)} \frac{dz}{w}$

If one of Q_i, say, Q_1, is ∞, then it suffices to take $\frac{w - w(\widetilde{Q}_2)}{z - z(Q_2)} \frac{dz}{w}$

Furthermore, the above expressions even make sense if one of Q_1, Q_2, or both are the ramification points. In fact, the numerator is solved to be $\frac{w_1 - w_2}{z_2 - z_1} z + \frac{z_2 w_1 - z_1 w_2}{z_2 - z_1} + w$ ($w_1 = w(\widetilde{Q}_1)$, $z_1 = z(Q_1)$, $w_2 = w(\widetilde{Q}_2)$, $z_2 = z(Q_2)$). If $w_1 = 0$, this just gives $\frac{w_2(z - z_1)}{z_1 - z_2} + w$, but $(z - z_1)$ is a zero of order 2 at the ramification point while w is a 0 of order 1 at z_1. Similarly if both are ramification points, then the numerator just gives w, which vanishes to order 1 at Q_1 and Q_2.

We may also construct meromorphic forms with a double pole at any $Q \in \hat{X}$.

If $Q \neq 0, 1, \lambda$ or ∞, then $w(\widetilde{Q}) = -w(Q) \neq 0$ we may try $\frac{\alpha z + \beta w + \nu}{(z - z(Q))^2} \frac{dz}{w}$ such that $\alpha z + \beta w + \nu$ vanishes to order 2 at $\widetilde{Q}$, where $\widetilde{Q}, Q$ are the two distinct points on $\hat{X}$ lying over $z(Q)$. Consider:

$$\alpha z+\beta w+\nu = \big((z(Q)-1)(z(Q)-\lambda)+z(Q)(z(Q)-1)+z(Q)(z(Q)-\lambda)\big)(z-z(Q)) - 2w(\widetilde{Q})(w-w(\widetilde{Q}))$$

We claim that the above expression vanishes to order ≥ 2 at $\widetilde{Q}$.

Indeed $(w-w(\widetilde{Q})+w(\widetilde{Q}))^2 = z(z-1)(z-\lambda) = (z-z(Q)+z(Q))(z-z(Q)+z(Q)-1)(z-z(Q)+z(Q)-\lambda)$

$$\begin{aligned}(w-w(\widetilde{Q}))^2+2w(\widetilde{Q})(w-w(\widetilde{Q}))+w(\widetilde{Q})^2 = {} & z(Q)(z(Q)-1)(z(Q)-\lambda)\\ & +\big((z(Q)-1)(z(Q)-\lambda)+z(Q)(z(Q)-1)+z(Q)(z(Q)-\lambda)\big)(z-z(Q))\\ & +(z(Q)+z(Q)-1+z(Q)-\lambda)(z-z(Q))^2\\ & +(z-z(Q))^3\end{aligned}$$

(note that $z(Q)=z(\widetilde{Q})$, $w(\widetilde{Q})^2 = z(\widetilde{Q})(z(\widetilde{Q})-1)(z(\widetilde{Q})-\lambda)$)

$$\begin{aligned}\Longrightarrow \ & \big((z(Q)-1)(z(Q)-\lambda)+z(Q)(z(Q)-1)+z(Q)(z(Q)-\lambda)\big)(z-z(Q)) - 2w(\widetilde{Q})w\\ & = (w-w(\widetilde{Q}))^2 - (z(Q)+z(Q)-1+z(Q)-\lambda)(z-z(Q))^2-(z-z(Q))^3\\ & = O(|z-z(\widetilde{Q})|^2)\end{aligned}$$

since $w-w(\widetilde{Q})$ is locally a holomorphic function in $z-z(Q)$, i.e. $w-w(\widetilde{Q})=(z-z(Q))\cdot h(z-z(Q))$.

Moreover, since in our case $w(\widetilde{Q})=-w(Q)\neq 0$,

$$\alpha z+\beta w+\nu \,\big|_{(z(Q),\,w(Q))} = -4w(\widetilde{Q})^2 \neq 0$$

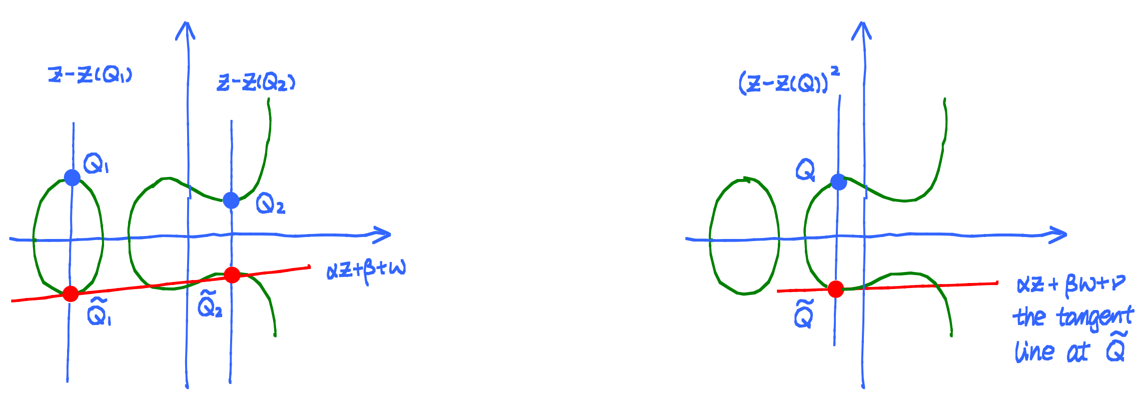
and $O(|\alpha z+\beta w+\nu|) = O(|z|^{\frac{3}{2}})$ $(|z|\to\infty)$, while $O(|z-z(Q)|^2) = O(|z|^2)$ $(|z|\to\infty)$.

It follows that $\frac{\alpha z+\beta+\nu}{(z-z(Q))^2}\frac{dz}{w}$ satisfies the required conditions.

If $z(Q)=0,1,$ or λ, it suffices to take $\frac{1}{z-z(Q)}\frac{dz}{w}$, since $(z-z(Q))$ vanishes to order 2 at these ramification points.

If $z(Q)=\infty$, it suffices to take $z\frac{dz}{w}$, since z has an order 2 pole at ∞.

In general, the above constructions can be visualized by considering the following picture of tori (as elliptic curves)

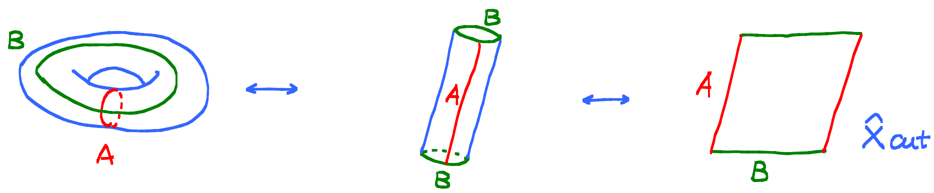


For the next step, we need to deal with $\int^{z} \omega_{P_0P_i}$ and $\int^{z} \omega$, and some computation needs to be done.

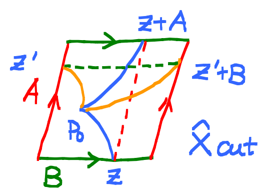
Method of abelian integral (Riemann)

Pick some fixed A, B cycles on $\hat{X}$, $P_0 \in \hat{X}$

$$f(z) \triangleq \int_{P_0}^{z} \omega \quad (\mathrm{I})$$



On the right hand side picture, $f(z)$ is well-defined on $\hat{X}_{cut}$ since $\hat{X}_{cut}$ is simply connected. However, it may not be doubly periodic.



i.e. $f(z) \neq f(z+A)$ and $f(z) \neq f(z+B)$. Thus it may not correspond to a function on $\hat{X}$. However, the differences are under control:

$$f(z+A) - f(z) = \int_{z}^{z+A} \omega = \oint_A \omega$$

$$f(z'+B) - f(z') = \int_{z'}^{z+B} \omega = \oint_B \omega$$

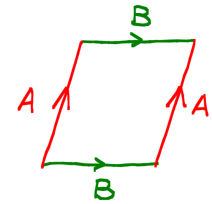
Example of using abelian integral

Let f be the abelian integral (I). Consider $\frac{i}{2}\int_{\hat{X}} \omega \wedge \hat{\omega}$. In local coordinate systems we may write $\omega = \psi(t)dt$. Then $\omega \wedge \bar{\omega} = |\psi(t)|^2 dt \wedge d\bar{t} = |\psi(t)|^2(-2i\,dt_1 \wedge dt_2)$ $\quad t = t_1 + it_2$

$\Rightarrow \frac{i}{2}\omega\wedge\bar{\omega} = |\psi(t)|^2 dt_1 \wedge dt_2$, thus $\frac{i}{2}\int_{\hat{X}} \omega \wedge \hat{\omega} > 0$.

Moreover, on $\hat{X}_{cut}$, $0 < \frac{i}{2}\int_{\hat{X}} \omega\wedge\bar{\omega} = \frac{i}{2}\int_{\hat{X}_{cut}} d(f(t)\cdot\bar{\omega})$ (The last equality holds because $d(f(z)\bar{\omega}) = (df(z))\cdot\bar{\omega} + f(z)d\bar{\omega} = (\frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial \bar{t}}d\bar{t})\psi(t)dt + f(t)(\frac{\partial \bar{\psi}}{\partial t}dt + \frac{\partial \bar{\psi}}{\partial \bar{t}})d\bar{t}$ and $\frac{\partial f}{\partial \bar{t}} = \frac{\partial \bar{f}}{\partial t} = 0$ since f is holomorphic; or it can be seen topologically since $d\omega = 0$ and $\omega|\hat{X}_{cut} = df$ since $\hat{X}_{cut}$ is contractable)

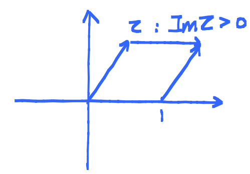
By Stokes' thm:



$$\begin{aligned}
0 < \int_{\hat{X}} \omega\wedge\hat{\omega} &= \tfrac{i}{2}\int_{\hat{X}_{cut}} d(f(t)\bar{\omega}) = \tfrac{i}{2}\int_{\partial\hat{X}_{cut}} f(t)\bar{\omega}\\
&= \tfrac{i}{2}\Big\{\int_A f(z)\bar{\omega}(z) - \int_A f(z+B)\bar{\omega}(z+B)\\
&\qquad + \int_B f(z+A)\bar{\omega}(z+A) - \int_B f(z)\bar{\omega}(z)\Big\}\\
&= \tfrac{i}{2}\Big\{-\oint_A \bar{\omega}\oint_B \omega + \oint_B \bar{\omega}\oint_A \omega\Big\}\\
&= \mathrm{Im}\Big(\oint_A \bar{\omega}\oint_B \omega\Big)
\end{aligned}$$

The second last equality holds because $\omega, \bar{\omega}$ are both defined on $\hat{X}$, thus must agree on boundaries of $\hat{X}_{cut}$, and $f(z+A)-f(z) = \oint_A \omega$, $f(z+B)-f(z) = \oint_B \omega$

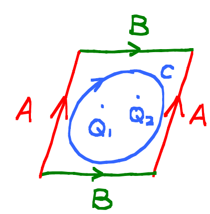
If we normalize $\oint_A \omega = 1 \Rightarrow \mathrm{Im}\int_B \omega > 0$, proving that the image of the Abel map is a genuine torus.



Example of method of abelian differential

Let $\omega_{Q_1Q_2}(z)$ be a meromorphic differential with simple poles at Q_1, Q_2, whose residues are $+1$ at Q_2 and -1 at Q_1. Fix cycles A, B ($\not\ni Q_1, Q_2$) and normalize $\omega_{Q_1Q_2}$ so that $\oint_A \omega_{Q_1Q_2} = 0$. This can be done since we have assumed $\oint_A \omega = 1$, and substracting $\oint_A \omega_{Q_1Q_2}\cdot\omega$ from $\omega_{Q_1Q_2}$ won't affect its poles.

Consider the integral $\oint_C f\cdot\omega_{Q_1Q_2}$:



Deforming C towards Q_1 and Q_2 gives:

$$\oint_C f\omega_{Q_1Q_2} = 2\pi i\sum_{Q_1,Q_2} \mathrm{Res}(f\cdot\omega_{Q_1Q_2})(Q_i) = 2\pi i(f(Q_2)-f(Q_1))$$

$$(\equiv 2\pi i\,(A(Q_2)-A(Q_1)) \mod (\mathbb{Z}+\mathbb{Z}\tau)\,)$$

Deforming C towards $\partial\hat{X}_{cut}$ gives:

$$\begin{aligned}
\oint_C f\omega_{Q_1Q_2} &= \int_A (f(z)-f(z+B))\omega_{Q_1Q_2} - \int_B (f(z)-f(z+A))\omega_{Q_1Q_2}\\
&= -\oint_A \omega_{Q_1Q_2}\oint_B \omega + \oint_B \omega_{Q_1Q_2}\oint_A \omega\\
&= \oint_B \omega_{Q_1Q_2} \quad \text{since we normalized } \oint_A \omega_{Q_1Q_2} = 0,\ \oint_A \omega = 1.
\end{aligned}$$

$$\Rightarrow \frac{1}{2\pi i}\oint_B \omega_{Q_1Q_2} = f(Q_2)-f(Q_1)$$

Proof of Abel's thm.

"$\Rightarrow$" Assume a meromorphic function φ with the desired property exists. Then $\frac{d\varphi}{\varphi}$ is a meromorphic form $\Rightarrow 0 = \sum \mathrm{Res}(\frac{d\varphi}{\varphi}) = M-N$.

To see the other property, observe that $\frac{d\varphi}{\varphi} = \sum_{i=1}^{N} \omega_{P_0P_i} - \sum_{j=1}^{N} \omega_{P_0Q_j} + \lambda\omega$ for some $\lambda \in \mathbb{C}$. This is because:

Lemma: $\tilde{\omega}$ is holomorphic on $\hat{X} \Rightarrow \tilde{\omega} = \lambda\omega$.

Pf: First of all, we may substract $(\oint_A \tilde{\omega})\cdot\omega$ from $\tilde{\omega}$ and assume that $\oint_A \tilde{\omega} = 0$

Consider the abelian integral of $\tilde{\omega}$, we obtain:

$$0 \leqslant \frac{i}{2}\int_{\hat{X}_{cut}} \tilde{\omega}\wedge\overline{\tilde{\omega}} = \operatorname{Im}(\oint_A \tilde{\omega} \cdot \overline{\oint_B \tilde{\omega}})$$

But $\oint_A \tilde{\omega} = 0 \Rightarrow \frac{i}{2}\int_{\hat{X}_{cut}} \tilde{\omega}\wedge\overline{\tilde{\omega}} = 0$. Locally, $\tilde{\omega} = \tilde{\varphi}(t)dt \Rightarrow \frac{i}{2}\tilde{\omega}\wedge\overline{\tilde{\omega}} = |\tilde{\varphi}(t)|^2 dt_1\wedge dt_2$

Thus $\varphi(t) \equiv 0 \Rightarrow \tilde{\omega} \equiv 0$. $\square$ of lemma.

Now we can evaluate the periods of $\oint_C \frac{d\varphi}{\varphi}$. This must be $2\pi i \cdot n$ for some $n \in \mathbb{Z}$ since anyway it's to calculate the residues within (or outside, which doesn't matter) C

$$2\pi i n = \oint_A \frac{d\varphi}{\varphi} = \oint_A \sum_{i=1}^{N}(\omega_{P_0P_i} - \omega_{P_0Q_i}) + \oint_A c\cdot\omega = c\cdot\oint_A \omega = c$$

since we have assumed that $\oint_A \omega_{P_0P_i} = 0$. On the other hand:

$$2\pi i m = \oint_B \frac{d\varphi}{\varphi} = \oint_B \sum_{i=1}^{N}(\omega_{P_0P_i} - \omega_{P_0Q_i}) + \oint_B c\cdot\omega$$

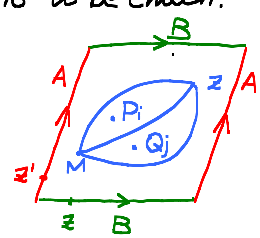
$$= 2\pi i \cdot \sum_{i=1}^{N}(f(P_i) - f(Q_i)) + c\tau$$

$$\Rightarrow \sum_{i=1}^{N} f(Q_i) = \sum_{i=1}^{N} f(P_i) + (n + m\tau) \equiv \sum_{i=1}^{N} f(Q_i) \mod(\mathbb{Z} + \mathbb{Z}\tau)$$

"$\Leftarrow$" Assume Abel's relation holds, i.e. $\sum_{i=1}^{N} f(P_i) = \sum_{j=1}^{N} f(Q_j) + n + m\tau$.

Define $\varphi(z) = \exp(\int_M^z (\sum_{j=1}^{N}(\omega_{P_0P_j} - \omega_{P_0Q_j}) + c\omega))$, where c is to be chosen.

For φ to be a well-defined meromorphic function on $\hat{X}$, it's sufficient and necessary that:



(1). $\varphi(z)$ is independent of choices of path.

(2). $\varphi(z+A) = \varphi(z)$

(3). $\varphi(z+B) = \varphi(z)$

(1) is easy since choosing different path amounts to adding $2\pi i\cdot$(residues of P_i, Q_j's) which is an integral multiple of $2\pi i$ and doesn't affect the value of φ.

(2): we have assumed that $\oint_A \omega_{P_0P_i} = 0$ and $\oint_A \omega = 1$ thus it sufficient if $c = 2\pi i \cdot k$, $k \in \mathbb{Z}$

(3): $\sum_{j=1}^{N} \oint_B (\omega_{P_0P_i} - \omega_{P_0Q_i}) = 2\pi i \cdot \sum_{j=1}^{N} (f(P_i) - f(Q_j))$, $\oint_B \omega = \tau$. By our assumption $2\pi i \sum_{j=1}^{N} (f(P_j) - f(Q_j)) = 2\pi i n + 2\pi i m \tau$. Choose $c = 2\pi i m$ and condition (3) and (2) are both satisfied. $\square$

§3. Function Theory on Tori

We have seen that on the R.S. $\hat{X}$: $w^2 = z(z-1)(z-\lambda)$, there are

(1) A holomorphic form $\omega = \frac{dz}{\sqrt{z(z-1)(z-\lambda)}}$

(2) $\forall\, P, Q \in \hat{X}$, $P \neq Q$, $\exists$ a meromorphic form ω_{PQ} with poles at P, Q.

(3) $\forall\, P \in \hat{X}$, $\exists$ a meromorphic form ω_P with a double pole at P.

We have also shown that $\forall\, P \in \hat{X}$, $P \mapsto A(P) = \int_{P_0}^{P} \omega \in \mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau$, where
$\tau = \left(\int_0^\lambda \frac{dz}{\sqrt{z(z-1)(z-\lambda)}} \right) / \left(\int_0^1 \frac{dz}{\sqrt{z(z-1)(z-\lambda)}} \right)$

Several View Points

1). Abelian Integral (done in §2 already)

2). Weierstrass function $\wp(z)$, $\zeta(z)$, $\sigma(z)$

3). Jacobi theta function

4). $\bar{\partial}$- construction (P.D.E.)

• Function Theory According To Weierstrass

Let $\mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau$ be a given complex torus, $\operatorname{im}\tau > 0$. A function φ on $\mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau$ is simply a function on $\mathbb{C}$ satisfying a doublely periodic condition:

$$\begin{cases} \varphi(z+1) = \varphi(z) \\ \varphi(z+\tau) = \varphi(z) \end{cases}$$

A holomorphic 1-form : $z \mapsto z+m+n\tau$ by $\mathbb{Z}\oplus\mathbb{Z}$ action. But dz is invariant under this action $\Rightarrow$ dz is a holomorphic 1-form on $\hat{X}$.

Suppose $P=0$, we want to construct a meromorphic form with a double pole at P. In view of the form dz, we can identify forms and functions, which is possible because dz has neither 0's nor poles, $f(z) \longleftrightarrow f(z)dz$. Thus we need to construct a function with exactly a double pole at 0 and which is doublely periodic :

Our first try would be to average out $\frac{1}{z^2}$ over the lattice $L \triangleq \mathbb{Z}+\mathbb{Z}\tau$

$\sum_{\omega\in\mathcal{L}} \frac{1}{(z+\omega)^2}$, but unfortunately it doesn't converge. ($\iint\cdots\int_{\mathbb{R}^n} \frac{dx}{1+\|x\|^p} < \infty$ if $p>n$)

Define $\wp(z) \triangleq \frac{1}{z^2} + \sum_{\omega\in\mathcal{L}^*} \left\{ \frac{1}{(z+\omega)^2} - \frac{1}{\omega^2} \right\}$ $\mathcal{L}^* = \mathcal{L}\setminus\{0\}$. Note that when $|\omega|\to\infty$, $\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2} = \frac{1}{\omega^2}\left(\frac{1}{(1+\frac{z}{\omega})^2} - 1\right) = O(\frac{1}{|\omega|})$, $\forall z$ fixed. Thus the sum converges.

From the discussion above, the series converges for any z and defines a meromorphic function with poles on $\mathcal{L}$.

Claim: $\wp(z)$ is doublely periodic.

Indeed, we can compute that $\wp'(z) = -\frac{2}{z^3} - \sum_{\omega\in\mathcal{L}^*} \frac{2}{(z+\omega)^3} = -2\sum_{\omega\in\mathcal{L}} \frac{1}{(z+\omega)^3}$

$\Rightarrow$ $\wp'(z)$ is doublely periodic : $\wp'(z+1) = \wp'(z)$, $\wp'(z+\tau) = \wp'(z)$.

But this implies $\frac{d}{dz}(\wp(z+1) - \wp(z)) = 0$, $\frac{d}{dz}(\wp(z+\tau) - \wp(z)) = 0$.

$\Rightarrow$ $\wp(z+1) = \wp(z) + C_1$, $\wp(z+\tau) = \wp(z) + C_2$.

Moreover, since $\wp(z)$ is even (the lattice is symmetric w.r.t. 0), taking z to be $-\frac{1}{2}$, $-\frac{\tau}{2}$ respectively gives $\wp(\frac{1}{2}) = \wp(-\frac{1}{2}) + C_1$, $\wp(\frac{\tau}{2}) = \wp(-\frac{\tau}{2}) + C_2 \Rightarrow C_1 = C_2 = 0$.

Observe that on $\mathbb{C}$ we have the function z which is holomorphic with a simple zero. Thus functions with given zero's and poles can immediately be written as $f(z) = \frac{\prod(z-p_i)}{\prod(z-q_j)}$.

Is there an analogue for such a function on the torus? i.e. a function with a single 0 ? Not true by maximal modulus principle. But there is an adequete replacement.

Idea: 1). Integrate $\wp(z)$ twice to get a $\log z$ and take exponential

2). Integrals give rise to Abelian integrals, so we need to keep track of the periods: $(\varphi(z+1) - \varphi(z),\ \varphi(z+\tau) - \varphi(z))$.

Integral of $\wp(z)$.

Consider $\frac{1}{z} + \sum_{\omega\in\mathcal{L}^*} \left\{ \frac{1}{z+\omega} + \frac{z}{\omega^2} \right\}$, which is the formal integration of $-\wp(z)$. But again this has convergence problems. How to correct this?

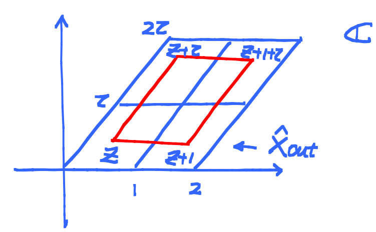
$$\frac{1}{z+w} = \frac{1}{w}\cdot\frac{1}{(1+\frac{z}{w})} = \frac{1}{w}\left(1-\frac{z}{w}+\frac{z^2}{w^2}-\cdots\right) \Rightarrow \frac{1}{z+w}+\frac{z}{w^2} = \frac{1}{w} + \underbrace{O\left(\frac{1}{|w|^3}\right)}_{\text{converges}}.$$

Define $\zeta(z) = \frac{1}{z} + \sum_{w\in\mathcal{L}^*}\left\{\frac{1}{z+w} - \frac{1}{w} + \frac{z}{w^2}\right\}$, which is meromorphic on the whole plane and with simple poles at $\mathcal{L}$.

Clearly, $\zeta'(z) = -\wp(z)$ (we are just substracting constants from the formal anti-derivative of $\wp(z)$). We need to determine $\zeta(z+1)-\zeta(z)$, $\zeta(z+\tau)-\zeta(z)$

But $\frac{d}{dz}(\zeta(z+1)-\zeta(z)) = -\wp(z+1)+\wp(z) = 0 \Rightarrow \zeta(z+1) = \zeta(z)+\eta_1$

Similarly $\zeta(z+\tau) = \zeta(z) + \eta_2$



Lemma: $\tau\eta_1 - \eta_2 = 2\pi i$

Pf: $\int_{\partial\hat{\gamma}_{out}} \zeta(z)dz = 2\pi i\cdot \operatorname{Res}\zeta(z) = 2\pi i$

On the other hand:

$$\begin{aligned}\int_{\partial\hat{\gamma}_{out}}\zeta(z)dz &= \int_z^{z+1}\zeta(z)dz + \int_{z+1}^{z+1+\tau}\zeta(z)dz \\ &\quad - \int_{z+\tau}^{z+1+\tau}\zeta(z)dz - \int_z^{z+\tau}\zeta(z)dz \\ &= \int_z^{z+1}(\zeta(z)-\zeta(z+\tau))dz + \int_z^{z+\tau}(\zeta(z+1)-\zeta(z))dz \\ &= -\eta_2 + \tau\eta_1\end{aligned}$$

$\square$

Next, we integrate $\zeta(z)$ and take exponential, thus the resulting $2\pi i n$'s from integrating $\zeta(z)$ won't affect the result:

$$e^{\int\zeta(z)dz} = z\prod_{w\in\mathcal{L}^*}(z+w)\,e^{(-\frac{z}{w}+\frac{z^2}{2w^2})}$$

which has convergence problem again. Instead:

Define $\sigma(z) \triangleq z\prod_{w\in\mathcal{L}^*}(1+\frac{z}{w})\,e^{(-\frac{z}{w}+\frac{z^2}{2w^2})}$.

Here we factored out $\prod_{w\in\mathcal{L}^*} w$, a "constant", which made the whole thing diverge. The function converges now since if we take a cut on $\mathbb{C}$, take log and obtain

$$\log z + \sum_{w\in\mathcal{L}^*}\left(\log(1+\frac{z}{w}) - \frac{z}{w} + \frac{z^2}{2w^2}\right) = \log z + \sum_{w\in\mathcal{L}^*}\left(\frac{z}{w} - \frac{z^2}{2w^2} + O(\frac{1}{|w|^3}) - \frac{z}{w} + \frac{z^2}{2w^2}\right)$$

$$= \log z + \underbrace{\sum_{w\in\mathcal{L}^*} O\left(\frac{1}{|w|^3}\right)}_{\text{Converges!}}.$$

Clearly $\sigma(z)$ is holomorphic on $\mathbb{C}$, with simple 0's at $\mathcal{L}$. Observe that $\sigma'(z)/\sigma(z) = \zeta(z)$, thus

$$\frac{\sigma'(z+1)}{\sigma(z+1)} - \frac{\sigma'(z)}{\sigma(z)} = \zeta(z+1) - \zeta(z) = \eta_1$$

$$\Rightarrow \log\sigma(z+1) - \log\sigma(z) = \eta_1 z + C_1 \quad (\mathrm{mod}\ 2\pi i\mathbb{Z})$$

$$\Rightarrow \sigma(z+1) = \sigma(z)e^{\eta_1 z + C_1}$$

Taking $z = -\frac{1}{2}$, and observing that $\sigma(z)$ is odd, we have:

$$\sigma(\tfrac{1}{2}) = \sigma(-\tfrac{1}{2})e^{-\frac{1}{2}\eta_1 + C_1} = -\sigma(\tfrac{1}{2})e^{-\frac{1}{2}\eta_1 + C_1}$$

$$(\sigma(\tfrac{1}{2}) \neq 0) \Rightarrow \sigma(z+1) = -\sigma(z)e^{\eta_1(z+\frac{1}{2})}$$

Similarly, we have $\sigma(z+\tau) = -\sigma(z)e^{\eta_2(z+\frac{\tau}{2})}$.

Proof of Abel's Theorem (second proof)

"Given $\sum_{i=1}^{N} A(P_i) = \sum_{j=1}^{N} A(Q_j) \Leftrightarrow \exists f$ meromorphic with zero's at P_i and poles at Q_j".

With the $\sigma(z)$ function, we may try $f(z) = \frac{\prod_{i=1}^{N}(\sigma(z-P_i))}{\prod_{j=1}^{N}(\sigma(z-Q_j))}$ and see if they descend to a function on $\hat{X}$, i.e. if it's doubly periodic on $\mathbb{C}$:

$$? \begin{cases} f(z+1) = f(z) \\ f(z+\tau) = f(z) \end{cases}$$

But

$$\begin{aligned} f(z+1) &= \prod_{i=1}^{N}(\sigma(z+1-P_i)) / \prod_{j=1}^{N}((\sigma(z+1-Q_j)) \\ &= [(-1)^N \prod_{i=1}^{N}\sigma(z-P_i)e^{\eta_1(z-P_i+\frac{1}{2})}] / [(-1)^N \prod_{j=1}^{N}\sigma(z-Q_j)e^{\eta_1(z-Q_j+\frac{1}{2})}] \\ &= (\prod_{i=1}^{N}\sigma(z-P_i) / \prod_{j=1}^{N}\sigma(z-Q_j))e^{\eta_1(\sum Q_j - \sum P_i)} \end{aligned}$$

The Abel map in this case is $A(p) = \int_0^p dz = p$, thus $\sum_{i=1}^{N} A(P_i) = \sum_{j=1}^{N} A(Q_j)$
$\Rightarrow \sum_{j=1}^{N} Q_j - \sum_{i=1}^{N} P_i = n + m\tau$. It is not a priori true that $\eta_1(n+m\tau) = 0$.

The correct solution is thus change different P_i, Q_j's, for instance we may take $Q_1' = Q_1 - n - m\tau$, $Q_i' = Q_i$, $i = 2, \cdots N$ and take

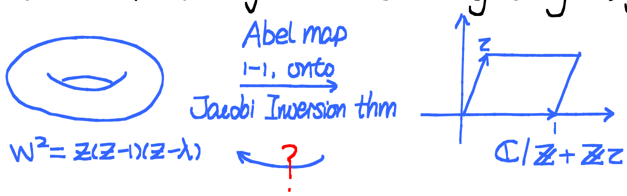
$$\bullet\ f(z) \triangleq \frac{\prod_{i=1}^{N}(\sigma(z-P_i))}{\prod_{j=1}^{N}(\sigma(z-Q_j'))}$$

which would then descend to $\hat{X}$.

Similar as in the previous section, we may produce:

- A form with a double pole at P: $\omega_P(z) \triangleq \wp(z-P)dz$. Since $\wp(z)$ is already a meromorphic function on $\mathbb{C}$ with a double pole at 0 which descends to $\hat{X}$.
- A form with 2 simple poles at P, Q: $\omega_{PQ} \triangleq (\zeta(z-P) - \zeta(z-Q))dz$. Although $\zeta(z)$ is not a function on $\hat{X}$, the difference is. Since $\zeta(z+1) - \zeta(z) = \eta_1$; $\zeta(z+\tau) - \zeta(z) = \eta_2$.

Recall that the Riemann surface was oringinally defined by $w^2 = z(z-1)(z-\lambda)$



First: the lattice $\mathcal{L}$ is given, we may express $\wp(z)$ as a Laurent series:

$$\begin{aligned}
\wp(z) &= \frac{1}{z^2} + \sum_{w\in\mathcal{L}^*}\left(\frac{1}{(w+z)^2} - \frac{1}{w^2}\right) \\
&= \frac{1}{z^2} + \sum_{w\in\mathcal{L}^*}\left(\frac{1}{w^2}\left(\frac{1}{1+\frac{z}{w}}\right)^2 - \frac{1}{w^2}\right) \\
&= \frac{1}{z^2} + \sum_{w\in\mathcal{L}^*}\left(\frac{1}{w^2}\sum_{k=0}^{\infty}(-1)^k(k+1)\left(\frac{z}{w}\right)^k - \frac{1}{w^2}\right) \\
&= \frac{1}{z^2} + \sum_{w\in\mathcal{L}^*}\left(\sum_{\ell=1}^{\infty}(2\ell+1)\frac{z^{2\ell}}{w^{2\ell+2}}\right) \quad \text{(odd order terms get cancelled since} \\
&= \frac{1}{z^2} + \sum_{k=1}^{\infty}(2k+1)z^{2k}\sum_{w\in\mathcal{L}^*}\frac{1}{w^{2k+2}} \qquad \mathcal{L}^* \text{ is symmetric: } w\in\mathcal{L}^* \Rightarrow -w\in\mathcal{L}^*)
\end{aligned}$$

Define $G_k(\mathcal{L}) \triangleq \sum_{w\in\mathcal{L}^*}\frac{1}{w^{2k}}$, the Einsenstein series, then:

$$\wp(z) = \frac{1}{z^2} + \sum_{k=1}^{\infty}(2k+1)G_{k+1}z^{2k}$$

Differentiate, we obtain:

$$\begin{aligned}
\wp'(z) &= -\frac{2}{z^3} + \sum_{k=1}^{\infty}(2k+1)2k\,G_{k+1}z^{2k-1} \\
&= -\frac{2}{z^3} + 6G_2 z + 20G_3 z^3 + O(z^5)
\end{aligned}$$

$$\Rightarrow (\wp'(z))^2 = \frac{4}{z^6} - \frac{24G_2}{z^2} - 80G_3 + O(z^2)$$

And we have:

$$\begin{aligned}
\wp(z)^3 &= \left(\frac{1}{z^2} + 3G_2z^2 + 5G_3z^4 + O(z^6)\right)^3 \\
&= \left(\frac{1}{z^4} + 6G_2 + 10G_3z^2 + O(z^4)\right)\left(\frac{1}{z^2} + 3G_2z^2 + 5G_3z^4 + O(z^6)\right) \\
&= \frac{1}{z^6} + 9G_2\frac{1}{z^2} + 15G_3 + O(z^2)
\end{aligned}$$

$$\Rightarrow (\wp'(z))^2 - 4(\wp(z))^3 = -60G_2\frac{1}{z^2} - 140G_3 + O(z^2)$$

$$\Rightarrow (\wp'(z))^2 - 4(\wp(z))^3 + 60G_2\wp(z) + 140G_3 = O(z^2) = 0 \quad \text{(Liouville's theorem)}$$

Define $g_2 = 60G_2$, $g_3 = 140G_3$, then we conclude that:

$$(\wp'(z))^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$$

Compared with $w^2 = z(z-1)(z-\lambda)$, which may be changed into the form

$$w^2 = 4z^3 - g_2 z - g_3$$

by a linear transformation. Hence the inverse map is given by:

$$z = \wp(s) \quad w = \wp'(s)$$

Remark: the Weierstrass $\wp(z)$ function answers an old question from calculous: What is an elliptic integral?

$$z = \int^z \frac{\wp'}{\sqrt{4\wp^3 - g_2\wp - g_3}}\,dz = \int^{\wp(z)} \frac{du}{\sqrt{4u^3 - g_2 u - g_3}} = E(\wp(z))$$

i.e. E (elliptic integral) is the inverse of Weierstrass function. Compare with the more familiar:

$$\int^u \frac{du}{\sqrt{1-u^2}} = \arcsin u\ , \quad \text{and} \quad \arcsin(\sin z) = z$$

- Function Theory According To Jacobi
  - Weierstrass: easy to follow, not easy to generalize to other R.S.
  - Jacobi: generalize well to the other R.S.'s. The key notion is the "theta functions"
  - Similar as above, we need to construct:
    1). A holomorphic form $\omega = dz$
    2). Meromorphic forms with simple poles at $P \neq Q$: ω_{PQ}.
    3). Meromorphic forms with a double pole at P: ω_P

Fix $\mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau$: $(\operatorname{Im}\tau > 0)$

- Define $\Theta(z|\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau + 2\pi i n z}$, $z \in \mathbb{C}$

This series converges for any z, because $|e^{\pi i n^2\tau + 2\pi i n z}| = e^{-\pi n^2 \operatorname{Im}\tau - 2\pi n \operatorname{Im} z}$ decays like a Gaussian w.r.t. $|n|$, and $\Theta(z|\tau)$ is holomorphic in z (and τ).

Key Transformations:

(I). $\Theta(z+1|\tau) = \Theta(z|\tau)$.

(II). $\Theta(z+\tau|\tau) = e^{-2\pi i z - \pi i \tau}\,\Theta(z|\tau)$. Note in particular that $-2\pi i z - i\pi\tau$ is linear in z.

Pf: $\Theta(z+\tau) = \sum_{n=-\infty}^{\infty} e^{i\pi n^2\tau + 2\pi i n(z+\tau)} = \sum_{n=-\infty}^{\infty} e^{i\pi\tau(n^2+2n+1-1)+2\pi i n z}$

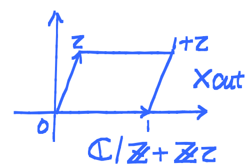
$$= \sum_{n=-\infty}^{\infty} e^{i\pi\tau(n+1)^2 - i\pi\tau - 2\pi i z + 2\pi i(n+1)z}$$

$$= e^{-2\pi i z - \pi i\tau}\sum_{n=-\infty}^{\infty} e^{i\pi\tau(n+1)^2+2\pi i(n+1)z}$$

$$= e^{-2\pi i z-\pi i \tau}\,\Theta(z)$$

Thm. $\Theta(z|\tau)=0 \Leftrightarrow z = \frac{1+\tau}{2} \pmod{\mathcal{L}}$, where $\mathcal{L}=\mathbb{Z}+\mathbb{Z}\tau$

Pf: We count the number of zero's inside a fundamental parallelogram. i.e. we compute the integral $\frac{1}{2\pi i}\int_{\partial X_{out}} \frac{\Theta'(z)}{\Theta(z)}dz = \#\{$ zero's inside a fundamental parallelogram.



$$\frac{1}{2\pi i}\oint_{\partial X_{out}}\frac{\Theta'(z)}{\Theta(z)}dz = \frac{1}{2\pi i}\left\{\int_0^1 \frac{\Theta'}{\Theta}dz + \int_1^{1+\tau}\frac{\Theta'}{\Theta}dz - \int_\tau^{1+\tau}\frac{\Theta'}{\Theta}dz - \int_0^\tau \frac{\Theta'}{\Theta}dz\right\}$$

$$= \frac{1}{2\pi i}\left\{\int_0^1\left(\frac{\Theta'}{\Theta}(z) - \frac{\Theta'}{\Theta}(z+\tau)\right)dz - \int_0^\tau\left(\frac{\Theta'}{\Theta}(z) - \frac{\Theta'}{\Theta}(z+1)\right)dz\right.$$

Now by (I) and (II). $\Theta'/\Theta(z) - \Theta'/\Theta(z+\tau) = (\ln\Theta(z))' - (\ln\Theta(z+\tau))' = 2\pi i$

$\frac{\Theta'}{\Theta}(z) - \frac{\Theta'}{\Theta}(z+1) = (\ln\Theta(z))'-(\ln\Theta(z+1))' = 0$

$\Rightarrow \frac{1}{2\pi i}\oint_{\partial X_{out}}\frac{\Theta'}{\Theta}dz = 1$

To find where the zero is, consider $\tilde\Theta(z|\tau) \triangleq \Theta(z+\frac{1+\tau}{2}|\tau)$.

$$\tilde\Theta(z|\tau) = \sum_{n=-\infty}^{\infty} e^{i\pi n^2\tau + 2\pi i n(z+\frac{1+\tau}{2})} = \sum_{n=-\infty}^{\infty} e^{i\pi n^2\tau + i\pi n\tau + \pi i n + 2\pi i n z}$$

$$= \sum_{n=-\infty}^{\infty} e^{i\pi\tau(n^2+n+\frac14) - \frac14 i\pi\tau + 2\pi i(n+\frac12)z - \pi i z + \pi i n}$$

$$= e^{-\frac14 i\pi\tau - \pi i z}\sum_{n=-\infty}^{\infty} e^{i\pi\tau(n+\frac12)^2 + 2\pi i(n+\frac12)z}\cdot(-1)^n$$

Evaluated at 0, we have: $\tilde\Theta(0|\tau) = e^{-\frac14 i\pi\tau}\sum_{n=-\infty}^{\infty} e^{\frac{i\pi\tau}{4}(2n+1)^2}(-1)^n$

But $(2n+1)^2 = (2k+1)^2$ $(k\geq 0) \Rightarrow 2n+1 = \pm(2k+1) \Rightarrow n=k$ or $-k-1$, which are of different parity. $\Rightarrow \tilde\Theta(0|\tau)=0$. The result follows. □

Rmk: The factor of $\Theta(z+\tau|\tau) = e^{-2\pi i z - \pi i\tau}\cdot\Theta(z|\tau)$ shows that $\Theta(z|\tau)$ is a section of a line bundle $\mathcal{L}$ on $\hat X$, and the above computation shows that $c_1(\mathcal{L})=1$.

Rmk: Define $\Theta_1(z|\tau) \triangleq e^{\pi i\frac{\tau}{4} + \pi i(z+\frac12)}\,\Theta(z+\frac{1+\tau}{2}|\tau)$, then Θ_1 is an odd function. This again explains that $\Theta(z+\frac{1+\tau}{2}|\tau)$ has a zero at 0. Moreover Θ_1 satisfies the following transformation relations:

(III). $\Theta_1(z+1|\tau) = -\Theta_1(z|\tau)$

(IV). $\Theta_1(z+\tau|\tau) = -e^{-i\pi\tau-2\pi i z}\Theta_1(z|\tau)$

Pf: By definition, $\Theta_1(z) = e^{\frac{\pi i}{2}}\sum_{n=-\infty}^{\infty} e^{i\pi\tau(n+\frac{1}{2})^2+2\pi i(n+\frac{1}{2})z+n\pi i}$

$$= \sum_{n=-\infty}^{\infty} e^{i\pi\tau(n+\frac{1}{2})^2+2\pi i(n+\frac{1}{2})z+\pi i(n+\frac{1}{2})}$$

Thus $\Theta_1(-z) = \sum_{n=-\infty}^{\infty} e^{i\pi\tau(-(n+\frac{1}{2}))^2+2\pi i(-(n+\frac{1}{2}))z+\pi i(-(n+\frac{1}{2}))+(2n+1)\pi i}$

$$=(-1)\sum_{k=-\infty}^{\infty} e^{i\pi\tau(k+\frac{1}{2})^2+2\pi i(k+\frac{1}{2})z+\pi i(k+\frac{1}{2})}$$

$$= -\Theta_1(z)$$

For the transformation relations, we note that

$$\Theta_1(z) = e^{\frac{1}{4}\pi\tau+\pi i(z+\frac{1}{2})}\tilde{\Theta}(z|\tau) = e^{\frac{\pi}{2}i}\sum_{n=-\infty}^{\infty} e^{i\pi\tau(n+\frac{1}{2})^2+2\pi i(n+\frac{1}{2})z}(-1)^n$$

Hence: $\Theta_1(z+1) = e^{\frac{\pi}{2}i}\sum_{n=-\infty}^{\infty} e^{i\pi\tau(n+\frac{1}{2})^2+2\pi i(n+\frac{1}{2})z+2\pi i(n+\frac{1}{2})}\cdot(-1)^n$

$$= -e^{\frac{\pi}{2}i}\sum_{n=-\infty}^{\infty} e^{i\pi\tau(n+\frac{1}{2})^2+2\pi i(n+\frac{1}{2})z}\cdot(-1)^n = -\Theta_1(z).$$

$$\Theta_1(z+\tau) = e^{\frac{\pi}{2}i}\sum_{n=-\infty}^{\infty} e^{i\pi\tau(n+\frac{1}{2})^2+2\pi i(n+\frac{1}{2})(z+\tau)}(-1)^n$$

$$= e^{\frac{\pi}{2}i}\sum_{n=-\infty}^{\infty} e^{i\pi\tau(n+\frac{1}{2})^2+2\pi i\tau(n+\frac{1}{2})+\pi i\tau-\pi i\tau+2\pi i(n+\frac{1}{2}+1)z-2\pi i z}\cdot(-1)(-1)^{n+1}$$

$$= -e^{\frac{\pi}{2}i}\left(\sum_{n=-\infty}^{\infty} e^{i\pi\tau(n+1+\frac{1}{2})^2+2\pi i(n+\frac{1}{2}+1)z}(-1)^{n+1}\right)\cdot e^{-\pi i\tau-2\pi i z}$$

$$= -e^{-\pi i\tau-2\pi i z}\,\Theta_1(z)$$

□

Proof of Abel's Theorem (third proof)

"Given $\sum_{i=1}^{N} A(P_i) = \sum_{j=1}^{N} A(Q_j) \Leftrightarrow \exists f$ meromorphic with zero's at P_i and poles at Q_j".

Now given $\Sigma P_i = \Sigma Q_j + n + m\tau$, and to construct f, we now use Θ-function:

For a first attempt, try $f = \left(\prod_{i=1}^{N}\Theta_1(z-P_i)\right)/\left(\prod_{j=1}^{N}\Theta_1(z-Q_j)\right)$

$$f(z+1) = \frac{\prod_{i=1}^{N}\Theta_1(z+1+\frac{1+\tau}{2}-Q_j)}{\prod_{j=1}^{N}\Theta_1(z+1+\frac{1+\tau}{2}-Q_j)} = \frac{(-1)^N\prod_{i=1}^{N}\Theta_1(z+\frac{1+\tau}{2}-P_i)}{(-1)^N\prod_{j=1}^{N}\Theta_1(z+\frac{1+\tau}{2}-Q_j)} = f(z)$$

$$f(z+\tau) = \frac{\prod_{i=1}^{N}\Theta_1(z+\tau+\frac{1+\tau}{2}-P_i)}{\prod_{j=1}^{N}\Theta_1(z+\tau+\frac{1+\tau}{2}-Q_j)}$$

$$= \frac{(-1)^N\prod_{i=1}^{N}\Theta_1(z+\frac{1+\tau}{2}-P_i)\cdot e^{-2\pi i(z-P_i)-\pi i\tau}}{(-1)^N\prod_{j=1}^{N}\Theta_1(z+\frac{1+\tau}{2}-Q_j)\cdot e^{-2\pi i(z-Q_j)-\pi i\tau}}$$

$$= e^{2\pi i(\Sigma Q_j-\Sigma P_i)} f(z)$$

So again we may replace one of the points, say, Q_1 by $\widetilde{Q}_1+n+m\tau$, so that $\Sigma P_i = \Sigma Q_j$, and the result follows. $\square$

Meromorphic Forms:

- We can also construct $\omega_{PQ}(z)$, a meromorphic form with simple poles at $P\neq Q$ as follows:

Since θ_1 satisfies the transformation rules III and IV, $(\ln\theta_1)'=\theta_1'/\theta_1$ will be a meromorphic function with a simple pole at the origin, and furthermore it satisfies:

$$\theta_1'(z+1)/\theta_1(z+1) = \theta_1'(z)/\theta_1(z)$$

$$\theta_1'(z+\tau)/\theta_1(z+\tau) = -2\pi i + \theta_1'(z)/\theta_1(z)$$

$\Rightarrow \frac{\theta_1'(z-P)}{\theta_1(z-P)} - \frac{\theta_1'(z-Q)}{\theta_1(z-Q)}$ is doubly periodic with simple poles at P and Q, and thus defines the required function on $\hat{X}$.

- To construct ω_P with a double pole at P, note that:

$(\ln\theta_1)''(z) = (\frac{\theta_1'}{\theta_1})'(z)$ has a double pole at P and is doublely periodic on $\mathbb{C}$, thus it suffices to consider $-(\frac{d^2}{dz^2}\ln\theta_1)(z-P)$.

- We can also connect Jacobi's theory with Weierstrass's theory, and we have the following identities.

$$\wp(z) = -\frac{d^2}{dz^2}\log\theta_1(z) + c(\tau)$$

Indeed, $\wp(z) - (-\frac{d^2}{dz^2}\log\theta_1(z))$ is holomorphic on $\mathbb{C}$ and doubly periodic.

$$\sigma(z) = e^{\frac{1}{2}\eta_1 z^2}\theta_1(z|\tau)/\theta_1'(0|\tau)$$

Pf: We first show that: $\frac{\sigma'(z)}{\sigma(z)} = \eta_1 z + \frac{\theta_1'(z)}{\theta_1(z)}$..

Indeed, we have $\frac{\sigma'(z)}{\sigma(z)} = \zeta(z)$ and $\zeta(z+1) = \zeta(z)+\eta_1$ $\zeta(z+\tau) = \zeta(z)+\eta_2$ $\tau\eta_1-\eta_2=2\pi i$ as proved before. similarly we have:

$$\begin{cases} \frac{\theta_1'}{\theta_1}(z+1) + \eta_1(z+1) - \frac{\theta_1'}{\theta_1}(z) - \eta_1 z = \eta_1 \\ \frac{\theta_1'}{\theta_1}(z+\tau) + \eta_1(z+\tau) - \frac{\theta_1'}{\theta_1}(z) - \eta_1 z = -2\pi i + \eta_1\tau = \eta_2 \end{cases}$$

$\Rightarrow (\eta_1 z - \frac{\hat{\theta}'}{\hat{\theta}}(z)) - \zeta(z)$ is doubly periodic and holomorphic, thus constant

But we also know that $\zeta(z) \to \frac{1}{z}$ and $\eta_1 z - \frac{\theta_1'}{\theta_1}(z) \to \frac{1}{z}$ where $z\to 0$ and both have no constant terms.

$\Rightarrow \eta_1 z - \frac{\theta_1'}{\theta_1}(z) = \zeta(z)$.

It follows that $C \cdot \sigma(z) = e^{\frac{1}{2}\eta_1 z^2} \cdot \theta_1(z|\tau)$. To specify C, we note that $\sigma'(0)=1$ and $\theta_1(z|\tau)(0)=0 \Rightarrow C = \lim_{z\to 0} e^{\frac{1}{2}\eta_1 z^2} \cdot \theta_1(z|\tau)/\sigma(z) = \theta_1'(0|\tau)$.

The formula follows. $\square$

Special Properties of θ - Functions (two important formulae)

- Product representation: set $q = e^{\pi i \tau}$

$$\theta(z|\tau) = \prod_{n=1}^{\infty}(1-q^{2n})\prod_{m=0}^{\infty}(1+q^{2m+1}e^{2\pi i z})(1+q^{2m+1}e^{-2\pi i z}) \qquad (1)$$

- Observation: since τ, $-\frac{1}{\tau}$ are parameters for the same lattice (differ by an $SL(2,\mathbb{Z})$ - action: $\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$), there should be some relation between $\theta(z|\tau)$ and $\theta(z|-\frac{1}{\tau})$. Indeed, there is the modular transformation law:

$$\theta(z|-\tfrac{1}{\tau}) = \sqrt{\tfrac{\tau}{i}}\, e^{i\pi\tau z^2}\theta(\tau z|\tau) \qquad (2)$$

where $\sqrt{t}$ is the main branch which is positive for $t>0$. In particular:

$$\theta(0|-\tfrac{1}{\tau}) = \sqrt{\tfrac{\tau}{i}}\,\theta(0|\tau) \quad \text{(duality)}$$

Proof of the product presentation.

Step1: Denote the R.H.S. of (1) by $T(z|\tau)$, we will first show that

$$\theta(z|\tau) = C(q)T(z|\tau) \text{ for some constant } C(q). \qquad (1)'$$

First of all, T vanishes when $1+qe^{-2\pi i z}=0$, or $e^{\tau\pi i - 2\pi i z} = e^{\pi i}$. In particular it vanishes when $z = \frac{1+\tau}{2}$.

Secondly, T transforms similarly as θ does under translations by $\mathcal{L}$, i.e.

$$\begin{cases} T(z+1|\tau) = T(z|\tau) & (3) \\ T(z+\tau|\tau) = e^{-2\pi i z - \pi i \tau}T(z|\tau) & (4) \end{cases}$$

Then it follows that T vanishes at every $\frac{1+\tau}{2}+\mathcal{L}$ and $\frac{T}{\theta}$ is a holomorphic, doubly periodic function on $\mathbb{C}$, thus must be constant.

Now let's check (3) and (4):

(3) follows easily since it's invariant termwise.

$$
\begin{aligned}
(4):\ T(z+\tau|\tau) &= \prod_{n=1}^{\infty}(1-q^{2n})(1+q^{2n-1}e^{2\pi i(z+\tau)})(1+q^{2n-1}e^{-2\pi i(z+\tau)}) \\
&= \prod_{n=1}^{\infty}(1-q^{2n})(1+q^{2n+1}e^{2\pi i z})(1+q^{2n-3}e^{-2\pi i z}) \\
&= \Big(\prod_{n=1}^{\infty}(1-q^{2n})(1+q^{2n-1}e^{2\pi i z})(1+q^{2n-1}e^{-2\pi i z})\Big)\,(1+q^{-1}e^{-2\pi i z})/(1+qe^{2\pi i z}) \\
&= T(z|\tau)(qe^{2\pi i z})^{-1}\Big(\frac{1+q^{-1}e^{-2\pi i z}}{1+q^{-1}e^{-2\pi i z}}\Big) \\
&= e^{-2\pi i z-\pi i\tau}\,T(z|\tau)
\end{aligned}
$$

Step 2. Firstly we shall show that $c(q)=1$.

Claim: $c(q)=c(q^4)$. (5)

Then it follows that $c(q)=c(q^4)=c(q^{16})=\cdots=c(q^{n^2})=\cdots$ Since $\operatorname{Im}\tau>0$, $|q^{n^2}|=e^{-n^2\pi\operatorname{Im}\tau}<1$
$\Rightarrow c(q)=\lim_{q\to0}c(q)$. But in both cases, $\Theta(0|\tau)\to1$, $T(0|\tau)\to1$, when $n^2\operatorname{Im}\tau\to\infty$
$\Rightarrow c(q)=\lim_{q\to0}c(q)=1$.

Now we prove (5). Apply (1)$'$ at $z=\frac{1}{2}$, we obtain that:

$$\Theta(\tfrac{1}{2}|\tau)=\sum_{n\in\mathbb{Z}}q^{n^2}(-1)^n.$$

and $T(\frac{1}{2}|\tau)=\prod_{n\geq1}(1-q^{2n})(1-q^{2n-1})^2=\prod_{n\geq1}(1-q^n)(1-q^{2n-1})$

$\Rightarrow c(q)=\big(\sum_{n\in\mathbb{Z}}q^{n^2}(-1)^n\big)\big/\prod_{n\geq1}(1-q^n)(1-q^{2n-1})$. (6)

Again apply (1)$'$ at $z=\frac{1}{4}$ we obtain that:

$$\Theta(\tfrac{1}{4}|\tau)=\sum_{n\in\mathbb{Z}}q^{n^2}i^n$$

But observe that $n=\pm(2k+1)$, $k\in\mathbb{N}\Rightarrow q^{(2k+1)^2}i^{2k+1}+q^{(-(2k+1))^2}i^{-(2k+1)}=q^{2k+1}(-1)^k(i+i^{-1})=0$

$$\Rightarrow \Theta(\tfrac{1}{4}|\tau)=\sum_{n\text{ even}}q^{n^2}i^n=\sum_{n\geq1}q^{4n^2}(-1)^n$$

$$
\begin{aligned}
T(\tfrac{1}{4}|\tau) &= \prod_{n\geq1}(1-q^{2n})(1+q^{2n-1}i)(1-q^{2n-1}i) \\
&= \prod_{n\geq1}(1-q^{2n})(1+q^{4n-2}) \\
&= \prod_{n\geq1}(1-q^{4n})(1-q^{4n-2})(1+q^{4n-2}) \\
&= \prod_{n\geq1}(1-q^{4n})(1-q^{8n-4}) \\
&= \prod_{n\geq1}(1-(q^4)^n)(1-(q^4)^{2n-1})
\end{aligned}
$$

$\Rightarrow c(q)=\big(\sum_{n\geq1}(q^4)^n(-1)^n\big)\big/\prod_{n\geq1}(1-(q^4)^n)(1-(q^4)^{2n-1})$ (7)

Compare (6) and (7), we obtain that $c(q)=c(q^4)$ as asserted, and finishes the proof of the product presentation. $\square$

Proof of the modular transformation.

Note that $\Theta(z|\tau)$ is holomorphic in both z and τ, it suffices to prove for $z\in\mathbb{R}$ and $\tau=i\tau_2$, $\tau_2>0$.

Now the R.H.S. of (2) reads:

$$\begin{aligned}\sqrt{\tau_2}\,e^{-\pi\tau_2z^2}\Theta(iz\tau_2|i\tau_2)&=\sqrt{\tau_2}\,e^{-\pi\tau_2z^2}\sum_{n\in\mathbb{Z}}e^{-\pi\tau_2n^2+2\pi i(iz\tau_2)n}\\&=\sqrt{\tau_2}\,e^{-\pi\tau_2z^2}\sum_{n\in\mathbb{Z}}e^{-\pi\tau_2n^2-2\pi z\tau_2n}\\&=\sqrt{\tau_2}\sum_{n\in\mathbb{Z}}e^{-\pi\tau_2(z+n)^2}\end{aligned}$$

Thus (2) $\Leftrightarrow$ $\sum_{n\in\mathbb{Z}}e^{-\frac{\pi}{\tau_2}n^2+2\pi inz}=\sqrt{\tau_2}\sum_{n\in\mathbb{Z}}e^{-\pi\tau_2(z+n)^2}$, which is actually a special case of the Poisson summation formula:

Thm. (Poisson Summation Formula) Let f be a smooth, rapidly decaying function. Define the Fourier transform $\hat{f}(\xi)=\int_{\mathbb{R}}e^{-2\pi ix\xi}f(x)dx$. Then we have:

$$\sum_{n\in\mathbb{Z}}f(\theta+n)=\sum_{n\in\mathbb{Z}}e^{2\pi in\theta}\hat{f}(n)\qquad\forall\,\theta\in\mathbb{R}.$$

In particular, let $\theta=0$, then we have:

$$\sum_{n\in\mathbb{Z}}f(n)=\sum_{n\in\mathbb{Z}}\hat{f}(n)$$

"Rapidly decaying" guarantees that $\sum_{n\in\mathbb{Z}}f(\theta+n)$ converges for all $\theta\in\mathbb{R}$. For example take the Gaussian: $f=e^{-\frac{\pi}{2}x^2}$, $\hat{f}(\xi)=\sqrt{2}\,e^{-2\pi\xi^2}$. Thus apply this formula to our problem: $f=e^{-\pi\tau_2z^2}$, $z\in\mathbb{R}$ $\Rightarrow$ $\hat{f}(\xi)=\frac{1}{\sqrt{\tau_2}}e^{-\frac{\pi\xi^2}{\tau_2}}$

$$\Rightarrow\sum_{n\in\mathbb{Z}}e^{-\pi\tau_2(z+n)^2}=\sum_{n\in\mathbb{Z}}e^{2\pi inz}\frac{1}{\sqrt{\tau_2}}e^{-\frac{\pi n^2}{\tau_2}}$$

$$\Rightarrow\sqrt{\tau_2}\sum_{n\in\mathbb{Z}}e^{-\pi\tau_2(z+n)^2}=\sum_{n\in\mathbb{Z}}e^{-\frac{\pi n^2}{\tau_2}+2\pi inz},\quad\text{as desired.}$$

Proof of Poisson summation formula.

For a rapidly decaying function f, we may define $\psi(\theta)=\sum_{n\in\mathbb{Z}}f(n+\theta)$. Moreover $\psi(\theta+1)=\psi(\theta)$. Thus ψ is a smooth periodic function, which can be expanded into Fourier series: $\psi(\theta)=\sum_n C_ne^{2\pi in\theta}$, where $C_n=\int_0^1\psi(\theta)e^{-2\pi in\theta}d\theta$

But
$$\begin{aligned}C_n&=\int_0^1e^{-2\pi in\theta}\Big(\sum_{k\in\mathbb{Z}}f(k+\theta)\Big)d\theta=\sum_{k\in\mathbb{Z}}\int_0^1e^{-2\pi in\theta}f(k+\theta)\,d\theta\\&=\sum_{k\in\mathbb{Z}}\int_k^{k+1}e^{-2\pi in\theta}f(\theta)d\theta=\int_{\mathbb{R}}e^{-2\pi in\theta}f(\theta)=\hat{f}(n)\end{aligned}$$

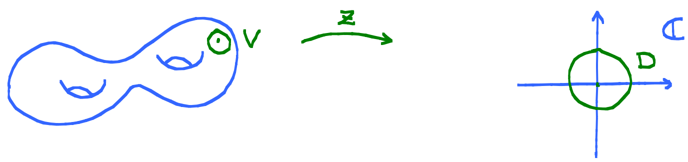
It follows that: $\psi(\theta)=\sum_{n\in\mathbb{Z}}\hat{f}(n)e^{2\pi in\theta}$. Hence by definition of ψ we have:

$$\sum_{n\in\mathbb{Z}}f(n+\theta)=\sum_{n\in\mathbb{Z}}\hat{f}(n)e^{2\pi in\theta}.$$

$\square$

Function Theory Using PDE

Goal: To construct holomorphic and meromorphic forms on a surface $\hat{X}$, on which every point has locally holomorphic coordinates.

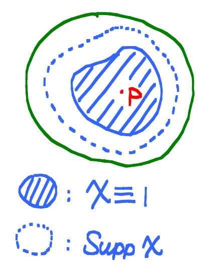


Idea: Suppose we want to construct a form ω_p meromorphic on $\hat{X}$ with a double pole at p. Using local coordinate charts, we have:

$$\begin{array}{ccc} \text{neighborhood } V \text{ of } P & \longleftrightarrow & \text{Disk } D \text{ in } \mathbb{C} \\ P & \leftrightarrow & 0 \\ z & \leftrightarrow & t(z) \end{array}$$

(1). Take $\frac{dt}{t^2}$ to obtain a form $\tilde{\omega}$ on V.

(2). Extend $\tilde{\omega}$ to $\hat{X}$ by considering $\chi\cdot\tilde{\omega}$, where $\chi \in C_0(V)$, $\chi \equiv 1$ in a smaller neighborhood of P in V. But $\chi\cdot\tilde{\omega}$ is no longer meromorphic on $\hat{X}$. We shall correct $\chi\tilde{\omega}$ by substracting a form ψ, so that $\bar{\partial}\psi = \bar{\partial}(\chi\tilde{\omega})$, where $\bar{\partial} = \frac{\partial}{\partial \bar{z}}$.



More precisely, we proceed as follows:

Thm. (Hodge Decomposition – a simple version). Let Φ be any smooth $(0,1)$-form on $\hat{X}$, locally $\Phi = \Phi(z)d\bar{z}$. Then there exists a smooth function $f \in C^\infty(\hat{X})$ and a $(0,1)$-form Φ_0 s.t.

$$\Phi = \frac{\partial f}{\partial \bar{z}} d\bar{z} + \Phi_0 \quad \text{and} \quad \frac{\partial \Phi_0}{\partial \bar{z}} = 0$$

Observation: $\overline{\Phi_0}$ is a holomorphic $(1,0)$-form and we shall show that the space of holomorphic $(1,0)$-forms is finite dimensional.

Assuming the Hodge decomposition thm, consider the following form

$$\Phi \triangleq \frac{\partial}{\partial \bar{z}}(\chi(z)\frac{1}{z})d\bar{z} \in C^\infty(\hat{X}\setminus\{p\}).$$

on $\hat{X}$. Since $\frac{\partial}{\partial \bar{z}}(\frac{1}{z}) = 0$ for $z \neq p$, we may just extend it by 0 at p, and 0 outside V, then it's a smooth $(0,1)$-form on $\hat{X}$.

By Hodge decomposition, we may write it as $\Phi = \partial_{\bar z} f \cdot d\bar z + \Phi_0$, with Φ_0 an anti-holomorphic $(0,1)$-form. Now we try the form $\omega \triangleq \partial_z(\chi \cdot \frac{1}{z} - f(z))\,dz$.

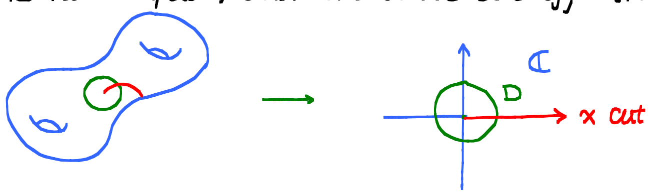
Claim: ω is holomorphic.

This is because $\frac{\partial}{\partial \bar z}\omega = \partial_z(\partial_{\bar z}(\chi \cdot \frac{1}{z} - f(z)))\,dz = \partial_z(\Phi - \partial_{\bar z} f\, d\bar z)\,dz = \partial_z \Phi_0\, dz = 0$.

Moreover we may construct a form with simple poles at $P \neq Q$.

Observation: how can one construct a form with a single pole?

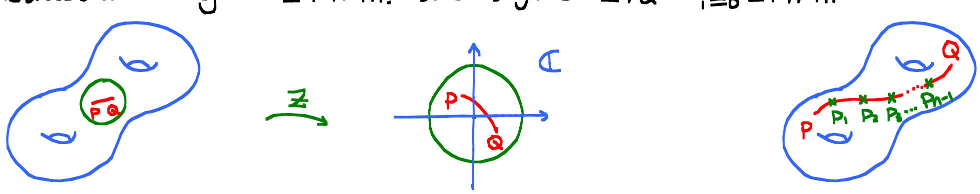
Naively, one may try $\partial_{\bar z}(\chi \cdot \ln z)$. But $\ln z$ has singularities along a "long" cut, which is not compact, and the above "cut off" trick doesn't apply. (supp χ is compact!)



However, if $P \neq Q$ are inside a same coordinate chart, we may introduce a cut by connecting P and Q, which is compact and the "cut off" technique applies:

$$\Phi_{PQ} \triangleq \partial_{\bar z}\left(\chi(z) \cdot \ln\left(\frac{z-P}{z-Q}\right)\right)$$

If $P \neq Q$ are not in the same chart, we may just take a sequence $\{P_i\}_{i=0}^{n}$, $P_0 = P$, $P_n = Q$, and each neighboring ones are in a same chart. Then apply the above construction to get $\Phi_{P_iP_{i+1}}$, and define $\Phi_{PQ} = \sum_{i=0}^{n-1} \Phi_{P_iP_{i+1}}$



Proof of Hodge decomposition.

More general statements will be proven latter, which holds for any dimension. At the present time, we give an explicit proof for $\mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau$, using explicit formula.

Observation: on $\mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau$, forms can be identified with doublely periodic functions on $\mathbb{C}$ via the correspondence: $\Phi(z)\,d\bar z \longleftrightarrow \Phi(z)$. Thus the problem becomes:

Given a function Φ, $? \exists f$, $\Phi = \frac{\partial}{\partial \bar z} f$.

1). Solving $\bar\partial$-equation in $\mathbb{C}$

Let Φ be a smooth function with compact support. Define $f(z) \triangleq \frac{1}{2\pi i}\iint_{\mathbb{C}} \frac{\Phi(\xi)}{\xi - z}\, d\xi \wedge d\bar\xi$.

Then $\frac{\partial f}{\partial \bar z} = \Phi$.

Pf:
$$\begin{aligned}
\frac{\partial f}{\partial \bar z} &= \frac{1}{2\pi i}\frac{\partial}{\partial \bar z}\iint_{\mathbb{C}} \frac{\Phi(\xi)}{\xi - z}\, d\xi\wedge d\bar\xi \\
&= \frac{1}{2\pi i}\frac{\partial}{\partial \bar z}\iint_{\mathbb{C}} \frac{\Phi(w+z)}{w}\, dw\wedge d\bar w \qquad (w = \xi - z) \\
&= \frac{1}{2\pi i}\iint_{\mathbb{C}} \frac{1}{w}\frac{\partial}{\partial \bar z}(\Phi(w+z))\, dw\wedge d\bar w \\
&= \frac{1}{2\pi i}\iint_{\mathbb{C}} \frac{1}{w}\frac{\partial}{\partial \bar w}(\Phi(w+z))\, dw\wedge d\bar w \qquad \left(\frac{\partial}{\partial \bar z}\Phi(z+w) = \frac{\partial}{\partial(\overline{z+w})}\Phi(z+w) = \frac{\partial}{\partial \bar w}\Phi(z+w)\right) \\
&= \frac{1}{2\pi i}\lim_{\varepsilon\to 0}\iint_{|w|\ge\varepsilon} \frac{1}{w}\frac{\partial \Phi}{\partial \bar w}(w+z)\, dw\wedge d\bar w \\
&= \frac{1}{2\pi i}\lim_{\varepsilon\to 0}\iint_{|w|\ge\varepsilon} -d\left(\frac{1}{w}\Phi(w+z)\,dw\right) \\
&= \frac{1}{2\pi i}\lim_{\varepsilon\to 0}\oint_{|w|=\varepsilon} \frac{1}{w}\Phi(w+z)\,dw \\
&= \frac{1}{2\pi i}\, 2\pi i\, \Phi(z) \\
&= \Phi(z).
\end{aligned}$$
$\square$

Using this result, we may also solve the equation $\Delta g = \Phi$, $\Delta = \partial_z\partial_{\bar z}$, where Φ is a smooth function with compact support. Indeed, just take

$$g(z) = \frac{1}{2\pi i}\iint_{\mathbb{C}} \log|z-w|^2 \cdot \Phi(w)\, dw\wedge d\bar w$$

This is because:

$$\partial_z g = \frac{1}{2\pi i}\iint_{\mathbb{C}} \frac{1}{z-w}\,\Phi(w)\, dw\wedge d\bar w$$

and $\partial_{\bar z}\partial_z g = \Phi(z)$

2). The torus case: $\mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau$

- Question: Given a C^∞-function φ on $\mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau$, when can we solve:
  - $\frac{\partial f}{\partial \bar z} = \varphi$ (f doubly periodic w.r.t. $\mathbb{Z}+\mathbb{Z}\tau$)
  - $\Delta g = \varphi$ ($\Delta = \partial_z\partial_{\bar z}$, again g doubly periodic)

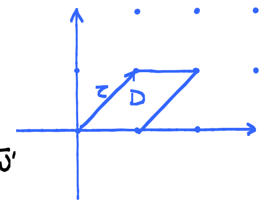
  Since on $\mathbb{C}$, $\varphi \in C_0^\infty(\mathbb{C})$, we can solve these equations by explicit formula:
  - $f(z) = \frac{1}{2\pi i}\iint_{\mathbb{C}} \frac{\varphi(w)}{z-w}\, dw\wedge d\bar w$
  - $g(z) = \frac{1}{2\pi i}\iint_{\mathbb{C}} (\log|z-w|^2)\,\varphi(w)\, dw\wedge d\bar w$

Returning to the torus case $\mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau$, we try a formula of similar kind:

$$g(z) = \frac{1}{2\pi i}\iint_{\mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau} G(z-w)\,\varphi(w)\, dw\wedge d\bar w.$$

Observations:

By $\iint_{\mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau}$, we mean an integral over a fundamental domain D. Furthermore, since:



$$\iint_{D+m+n\tau} G(z-\omega)\varphi(\omega)d\omega\wedge d\bar{\omega} = \iint_D G(z-m-n\tau-\omega')\varphi(\omega'+m+n\tau)d\omega'\wedge d\bar{\omega}'$$
$$= \iint_D G(z-m-n\tau-\omega')\varphi(\omega')d\omega'\wedge d\bar{\omega}',$$

when $G(z-\omega)$ is doubly periodic, the integral is doubly periodic and $g(z)$ is well-defined. Thus we look for $G(z)$ doubly periodic and $G(z)\sim \log|z|^2$ for $z\sim 0$.

Try $\log|\Theta_1(z|\tau)|^2$. Note that $\Theta_1(z|\tau) = \Theta_1(0|\tau) + z\Theta_1'(0|\tau) + z^2E(z)$, $E(z)$ holo. or $\Theta_1 = z(\Theta_1'(0|\tau) + zE(z))$ as $\Theta_1(0|\tau)=0$. Thus we try $\log\left|\frac{\Theta_1(z|\tau)}{\Theta_1'(0|\tau)}\right|^2$

Recall that: $\Theta_1(z+1|\tau) = -\Theta_1(z|\tau)$

$$\Theta_1(z+\tau|\tau) = -e^{-\pi i\tau - 2\pi i z}\Theta_1(z|\tau)$$

Clearly: $\begin{cases} \dfrac{|\Theta_1(z+1|\tau)|^2}{|\Theta_1'(0|\tau)|^2} = \dfrac{|\Theta_1(z|\tau)|^2}{|\Theta_1'(0|\tau)|^2} \\ \dfrac{|\Theta_1(z+\tau|\tau)|^2}{|\Theta_1'(0|\tau)|^2} = |e^{-\pi i\tau-2\pi i z}|^2\dfrac{|\Theta_1(z|\tau)|^2}{|\Theta_1'(0|\tau)|^2} = |e^{\pi\tau_2+2\pi y}|^2\dfrac{|\Theta_1(z|\tau)|^2}{|\Theta_1'(0|\tau)|^2} \end{cases}$, $\tau=\tau_1+i\tau_2$, $z=x+iy$.

$$\Rightarrow \begin{cases} \log\dfrac{|\Theta_1(z+1|\tau)|^2}{|\Theta_1'(0|\tau)|^2} = \log\dfrac{|\Theta_1(z|\tau)|^2}{|\Theta_1'(0|\tau)|^2} \\ \log\dfrac{|\Theta_1(z+\tau|\tau)|^2}{|\Theta_1'(0|\tau)|^2} = 2\pi\tau_2 + 4\pi y + \log\dfrac{|\Theta_1(z|\tau)|^2}{|\Theta_1'(0|\tau)|^2} \end{cases}$$

Thus we may try instead $\log\dfrac{|\Theta_1(z|\tau)|^2}{|\Theta_1'(0|\tau)|^2} - \dfrac{2\pi}{\tau_2}y^2$. Under the transformation $z\mapsto z+\tau$ (or $x+yi \mapsto x+\tau_1+(y+\tau_2)i$, $y\mapsto y+\tau_2$):

$$\frac{2\pi}{\tau_2}y^2 \mapsto \frac{2\pi}{\tau_2}(y^2+2y\tau_2+\tau_2^2) = \frac{2\pi}{\tau_2}y^2 + 4\pi y + 2\pi\tau_2$$

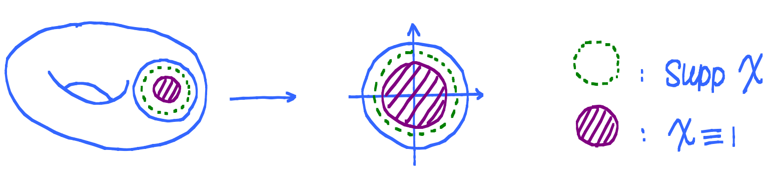
which cancels the extra factor.

Def: (Green's function on torus). $G(z) = \log\dfrac{|\Theta_1(z|\tau)|^2}{|\Theta_1'(0|\tau)|^2} - \dfrac{2\pi}{\tau_2}(\operatorname{Im} z)^2$.

Set $g(z) = \frac{1}{2\pi i}\iint_{\mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau} G(z-\omega)\varphi(\omega)d\omega\wedge d\bar{\omega}$ $(*)$. We compute $\Delta(g(z)) = \partial_{\bar{z}}\partial_z g(z)$

We shall bring ourselves back to the complex plane case as follows:

Fix an arbitrary $z\in\mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau$, and let $\chi_\omega\in C^\infty(\mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau)$, $\chi(\omega)\equiv 1$ in a neighborhood of z; $\chi(\omega)=0$ outside a neighborhood of z:



$$\iint_{\mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau} G(z-w)\varphi(w)\,dw\wedge d\bar{w} = \iint G(z-w)(\chi(w)\varphi(w))\,dw\wedge d\bar{w} + \iint G(z-w)(1-\chi(w))\varphi(w)\,dw\wedge d\bar{w}$$

$G(z-w)(1-\chi(w))\varphi(w)$ is C^∞ on $\mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau$ since singularity of G occurs only when $w=z$

Thus: $\Delta_z \iint_{\mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau} G(z-w)(1-\chi)\cdot\varphi(w)\,dw\wedge d\bar{w} = \iint (\Delta_z G(z-w))(1-\chi)\varphi(w)\,dw\wedge d\bar{w}$

$$= \iint -\frac{\pi}{\tau_2}(1-\chi(w))\varphi(w)\,dw\wedge d\bar{w}$$

(In the last equality, we used the fact that:

$$\partial_z\partial_{\bar{z}}(G(z-w)) = \partial_z\partial_{\bar{z}}\left(\log\frac{|\Theta_1(z-w|\tau)|^2}{|\Theta_1'(0|\tau)|^2} - \frac{2\pi}{\tau_2}(\mathrm{Im}(z-w)^2)\right) = -\frac{\pi}{\tau_2}\ ).$$

To calculate $\Delta_z \iint G(z-w)(\chi(w)\varphi(w))\,dw\wedge d\bar{w}$, observe that for $w \in \mathrm{Supp}\,\chi$ and hence near z, we have $\left|\frac{\Theta_1'(z-w|\tau)}{\Theta_1'(0|\tau)}\right|^2 = |z-w|^2\,|1+(z-w)h(z-w)|^2$, h holo.

$$\Rightarrow \log\left|\frac{\Theta_1'(z-w|\tau)}{\Theta_1'(0|\tau)}\right|^2 = \log|z-w|^2 + \log|1+(z-w)h(z-w)|^2$$

$$\Rightarrow G(z-w) = \log|z-w|^2 + \log|1+(z-w)h(z-w)|^2 - \frac{2\pi}{\tau_2}\mathrm{Im}(z-w)^2$$

$$\Rightarrow \Delta_z \iint G(z-w)(\chi(w)\varphi(w))\,dw\wedge d\bar{w} = \Delta_z \iint \log|z-w|^2\,\chi\cdot\varphi(w)\,dw\wedge d\bar{w} + 0$$

$$+ \iint \Delta_z\left(-\frac{2\pi}{\tau_2}\mathrm{Im}(z-w)^2\right)\chi\varphi(w)\,dw\wedge d\bar{w}$$

and by the $\mathbb{C}$-case $\Delta_z \iint \log|z-w|^2\,\chi\cdot\varphi(w)\,dw\wedge d\bar{w} = \varphi(z)$

Altogether:

$$\Delta g = \varphi(z) - \frac{1}{2\tau_2 i}\iint_{\mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau}\varphi(w)\,dw\wedge d\bar{w} \quad (**)$$

and we make the following observations:

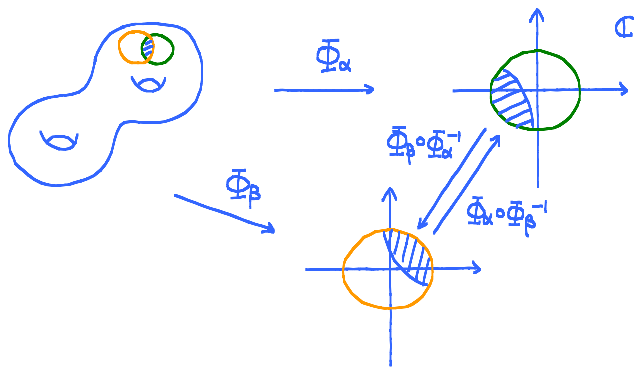
- If $\iint_{\mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau}\varphi(w)\,dw\wedge d\bar{w} = 0$, $\Delta g = \varphi$ admits a solution and is given explicitly by the formula $(*)$.
- Furthermore, if the equation $\Delta g = \varphi$ admits a solution, $\iint_{\mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau}\varphi\,dw\wedge d\bar{w} = 0$
Indeed $\iint \varphi\,dz\wedge d\bar{z} = \iint \partial_z\partial_{\bar{z}}\,g(z)\,dz\wedge d\bar{z} = \iint d(\partial_{\bar{z}}g(z)\,d\bar{z}) = 0$, by Stokes thm.
- The space $\{\varphi \mid \Delta g = \varphi \text{ is solvable}\}$ has codimension 1 in $C^\infty(\mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau)$
- Furthermore, $(**) \Rightarrow \varphi(z) = \partial_{\bar{z}}\partial_z g + \varphi_0 = \partial_{\bar{z}} f + \varphi_0$, $f = \partial_z g$ and φ_0 a constant.
This is exactly the Hodge decomposition thm on $\mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau$.

§4. General Theory

Def: X is called a Riemann surface (R.S.) if $X = \cup_\alpha X_\alpha$ with the property $\Phi_\alpha : X_\alpha \longrightarrow D \subseteq \mathbb{C}$ (D: unit disk) and $\forall \alpha, \beta$

$$\Phi_\beta \circ \Phi_\alpha^{-1} : \Phi_\alpha(X_\alpha \cap X_\beta) \longrightarrow \Phi_\beta(X_\alpha \cap X_\beta)$$

is holomorphic, 1-1, $(\Phi_\beta \circ \Phi_\alpha^{-1})'(z) \neq 0$ and with holomorphic inverse.



For notational sanity, we will use $X_\alpha \ni z \overset{\Phi_\alpha}{\longmapsto} z_\alpha \in D$.

Def. of line bundles (generalization of differential forms)

A line bundle L is an assignment : $L \longleftrightarrow \{t_{\alpha\beta}\}$, where $t_{\alpha\beta}$ are functions on $X_\alpha \cap X_\beta$, satisfying the following cocycle conditions.

$$\begin{cases} t_{\alpha\beta} \cdot t_{\beta\gamma} \equiv t_{\alpha\gamma} & \text{on } X_\alpha \cap X_\beta \cap X_\gamma \\ t_{\alpha\alpha} = 1 \end{cases}$$

L is said to be C^∞ if each $t_{\alpha\beta}$ is smooth

L is said to be holomorphic if each $t_{\alpha\beta}$ is holomorphic

L is said to be antiholomorphic if each $t_{\alpha\beta}$ is antiholomorphic.

To a line bundle corresponds its space of smooth sections. $L \longleftrightarrow \Gamma(X, L)$.

$\Gamma(X, L) = \{(\varphi_\alpha) \mid \varphi_\alpha : X_\alpha \to \mathbb{C},\ C^\infty\text{-function on } X_\alpha \text{ and } \varphi_\alpha = t_{\alpha\beta}\varphi_\beta \text{ on } X_\alpha \cap X_\beta\}$

To a holomorphic line bundle there is also the space of holomorphic sections:

$H^0(X, L) = \{(\varphi_\alpha) \mid \varphi_\alpha \text{ holomorphic on } X_\alpha \text{ and } \varphi_\alpha = t_{\alpha\beta}\varphi_\beta \text{ on } X_\alpha \cap X_\beta\}$

Examples:

(1). L = the trivial bundle $\longleftrightarrow \{t_{\alpha\beta} \equiv 1,\ \forall \alpha, \beta\}$

Then $\Gamma(X,L)=\{(\varphi_\alpha),\ \varphi_\alpha$ functions on X_α ; $\varphi_\alpha=\varphi_\beta$ on $X_\alpha\cap X_\beta\}$
$= \{$ all functions on $X\}$

(2). The canonical bundle $\Lambda^{1,0}(X)$ (or K_X)
$\Lambda^{1,0}(X)=\{t_{\alpha\beta}=\frac{\partial z_\beta}{\partial z_\alpha}$ on $X_\alpha\cap X_\beta\}$. The chain rule $\Rightarrow \{t_{\alpha\beta}\}$ satisfy the cocycle condition
$\Gamma(X,L)=\{(\varphi_\alpha):\ \varphi_\alpha$ defined on X_α and $\varphi_\alpha=\frac{\partial z_\beta}{\partial z_\alpha}\varphi_\beta\}$
$=\{(\varphi_\alpha dz_\alpha),\ \ \varphi_\alpha dz_\alpha=\varphi_\beta\frac{\partial z_\beta}{\partial z_\alpha}dz_\alpha=\varphi_\beta dz_\beta$ on $X_\alpha\cap X_\beta\}$
$=\{$ Smooth $(1,0)$-forms on $X\}$
In general, for arbitrary complex manifold of dim n. $X=\cup_\alpha X_\alpha$ and $\Phi_\alpha: X_\alpha\to D\subseteq\mathbb{C}^n$
$z\mapsto z_\alpha=(z_\alpha^j)_{j=1}^n\in\mathbb{C}^n$, we may similarly define a rank n vector bundle $\Lambda^{1,0}(X)$:
$\Lambda^{1,0}(X)=\{t_{\alpha\beta}=(\frac{\partial z_\beta^j}{\partial z_\alpha^k})_{n\times n}$, holomorphic functions taking value in $GL_n(\mathbb{C})\}$.
In $\dim_{\mathbb{C}}X=1$, we will use the notation K_X synonymously.

(3). The holo. tangent bundle
$L\leftrightarrow\{t_{\alpha\beta}=(\frac{\partial z_\beta}{\partial z_\alpha})^{-1}=\frac{\partial z_\alpha}{\partial z_\beta}$ on $X_\alpha\cap X_\beta\}$
$\Rightarrow\Gamma(X,L)=\{\varphi_\alpha:\ \varphi_\alpha=\frac{\partial z_\alpha}{\partial z_\beta}\varphi_\beta$ on $X_\alpha\cap X_\beta\}$
$=\{(\varphi_\alpha\frac{\partial}{\partial z_\alpha}),\ \ \varphi_\alpha\frac{\partial}{\partial z_\alpha}=\varphi_\beta\frac{\partial z_\alpha}{\partial z_\beta}\frac{\partial}{\partial z_\alpha}=\varphi_\beta\frac{\partial}{\partial z_\beta}\}$

More generally, given a bundle L on X. $L\longleftrightarrow\{t_{\alpha\beta}\}$, we can define another bundle $L^k\longleftrightarrow\{(t_{\alpha\beta})^k\}$, $k\in\mathbb{Z}$.
E.g. $L=K_X$, $L^3\longleftrightarrow\{(\frac{\partial z_\beta}{\partial z_\alpha})^3\}$ and $\Gamma(X,L)=\{(\varphi_\alpha):\ \varphi_\alpha$ defined on X_α with $\varphi_\alpha(dz_\alpha)^3=\varphi_\beta(dz_\beta)^3\}$.

Covariant derivatives, metrics, and curvature on line bundles

Question: Can we differentiate sections of L ?
$\varphi\in\Gamma(X,L)\longleftrightarrow\{(\varphi_\alpha(z)):\ \varphi_\alpha: X_\alpha\longrightarrow\mathbb{C},\ \ \varphi_\alpha=t_{\alpha\beta}\varphi_\beta\}$

A naive attempt: Just take $\{\frac{\partial\varphi_\alpha}{\partial z_\alpha}\}$, can this be glued back to a section of some line bundle ?

$$\frac{\partial \varphi_\alpha}{\partial \bar z_\alpha} = \frac{\partial (t_{\alpha\beta}\varphi_\beta)}{\partial \bar z_\alpha} = \frac{\partial t_{\alpha\beta}}{\partial \bar z_\alpha}\varphi_\beta + \frac{\partial \varphi_\beta}{\partial \bar z_\alpha} t_{\alpha\beta} = \frac{\partial t_{\alpha\beta}}{\partial \bar z_\alpha}\varphi_\beta + \frac{\partial \varphi_\beta}{\partial \bar z_\beta}\frac{\partial \bar z_\beta}{\partial \bar z_\alpha} t_{\alpha\beta}$$

Observe that if L holomorphic, then $\frac{\partial t_{\alpha\beta}}{\partial \bar z_\alpha} = 0$, thus

$$\frac{\partial \varphi_\alpha}{\partial \bar z_\alpha} = t_{\alpha\beta}\frac{\partial \bar z_\beta}{\partial \bar z_\alpha}\frac{\partial \varphi_\beta}{\partial \bar z_\beta}.$$

i.e. $(\frac{\partial \varphi_\alpha}{\partial \bar z_\alpha})$ is a section of $L \otimes \overline{\Lambda^{1,0}}(X) = L \otimes \Lambda^{0,1}(X)$. This is the so called Chern connection.

Def. The Chern connection is defined by:

$$\bar\partial : \Gamma(X, L) \longrightarrow \Gamma(X, L\otimes\Lambda^{0,1}), \quad \varphi = \{\varphi_\alpha\} \longmapsto \bar\partial\varphi = \{\frac{\partial \varphi_\alpha}{\partial \bar z_\alpha}\}$$

Def. A metric on $L = \{t_{\alpha\beta}\}$ is an assignment h:

$h = \{(h_\alpha) : h_\alpha : X_\alpha \longrightarrow \mathbb{R}^{>0}$, smooth, strictly positive on X_α satisfying $h_\alpha = |t_{\alpha\beta}|^{-2} h_\beta$ on $X_\alpha \cap X_\beta\}$

Equivalently, h is a strictly positive section in $L^{-1}\otimes \bar L^{-1}$. There are plenty of such sections by glueing local sections.

Def: Given $\varphi \in \Gamma(X, L)$, $\varphi = \{\varphi_\alpha\}$, we can define its norm by:

$$\|\varphi\| \triangleq \varphi_\alpha \bar\varphi_\alpha h_\alpha$$

Well-defined since on $X_\alpha \cap X_\beta$, $\varphi_\alpha \bar\varphi_\alpha h_\alpha = t_{\alpha\beta}\varphi_\beta \cdot \bar t_{\alpha\beta}\bar\varphi_\beta |t_{\alpha\beta}|^{-2} h_\beta = \varphi_\beta \bar\varphi_\beta h_\beta$.

Given a metric h on L, we can define the covariant derivative ∇ as follows:

$$\nabla_z : \Gamma(X, L) \longrightarrow \Gamma(X, L\otimes\Lambda^{1,0}),$$
$$\varphi = \{\varphi_\alpha\} \longmapsto h_\alpha^{-1}\frac{\partial}{\partial z_\alpha}(h_\alpha\varphi_\alpha)$$

Indeed, $\{h_\alpha\varphi_\alpha\} \in \Gamma(X, L^{-1}\otimes\bar L^{-1}\otimes L) \cong \Gamma(X, \bar L^{-1})$ is a section of an antiholomorphic bundle and $\frac{\partial}{\partial z_\alpha}(h_\alpha\varphi_\alpha)$ is thus a well-defined section of $\Gamma(X, \bar L^{-1}\otimes\Lambda^{1,0})$. Hence $\{h_\alpha^{-1}\frac{\partial}{\partial z_\alpha}(h_\alpha\varphi_\alpha)\}$ is a well-defined section of $\Gamma(X, L\otimes\bar L\otimes\bar L^{-1}\otimes\Lambda^{1,0}) = \Gamma(X, L\otimes\Lambda^{1,0})$.

Another way of writing ∇_z would be useful: $\nabla_z\varphi = \{\partial_{z_\alpha}\varphi_\alpha + h_\alpha^{-1}(\partial_{z_\alpha}h_\alpha)\varphi_\alpha\}$ $= \{\partial_{z_\alpha}\varphi_\alpha + \partial_{z_\alpha}(\log h_\alpha)\varphi_\alpha\}$. Define $\Gamma_\alpha = \partial_{z_\alpha}(\log h_\alpha)$. then $\nabla_z\varphi = \{\partial_{z_\alpha}\varphi_\alpha + \Gamma_\alpha\varphi_\alpha\}$

- Commutation rules:

Fix $L \longrightarrow X$ a holomorphic line bundle $L = \{t_{\alpha\beta}\}$, metric $h = \{h_\alpha\}$, $\varphi = \{\varphi_\alpha\} \in \Gamma(X, L)$

$$\nabla_z\nabla_{\bar z}\varphi-\nabla_{\bar z}\nabla_z\varphi=\nabla_z(\partial_{\bar z}\varphi)-\nabla_{\bar z}(h^{-1}\partial_z(h\varphi))$$
$$=h^{-1}\partial_z(h(\partial_{\bar z}\varphi))-\partial_{\bar z}(h^{-1}\partial_z(h\varphi))$$
$$=\partial_z\partial_{\bar z}\varphi+\Gamma\cdot\partial_{\bar z}\varphi-\partial_{\bar z}(\Gamma\varphi+\partial_z\varphi)$$
$$=-(\partial_{\bar z}\Gamma)\varphi$$

Note that $\nabla_z\varphi\in\Gamma(X,L\otimes\Lambda^{1,0})$, a section of a holomorphic line bundle, to which $\partial_{\bar z}$ applies; $\partial_{\bar z}\varphi\in\Gamma(X,L\otimes\Lambda^{0,1})$, $h\partial_{\bar z}\varphi\in\Gamma(X,L^{-1}\otimes\bar L^{-1}\otimes L\otimes\Lambda^{0,1})\cong\Gamma(X,\bar L^{-1}\otimes\Lambda^{0,1})$, i.e. a section of an anti-holomorphic bundle, to which ∂_z applies.

Hence we obtain:

$$[\nabla_z,\nabla_{\bar z}]\varphi=F_{\bar z z}\cdot\varphi$$

where $F_{\bar z z}=-\partial_{\bar z}\Gamma=-\partial_{\bar z}\partial_z\log h$, which is called the curvature of the line bundle L w.r.t. h.

Observation: $F_{\bar z z}=\{F_{\bar z_\alpha z_\alpha}\}$ is a section of $\Lambda^{1,1}\cong\Lambda^{1,0}\otimes\Lambda^{0,1}$.

Explicitly, $\partial_{\bar z_\alpha}\partial_{z_\alpha}\log h_\alpha=\frac{\partial\bar z_\beta}{\partial\bar z_\alpha}\cdot\frac{\partial z_\beta}{\partial z_\alpha}\partial_{\bar z_\beta}\partial_{z_\beta}(\log h_\beta-\log t_{\alpha\beta}-\log\bar t_{\alpha\beta})$

$$=\frac{\partial\bar z_\beta}{\partial\bar z_\alpha}\cdot\frac{\partial z_\beta}{\partial z_\alpha}\partial_{\bar z_\beta}\partial_{z_\beta}\log h_\beta$$

It's not cohomologically trivial since in general $\{(\partial_{z_\alpha}\log h_\alpha)dz_\alpha\}$ is not a global form on X.

- Basic residue formula for holomorphic line bundles.

For any $L\to X$, holomorphic line bundle, any metric h and any meromorphic section $\varphi=\{\varphi_\alpha\}$, i.e. φ_α is a meromorphic function on X_α:

$$\#\{\text{zeroes of }\varphi\}-\#\{\text{poles of }\varphi\}=\frac{i}{2\pi}\int_X F_{\bar z z}\,dz\wedge d\bar z$$

Observations:

(1). L.H.S. doesn't depend on the metric chosen.

(2). R.H.S. doesn't depend on the meromorphic section chosen.

(3). Meromorphic sections always exist (to be proven). We want to know if holomorphic sections exist. If $\frac{i}{2\pi}\int_X F_{\bar z z}\,dz\wedge d\bar z<0$, then for any φ, $\#\{\text{poles of }\varphi\}>0\Rightarrow\nexists$ holomorphic sections. Thus the sign of curvature is of great importance.

Def. (The 1st Chern class) $C_1(L)\triangleq\frac{i}{2\pi}\int_X F_{\bar z z}\,dz\wedge d\bar z$.

Proof of the formula.

We compute the R.H.S., Locally:

$$F_{\bar{z}z}\, dz\wedge d\bar{z} = -\partial_z\partial_{\bar{z}} \log h \; dz\wedge d\bar{z} = -d((\partial_{\bar{z}} \log h) d\bar{z})$$

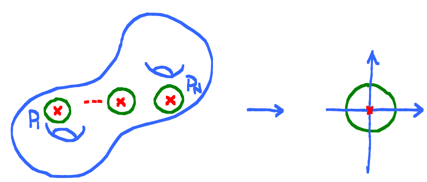
Notice that $\partial_{\bar{z}} \log h\, d\bar{z}$ is not a section of $\Lambda^{1,0}$. Note that:

$$F_{\bar{z}z}\, dz\wedge d\bar{z} = -\partial_z\partial_{\bar{z}}(\log\|\varphi\|^2)\cdot dz\wedge d\bar{z},$$

away from the zeroes and poles of φ, which are denoted as $\{P_1,\cdots,P_N\}$. Take ε small enough, so that each disk $D(P_i,\varepsilon)(\triangleq D_i)$ contains only one P_i. Thus

$$\int_X F_{\bar{z}z}\, dz\wedge d\bar{z} = \lim_{\varepsilon\to 0}\int_{X\setminus\bigcup_{i=1}^{N} D_i} -\partial_z\partial_{\bar{z}}(\log\|\varphi\|^2)\cdot dz\wedge d\bar{z}$$
$$= \lim_{\varepsilon\to 0}\sum_{i=1}^{N}\oint_{\partial D_i}\partial_{\bar{z}}(\log\|\varphi\|^2)d\bar{z}$$

On each D_i (contained in a coordinate chart), $\varphi = z^M\Phi(z)$, Φ: holo, $\Phi(0)\neq 0$.



D_i: contained in a chart

$$\oint_{\partial D_i}\partial_{\bar{z}}(\log\|\varphi\|^2)d\bar{z} = \oint_{\partial D_i}\partial_{\bar{z}}(\log|z|^{2M}+\log(h|\Phi(z)|^2))d\bar{z}$$
$$= \oint_{\partial D_i}\left(\frac{M}{\bar{z}} + \partial_{\bar{z}}\log h|\Phi(z)|^2\right)d\bar{z}$$
$$= -2\pi i M + O(\varepsilon) \longrightarrow -2\pi i M \quad (\varepsilon\to 0)$$
$$\Rightarrow \frac{i}{2\pi}\int_X F_{\bar{z}z}\,dz\wedge d\bar{z} = \sum_{i=1}^{N} M_i = \#\{\text{zeroes of } \varphi\} - \#\{\text{poles of } \varphi\} \qquad \square$$

- The Riemann-Roch Theorem

Notation: $H^0(X,L)\triangleq\{\text{holomorphic sections of } L\} = \{\varphi=(\varphi_\alpha) \mid \partial_{\bar{z}_\alpha}\varphi_\alpha = 0\}$

For any line bundle $L\to X$, we have the following formula (Riemann-Roch):

$$\dim H^0(X,L) - \dim H^0(X, L^{-1}\otimes K_X) = C_1(L) + \tfrac{1}{2}\chi(X)$$

where $K_X = \Lambda^{1,0}(X)$, $\chi(X)\triangleq C_1(K_X^{-1})$ = the Euler characteristic of X.

Furthermore, we shall prove the following (Gauss-Bonnet):

If X is a surface with g holes, then $\chi(X) = 2-2g$

Corollaries:

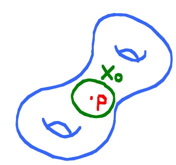
1). Let X be a Riemann surface with g holes. Take $L = K_X$, then we have $K_X^{-1} \otimes K_X$ is trivial, thus by R.R. Thm:

$$\dim H^0(X, L) - 1 = -\chi(X) + \tfrac{1}{2}\chi(X) = g - 1$$
$$\Rightarrow \dim H^0(X, L) = g$$

E.g. On the torus, we exhibited $\omega = \frac{dz}{\sqrt{z(z-1)(z-\lambda)}}$, which is a basis of $H^0(X, K_X)$.

2). Use of point bundles

Fix a R.S. X and $p \in X$. We construct a line bundle in the following way: Pick a coordinate system in a neighborhood X_0 of p, and set $X_\infty = X \setminus \{p\}$. Let L be $\{t_{0\infty}(z) = z$ on $X_0 \cap X_\infty\}$. This defines a holomorphic line bundle which admits a holomorphic section $1_p = 1$ on X_∞; z on X_0, and $1_p|_{X_0} = z = z \cdot 1$ $= t_{0\infty} \cdot 1_p|_{X_\infty}$. Thus 1_p is a section of $L \triangleq [P]$, which is holomorphic, and has exactly 1 zero at p. In particular $C_1(L) = \#(\text{zeroes of } 1_p) - \#(\text{poles of } 1_p) = 1$.



Similarly, given any integer $n \in \mathbb{Z}$, we may define $[nP]$ by the transition function $\{z^n\}$, and a holomorphic / meromorphic section 1_{nP} depending on $n \geq 0$ / $n < 0$. It follows that $C_1([nP]) = n$.

More generally, given $P_1, \cdots, P_M$ on X and integers $n_1, \cdots, n_M$, we may construct $[n_1 P_1 + \cdots + n_M P_M] = L \triangleq [n_1 P_1] \otimes \cdots \otimes [n_M P_M]$, and $C_1(L) = \sum_{i=1}^{M} n_i$

Proof that any holomorphic line bundle admits a non-trivial meromorphic section.

Pick a point P, consider $L \otimes [nP]$, $C_1(L \otimes [nP]) = C_1(L) + n$. By R.R Thm.

$$\dim H^0(X, L \otimes [nP]) - \dim H^0(X, L^{-1} \otimes [-nP] \otimes K) = C_1(L) + n - \tfrac{1}{2}\chi(X) > 0 \quad (n \gg 0)$$

$\Rightarrow \dim H^0(X, L \otimes [nP]) > 0 \quad (n \gg 0)$

$\Rightarrow \exists\, \varphi \neq 0$, φ a section of $L \otimes [nP]$, φ holomorphic.

$\Rightarrow \psi = \varphi \cdot 1_{-nP}$ is a non-zero meromorphic section of L.

3). Given any two points $P_1 \neq P_2$, we show that R.R. $\Rightarrow \exists$ meromorphic form φ with simple poles P_1 and P_2.

Consider the bundle $L = [-P_1 - P_2] \Rightarrow C_1(L) = -2$ and L can't admit any holomorphic section.

$R.R \Rightarrow 0 - \dim H^0([P_1+P_2]\otimes K_X) = -2+1-g$

$\Rightarrow \dim H^0([P_1+P_2]\otimes K_X) = g+1.$

Let $\psi_1, \cdots, \psi_g$ be a basis of $H^0(X, K_X)$, then $1_{[P_1+P_2]}\cdot\psi_1, \cdots, 1_{[P_1+P_2]}\cdot\psi_g$ are linearly independent sections of $[P_1+P_2]\otimes K_X$. Let Φ be a holomorphic section of the bundle which is linearly independent of $1_{[P_1+P_2]}\cdot\psi_1, \cdots, 1_{[P_1+P_2]}\cdot\psi_g$, and set $\varphi = 1_{[-P_1-P_2]}\cdot\Phi$ is a section we need. (Potentially φ may not have 2 poles, then it would have no pole at all since $\sum_{p=p_i}\mathrm{Res}(\varphi)(p) = 0$. In that case φ is a holomorphic section and $1_{[P_1+P_2]}\cdot\varphi = \Phi$ would be holomorphic thus a linear combination of $\psi_1, \cdots, \psi_g$, which contradicts the fact that Φ being linearly independent from $1_{[P_1+P_2]}\cdot\psi_1, \cdots, 1_{[P_1+P_2]}\cdot\psi_g$.)

Proof of Riemann-Roch Theorem

- Basic idea behind index theorems - heuristic point of view

Assume D is a linear operator between two vector spaces: $D: H_1 \longrightarrow H_2$ where H and H_1 are Hilbert spaces (in general, $H_1 \neq H_2$, for instance, $\bar\partial: \Gamma(X, L) \longrightarrow \Gamma(X, L\otimes\Lambda^{0,1})$, $\varphi \mapsto \bar\partial\varphi$). Since the spectrum of D is not available. Consider $\Delta_- = D^\dagger D$, where $D^\dagger$ is the adjoint of D (which is defined by $\langle Du, v\rangle = \langle u, D^\dagger v\rangle$, $\forall u \in H_1, v\in H_2$), $\Delta_-: H_1 \longrightarrow H_1$, self-adjoint. Hence we can talk about the spectrum of Δ_-. Spectrum $\Delta_- = \{\lambda_n^- : \exists\, 0 \neq \varphi_n \in H_1, \Delta_-\varphi_n = \lambda_n^-\varphi_n\}$. Furthermore, $\ker\Delta_- = \{\varphi: D^\dagger D\varphi = 0\} = \{\varphi: \langle\varphi, D^\dagger D\varphi\rangle = 0\} = \{\varphi: \|D\varphi\| = 0\} = \{\varphi: D\varphi = 0\} = \ker D$.

Another natural operator: $\Delta_+ = DD^\dagger: H_2 \longrightarrow H_2$ and again, we define the spectrum $\Delta_+ = \{\lambda_n^+ \mid \exists\psi\in H_2, \Delta_+\psi = \lambda_n^+\psi\}$, and similarly $\ker\Delta_+ = \ker D_+$.

A simple observation: the spectra of Δ_- and Δ_+ are the same (counted with multiplicity) are the same except for the kernels. Indeed, $D^\dagger D\varphi = \lambda_n^-\varphi$ $\Rightarrow \Delta^+(D\varphi) = DD^\dagger D\varphi = \lambda_n^- D\varphi$. i.e. $\forall\lambda_n^-$ a non-zero eigen-value of Δ_-, λ_n^- is also an eigenvalue of Δ^+.

Consider, $\forall t>0$, $Tr(e^{-t\Delta_-}) - Tr(e^{-t\Delta_+})$. Here we assume that Δ_-, Δ_+ can be diagonalized, i.e. $\exists$ an o.n.b. $\{\varphi_n^-\}$ of H_1 with $\Delta_-\varphi_n^- = \lambda_n^-\varphi_n^-$, an o.n.b. $\{\varphi_n^+\}$ of H_2 with $\Delta_+\varphi_n^+ = \lambda_n^+\varphi_n^+$. We may define $f(\Delta_-)\varphi_n^- = f(\lambda_n^-)\varphi_n^-$:

e.g. $e^{-t\Delta_-} = \sum \frac{(-t)^n}{n!}(\Delta_-)^n$, then $e^{-t\Delta_-}\varphi_n^- = e^{-t\lambda_n^-}\varphi_n^-$.

In general, $\varphi = \sum c_n\varphi_n^-$, $f(\Delta_-)\varphi = \sum c_n f(\Delta_-)\varphi_n^- = \sum c_n f(\lambda_n^-)\varphi_n^-$. Note that $\varphi\in H$ $\Rightarrow \sum|c_n|^2<\infty$. for $f(\Delta_-)\varphi\in H$, we need $\sum|c_n|^2|f(\lambda_n^-)|^2<\infty$. In our case it's good since $\lambda_n>0$, $e^{-t\lambda_n}<1$.

$Tr(f(\Delta_-)) \triangleq \sum_n f(\lambda_n^-)$ if it converges. Then

$$Tr(e^{-t\Delta_-}) - Tr(e^{-t\Delta_+}) = \sum e^{-t\lambda_n^-} - \sum e^{-t\lambda_n^+} = \dim\ker D - \dim\ker D^+.$$

Our next observation is that, the LHS can be computed using perturbation theory. More precisely, set $P = e^{-t\Delta}$, then

$$\begin{cases} (\frac{\partial}{\partial t} + \Delta)P = 0 \\ P|_{t=0} = Id, \end{cases} \qquad \text{(heat equation)}$$

and solutions of the heat equation can be computed asymptotically for small t.

Details

Let $L\to X$ be a holomorphic line bundle. We want to set up:

$$\bar\partial: \Gamma(X,L)\longrightarrow\Gamma(X, L\otimes\Lambda^{0,1})$$

- Inner products on $\Gamma(X,L)$ and $\Gamma(X, L\otimes\Lambda^{0,1})$

Pick a metric h on L, and a metric g on $(\Lambda^{1,0})^{-1} = \{f_\alpha\frac{\partial}{\partial z_\alpha}\}$ (the holomorphic tangent bundle. Recall that for a holo. line bundle L, a metric is a positive section of $L^{-1}\otimes\bar L^{-1}$. Thus g is a positive section of $\Lambda^{1,0}\otimes\overline{\Lambda^{1,0}}\cong\Lambda^{1,1}$, a positive $(1,1)$-form; we shall write $g_{\bar z z}$ to stress it's a $(1,1)$-form).

Def. $\forall\ \varphi,\psi\in\Gamma(X,L)$, $\Phi,\Psi\in\Gamma(X,L\otimes\Lambda^{0,1})$, we define inner products

$$\langle\varphi,\psi\rangle \triangleq \int_X h\varphi\bar\psi g_{\bar z z}$$
$$\langle\Phi,\Psi\rangle \triangleq \int_X h\Phi\bar\Psi$$

Here note that $h\varphi\bar\psi g_{\bar z z}\in\Gamma(X, L^{-1}\otimes\bar L^{-1}\otimes L\otimes\bar L\otimes\Lambda^{1,1}) = \Gamma(X,\Lambda^{1,1})$

$$h\Phi\bar\Psi\in\Gamma(X, L^{-1}\otimes\bar L^{-1}\otimes L\otimes\Lambda^{0,1}\otimes\bar L\otimes\Lambda^{1,0}) = \Gamma(X,\Lambda^{1,1}),$$

hence both are $(1,1)$ forms, and thus $\int_X$ makes sense.

- Formal adjoint $\bar\partial^+$ of $\bar\partial$

$\bar\partial^+ : \Gamma(X, L\otimes\Lambda^{0,1}) \to \Gamma(X, L)$ w.r.t. the inner products above is defined by

$$<\bar\partial\varphi, \Phi> = <\varphi, \bar\partial^+\Phi>$$

Explicitly: $\int_X h\,\partial_{\bar z}\varphi\cdot\bar\Phi = \int_X h\cdot\varphi\cdot\overline{\bar\partial^+\Phi}\, g_{\bar z z}$

The L.H.S. $= \int_X \partial_{\bar z}\varphi\cdot h\bar\Phi = -\int_X \varphi\ \partial_{\bar z}(h\bar\Phi)$ (integration by parts)

$= -\int_X h\varphi\,\overline{g^{z\bar z}h^{-1}\partial_z(h\Phi)}\, g_{\bar z z}$ $\quad (g^{z\bar z}\cdot g_{\bar z z} = 1)$

$\Rightarrow \quad \bar\partial^+\Phi = -g^{\bar z z}\nabla_z\Phi.$

(Indeed, $g^{\bar z z}\nabla_z\Phi \in \Gamma(X, (\Lambda^{1,1})^{-1}\otimes L\otimes\Lambda^{0,1}\otimes\Lambda^{1,0}) = \Gamma(X, L)$ so this is a valid def.)

Then we define $\Delta_- = \bar\partial^+\bar\partial : \Gamma(X, L) \to \Gamma(X, L)$

$\Delta_+ = \bar\partial\bar\partial^+ : \Gamma(X, L\otimes\Lambda^{0,1}) \to \Gamma(X, L\otimes\Lambda^{0,1})$

Thm. (Spectral Decomposition of Laplacian Δ_-)

(1). $\exists\{\varphi_n \in \Gamma(X, L), n\in\mathbb{Z}_+\}$ an o.n.b. of $H_1 = \overline{\Gamma(X, L)}$ which consists only of eigenfunctions for Δ_-, i.e. $\Delta_-\varphi_n = \lambda_n^-\varphi_n$.

(2). $\lambda_n^- \geq 0$ and $\lambda_n^- \to \infty$ at least at polynomial growth rate as $n\to\infty$, i.e. $C\cdot n^\nu \geq \lambda_n \geq D\cdot n^\delta$ for some $C, D, \nu, \delta > 0$.

(3). For any $k\in\mathbb{Z}_+$, $\exists\,\delta_k$ s.t. $\|\varphi_n\|_{C^k} \leq A_k|\lambda_n|^{\delta_k}$ for some constant A_k.

Here $\|\varphi\|_{C^0} \triangleq \sup_X |\varphi_n|^2 h$; $\|\varphi\|_{C^1} \triangleq \|\varphi\|_{C^0} + \sup_X|\nabla_z\varphi|^2 h g^{z\bar z} + \sup_X|\nabla_{\bar z}\varphi|^2 h g^{z\bar z}$, etc.

(4). Each eigenvalue λ_n^- occurs with finite multiplicity. ($\Rightarrow \ker\Delta^-$ is finite dimensional).

Assuming this thm, we define $e^{-t\Delta_-}$ and its trace as follows:

Def. $u\in\overline{\Gamma(X, L)}$, write $u = \sum_{n=0}^\infty u_n\varphi_n$, $u_n = <u, \varphi_n>$, with the series converging in the L^2-sense w.r.t. $<,>$ on H_1. Define:

$$e^{-t\Delta_-}u \triangleq \sum_{n=0}^\infty e^{-t\lambda_n^-}u_n\varphi_n \qquad (*).$$

The R.H.S is a well-defined element of H_1, in view of the

Thm. (Riesz-Fisher) $\sum_{n=0}^\infty u_n\varphi_n$ converges in H_1 iff $\sum_{n=0}^\infty |u_n|^2 < \infty$.

Pf: $\{\sum_{n=0}^N u_n\cdot\varphi_n\}$ converges iff it's Cauchy iff $\|\sum_{n=N}^M u_n\|^2 < \varepsilon$ for $M, N \gg 0$ iff $\sum_{n=0}^\infty |u_n|^2 < +\infty$. $\square$

Observe that $\{\varphi_n\}$ is also an o.n.b. of eigen-functions for $e^{-t\Delta^-}$. Indeed

$$e^{-t\Delta^-}\varphi_n = (e^{-t\lambda_n^-})\varphi_n$$

with eigenvalues $\{e^{-t\lambda_n^-}\}$. It follows from our thm that

$$\mathrm{Tr}(e^{-t\Delta^-}) = \sum_n e^{-t\lambda_n^-} < \infty, \quad \forall t>0$$

since $\lambda_n^- \geq c\cdot n^\delta$ for $n \gg 0$.

It follows that
$$\dim\ker\bar\partial - \dim\ker\bar\partial^+ = \dim\ker\Delta^- - \dim\ker\Delta^+ = \mathrm{Tr}(e^{-t\Delta^-}) - \mathrm{Tr}(e^{-t\Delta^+})$$

Evaluating $\mathrm{Tr}(e^{-t\Delta^-})$

1). Expressing it as a kernel.

$\forall u\in\Gamma(X,L)$.
$$\begin{aligned}(e^{-t\Delta^-}u)(z) &= \sum_n e^{-t\lambda_n^-}u_n\varphi_n(z)\\ &= \sum_n e^{-t\lambda_n^-}\left(\int_X hu\bar\varphi_n g_{\bar w w}\tfrac{i}{2}dw\wedge d\bar w\right)\varphi_n(z)\\ &= \int_X\left(\sum_n e^{-t\lambda_n^-}\varphi_n(z)\bar\varphi_n(w)hg_{\bar w w}(w)\right)u(w)\tfrac{i}{2}dw\wedge d\bar w\end{aligned}$$

Note that we may interchange sum and integral since $\|\varphi_n\|_{C^k} \leq A_k|\lambda_n|^{\delta_k}$ and thus the sum converges.

$\Rightarrow (e^{-t\Delta^-}u)(z) = \int_X K_t(z,w)u(w)\frac{i}{2}dw\wedge d\bar w$, where

$$K_t(z,w) = \Sigma_n e^{-t\lambda_n^-}\underbrace{\varphi_n(z)}_{\in L_z}\underbrace{\overline{\varphi_n(w)}}_{\bar L_w}\underbrace{h(w)}_{L_w^{-1}\otimes\bar L_w^{-1}}\underbrace{g_{\bar w w}}_{\Lambda^{1,1}_w} \quad \text{(heat kernel)}$$

$\Rightarrow K_t(z,z) \in \Gamma(X,\Lambda^{1,1})$

$$\begin{aligned}\Rightarrow \int_X K_t(z,z)\tfrac{i}{2}dz\wedge d\bar z &= \int_X\sum_n e^{-t\lambda_n^-}|\varphi_n(z)|^2h(z)g_{\bar z z}\tfrac{i}{2}dz\wedge d\bar z\\ &= \sum_n e^{-t\lambda_n^-}\int_X|\varphi_n(z)|^2h(z)g_{\bar z z}\tfrac{i}{2}dz\wedge d\bar z\\ &= \sum_n e^{-t\lambda_n^-}\|\varphi_n\|\\ &= \sum_n e^{-t\lambda_n^-}\\ &= \mathrm{Tr}\, e^{-t\Delta^-}\end{aligned}$$

2). Determine $K_t(z,z)$ for small t.

Claim: (a) There exists an asymptotic expansion of $K_t(z,z)$, in the sense that

$$K_t(z,z) = \frac{u_{\bar z z}}{t} + v_{\bar z z} + O(t)$$

and $\quad \left\|K_t(z,z) - \frac{u_{\bar z z}}{t} - v_{\bar z z}\right\|_{C^m} \leq A_m\cdot t \quad (t\ll 1)$

Clearly $u_{\bar z z}\in\Gamma(X,\Lambda^{1,1})$ and so is $v_{\bar z z}$.

(b). $U_{\bar{z}z}$, $U_{\bar{z}z}$ depend only on h of L and g of $\Lambda^{-1,0}$.

Key observation: $U_{\bar{z}z}$ doesn't depend on the derivatives of h and g. while $U_{z\bar{z}}$ depends on derivatives of h, g up to order 2. (This will follow from heat kernel equation theory to be explained below).

$$\begin{cases} U_{\bar{z}z} = a\, g_{\bar{z}z} \\ U_{\bar{z}z} = b F^{h}_{\bar{z}z} + c F^{g}_{\bar{z}z} \end{cases}$$

(This can be guessed from power counting)

We write specifically for Δ^- and Δ^+

kernel of Δ^-: $\quad K_t^-(z,z) = a^- \dfrac{g_{\bar{z}z}}{t} + b^- F^h_{\bar{z}z} + c^- F^g_{\bar{z}z}$

kernel of Δ^+: $\quad K_t^+(z,z) = a^+ \dfrac{g_{\bar{z}z}}{t} + b^+ F^h_{\bar{z}z} + c^+ F^g_{\bar{z}z}$

$$\Rightarrow \operatorname{Tr}(e^{-t\Delta^-}) - \operatorname{Tr}(e^{-t\Delta^+}) = (a^- - a^+)\frac{\operatorname{Vol}(X)}{t} + (b^- - b^+)C_1(L) + (c^- - c^+)C_1(K_X^{-1})$$

Clearly $a^- = a^+$ otherwise the L.H.S. would blow-up.

$$\Rightarrow \operatorname{Tr}(e^{-t\Delta^-}) - \operatorname{Tr}(e^{-t\Delta^+}) = \beta C_1(L) + \gamma C_1(K_X^{-1})$$

We can even determine the coefficients β and γ.

- Serre duality : We know that $\operatorname{Tr}(e^{-t\Delta^-}) - \operatorname{Tr}(e^{-t\Delta^+}) = \dim\ker\bar\partial - \dim\ker\bar\partial^+$ but how does $\ker\bar\partial^+ \sim H^0(L^{-1}\otimes K_X)$?

Claim: $\Phi \mapsto h\bar\Phi$ gives an isomorphism: $\Gamma(X, L\otimes\Lambda^{0,1}) \longrightarrow \Gamma(X, L^{-1}\otimes K_X)$

Indeed $\Gamma(X, L\otimes\Lambda^{0,1}) \ni \Phi \mapsto h\bar\Phi \in \Gamma(X, \bar{L}\otimes\Lambda^{1,0}\otimes L^{-1}\otimes\bar{L}^{-1}) = \Gamma(X, L^{-1}\otimes\Lambda^{1,0})$.

Moreover, $\Phi \in \ker\bar\partial^+ \Leftrightarrow \bar\partial^+\Phi = 0 \Leftrightarrow g^{\bar{z}z}h^{-1}(\partial_z h\Phi) = 0$

$\Leftrightarrow \partial_z h\Phi = 0 \Leftrightarrow \partial_{\bar{z}} h\bar\Phi = 0$ (h is positive, thus real)

$\Leftrightarrow h\bar\Phi \in H^0(X, L^{-1}\otimes K_X)$.

Thus $\dim\ker\bar\partial^+ = \dim H^0(X, L^{-1}\otimes K_X)$.

Now $\dim\ker\bar\partial - \dim\ker\bar\partial^+ = \dim H^0(X,L) - \dim H^0(X, L^{-1}\otimes K_X)$

$= \beta C_1(L) + \gamma C_1(K_X^{-1})$

Later we will see that β and γ are universal constants.

- Special cases

1). $L = $ trivial. $\Rightarrow 1 - \dim H^0(X, K_X) = 0 + \gamma C_1(K_X^{-1})$

2). $L = K_X \quad \Rightarrow \dim H^0(X, K_X) - 1 = \beta C_1(K_X) + \gamma \cdot C_1(K_X^{-1})$

Combining 1), 2) $\Rightarrow \gamma = \frac{\beta}{2}$. To determine the over all value (assuming Gauss-Bonnet thm.), consider $X = S^2$, $C_1(K_X^{-1}) = 2 - 2g = 2$. $\Rightarrow C_1(K_X) = -2$ and $H^0(X, K_X) = 0$.

$\Rightarrow 1 - 0 = \gamma C_1(K_X^{-1}) = 2\gamma \Rightarrow \gamma = \frac{1}{2}$. Hence Riemann-Roch follows.

- Road map and summary: three main ingredients.

(1). Spectral theory of $\Delta_- = \bar\partial^\dagger \bar\partial$, $\Delta_+ = \bar\partial \bar\partial^\dagger$

(2). Small time expansion of $K_t^\pm(z, z)$

(3). Gauss-Bonnet $\chi(X) = \frac{i}{2\pi} \int_X F_{\bar z z} \, dz \wedge d\bar z = 2 - 2g$. $g = \#$ holes.

$F_{\bar z z}$: curvature of $g_{\bar z z}$.

Proof of Gauss-Bonnet

- Elements of Riemannian Geometry

X: (real) manifold of dim n. $X = \bigcup X_\alpha$. $\Phi_\alpha : X_\alpha \longrightarrow B \subseteq \mathbb{R}^n$. $\Phi_\alpha \circ \Phi_\beta^{-1}$ smooth with smooth metric. Jacobian $\neq 0$.

Locally, on each chart, $\{x^i, i = 1, \cdots, n\}$ be local charts. A metric is

$$ds^2 = \Sigma_{ij} \, g_{ij} \, dx^i dx^j$$

with $g_{ij}(x)$ a symmetric, positive definite metric, transforming in such a way as to make ds^2 invariant. i.e. on $X_\alpha \cap X_\beta$ $g_{ij}(x_\alpha) dx_\alpha^i dx_\alpha^j = g_{kl}(x_\beta) dx_\beta^k dx_\beta^l$

$\Leftrightarrow g_{kl}^\beta(x_\beta) \frac{\partial x_\beta^k}{\partial x_\alpha^i} \frac{\partial x_\beta^l}{\partial x_\alpha^j} = g_{ij}^\alpha(x_\alpha)$.

Levi-Civita Connection

A vector field V on X corresponds to $(V^i(x_\alpha) \mid i = 1, \cdots, n)$, i.e. a vector valued function on X_α, transforming in such a way that $\{V^i \frac{\partial}{\partial x_\alpha^i}\}$ is invariant.

i.e. on $X_\alpha \cap X_\beta$, $V_\alpha^i(x_\alpha) \frac{\partial}{\partial x_\alpha^i} = V_\beta^j(x_\beta) \frac{\partial}{\partial x_\beta^j} \Leftrightarrow V_\alpha^i(x_\alpha) = V_\beta^j \frac{\partial x_\alpha^i}{\partial x_\beta^j}$

Fix a metric $ds^2 = g_{ij} dx^i dx^j$

Claim: $\exists! \{\Gamma_{ij}^k \mid i, j, k = 1, \cdots, n\}$ so that $((\nabla_m V)^i \triangleq \partial_m V^i + \Gamma_{mk}^i V^k)$ is a vector field and: 1). $\partial_m\langle V, W\rangle = \langle \nabla_m V, W\rangle + \langle V, \nabla_m W\rangle$

2). $\Gamma_{ij}^k = \Gamma_{ji}^k$ (torsion free)

Indeed Γ_{ij}^k satisfying these two conditions are computed to be

$$\Gamma_{ij}^k = \frac{1}{2} g^{k\ell}(\partial_i g_{\ell j} + \partial_j g_{i\ell} - \partial_\ell g_{ij}).$$

Proof. 1) $\Rightarrow \partial_m(g_{k\ell}V^k W^\ell) = (\partial_m V^k) g_{k\ell} W^\ell + g_{k\ell}\Gamma_{mp}^k V^p W^\ell + V^k g_{k\ell}(\partial_m W^\ell) + g_{k\ell} V^k \Gamma_{mp}^\ell W^p$

$\Rightarrow (\partial_m g_{k\ell}) V^k W^\ell = g_{k\ell}\Gamma_{mp}^k V^p W^\ell + g_{k\ell}\Gamma_{mp}^\ell V^k W^p = g_{p\ell}\Gamma_{mk}^p V^k W^\ell + g_{kp}\Gamma_{m\ell}^p V^k W^\ell$

$\Rightarrow \partial_m g_{k\ell} = g_{p\ell}\Gamma_{mk}^p + g_{kp}\Gamma_{m\ell}^p.$ (a)

Permuting m, k, ℓ, we obtain:

$\partial_k g_{\ell m} = g_{pm}\Gamma_{k\ell}^p + g_{\ell p}\Gamma_{km}^p$ (b)

$\partial_\ell g_{mk} = g_{pk}\Gamma_{\ell m}^p + g_{mp}\Gamma_{\ell k}^p$ (c)

(b)+(c)−(a), using 2), we obtain:

$2 g_{pm}\Gamma_{k\ell}^p = \partial_\ell g_{mk} + \partial_k g_{\ell m} - \partial_m g_{k\ell}$

Multiplying both sides by $\frac{1}{2}g^{qm}$ and summing over p, we obtain:

$$\Gamma_{k\ell}^q = \frac{1}{2} g^{qm}(\partial_\ell g_{mk} + \partial_k g_{\ell m} - \partial_m g_{k\ell})$$

This computation also establishes the uniqueness of $\{\Gamma_{ij}^k\}$ once the metric is fixed. □

The ∇_m's don't commute, and we have:

$$[\nabla_\ell, \nabla_m] V^i = R_{\ell m}{}^i{}_p V^p.$$

Def. (Curvatures)

1). $R_{\ell m}{}^i{}_p = \partial_\ell \Gamma_{mp}^i - \partial_m \Gamma_{\ell p}^i + \Gamma_{\ell q}^i \Gamma_{mp}^q - \Gamma_{mq}^i \Gamma_{\ell p}^q$ is the Riemannian curvature

2). $R_{mp} = R_{\ell m}{}^\ell{}_p$ is the Ricci curvature.

3). $R = g^{pm} R_{mp}$ is the scalar curvature.

Note that our convention for raising and lowering indices: $V^i \longrightarrow V_j \triangleq V^i g_{ij}$. for example, $R = R^p{}_p = g^{pm}R_{mp} = R_m{}^m$.

Example: X a Riemann surface, with a metric $g_{\bar{z}z}$ on $\Lambda^{-1,0}$ gives rise to a Riemannian metric on X. We set up a dictionary.

- Dictionary. On a holomorphic chart $z = x^1 + ix^2 \leftrightarrow (x^1, x^2)$

1). $V \in \Gamma(X, \Lambda^{-1,0})$, $V = V\partial_z$. $\longleftrightarrow$ v: vector field $V+\bar{V}$

Explicitly, $V\partial_z = (V^1 + iV^2)\cdot\frac{1}{2}(\partial_{x^1} - i\partial_{x^2}) = \frac{1}{2}(V^1\partial_{x^1} + V^2\partial_{x^2}) + \frac{i}{2}(V^2\partial_{x^1} - V^1\partial_{x^2})$

$\Rightarrow v = V^1\partial_{x^1} + V^2\partial_{x^2}$. Conversely, define $J: V^1\partial_{x^1} + V^2\partial_{x^2} \mapsto V^2\partial_{x^1} - V^1\partial_{x^2}$,

then $V = \frac{1}{2}(v + iJv)$.

2). A metric $g_{\bar{z}z}$ on $\Lambda^{-1,0}$: $\|V\| = V\bar{V}g_{\bar{z}z}$. Define $\|v\|^2 \triangleq \|V\|^2 = ((V^1)^2 + (V^2)^2)\, g_{\bar{z}z}$

Compared with $\|v\|^2 = \Sigma\, g_{ij} v^i v^j \Rightarrow ds^2 = g_{ij}dx^i dx^j = g_{\bar{z}z}((dx^1)^2 + (dx^2)^2)$

Thus $(g_{\bar{z}z}) \longleftrightarrow ds^2 = g_{\bar{z}z}((dx^1)^2 + (dx^2)^2)$

3). Connection

$\nabla_z V$	$\longleftrightarrow$	$\nabla_1 v = (\nabla_z + \nabla_{\bar{z}})(V + \bar{V})$
(Chern connection defined by $g_{\bar{z}z}$)		Levi-Civita connection w.r.t. $ds^2 = g_{\bar{z}z}((dx^1)^2 + (dx^2)^2)$

Dependence of Γ_{ij}^k and $Rm^i{}_p$ on the metric g_{ij}.

We determine the variations of Γ_{ij}^k and $Rm^i{}_p$ under variations of g. (The derivative (linear approximation) is simpler than the function itself). Since $g_{ij} \rightsquigarrow \Gamma_{ij}^k$, $\delta g_{ij} \rightsquigarrow \delta\Gamma_{ij}^k$.

Key variation formulae: (to be proven later)

1). $\delta\Gamma_{ij}^k = \frac{1}{2}g^{kl}(\nabla_j \delta g_{il} + \nabla_i \delta g_{lj} - \nabla_l \delta g_{ij})$

2). $\delta R_{jm}{}^l{}_k = \nabla_j(\delta\Gamma_{mk}^l) - \nabla_m(\delta\Gamma_{jk}^l)$

3). $\delta R_{ml} = \nabla_j(\delta\Gamma_{ml}^j) - \frac{1}{2}\nabla_m\nabla_l(g^{jp}\delta g_{pj})$

4). $\delta R = -\delta g^{lm}R_{ml} + \nabla^j\nabla^l\delta g_{lj} - \Delta(g^{ml}\delta g_{lm})$

where $\delta g^{ij} \triangleq (\delta g_{kl})g^{ki}g^{lj}$.

Using these formula, we claim:

Define the functional on the space of all Riemannian metrics on X.

$$g_{ij} \mapsto I(g) = \frac{1}{2\pi}\int_X (R\sqrt{g})\,dx\,,$$

then $\delta I(g) = 0$ for any δg_{ij}.

Pf of claim:

$$\delta I(g) = \frac{1}{2\pi}\int_X \delta(R\sqrt{g})\,dx = \frac{1}{2\pi}\int_X (\delta R\cdot\sqrt{g} + \tfrac{1}{2}\delta g/\sqrt{g}\cdot R)\,dx$$

where $g = \det(g_{ij})$. Note that $\delta\log(\det(g_{ij})) = g^{ij}\delta g_{ji}$.

(Indeed, we may assume that $(g_{ij}) = \mathrm{diag}(\lambda_1,\cdots,\lambda_n) \Rightarrow \log(\det(g_{ij})) = \Sigma\log\lambda_i \Rightarrow \delta\log\det(g_{ij})$
$= \Sigma\frac{\delta\lambda_i}{\lambda_i} = tr((g_{ij})^{-1}\delta(g_{ij})) = g^{ji}\delta g_{ij}$.)

$$\Rightarrow \delta I(g) = \frac{1}{2\pi}\int_X (\delta R + \tfrac{1}{2}(\delta\log g)\cdot R)\cdot\sqrt{g}\,dx$$
$$= \frac{1}{2\pi}\int_X (-\delta g^{lm}R_{ml} + \nabla^j\nabla^l\delta g_{lj} - \Delta(g^{ml}\delta g_{lm}) + \tfrac{1}{2}g^{ij}\delta g_{ji}R)\sqrt{g}\,dx$$
$$= \frac{1}{2\pi}\int_X (-\delta g^{lm}(R_{ml} - \tfrac{1}{2}g_{ml}R) + \nabla^j\nabla^l\delta g_{lj} - \Delta(g^{ml}\delta g_{ml}))\sqrt{g}\,dx$$

Observation: In case $\dim X = 2$ $\;-R_{ml} + \frac{1}{2}R\cdot g_{ml} = 0$ and $\nabla^j\nabla^l\delta g_{lj} - \Delta(g^{ml}\delta g_{ml})$ is exact thus the integral vanishes.

The first identity is a consequence of the symmetries of the Riemannian curvature tensor:

Recall that $[\nabla_j, \nabla_k]V^l = R_{jk}{}^l{}_m V^m$, and

- $R_{jkpm} = -R_{kjpm} = -R_{jkmp}$
- $R_{jkpm} = R_{pmjk}$.

In dimension 2, we calculate the Ricci tensor $R_{ij} = R_{ilj}{}^l = g^{lp}R_{iljp}$:

$R_{11} = g^{lp}R_{1l1p} = g^{22}R_{1212}$; $\quad R_{12} = g^{lp}R_{1l2p} = g^{21}R_{1221} = -g^{21}R_{1212} = R_{21}$

$R_{22} = g^{lp}R_{2l2p} = g^{11}R_{2121} = g^{11}R_{1212}$

$$\Rightarrow \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} = R_{1212}\begin{pmatrix} g^{22} & -g^{21} \\ -g^{12} & g^{11} \end{pmatrix} = \frac{1}{g}R_{1212}\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$

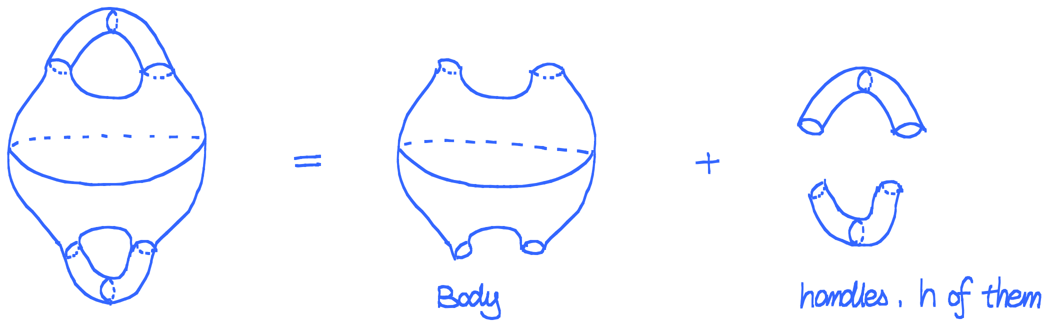
$$\Rightarrow R_{ml} - \tfrac{1}{2}Rg_{ml} = \tfrac{1}{g}R_{1212}(g_{ml} - \tfrac{1}{2}g^{ij}g_{ij}g_{ml}) = 0$$

Now the Gauss-Bonnet formula follows easily from this result:

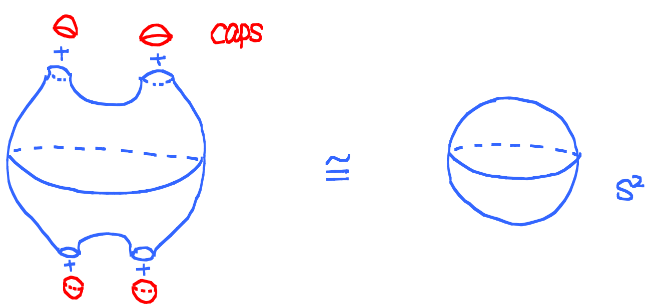
$$\frac{1}{2\pi}\int_X R\sqrt{g}\,dx = 2 - 2h$$

Here we temporarily write $h = \#$ holes of X, $g = \det(g_{ij})$

Decompose X as follows: (surgury)



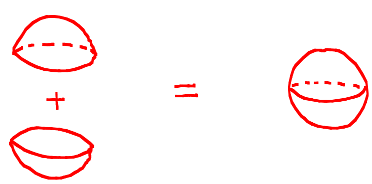
Now $\frac{1}{2\pi}\int_X R\sqrt{g}\,dx = \frac{1}{2\pi}\int_{\text{Body}} R\sqrt{g}\,dx + \sum \frac{1}{2\pi}\int_{\text{Handle}} R\sqrt{g}\,dx$. We calculate each integral :



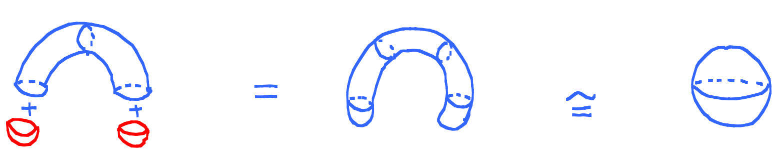
$$\frac{1}{2\pi}\int_{\text{Body}} R\sqrt{g}\,dx = \frac{1}{2\pi}\int_{\text{Body}+2h\text{ caps}} R\sqrt{g}\,dx - \frac{1}{2\pi}\int_{2h\text{ caps}} R\sqrt{g}\,dx .$$

By the previous claim, $\frac{1}{2\pi}\int_{\text{Body}+2h\text{ caps}} R\sqrt{g}\,dx = \frac{1}{2\pi}\int_{S^2} R'\sqrt{g'}\,dx$, where (g'_{ij}), g', R' are derived from the standard metric on S^2 as the unit sphere, and a direct calculation shows that $R' = 1$, thus $\frac{1}{2\pi}\int_{S^2} R'\sqrt{g'}\,dx = 4\pi/2\pi = 2$.

Moreover, for each cap $\frac{1}{2\pi}\int_{\text{cap}} R\sqrt{g}\,dx = \frac{1}{2}\left(\frac{1}{2\pi}\int_{\text{Sphere}} R\sqrt{g}\right) = \frac{1}{2}\cdot 2 = 1$



Lastly, for each handle,



$$\frac{1}{2\pi}\int_{\text{handle}} R\sqrt{g}\,dx = \frac{1}{2\pi}\int_{\text{handle}+2\text{caps}} R\sqrt{g}\,dx - \frac{1}{2\pi}\int_{2\text{caps}} R\sqrt{g}\,dx$$
$$= 2 - 2\cdot 1 = 0$$

Altogether : $\frac{1}{2\pi}\int_X R\sqrt{g}\,dx = \frac{1}{2\pi}\int_{\text{Body}} R\sqrt{g}\,dx + \sum \frac{1}{2\pi}\int_{\text{Handle}} R\sqrt{g}\,dx$.
$$= 2 - 2h$$

Main valuation formulae.

Now let's derive our key variational formulae. First of all, assuming 1) and 2)

1). $\delta\Gamma_{ij}^{k} = \frac{1}{2} g^{kl}(\nabla_j \delta g_{il} + \nabla_i \delta g_{lj} - \nabla_l \delta g_{ij})$

2). $\delta R_{jm}{}^{l}{}_{k} = \nabla_j(\delta\Gamma_{mk}^{l}) - \nabla_m(\delta\Gamma_{jk}^{l})$

we shall evaluate δR_{mk} and δR:

$$\delta R_{mk} = \delta(R_{jm}{}^{j}{}_{k}) = \delta(\nabla_j \delta\Gamma_{mk}^{j} - \nabla_m \delta\Gamma_{jk}^{j})$$

However, note that $\delta\Gamma_{jk}^{j} = \frac{1}{2} g^{jp}(\nabla_k \delta g_{jp} + \nabla_j \delta g_{pk} - \nabla_p \delta g_{jk})$

$$= \frac{1}{2} g^{jp} \nabla_k \delta g_{jp}$$

$$= \frac{1}{2} \nabla_k (g^{jp} \delta g_{jp})$$

The second equality holds because $\nabla_j \delta g_{pk} - \nabla_p \delta g_{jk}$ is antisymmetric in p and j while g^{jp} is symmetric. The last equality holds since $\nabla_j g_{pk} = 0$, which follows from

$$\partial_j g_{pk} = g_{pl}\Gamma_{jk}^{l} + g_{lk}\Gamma_{jp}^{l}$$

(known as Ricci's lemma, see the general remark on covariant differentiation of tensors.)

Thus $\delta R_{mk} = \nabla_j(\delta\Gamma_{mk}^{j}) - \frac{1}{2}\nabla_m\nabla_k(g^{jp}\delta g_{jp})$, as claimed in 3).

Next. $\delta R = \delta(g^{km} R_{mk})$

$$= (\delta g^{km}) \cdot R_{mk} + g^{km}\delta R_{mk}$$

$$= -g^{kp}(\delta g_{pq}) g^{qm} R_{mk} + g^{km}(\nabla_j(\delta\Gamma_{mk}^{j}) - \tfrac{1}{2}\nabla_m\nabla_k(g^{jp}\delta g_{jp})).$$

(Here we used that $G \cdot G^{-1} = Id \Rightarrow (\delta G) \cdot G^{-1} + G \cdot \delta(G^{-1}) = 0 \Rightarrow \delta G^{-1} = -G^{-1}(\delta G) G^{-1}$).

$$= -\delta g^{km} R_{mk} + \nabla_j(g^{km}\delta\Gamma_{mk}^{j}) - \tfrac{1}{2}\Delta(g^{jp}\delta g_{jp})$$

(where by definition, $\Delta f \triangleq g^{km}\nabla_m\nabla_k f = g^{km}\nabla_m(\partial_k f)$, for any smooth f).

We compute $g^{km}\delta\Gamma_{mk}^{j}$:

$$g^{km}\delta\Gamma_{mk}^{j} = g^{km} \cdot \tfrac{1}{2} g^{jp}(\nabla_k \delta g_{mp} + \nabla_m \delta g_{pk} - \nabla_p \delta g_{mk})$$

$$= g^{km} g^{jp} \nabla_k \delta g_{mp} - \tfrac{1}{2} g^{jp}\nabla_p(g^{km}\delta g_{mk})$$

(since $\nabla_k \delta g_{mp} + \nabla_m \delta g_{pk}$ is symmetric in k, p and so is g^{km}; $\nabla_p g^{km} = 0$.)

$$= \nabla_k(\delta g^{kj}) - \tfrac{1}{2} g^{jp}\nabla_p(g^{km}\delta g_{mk})$$

Plugging in, we obtain:

$$\delta R = -\delta g^{km} R_{mk} + \nabla_j\nabla_k(\delta g^{kj}) - \Delta(g^{jp}\delta g_{pj})$$

which is the key variational formula 4), with the term $\nabla_j\nabla_k(\delta g^{kj}) - \Delta(g^{jp}\delta g_{pj})$ being an exact term.

Thirdly, we calculate $\delta R_{jm}{}^{\ell}{}_{k}$. One way of doing this is via the formula

$$R_{jm}{}^{\ell}{}_{k} = \partial_j \Gamma^{\ell}_{mk} - \partial_m \Gamma^{\ell}_{jk} + \Gamma^{\ell}_{jp}\Gamma^{p}_{mk} - \Gamma^{\ell}_{mp}\Gamma^{p}_{jk}.$$

Instead, we shall use the defining equation below to simplify computations:

$$[\nabla_j, \nabla_m] V^{\ell} = R_{jm}{}^{\ell}{}_{k} V^{k}$$

$\Rightarrow \delta([\nabla_j, \nabla_m] V^{\ell}) = \delta(R_{jm}{}^{\ell}{}_{k} V^{k}) = (\delta R_{jm}{}^{\ell}{}_{k}) V^{k}$, since $\{V^k\}$ is a fixed vector field thus independent of variations. On the other hand:

$$\begin{aligned}\delta([\nabla_j, \nabla_m] V^{\ell}) &= \delta(\nabla_j \nabla_m V^{\ell} - \nabla_m \nabla_j V^{\ell}) \\ &= (\delta\nabla_j)\nabla_m V^{\ell} + \nabla_j(\delta\nabla_m)V^{\ell} - (\delta\nabla_m)\nabla_j V^{\ell} - \nabla_m(\delta\nabla_j)V^{\ell} \\ &= ((\delta\nabla_j)\nabla_m V^{\ell} - \nabla_m(\delta\nabla_j)V^{\ell}) + (\nabla_j(\delta\nabla_m)V^{\ell} - (\delta\nabla_m)\nabla_j V^{\ell}) \\ &= [\delta\nabla_j, \nabla_m]V^{\ell} + [\nabla_j, \delta\nabla_m]V^{\ell}\end{aligned}$$

Observation: (to be shown in the remark after the verification of these formulae)
If W^{ℓ}_{m} is a (1,1)-tensor, then $\nabla_j(W^{\ell}_{m}) = \partial_j W^{\ell}_{m} + \Gamma^{\ell}_{jk} W^{k}_{m} - \Gamma^{k}_{jm} W^{\ell}_{k}$. Hence viewing $\delta\nabla_j$ as an (infinitesimal) difference of two connections, which is a tensor, we have:

$$(\delta\nabla_j)\nabla_m V^{\ell} = \underline{(\delta\Gamma^{\ell}_{jk})\nabla_m V^{k}} - \delta\Gamma^{k}_{jm}\nabla_k V^{\ell}.$$

while $\nabla_m(\delta\nabla_j)V^{\ell} = \nabla_m(\delta\Gamma^{\ell}_{jk}V^{k}) = (\nabla_m\delta\Gamma^{\ell}_{jk})\cdot V^{k} + \underline{\delta\Gamma^{\ell}_{jk}\cdot\nabla_m V^{k}}$. Note the underlined terms cancel when substracting. Furthermore, the expression

$$[\delta\nabla_j, \nabla_m]V^{\ell} + [\nabla_j, \delta\nabla_m]V^{\ell} = [\delta\nabla_j, \nabla_m]V^{\ell} - [\delta\nabla_m, \nabla_j]V^{\ell}$$

is anti-symmetric in j, m. It follows that $-\delta\Gamma^{k}_{jm}\nabla_k V^{\ell}$, which is symmetric in j, m, will eventually get cancelled in the final expression. Hence

$$\delta R_{jm}{}^{\ell}{}_{k} V^{k} = (-(\nabla_m\delta\Gamma^{\ell}_{jk}) + \nabla_j\delta\Gamma^{\ell}_{mk})V^{k}$$

Since this is true for arbitrary vector fields, we obtain 2).

$$\delta R_{jm}{}^{\ell}{}_{k} = -(\nabla_m\delta\Gamma^{\ell}_{jk}) + \nabla_j\delta\Gamma_{mk}$$

Finally, it remains to show 1), the variation of Γ^{k}_{ij}.

Recall that $\Gamma^{k}_{ij} = \frac{1}{2}g^{k\ell}(\partial_i g_{\ell j} + \partial_j g_{i\ell} - \partial_\ell g_{ij})$, consequently:

$$\delta\Gamma^{k}_{ij} = \tfrac{1}{2}g^{k\ell}(\partial_i\delta g_{\ell j} + \partial_j\delta g_{i\ell} - \partial_\ell\delta g_{ij}) - \tfrac{1}{2}g^{ks}\delta g_{sr}g^{r\ell}(\partial_i g_{\ell j} + \partial_j g_{i\ell} - \partial_\ell g_{ij})$$

Since we know that $\delta\Gamma^{k}_{ij}$ should be an (infinitesimal) tensor, the terms as $\partial_i\delta g_{\ell j}$ will have to be replace by tensors like $\nabla_j\delta g_{\ell j}$, and formula 1) follows. More explictly

$$\tfrac{1}{2}g^{ks}\delta g_{sr}g^{r\ell}(\partial_i g_{\ell j} + \partial_j g_{i\ell} - \partial_\ell g_{ij}) = g^{ks}\delta g_{sr}\Gamma^{r}_{ij}$$

and from taking covariant derivative of (2,0)-tensor δg_{ij}, we have:

$$\partial_i(\delta g_{\ell j}) = \nabla_i(\delta g_{\ell j}) + \underline{\Gamma_{i\ell}^s \delta g_{sj}} + \Gamma_{ij}^s \delta g_{\ell s}$$
$$\partial_j(\delta g_{i\ell}) = \nabla_j(\delta g_{i\ell}) + \Gamma_{ji}^s \delta g_{s\ell} + \underline{\underline{\Gamma_{j\ell}^s \delta g_{is}}}$$
$$-\partial_\ell(\delta g_{ij}) = -\nabla_\ell(\delta g_{ij}) - \underline{\Gamma_{\ell i}^s \delta g_{sj}} - \underline{\underline{\Gamma_{\ell j}^s \delta g_{is}}}$$

Summing up, it gives:

$$\delta\Gamma_{ij}^k = \tfrac{1}{2} g^{k\ell}(\nabla_i \delta g_{\ell j} + \nabla_j \delta g_{i\ell} - \nabla_\ell \delta g_{ij}) + g^{k\ell}\delta g_{\ell s}\Gamma_{ij}^s - g^{ks}\delta g_{sr}\Gamma_{ij}^r$$
$$= \tfrac{1}{2} g^{k\ell}(\nabla_i \delta g_{\ell j} + \nabla_j \delta g_{i\ell} - \nabla_\ell \delta g_{ij})$$

and finishes the proof of i).

Rmk: Throughout, we used the covariant derivative of tensor fields, which is uniquely extended from covariant derivative of vector fields by the rule:

a). $\nabla_i T_{i_1\cdots i_s}^{j_1\cdots j_r}$ is a tensor of type $(r, s+1)$

b). $\nabla_i f = \partial_i f$, $\nabla_i V^j = \partial_i V^j + \Gamma_{ik}^j V^k$

c). $\nabla_i (T_{i_1\cdots i_s}^{j_1\cdots j_r} \cdot R_{\ell_1\cdots \ell_v}^{k_1\cdots k_u}) = (\nabla_i T_{i_1\cdots i_s}^{j_1\cdots j_r}) R_{\ell_1\cdots \ell_v}^{k_1\cdots k_u} + T_{i_1\cdots i_s}^{j_1\cdots j_r} \cdot \nabla_i R_{\ell_1\cdots \ell_v}^{k_1\cdots k_u}$

d). ∇_i commutes with contraction of indices.

For example: $(\partial_i V^j) W_j + V^j(\partial_i W_j) = \partial_i (V^j W_j)$ by b)

$= \nabla_i (V^j W_j)$ by c) and d)

$= (\nabla_i V^j) W_j + V^j(\nabla_i W_j)$

$= (\partial_i V^j) W_j + \Gamma_{ik}^j V^k W_j + V^j(\nabla_i W_j)$

$\Rightarrow V^j(\nabla_i W_j) = V^j \partial_i W_j - V^j \Gamma_{ij}^k W_k$

$\Rightarrow \nabla_i W_j = \partial_i W_j - \Gamma_{ij}^k W_k$.

Sketch of spectral decomposition of Laplacian.

Recall our setting: $L \to X$, holomorphic line bundle with a metric h. $g_{\bar{z}\bar{z}}$ is a metric on $\Lambda^{-1,0}$, we have $\bar\partial$ and its formal adjoint:

$$\Gamma(X, L) \underset{\bar\partial^+}{\overset{\bar\partial}{\rightleftarrows}} \Gamma(X, L\otimes\Lambda^{0,1})$$

adjoint in the sense that $\langle \bar\partial\varphi, \psi\rangle = \langle \varphi, \bar\partial^+\psi\rangle$.

- Key fact: A Priori Estimate: $\forall \varphi \in \Gamma(X, L)$ a smooth section

$$\|\varphi\|_{(2)} \le C\cdot(\|\Delta\varphi\|_{(0)} + \|\varphi\|_{(1)})$$

where $\|\varphi\|_{(s)}$ denotes the Sobolev norm defined below:

Def: (Sobolev norm $\|\varphi\|_{(s)}$)

$\|\varphi\|^2_{(0)} \triangleq \|\varphi\|^2$, the L^2- norm of φ.

$\|\varphi\|^2_{(1)} \triangleq \|\varphi\|^2_{(0)} + \|\nabla_z\varphi\|^2 + \|\nabla_{\bar z}\varphi\|^2$

$\|\varphi\|^2_{(2)} \triangleq \|\varphi\|^2_{(1)} + \|\nabla_z\nabla_z\varphi\|^2 + \|\nabla_{\bar z}\nabla_{\bar z}\varphi\|^2 + \|\nabla_{\bar z}\nabla_z\varphi\|^2 + \|\nabla_{\bar z}\nabla_{\bar z}\varphi\|^2$

...

Here $\|\cdot\|$ are various L^2-norms in different line bundles. For instance

$\nabla_{\bar z}\varphi \in \Gamma(X, L\otimes\Lambda^{1,0})$, $\|\nabla_{\bar z}\varphi\|^2 \triangleq \int_X \nabla_{\bar z}\varphi\, \overline{\nabla_{\bar z}\varphi}\, h$

$\nabla_z\nabla_{\bar z}\varphi \in \Gamma(X, L\otimes\Lambda^{0,1}\otimes\Lambda^{1,0})$, $\|\nabla_z\nabla_{\bar z}\varphi\|^2 \triangleq \int_X \nabla_z\nabla_{\bar z}\varphi\cdot\overline{\nabla_z\nabla_{\bar z}\varphi}\, h\, g^{z\bar z}\, dz d\bar z$

In general, in n-dimensional case $\|\varphi\|_{(s)} \triangleq \sum_{0\le p\le s}\int_X g^{i_1\bar i_1}\cdots g^{i_p\bar i_p}\, \nabla_{i_1}\cdots\nabla_{i_p}\varphi\cdot\overline{\nabla_{i_1}\cdots\nabla_{i_p}\varphi}\, h\, dv$.

where dv is the volume element. $i_1,\cdots,i_p \in \{1,\cdots,n,\bar 1,\cdots,\bar n\}$.

W.r.t. $\|\cdot\|_{(s)}$, we define $H_{(s)} \triangleq \overline{\Gamma(X,L)}$, the completion, the Sobolev space. (This is a stronger sense of convergence by giving restrictions on all $\le s$th derivatives).

A trivial observation: $\|\Delta\varphi\|_{(0)} \le C\|\varphi\|_{(2)}$.

Indeed, recall that $\bar\partial^+\varphi = -g^{z\bar z}\nabla_z\varphi$, $\Delta\varphi = -g^{z\bar z}\nabla_z\nabla_{\bar z}\varphi$, which is just a rescaling of $\nabla_z\nabla_{\bar z}\varphi$.

General a priori estimate: $\forall s\in\mathbb{N}\cup\{0\}$, $\exists C_s>0$ s.t. $\forall\varphi\in\Gamma(X,L)$,

$$\|\varphi\|^2_{(s+2)} \le C_s(\|\Delta\varphi\|^2_{(s)} + \|\varphi\|^2_{(s+1)})$$

Two basic lemmas about Sobolev spaces.

- Sobolev lemma: X: a manifold of real dimension n. Then $\forall\varphi\in\Gamma(X,L)$.
  (a). $\|\varphi\|_{C^k} \le C\cdot\|\varphi\|_{(k+s)}$, $\forall s>\frac{n}{2}$
  (b). $H_{(k+s)} \subseteq C^k(X,L)$
  (c). As a consequence of (b), $\bigcap_{s\ge0} H_{(s)} = \Gamma(X,L)$

- Rellich's lemma. X: compact manifold. Let $t>s$. Then a sequence $\{\varphi_j\}$ in $H_{(t)}$ with $\|\varphi_j\| \le C$ (bounded) has a subsequence converging in $H_{(s)}$. (AKA: $H_{(s)} \hookrightarrow H_{(t)}$ being compact).

Using A Priori Estimate:

Basic observations:

(a). We can extend $\Delta: \Gamma(X,L) \to \Gamma(X,L)$ to $\Delta: H_{(2)} \to H_{(0)}$, a continuous map. Namely, $\forall \varphi \in H_{(2)}$, $\varphi = \lim \varphi_n$ w.r.t. $\|\cdot\|_{(2)}$, then by the trivial observation made above:

$$\|\Delta\varphi_i - \Delta\varphi_j\|_{(0)} \leq C \cdot \|\varphi_i - \varphi_j\|_{(2)} \to 0 \quad \text{as } (i,j \to \infty).$$

(b). Define the range of Δ: $\text{Range}\Delta = \{\Delta\varphi \mid \varphi \in H_{(2)}\}$. Then a priori estimate implies that $\text{Range}\Delta$ is closed.

Claim: A priori estimate $\Rightarrow \exists C>0$ s.t. $\forall \varphi \in \Gamma(X,L)$, $\varphi \perp \text{Ker}\Delta$, we have

$$\|\varphi\|_{(2)} \leq C \cdot \|\Delta\varphi\|_{(0)}. \quad (*)$$

Then $\forall \Phi \in \text{Range}\Delta$, $\Phi_n = \Delta\varphi_n$, $\varphi_n \perp \text{Ker}\Delta$, and $\Phi_n \to \Phi$ in $H_{(2)}$. Now by $(*)$ $\|\varphi_n - \varphi_m\|_{(2)} \leq C \cdot \|\Delta\varphi_n - \Delta\varphi_m\|_{(0)} = C\|\Phi_n - \Phi_m\|_{(0)} \to 0 \Rightarrow \{\varphi_n\}$ converges in H_2, say, $\varphi_n \to \varphi$. Then $\Delta\varphi = \lim \Delta\varphi_n = \Phi$.

Pf of claim:

Otherwise, $\forall n \in \mathbb{N}$, $\exists \varphi_n \in \Gamma(X,L)$, $\varphi_n \perp \text{Ker}\Delta$ with $\|\varphi_n\|_{(2)} \geq n\|\Delta\varphi_n\|_{(0)}$. Define: $\psi_n \triangleq \varphi_n / \|\varphi_n\|_{(2)}$. Then $\|\Delta\psi_n\|_{(0)} \leq \frac{1}{n}$, $\|\psi_n\|_{(2)} = 1$.

By a priori estimate, $\|\psi_n - \psi_m\|_{(2)} \leq D \cdot (\|\Delta\psi_n - \Delta\psi_m\|_{(0)} + \|\psi_n - \psi_m\|_{(1)})$ for some $D>0$. Rellich's lemma $\Rightarrow \exists$ subsequence, which we may assume to be $\{\psi_n\}$ to start with, s.t. $\|\psi_n - \psi_m\|_{(1)} \to 0$ as $n,m \to \infty$. $\Rightarrow \psi_n \to \psi$ in $H_{(2)}$, with $\|\psi\|_{(2)} = \lim \|\psi_n\|_{(2)} = 1$. and $\Delta\psi = \lim_n \Delta\psi_n = 0 \Rightarrow \psi \in \text{Ker}\Delta \subseteq H_{(2)}$. On the other hand, $\psi_n \in (\text{Ker}\Delta)^\perp \Rightarrow \psi \in (\text{Ker}\Delta)^\perp \subseteq H_{(2)}$. It follows that $\psi = 0$. contradiction to $\|\psi\|_{(2)} = 1$.

Now $\text{Range}\Delta$ being closed allows us to construct an "inverse" of Δ as follows. By basic Hilbert space theory, $H_{(0)} = \text{Range}\Delta \oplus (\text{Range}\Delta)^\perp$. (In general, if we don't know that $\text{Range}\Delta$ is closed, we can only say that $H_{(0)} = \overline{\text{Range}\Delta} \oplus (\text{Range}\Delta)^\perp$.)

Given $\psi \in H_{(0)}(X,L)$. $\psi = \Delta\varphi + \psi^\perp$, and φ is unique by requiring that $\varphi \perp \text{Ker}\Delta$ (this will be explained in more detail in the next chapter). Then define $G\psi \triangleq \varphi$. i.e. $\Delta G = I - Pr_{(\text{Range}\Delta)^\perp}$.

Finally, $G: H_{(0)} \to H_{(2)} \hookrightarrow H_{(0)}$ is a self-adjoint operater on a Hilbert space and we may talk about eigenvalues of G. Riesz-Schauder thm $\Rightarrow \exists$ orthonormal basis of eigen-functions $\{\varphi_n\}$

In general, we can construct extensions of Δ to $H_{(s+2)} \to H_{(s)}$ and $G: H_{(s)} \longrightarrow H_{(s+2)}$, then the general a priori estimate shows that RangeΔ is closed and G is similarly constructed. Now:

$$G\varphi_n = \lambda_n \varphi_n \in H_{(0)} \supseteq H_{(2)}$$

$$\Rightarrow \varphi_n \in H_{(2)} \Rightarrow \varphi_n \in H_{(4)} \Rightarrow \cdots \Rightarrow \varphi_n \in \cap_n H_{(n)} = \Gamma(X,L),$$ by Sobolev's lemma.

A sketch of A Priori Estimate on Riemann surfaces.

$$\forall \varphi \in \Gamma(X,L), \quad \|\Delta\varphi\|_{(0)} = \int_X g^{\bar{z}z} \nabla_z \nabla_{\bar{z}} \varphi \, \overline{\nabla_z \nabla_{\bar{z}} \varphi} \, h \, g^{z\bar{z}} g_{z\bar{z}} \, dz d\bar{z}$$
$$= \int_X \nabla_z \nabla_{\bar{z}} \varphi \cdot \overline{\nabla_z \nabla_{\bar{z}} \varphi} \, h g^{\bar{z}z} dz d\bar{z}$$
$$= \|\nabla_z \nabla_{\bar{z}} \varphi\|^2$$

The remaining terms $\|\nabla_z \nabla_z \varphi\|^2$ can also be bounded by $\|\Delta\varphi\|_{(0)}$. For example:

$$\|\nabla_z \nabla_z \varphi\|^2 = \int_X \nabla_z \nabla_z \varphi \, \overline{\nabla_z \nabla_z \varphi} \, (g^{z\bar{z}})^2 h \, g_{z\bar{z}} \, dz d\bar{z}$$
$$= -\int_X \nabla_z \varphi \, \overline{\nabla_{\bar{z}} \nabla_z \nabla_z \varphi} \, g^{z\bar{z}} h \, dz d\bar{z} \quad \text{by integration by parts}$$
$$= -\int_X \nabla_z \varphi \, (\overline{\nabla_z \nabla_{\bar{z}} \nabla_z \varphi + R_{\bar{z}z} \nabla_z \varphi}) \, g^{z\bar{z}} h \, dz d\bar{z}$$
$$= \int_X \nabla_{\bar{z}} \nabla_z \varphi \, \overline{\nabla_{\bar{z}} \nabla_z \varphi} \, g^{z\bar{z}} h \, dz d\bar{z} + \text{terms in } \|\nabla_z \varphi\|^2$$
$$= \int_X \nabla_z \nabla_{\bar{z}} \varphi \cdot \nabla_z \nabla_{\bar{z}} \varphi \, g^{z\bar{z}} h \, dz d\bar{z} + \text{terms in } \|\nabla_{\bar{z}} \varphi\|^2 \text{ or } \|\nabla_z \varphi\|^2$$
$$\leq \|\Delta\varphi\|_{(0)} + C\|\varphi\|_{(1)}$$

$$\Rightarrow \|\varphi\|_{(2)} \leq C(\|\Delta\varphi\|_{(0)} + \|\varphi\|_{(1)}),$$ as asserted.

§5. Curvatures on Vector Bundles

Def. of Complex Manifolds

$X = \bigcup_\mu X_\mu$, $\phi_\mu : X_\mu \longrightarrow \mathbb{C}^n$: local coordinate charts , with $\phi_\mu \circ \phi_\nu^{-1}|_{\phi_\nu(X_\mu \cap X_\nu)}$ holomorphic, 1-1 and has invertible Jacobian.

Observation: Let $f : \Omega \longrightarrow \mathbb{C}$, $(z_1, \cdots, z_n) \mapsto f(z_1, \cdots, z_n)$, $\Omega \subseteq \mathbb{C}^n$. We say that f is holomorphic if f is holomorphic in each variable $z_1, \cdots, z_n$. Such f is characterized by:

Thm. (Hartog) f is holomorphic (in the above sense) iff f can be expanded as a power series near any point $\zeta \in \Omega$:

$$f(z) = \sum_{|\alpha|=0}^{\infty} C_\alpha (z - \zeta)^\alpha$$

for $|z - \zeta| < \varepsilon$, some $\varepsilon > 0$. □

Notation: α is the multi-index : $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$. $\zeta^\alpha \triangleq \zeta_1^{\alpha_1} \cdots \zeta_n^{\alpha_n}$. $|\alpha| \triangleq \alpha_1 + \cdots + \alpha_n$.

- There is no such characterization in the smooth category.

Notation: $F : \Omega \longrightarrow \mathbb{C}^n$. $F : (z_1, \cdots, z_n) \mapsto (f_1(z), \cdots, f_n(z))$. F is said to be holomorphic if each of its component f_i is holomorphic. The Jacobian of F is defined as:

$$\mathrm{Jac}(F) = \left(\frac{\partial f_i}{\partial z_j}\right)_{n \times n}$$

Holomorphic Vector Bundle

$E \xrightarrow{\pi} X$ is a holomorphic vector bundle of rank r

$\Leftrightarrow$ $X = \bigcup_\mu X_\mu$, with $\{(t_{\mu\nu}{}^\alpha{}_\beta)(z)$ on $X_\mu \cap X_\nu$, $1 \le \alpha, \beta \le r\}$ with

- $(t_{\mu\nu}{}^\alpha{}_\beta)_{r \times r}$ holomorphic and invertible.
- $t_{\mu\nu}{}^\alpha{}_\beta \, t_{\nu\rho}{}^\beta{}_\gamma = t_{\mu\rho}{}^\alpha{}_\gamma$ on $X_\alpha \cap X_\beta \cap X_\gamma$

(Recall that for holomorphic line bundles over a Riemann surface: $L \longrightarrow X$ $\Leftrightarrow$ $\{t_{\mu\nu}(z) : X_\mu \cap X_\nu \longrightarrow \mathbb{C}^*$, holomorphic ; $t_{\mu\nu}(z) t_{\nu\rho}(z) = t_{\mu\rho}(z)$ on $X_\mu \cap X_\nu \cap X_\rho\}$.)

(Smooth) Sections of $E \longrightarrow X$.

$\Gamma(X, E) \ni \varphi \Leftrightarrow \{\varphi_\mu^\alpha(z)$: smooth functions defined on X_μ, $1 \le \alpha \le r$, with $(\varphi_\mu^1, \cdots, \varphi_\mu^r)^t$ the $\mathbb{C}^r$-valued function satisfying $\varphi_\mu^\alpha(z) = t_{\mu\nu}{}^\alpha{}_\beta \, \varphi_\nu^\beta(z)$ on $X_\mu \cap X_\nu\}$.

Covariant Derivatives of Sections of Holomorphic Vector Bundles

$E \to X$: holomorphic vector bundle. $\Gamma(X,E) \ni \varphi = \{\varphi_\mu\}$

$$\bar\partial\varphi \triangleq \{\tfrac{\partial}{\partial \bar z^j}\varphi\, d\bar z^j\}$$
$$= \{\tfrac{\partial}{\partial \bar z_\mu^j}\varphi_\mu^\alpha d\bar z_\mu^j \text{ on } X_\mu\}$$

- How do they glue together?

Recall: $\varphi_\mu^\alpha = t_{\mu\nu}{}^\alpha{}_\beta\, \varphi_\nu^\beta$ on $X_\mu \cap X_\nu$.

$$\Rightarrow \frac{\partial}{\partial \bar z_\mu^j}\varphi_\mu^\alpha = \frac{\partial}{\partial \bar z_\mu^j}(t_{\mu\nu}{}^\alpha{}_\beta\, \varphi_\nu^\beta) = t_{\mu\nu}{}^\alpha{}_\beta \frac{\partial}{\partial \bar z_\mu^j}\varphi_\nu^\beta = t_{\mu\nu}{}^\alpha{}_\beta \frac{\partial \bar z_\nu^k}{\partial \bar z_\mu^j}\frac{\partial}{\partial \bar z_\nu^k}\varphi_\nu^\beta.$$

$$\Rightarrow \frac{\partial}{\partial \bar z_\mu^j}\varphi_\mu^\alpha = t_{\mu\nu}{}^\alpha{}_\beta \left(\frac{\partial}{\partial \bar z_\nu^k}\varphi_\nu^\beta\right)\frac{\partial \bar z_\nu^k}{\partial \bar z_\mu^j}$$

i.e. it's a section of $E\otimes\Lambda^{0,1}$, where $\Lambda^{0,1}$ is the vector bundle on X with transition functions $\tilde t_{\mu\nu}{}^k{}_j = \frac{\partial \bar z_\nu^k}{\partial \bar z_\mu^j}$ (transforms as row vectors, so that $\{\frac{\partial}{\partial \bar z_\mu^j}\varphi_\mu^\alpha d\bar z_\mu^j\}$ is invariant).

To differentiate in the z^j direction, we need a connection. An important connection is a unitary connection:

- Let $\{H_{\bar\beta\alpha}\}$ be a metric on $E\to X$, i.e. given $\varphi\in\Gamma(X,E)$, we associate it its length
$$\|\varphi\|^2 = \{\|\varphi\|_\mu^2 \triangleq (H_\mu)_{\bar\beta\alpha}(z)\varphi_\mu^\alpha\overline{\varphi_\nu^\beta} \text{ s.t. } \|\varphi\|_\mu^2 = \|\varphi\|_\nu^2 \text{ on } X_\mu\cap X_\nu\}$$

$$\Rightarrow (H_\mu)_{\bar\beta\alpha}\varphi_\mu^\alpha\overline{\varphi_\mu^\beta} = (H_\nu)_{\bar\gamma\delta}\varphi_\nu^\delta\overline{\varphi_\nu^\gamma} \quad \text{on } X_\mu\cap X_\nu$$

$$\Rightarrow (H_\mu)_{\bar\beta\alpha}t_{\mu\nu}{}^\alpha{}_\delta\varphi_\nu^\delta\,\overline{t_{\mu\nu}{}^\beta{}_\gamma}\,\overline{\varphi_\nu^\gamma} = (H_\nu)_{\bar\gamma\delta}\varphi_\nu^\delta\overline{\varphi_\nu^\gamma} \text{ on } X_\mu\cap X_\nu$$

Hence $(H_\mu)_{\bar\beta\alpha}t_{\mu\nu}{}^\alpha{}_\delta\,\overline{t_{\mu\nu}{}^\beta{}_\gamma} = (H_\nu)_{\bar\gamma\delta}$ (*)

Def. A metric on $E\to X$ is an assignment of $(H_\mu)_{\bar\beta\alpha}$ on X_μ s.t. (*) holds and $\|\varphi\|^2\geq 0$, "=" iff $\varphi=0$.

Observation: a short hand for a metric is that:

$$\|\varphi\|_\mu^2 = (\cdots\overline{\varphi_\mu^\alpha}\cdots)\left((H_\mu)_{\bar\alpha\beta}\right)\begin{pmatrix}\vdots\\ \varphi_\mu^\beta\\ \vdots\end{pmatrix}$$

so that if $\begin{pmatrix}\vdots\\ \varphi_\mu^\beta\\ \vdots\end{pmatrix} = (t_{\mu\nu}{}^\beta{}_\gamma)\begin{pmatrix}\vdots\\ \varphi_\nu^\gamma\\ \vdots\end{pmatrix}$, then

$$(H_\nu)_{\bar\gamma\delta} = \overline{t_{\mu\nu}{}^\beta{}_\gamma}\,(H_\mu)_{\bar\beta\alpha}\,t_{\mu\nu}{}^\alpha{}_\delta$$

i.e. if we define $(t_{\mu\nu}^\dagger)^{\bar\gamma}{}_{\bar\beta} = \overline{(t_{\mu\nu})^\beta{}_\gamma}$, then $H_\nu = t_{\mu\nu}^\dagger H_\mu t_{\mu\nu}$.

Now given a metric $H_{\bar\alpha\beta}$ on E, we can define the corresponding covariant derivative on $\Gamma(X,E)$:

Def: (Connection, unitary). $\varphi\in\Gamma(X,E)$, $\nabla\varphi \triangleq \{\nabla_j\varphi\, dz^j\}$, where $\nabla_j\varphi = H^{\alpha\bar\beta}\frac{\partial}{\partial z^j}(H_{\bar\beta\gamma}\varphi^\gamma)$.
Here $H^{\alpha\bar\beta}H_{\bar\beta\gamma} = \delta^\alpha{}_\gamma$. $\nabla: \Gamma(X,E)\longrightarrow\Gamma(X,E\otimes\Lambda^{1,0})$ is the unitary connection on E.

This definition makes sense since $\{H_{\bar\beta\gamma}\varphi^\gamma\}$ is a section of the bundle $E^\dagger$, which is anti-holomorphic and thus $\{\partial_j H_{\bar\beta\gamma}\varphi^\gamma\} \in \Gamma(X, E^\dagger \otimes \Lambda^{1,0})$. tensoring with $H^{\alpha\bar\beta}$ gives us a section of $E \otimes \Lambda^{1,0}$.

Now, $\nabla_j \varphi^\alpha = H^{\alpha\bar\beta}\{H_{\bar\beta\gamma}\partial_j\varphi^\gamma + (\partial_j H_{\bar\beta\gamma})\varphi^\gamma\}$

$$= \delta^\alpha{}_\gamma \partial_j \varphi^\gamma + (H^{\alpha\bar\beta}\partial_j H_{\bar\beta\gamma})\varphi^\gamma$$

$$= \partial_j\varphi^\alpha + (H^{\alpha\bar\beta}\partial_j H_{\bar\beta\gamma})\varphi^\gamma \qquad (*)$$

Denote $A_{j\gamma}^{\alpha} = H^{\alpha\bar\beta}\partial_j H_{\bar\beta\gamma}$, $1 \le \alpha, \beta \le r$, $1 \le j \le n$.

Def. (Connection, general form). A connection on E is an assignment $\{A_{j\gamma}^{\alpha}\}$ satisfying the requirement that $(*)$ defines a section in $\Gamma(X, E\otimes\Lambda^{1,0})$.

Thus, a unitary connection is a connection with $A_{j\gamma}^{\alpha} = H^{\alpha\bar\beta}\partial_j H_{\bar\beta\gamma}$. View $A_{j\gamma}^{\alpha}$ as entries of the matrix $A_j = H^{-1}\partial_j H$.

It's also convenient to introduce the connection form $A \triangleq A_j dz^j$. However, A is not globally defined, i.e. A_j doesn't transform as a global $(1,0)$-form under change of coordinates. This is readily seen since by $(*)$, $\nabla_j\varphi^\alpha$ transforms like a tensor while $\partial_j\varphi^\alpha$ doesn't.

In the special case of line bundles, $\operatorname{rank} E = 1$, $H = h$ is a complex scalar function, thus we have our former $A_j = h^{-1}\partial_j h = \partial_j(\log h)$.

Commutation Rules for Covariant Derivative.

We compute $\nabla_{\bar k}\nabla_j\varphi^\alpha - \nabla_j\nabla_{\bar k}\varphi^\alpha = \nabla_{\bar k}(\partial_j\varphi^\alpha + A_{j\gamma}^{\alpha}\varphi^\gamma) - \nabla_j(\partial_{\bar k}\varphi^\alpha)$

$$= \partial_{\bar k}\partial_j\varphi^\alpha + \partial_{\bar k}(A_{j\gamma}^{\alpha}\varphi^\gamma) - \partial_j\partial_{\bar k}\varphi^\alpha - A_{j\gamma}^{\alpha}\partial_{\bar k}\varphi^\gamma$$

$$= (\partial_{\bar k}A_{j\gamma}^{\alpha})\varphi^\gamma$$

Def. (Curvature of E w.r.t. $H_{\bar\alpha\beta}$)

$F_{\bar k j}{}^{\alpha}{}_{\gamma} \triangleq -\partial_{\bar k}A_{j\gamma}^{\alpha} = -\partial_{\bar k}(H^{\alpha\bar\beta}\partial_j H_{\bar\beta\gamma}) = [-\partial_{\bar k}(H^{-1}\partial_j H)]^\alpha{}_\gamma$ is the curvature associated with metric $H_{\bar\alpha\beta}$. Thus in summary, we have the key formula:

- $[\nabla_{\bar k}, \nabla_j]\varphi^\alpha = -F_{\bar k j}{}^{\alpha}{}_{\gamma}\varphi^\gamma$

Invariant Point of View.

Given a bundle $E \to X$, we can construct the bundle $\mathrm{End}(E) \to X$ as follows:

$\Gamma(X, \mathrm{End}E) \ni T \Leftrightarrow \{ T_{\mu\beta}^{\alpha}(z)$, smooth matrix-valued function defined on X_μ, satisfying the condition that $\{T_{\mu\beta}^{\alpha}\varphi_\mu^\beta\}$ is a globally defined section of $\Gamma(X,E)$ whenever $\varphi = \{\varphi_\mu^\alpha\} \in \Gamma(X,E)\}$

More explicitly, $T_{\mu\beta}^{\alpha}\varphi_\mu^\beta = t_{\mu\nu}{}^{\alpha}{}_{\gamma} T_{\nu\delta}^{\gamma}\varphi_\nu^\delta$ on $X_\mu \cap X_\nu$. But since $\varphi_\mu^\beta = t_{\mu\nu}{}^{\beta}{}_{\delta}\varphi_\nu^\delta$, we have

$$T_{\mu\beta}^{\alpha}\varphi_\mu^\beta = T_{\mu\beta}^{\alpha} t_{\mu\nu}{}^{\beta}{}_{\delta}\varphi_\nu^\delta = t_{\mu\nu}{}^{\alpha}{}_{\gamma} T_{\nu\delta}^{\gamma}\varphi_\nu^\delta$$

Hence:

$$T_{\mu\beta}^{\alpha} t_{\mu\nu}{}^{\beta}{}_{\delta} = t_{\mu\nu}{}^{\alpha}{}_{\gamma} T_{\nu\delta}^{\gamma}$$

Or more compactly, in matrix form: $T_\mu = t_{\mu\nu} T_\nu t_{\mu\nu}^{-1}$

Observation: $F_{\bar{k}j}$ is a section of $\Gamma(X, \mathrm{End}E)$. This follows by def. of $\mathrm{End}E$ and the key formula above.

We may also introduce the curvature form $F = F_{\bar{k}j}\, dz^j \wedge d\bar{z}^k$, which is a section of $\Gamma(X, \mathrm{End}E \otimes \Lambda^{1,1})$. First let's review:

Formalism of Differential Forms

Digression: de Rham complex

Recall that if X is a smooth manifold, then we have the notions:

- $\Gamma(X, \Lambda^p) = \{ \frac{1}{p!}\varphi_{i_1\cdots i_p} dx^{i_1}\wedge\cdots\wedge dx^{i_p} \}$
- The exterior derivative: $d: \Gamma(X,\Lambda^p) \longrightarrow \Gamma(X,\Lambda^{p+1})$.

Also recall that d is defined by:

(1). On functions, $df \triangleq \sum_i \frac{\partial f}{\partial x^i} dx^i$ in a local coordinate chart.

(2). On higher forms, it's extended by: $d(\frac{1}{p!}\sum_I \varphi_I dx^I) = \frac{1}{p!}\sum_{i,I} \frac{\partial \varphi_I}{\partial x^i} dx^i \wedge dx^I$

This local definition is well-defined globally. For instance, on different coordinate neighborhoods, $\sum \varphi_i dx^i = \sum \tilde{\varphi}_j dy^j \Rightarrow \varphi_i = \tilde{\varphi}_j \frac{\partial y^j}{\partial x^i}$. Thus, by def.

$$\begin{aligned}
d(\sum \varphi_i dx^i) &= \sum_{i,j} \frac{\partial \varphi_i}{\partial x^j} dx^j \wedge dx^i \\
&= \frac{1}{2}\sum_{i,j}\left(\frac{\partial \varphi_i}{\partial x^j} - \frac{\partial \varphi_j}{\partial x^i}\right) dx^j \wedge dx^i \\
&= \frac{1}{2}\sum_{i,j,k}\left(\frac{\partial \tilde{\varphi}_k}{\partial x^j}\frac{\partial y^k}{\partial x^i} + \tilde{\varphi}_k \frac{\partial^2 y^k}{\partial x^i \partial x^j} - \frac{\partial \tilde{\varphi}_k}{\partial x^i}\frac{\partial y^k}{\partial x^j} - \tilde{\varphi}_k \frac{\partial^2 y^k}{\partial x^j \partial x^i}\right) dx^j \wedge dx^i \\
&= \frac{1}{2}\sum_{i,j,k,\ell}\left(\frac{\partial \tilde{\varphi}_k}{\partial y^\ell}\frac{\partial y^\ell}{\partial x^j}\frac{\partial y^k}{\partial x^i} - \frac{\partial \tilde{\varphi}_k}{\partial y^\ell}\frac{\partial y^\ell}{\partial x^i}\frac{\partial y^k}{\partial x^j}\right) dx^j \wedge dx^i
\end{aligned}$$

$$= \frac{1}{2} \sum_{k,l} \left(\frac{\partial \hat{\varphi}_k}{\partial y^l} - \frac{\partial \hat{\varphi}_l}{\partial y^k} \right) dy^l \wedge dy^k$$
$$= d\left(\sum_k \hat{\varphi}_k dy^k \right).$$

Now in our case, $E \longrightarrow X$ is a vector bundle, we can similarly define, with a fixed connection A (associated with a Hermitian metric)

- $\Gamma(X, E) \ni \varphi \xrightarrow{d_A} d_A\varphi \triangleq \nabla_j \varphi\, dz^j + \nabla_{\bar{j}} \varphi\, d\bar{z}^j \in \Gamma(X, E \otimes \Lambda^1)$.
- $d_A \left(\frac{1}{p!q!} \sum \varphi_{\bar{J}I}\, dz^I d\bar{z}^J \right) \triangleq \frac{1}{p!q!} (d_A \varphi_{\bar{J}I})\, dz^I d\bar{z}^J \quad I = (i_1, \cdots, i_p) \;\; \bar{J} = (\bar{j}_1, \cdots, \bar{j}_q)$

In this notation, $\varphi_{\bar{J}I}\, dz^I d\bar{z}^J = \varphi_{\bar{j}_q \cdots \bar{j}_1 i_p \cdots i_1}\, dz^{i_1} \cdots dz^{i_p} d\bar{z}^{j_1} \cdots d\bar{z}^{j_q}$.

Now, in this formalism, we have the following basic identities.

- $d_A^2 \varphi = F \wedge \varphi$
- $F = dA + A \wedge A$
- $d_A F = 0$ (the 2^{nd}-Bianchi's identity)

Proof of Identities.

Take $\varphi \in \Gamma(X, E)$, we compute:

$$d_A \varphi = \nabla_{\bar{k}} \varphi\, d\bar{z}^k + \nabla_k \varphi\, dz^k$$
$$d_A^2 \varphi = (d_A \nabla_{\bar{k}} \varphi) d\bar{z}^k + (d_A \nabla_k \varphi) dz^k$$
$$= (\nabla_j \nabla_{\bar{k}} \varphi\, dz^j + \nabla_{\bar{j}} \nabla_{\bar{k}} \varphi\, d\bar{z}^j) d\bar{z}^k + (\nabla_j \nabla_k \varphi\, dz^j + \nabla_{\bar{j}} \nabla_k \varphi\, d\bar{z}^j) dz^k$$
$$= \underbrace{\nabla_{\bar{j}} \nabla_{\bar{k}} \varphi\, d\bar{z}^j d\bar{z}^k}_{①} + \underbrace{\nabla_j \nabla_{\bar{k}} \varphi\, dz^j d\bar{z}^k + \nabla_{\bar{j}} \nabla_k \varphi\, d\bar{z}^j dz^k}_{③} + \underbrace{\nabla_j \nabla_k \varphi\, dz^j dz^k}_{②}$$

$$① = \frac{1}{2} \left(\nabla_{\bar{j}} \nabla_{\bar{k}} \varphi\, d\bar{z}^j d\bar{z}^k - \nabla_{\bar{k}} \nabla_{\bar{j}} \varphi\, d\bar{z}^j d\bar{z}^k \right)$$
$$= \frac{1}{2} \left(\partial_{\bar{j}} \partial_{\bar{k}} \varphi\, d\bar{z}^j d\bar{z}^k - \partial_{\bar{k}} \partial_{\bar{j}} \varphi\, d\bar{z}^j d\bar{z}^k \right)$$
$$= 0$$

$$② = \frac{1}{2} (\nabla_j \nabla_k \varphi - \nabla_k \nabla_j \varphi)\, dz^j \wedge dz^k$$
$$= \frac{1}{2} \left(\partial_j (\partial_k \varphi + A_k \varphi) + A_j (\partial_k \varphi + A_k \varphi) - \text{terms}(j \leftrightarrow k) \right) dz^j \wedge dz^k$$
$$= \frac{1}{2} \Big(\underbrace{\partial_j \partial_k \varphi}_{\text{sym. in } j,k} + (\partial_j A_k) \varphi + \underbrace{A_k \partial_j \varphi + A_j \partial_k \varphi}_{\text{sym. in } j,k} + A_j A_k \varphi - \text{terms}(j \leftrightarrow k) \Big) dz^j \wedge dz^k$$

$= \frac{1}{2}[(\partial_j A_k - \partial_k A_j) + (A_j A_k - A_k A_j)]\varphi \, dz^j dz^k$

Claim

$= 0$ since A_j is induced from a Hermitian metric.

Proof of claim:

$\partial_j A_k = \partial_j (H^{-1}\partial_k H) = -H^{-1}(\partial_j H)H^{-1}\partial_k H$

(Recall that $H^{-1}H = I \Rightarrow (\partial_j H^{-1})H + H^{-1}\partial_j H = 0 \Rightarrow \partial_j H^{-1} = -H^{-1}\partial_j H H^{-1}$)

Hence $(\partial_j A_k - \partial_k A_j) + (A_j A_k - A_k A_j)$

$= -H^{-1}(\partial_j H)H^{-1}(\partial_k H) + H^{-1}(\partial_k H)H^{-1}(\partial_j H) + (H^{-1}\partial_j H)(H^{-1}\partial_k H) - (H^{-1}\partial_k H)(H^{-1}\partial_j H)$

$= 0$

③ $= \nabla_j \nabla_{\bar{k}} \varphi \, dz^j d\bar{z}^k + \nabla_{\bar{j}} \nabla_k \varphi \, d\bar{z}^j dz^k$

$= \nabla_j \nabla_{\bar{k}} \varphi \, dz^j d\bar{z}^k - \nabla_{\bar{k}} \nabla_j \varphi \, dz^j d\bar{z}^k$

$= [\nabla_j \; \nabla_{\bar{k}}]\varphi \, dz^j d\bar{z}^k$

$= F_{\bar{k}j} \varphi \, dz^j d\bar{z}^k$

Summarizing, we have: $d_A^2 \varphi = F_{\bar{k}j}\varphi \, dz^j d\bar{z}^k = F \wedge \varphi$

Next, we observe that:

$$\begin{aligned}
dA &= d(\Sigma A_j dz^j) \\
&= \Sigma \, dA_j dz^j \\
&= \Sigma \, \partial_{\bar{k}} A_j d\bar{z}^k dz^j + \partial_k A_j dz^k dz^j \\
&= \tfrac{1}{2} \Sigma (\partial_k A_j - \partial_j A_k) dz^k dz^j + \Sigma \partial_{\bar{k}} A_j d\bar{z}^k dz^j \\
&= \tfrac{1}{2} (A_j A_k - A_k A_j) dz^k dz^j + F_{\bar{k}j} d\bar{z}^k dz^j \qquad \text{(by the claim above)} \\
&= A_j A_k dz^k dz^j + F_{\bar{k}j} d\bar{z}^k dz^j \\
&= -A_j dz^j A_k dz^k + F_{\bar{k}j} d\bar{z}^k dz^j \\
&= -A \wedge A + F
\end{aligned}$$

$\Rightarrow F = dA + A \wedge A$, as asserted.

Thirdly, $dF = d(dA + A \wedge A)$

$$= dA\wedge A - A\wedge dA$$
$$=(F-A\wedge A)\wedge A - A\wedge(F-A\wedge A)$$
$$= F\wedge A - A\wedge F$$

Or equivalently. $dF + A\wedge F - F\wedge A = 0$.

Observation: $d_A F = dF + A\wedge F - F\wedge A$.

Recall that $F\in\Gamma(X, \mathrm{End}E\otimes\Lambda^{1,1})$, $F = F_{\bar{k}j}dz^j d\bar{z}^k$, then $d_AF = (d_AF_{\bar{k}j})dz^j d\bar{z}^k$. The observation comes from the simple fact a connection on E determines a connection on $\mathrm{End}E\cong E\otimes E^*$ (In fact, over the tensor algebra of E and E^*). Let $T\in\Gamma(X, \mathrm{End}E)$, $\varphi\in\Gamma(X, E)$, then $T\varphi\in\Gamma(X, E)$, and:

$$\nabla_j(T\varphi) = (\nabla_j T)\varphi + T\nabla_j\varphi$$
$$\Rightarrow (\nabla_j T)\varphi = \nabla_j(T\varphi) - T\nabla_j\varphi$$

Explicitly:

$$(\nabla_j T)\varphi = \partial_j(T\varphi) + A_jT\varphi - T\partial_j\varphi - TA_j\varphi$$
$$= (\partial_j T)\varphi + T\partial_j\varphi + A_jT\varphi - T\partial_j\varphi - TA_j\varphi$$
$$= (\partial_j T + A_jT - TA_j)\varphi$$
$$\Rightarrow \quad \nabla_j T = \partial_j T + A_jT - TA_j$$
$$\Rightarrow \quad d_A T = dT + A\wedge T - T\wedge A.$$

Componentwise: $\nabla_j T^\alpha{}_\beta = \partial_j T^\alpha{}_\beta + A_j{}^\alpha{}_\gamma T^\gamma{}_\beta - T^\alpha{}_\gamma A_j{}^\gamma{}_\beta$. Notice that $-A_j{}^\gamma{}_\beta$ comes from the connection of E^*. This finishes the proof of the basic equalities.

Special case of $E = T^{1,0}X$.

$\Gamma(X, T^{1,0}X) = \{V^j\partial_j \mid \text{holomorphic vector fields}\}$

$\Gamma(X, T^{1,0}X)\ni V \leftrightarrow V^j_\mu(z)$ on X_μ s.t $V^j_\mu(z) = \frac{\partial z^j_\mu}{\partial \tilde{z}^k_\nu} V^k_\nu(\tilde{z})$, i.e. $t_{\mu\nu}{}^j{}_k = \frac{\partial z^j_\mu}{\partial \tilde{z}^k}$.

In this case, the connection takes the form $A_j{}^\alpha{}_\beta = A_j{}^\ell{}_k$ ($j, k, \ell\in\{1,\cdots,n\}$), and the curvature takes the form $F_{\bar{k}j}{}^\alpha{}_\beta = F_{\bar{k}j}{}^\ell{}_m$ ($j, k, \ell, m\in\{1,\cdots,n\}$).

Def. A metric $g_{\bar{k}j}$ on $T^{1,0}$ is said to be Kähler if $\partial_\ell g_{\bar{k}j} = \partial_j g_{\bar{k}\ell}$.

This condition is invariant under change of coordinates because it's equivalent to the global condition $0 = d\omega = d(\Sigma g_{\bar{k}j}dz^j d\bar{z}^k)$.

For a Kähler metric $g_{\bar{k}j}$ (we denote the curvature for tangent bundles by R instead of K), we have:

$$R_{\bar{k}j\bar{p}m} = R_{\bar{p}m\bar{k}j} = R_{\bar{k}m\bar{p}j} \quad \text{(the 1st Bianchi identity)}$$

where $R_{\bar{k}j\bar{p}m} = g_{\bar{p}\ell} R_{\bar{k}j}{}^{\ell}{}_{m}$.

Proof. $R_{\bar{k}j\bar{p}m} = g_{\bar{p}\ell} R_{\bar{k}j}{}^{\ell}{}_{m}$

$$\begin{aligned}
&= g_{\bar{p}\ell}\,(-\partial_{\bar{k}}(g^{\ell\bar{q}}\partial_j g_{\bar{q}m})) \\
&= g_{\bar{p}\ell}\,(-(\partial_{\bar{k}} g^{\ell\bar{q}})\partial_j g_{\bar{q}m} - g^{\ell\bar{q}}\partial_{\bar{k}}\partial_j g_{\bar{q}m}) \\
&= g_{\bar{p}\ell}\,(g^{\ell\bar{s}}(\partial_{\bar{k}} g_{\bar{s}r}) g^{r\bar{q}}\cdot\partial_j g_{\bar{q}m} - g^{\ell\bar{q}}\partial_{\bar{k}}\partial_j g_{\bar{q}m}) \\
&= \partial_{\bar{k}} g_{\bar{p}r}\, g^{r\bar{q}}\partial_j g_{\bar{q}m} - \partial_{\bar{k}}\partial_j g_{\bar{p}m}
\end{aligned}$$

By def. of Kähler metrics, $(j \leftrightarrow m)$ $(\bar{k} \leftrightarrow \bar{p})$ doesn't change $R_{\bar{k}j\bar{p}m}$.

All of the above formalism extends trivially to the case of smooth bundles. Consider $E \longrightarrow X$ a smooth complex vector bundle over a smooth manifold. By def., a smooth vector bundle is defined by smooth transition functions $\{t_{\mu\nu}$: smooth invertible matrix valued functions on $X_\mu \cap X_\nu\}$.

Def: A connection on $E \longrightarrow X = \cup_\mu X_\mu$ is given by $A_\mu = A_j{}^{\alpha}{}_{\beta}\, dx^j$ $\;1 \le \alpha, \beta \le r$ on X_μ satisfying: $\Gamma(X, E) \ni \varphi = \{\varphi^\alpha\} \longmapsto \nabla_j \varphi = \partial_j \varphi^\alpha + A_j{}^{\alpha}{}_{\beta}\varphi^\beta \in \Gamma(X, E \otimes \Lambda^1)$.

Def: The curvature tensor is defined by:

$$[\nabla_i, \nabla_j]\varphi^\alpha = -F_{ij}{}^{\alpha}{}_{\beta}\,\varphi^\beta.$$

It follows from a simple computation that

$$F_{ij}{}^{\alpha}{}_{\beta} = -(\partial_i A_j{}^{\alpha}{}_{\beta} - \partial_j A_i{}^{\alpha}{}_{\beta} + A_i{}^{\alpha}{}_{\gamma} A_j{}^{\gamma}{}_{\beta} - A_j{}^{\alpha}{}_{\gamma} A_i{}^{\gamma}{}_{\beta}).$$

If X has more structure, say, X is a complex manifold, $x \mapsto (z, \bar{z})$, we are interested in a more special class of connections, which respect the complex structure of X as much as possible.

Def. $A = A_j{}^{\alpha}{}_{\beta}\, dx^j$ is called a Chern connection if $A_{\bar{j}}{}^{\alpha}{}_{\beta}\, d\bar{z}^j = 0$, i.e. $\nabla_{\bar{j}}\varphi^\alpha = \partial_{\bar{j}}\varphi^\alpha$.

Assume that A is a Chern connection, i.e. $A_{\bar{j}}$ is 0, we have

$$F_{\bar{k}j} = -(\partial_{\bar{k}} A_j - \partial_j \underbrace{A_{\bar{k}}}_{0} + \underbrace{A_{\bar{k}}}_{0} A_j - A_j \underbrace{A_{\bar{k}}}_{0})$$

$$= -\partial_{\bar{k}} A_j.$$

However, F_{ij} may not be 0. However, we have the following characterization:

Thm. (Newlander & Nirenberg) If A is a Chern connection with $F_{ij}=0$ ($F_{\bar{i}\bar{j}}=0$ by def. of Chern connection), then $E \to X$ admits a holomorphic structure.

For a proof, c.f. Donaldson-Kronheimer, The Geometry of Four Manifolds)

Characteristic Classes

$E \to X$: smooth complex vector bundle over compact X. Let A be any connection on E

Def: $\tilde{C}_m(A) = tr(\Lambda^m F) = tr(\underbrace{F\wedge F\wedge\cdots\wedge F}_{m}) \in \Gamma(X, \Lambda^{2m})$

where $F = F_{ij\beta}^{\alpha} dx^j dx^i \in \Gamma(X, \mathrm{End}E \otimes \Lambda^2)$.

Basic observation: $d\tilde{C}_m(A) = 0$

Indeed, $d\tilde{C}_m(A) = Tr(d(F\wedge\cdots\wedge F))$

$$= Tr(dF\wedge F\wedge\cdots\wedge F + F\wedge dF\wedge\cdots\wedge F + \cdots + F\wedge F\wedge\cdots\wedge dF)$$

$$= Tr(m\, dF\wedge F\wedge\cdots\wedge F) \qquad (\text{since } Tr(AB)=Tr(BA))$$

$$= Tr(m(F\wedge A - A\wedge F)\wedge F\wedge\cdots\wedge F) \qquad (2^{nd} \text{ Bianchi's identity})$$

$$= Tr(m(A\wedge F\wedge\cdots\wedge F - A\wedge F\wedge\cdots\wedge F)$$

$$= 0.$$

Def. We define the Chern classes of E as $[\tilde{C}_m(A)]$.

Claim: $[\tilde{C}_m(A)]$ is independent of choices of A. More precisely, if A' is any other connection, we can write:

$$\tilde{C}_m(A') - \tilde{C}_m(A) = d(m\int_0^1 Tr(B\wedge F_t^{m-1})dt)$$

where $B = A' - A$. $A_t = A + tB$ and F_t is the curvature of the connection A_t. Note that B is an $\mathrm{End}E$-valued 1-form. ($\because \forall \varphi \in \Gamma(X, E)$). $\partial_j\varphi + A_j\varphi$ and $\partial_j\varphi + A_j'\varphi$ are both global $\Rightarrow A_j\varphi - A_j'\varphi$ is global $\Rightarrow A_j - A_j' \in \Gamma(X, \mathrm{End}E)$.)

Proof.

$$\tilde{C}_m(A') - \tilde{C}_m(A) = \int_0^1 \frac{d\tilde{C}_m(A_t)}{dt}dt$$
$$= \int_0^1 \frac{d}{dt}(\mathrm{Tr}(F_t\wedge\cdots\wedge F_t))dt$$
$$= \int_0^1 \mathrm{Tr}(\dot{F}_t\wedge F_t\wedge\cdots\wedge F_t + F_t\wedge\dot{F}_t\wedge\cdots\wedge F_t + \cdots + F_t\wedge F_t\wedge\cdots\wedge\dot{F}_t)dt$$

Since $F_t = dA_t + A_t\wedge A_t \Rightarrow \dot{F}_t = d\dot{A}_t + \dot{A}_t\wedge A_t + A_t\wedge\dot{A}_t$.

$$\Rightarrow \dot{F}_t = dB + B\wedge A_t + A_t\wedge B$$

$$\Rightarrow \mathrm{Tr}(\dot{F}_t\wedge F_t\wedge\cdots\wedge F_t + F_t\wedge\dot{F}_t\wedge\cdots\wedge F_t + \cdots + F_t\wedge F_t\wedge\cdots\wedge\dot{F}_t)$$
$$= \mathrm{Tr}(m\dot{F}_t\wedge F_t\wedge\cdots\wedge F_t)$$
$$= \mathrm{Tr}(m(dB + B\wedge A_t + A_t\wedge B)\wedge F_t\wedge\cdots\wedge F_t)$$
$$= m\,\mathrm{Tr}(d(B\wedge F_t\wedge\cdots\wedge F_t) + B\wedge(\sum_{i=1}^{m-1}\Lambda^{i-1}F_t\wedge dF_t\wedge\Lambda^{m-i-1}F_t) + B\wedge A_t\wedge\Lambda^{m-1}F_t + A_t\wedge B\wedge\Lambda^{m-1}F_t)$$
$$= m\,\mathrm{Tr}(d(B\wedge\Lambda^{m-1}F_t) + B\wedge(\sum_{i=1}^{m-1}\Lambda^{i-1}F_t\wedge(F_t\wedge A_t - A_t\wedge F_t)\wedge\Lambda^{m-i-1}F_t) + B\wedge A_t\wedge\Lambda^{m-1}F_t$$
$$- B\wedge\Lambda^{m-1}F_t\wedge A_t)$$
$$= m\,\mathrm{Tr}(d(B\wedge\Lambda^{m-1}F_t) + B\wedge(\sum_{i=1}^{m-1}\Lambda^{i}F_t\wedge A_t\wedge\Lambda^{m-i-1}F_t - \sum_{i=0}^{m-2}\Lambda^{i}F_t\wedge A_t\wedge\Lambda^{m-i-1}F_t + A_t\wedge\Lambda^{m-1}F_t$$
$$- \Lambda^{m-1}F_t\wedge A_t))$$
$$= m\,\mathrm{Tr}(d(B\wedge\Lambda^{m-1}F_t) + B\wedge(\Lambda^{m-1}F_t\wedge A_t - A_t\wedge\Lambda^{m-1}F_t + A_t\wedge\Lambda^{m-1}F_t - \Lambda^{m-1}F_t\wedge A_t))$$
$$= m\,\mathrm{Tr}(d(B\wedge\Lambda^{m-1}F_t))$$
$$= d(m\,\mathrm{Tr}(B\wedge\Lambda^{m-1}F_t)).$$

Note that in the computation we used the fact that $T(ABC) = \mathrm{Tr}(BCA)$ and differential forms form a $\mathbb{Z}/2$-graded ring. □

Def. The ordinary Chern classes $C_m(E)$ are obtained from $\tilde{C}_m(E)$ as follows: The two basis of symmetric functions are related by polynomial relations:

i.e. if

$$\begin{cases} \sigma_1 = x_1 + \cdots + x_r \\ \sigma_2 = \sum_{i\neq j} x_i x_j \\ \cdots \\ \sigma_r = x_1\cdots x_r \end{cases} \qquad \begin{cases} S_1 = x_1 + \cdots + x_r \\ S_2 = x_1^2 + \cdots + x_2^2 \\ \cdots \\ S_r = x_1^n + \cdots + x_r^n \end{cases}$$

then $S_i = P_i(\sigma_1, \cdots, \sigma_r)$, $i = 1, \cdots, n$. For instance: $S_1 = \sigma_1$, $S_2 = \sigma_1^2 - \sigma_2$, . $\cdots$ Then:

$$C_i(E) \triangleq P_i(\tilde{C}_1(E), \cdots, \tilde{C}_r(E)).$$

Note that $C_1(E)$ is well-defined by the ring structure of de Rham cohomology.

Interlude: Maxwell equations, geometric interpretation.

$X = \mathbb{R}^{1,3}$, $ds^2 = -(dt)^2 + dx^2 + dy^2 + dz^2$, $L = \mathbb{R}^{1,3} \times \mathbb{C}$.

Let the connection be given by $A = A_j dx^j = \underbrace{-\varphi}_{\text{potential}} dx^0 + \underbrace{A_x dx + A_y dy + A_z dz}_{\text{vector potential}}$.

$$\begin{aligned} F &= dA + A \wedge A \\ &= dA \qquad\qquad (\text{since } A \text{ is just a 1-form } (U(1)\text{-connection}).) \\ &= \Sigma_\mu A_\mu dx^\mu \\ &= \tfrac{1}{2} \Sigma \left(\frac{\partial A_\mu}{\partial x^\nu} - \frac{\partial A_\nu}{\partial x^\mu} \right) dx^\nu \wedge dx^\mu \\ &= \tfrac{1}{2} F_{\mu\nu} dx^\nu \wedge dx^\mu , \end{aligned}$$

where we define the curvature $F_{\mu\nu} = \frac{1}{2}\left(\frac{\partial A_\mu}{\partial x^\nu} - \frac{\partial A_\nu}{\partial x^\mu} \right)$, which is also refered to as the field strength in physics litterature.

Def. The electric field $\vec{E} = (E_1, E_2, E_3)$ is defined by $E_j = F_{j0}$.

The magnetic field $\vec{B} = (B_1, B_2, B_3)$ is defined by $B_x = F_{yz}$, $B_y = F_{zx}$, $B_z = F_{xy}$.

We also write $F_{\mu\nu}$ in a matrix form:

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -F_{yx} & -F_{zx} \\ E_y & F_{yx} & 0 & -F_{zy} \\ E_z & F_{zx} & F_{zy} & 0 \end{pmatrix}$$

Recall that F satisfies Bianchi's identity: $d_A F = 0$

$$\begin{aligned} &\Rightarrow 0 = d_A F = dF + A \wedge F - F \wedge A = dF \\ &\Rightarrow 0 = \tfrac{1}{2} \Sigma\, dF_{\mu\nu}\, dx^\nu dx^\mu = \tfrac{1}{2} \frac{\partial F_{\mu\nu}}{\partial x^\rho} dx^\rho \wedge dx^\nu \wedge dx^\mu . \\ &\Rightarrow \partial_\rho F_{\mu\nu} + \partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} = 0 \quad , \forall \rho, \mu, \nu . \end{aligned}$$

In terms of the electric and magnetic fields:

(1). All ρ, μ, ν are space indices: $(\rho = x, \mu = y, \nu = z)$

$$\Rightarrow \partial_x F_{yz} + \partial_y F_{zx} + \partial_z F_{xy} = 0$$

i.e. $\partial_x B_x + \partial_y B_y + \partial_z B_z = 0$

or $\qquad \nabla \cdot \vec{B} = 0$

(2). One index is the time index

$$\begin{cases} \partial_t F_{xy} + \partial_x F_{y0} + \partial_y F_{0x} = 0 & \text{i.e. } \partial_t B_z + \partial_x E_y - \partial_y E_x = 0 \\ \partial_t F_{yz} + \partial_y F_{z0} + \partial_z F_{0y} = 0 & \text{i.e. } \partial_t B_x + \partial_y E_z - \partial_z E_y = 0 \\ \partial_t F_{zx} + \partial_z F_{x0} + \partial_x F_{0z} = 0 & \text{i.e. } \partial_t B_y + \partial_z E_x - \partial_x E_z = 0 \end{cases}$$

or $$\partial_t \vec{B} + \vec{\nabla} \times \vec{E} = 0.$$

Thus the Bianchi's identity accounts for 2 of Maxwell's equations. The other 2 equations arise from a variational principle, i.e. given a connection A, we can associate with it its action $I(A) = \int_{\mathbb{R}^{3,1}} |F|^2$. The electric magnetic fields are the ones that minimize this action. Namely:

$$\delta I(A) = \frac{\delta I}{\delta A} \cdot \delta A$$

and the critical points are the ones satisfying $\delta I / \delta A = 0$.

In the case of connections, the critical point equation is:

$$\nabla^\mu F_{\mu\nu} = 0 \quad (\text{Yang-Mills Equation})$$

This will be shown later. Explicitly in the present case, $\nabla^\mu F_{\mu\nu} = \partial_\rho (g^{\rho\mu} F_{\mu\nu})$, where $g^{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$, and thus:

$$g^{\rho\mu} F_{\mu\nu} = \begin{cases} -F_{0\nu} & \rho = 0 \\ F_{\rho\nu} & \rho = x, y, z. \end{cases}$$

(3). Take $\nu = 0$, then:

$$0 = \partial_\rho (g^{\rho\mu} F_{\mu 0}) = \partial_0 (-F_{00}) + \partial_x F_{x0} + \partial_y F_{y0} + \partial_z F_{z0}.$$

i.e. $$\partial_x E_x + \partial_y E_y + \partial_z E_z = 0$$

or $$\nabla \cdot \vec{E} = 0$$

(4). Take $\nu = x$, (or y, z), then

$$\begin{cases} 0 = \partial_\rho (g^{\rho\mu} F_{\mu x}) = \partial_0 (-F_{0x}) + \partial_x F_{xx} + \partial_y F_{yx} + \partial_z F_{zx} \\ 0 = \partial_\rho (g^{\rho\mu} F_{\mu y}) = \partial_0 (-F_{0y}) + \partial_x F_{xy} + \partial_y F_{yy} + \partial_z F_{zy} \\ 0 = \partial_\rho (g^{\rho\mu} F_{\mu z}) = \partial_0 (-F_{0z}) + \partial_x F_{xz} + \partial_y F_{yz} + \partial_z F_{zz} \end{cases}$$

i.e. $$\begin{cases} \partial_t E_x - (\partial_y B_z - \partial_z B_y) = 0 \\ \partial_t E_y - (\partial_z B_x - \partial_x B_z) = 0 \\ \partial_t E_z - (\partial_x B_y - \partial_y B_x) = 0 \end{cases}$$

or $\quad \partial_t \vec{E} - \nabla \times \vec{B} = 0$

Variational formula.

Given a variation $A \rightsquigarrow A + \delta A$ (note that δA is a 1-form). Recall that

$$F_{jk} = -(\partial_j A_k - \partial_k A_j + A_j A_k - A_k A_j)$$

$$\begin{aligned} \Rightarrow \delta F_{jk} &= -(\partial_j \delta A_k - \partial_k \delta A_j + \delta A_j \cdot A_k + A_j \delta A_k - \delta A_k \cdot A_j - A_k \delta A_j) \\ &= -(\partial_j \delta A_k + A_j \delta A_k - \delta A_k \cdot A_j - (\partial_k \delta A_j + A_k \delta A_j - \delta A_j A_k)) \\ &= -(\nabla_j \delta A_k - \nabla_k \delta A_j) \end{aligned}$$

Thus,
$$\begin{aligned} \delta I &= \delta \int_X \langle F, F \rangle \\ &= 2 \int_X \langle \delta F, F \rangle \\ &= 2 \int_X (\nabla_j \delta A_k - \nabla_k \delta A_j) g^{jl} g^{km} F_{ml} \\ &= 2 \int_X -\delta A_k \cdot g^{km} \nabla^l F_{ml} + \delta A_j g^{jl} \nabla^m F_{ml} \quad \text{(Integration by parts)} \\ &= 4 \int_X \delta A_k \cdot g^{km} \nabla^l F_{lm} \end{aligned}$$

The right hand side is linear in δA_k now, thus

$$\delta I / \delta A = 0 \Leftrightarrow \nabla^l F_{lm} = 0.$$

A basic example of complex manifolds : $\mathbb{CP}^n$.

Def. $\mathbb{CP}^n = \mathbb{C}^{n+1} \setminus \{0\} / \sim$, where $(\zeta^0, \cdots, \zeta^n) \sim \lambda (\zeta^0, \cdots, \zeta^n)$, $\forall \lambda \in \mathbb{C}^*$.

We use the following coordinate system, $\forall j = 0, \cdots, n$.

$$X_j \triangleq \{ [\zeta^0 : \cdots : \zeta^n] \mid \zeta_j \neq 0 \}$$

and we identify X^j with $\mathbb{C}^n$ via:

$$X_j \ni (\zeta^0, \cdots, \zeta^n) \longmapsto \left(\frac{\zeta^0}{\zeta^j}, \cdots, \widehat{\frac{\zeta^j}{\zeta^j}}, \cdots, \frac{\zeta^n}{\zeta^j} \right) \triangleq z \in \mathbb{C}^n$$

The transition functions, for instance, on $X_0 \cap X_n$, is given by:

$$\begin{array}{ccc} & [\zeta^0 : \cdots : \zeta^n] & \\ \swarrow & & \searrow \\ z \triangleq \left(\frac{\zeta^1}{\zeta^0}, \cdots \frac{\zeta^n}{\zeta^0} \right) & & \left(\frac{\zeta^0}{\zeta^n}, \cdots, \frac{\zeta^{n-1}}{\zeta^n} \right) \triangleq w \end{array}$$

then $w^0 = \frac{1}{z^n}$, $w^1 = \frac{z^1}{z^n}$, $w^2 = \frac{z^2}{z^n}$, $\cdots$.

Def. The universal line bundle L^{-1} assigns each point of $\mathbb{CP}^n$ the line it represents.

Note that the total space of $L^{-1} \setminus \{0\text{-section}\} \cong \mathbb{C}^{n+1}_*$. We trivialize L^{-1} in the following way:

On X_0, $(\zeta^0, \cdots, \zeta^n) \longmapsto ([\zeta^0 : \cdots : \zeta^n], \zeta^0) \in \mathbb{CP}^n \times \mathbb{C}$

On X_n, $(\zeta^0, \cdots, \zeta^n) \longrightarrow ([\zeta^0 : \cdots : \zeta^n], \zeta^n) \in \mathbb{CP}^n \times \mathbb{C}$

Since $\zeta^0 = (\zeta^0/\zeta^n) \cdot \zeta^n$, we define L^{-1} as the holomorphic line bundle with transition functions $t_{jk} = \zeta^j/\zeta^k$ on $X_j \cap X_k$, i.e.

$$\Gamma(X, L^{-1}) \ni \varphi \iff \{\varphi_j = \zeta^j/\zeta^k \cdot \varphi_k \text{ on } X_j \cap X_k\}.$$

An example of a meromorphic section is given as follows:

On X_0, set $\varphi_0 = 1$. This determines $\varphi_j = \frac{\zeta^j}{\zeta^0} \cdot 1 = \frac{\zeta^j}{\zeta^0}$ on $X_j \cap X_0$.

This is meromorphic section of L^{-1} with poles along the codimension 1 subvariety $\{\zeta_0 = 0\}$.

The universal bundle L^{-1} admits a natural metric:

The invariant way of defining it is if $\varphi = (\zeta^0, \cdots, \zeta^n)$, set

$$\|\varphi\|^2 = \sum_{i=0}^n |\zeta^i|^2$$

Locally, say, on X_0:

$$\|\varphi\|^2 = |\zeta^0|^2 (1 + |\tfrac{\zeta^1}{\zeta^0}|^2 + \cdots + |\tfrac{\zeta^n}{\zeta^0}|^2)$$
$$= |\zeta^0|^2 h(z).$$

Thus $h(z) = (1 + \|z\|^2)$, where $\|z\|^2 = |z_1|^2 + \cdots + |z_n|^2$ on each coordinate chart.

Curvature of L^{-1}.

By our previous formula:

$$\begin{aligned} F_{\bar{k}j} &= -\partial_{\bar{k}} \partial_j \log h \\ &= -\partial_{\bar{k}} \partial_j (\log(1 + \|z\|^2)) \\ &= -\partial_{\bar{k}} \left(\frac{\bar{z}^j}{1 + \|z\|^2}\right) \\ &= -\left\{ \frac{\delta_{kj}}{1 + \|z\|^2} - \frac{\bar{z}^j z^k}{(1 + \|z\|^2)^2} \right\} \end{aligned}$$

Observe that if $v = (v^1, \cdots, v^n)$ is a vector

$$F_{\bar{k}j} v^j \bar{v}^k = -\left\{ \frac{\|v\|^2}{1 + \|z\|^2} - \frac{\langle v, z \rangle \langle z, v \rangle}{(1 + \|z\|^2)^2} \right\}$$

$$= -\left\{ \frac{\|v\|^2 + \|v\|^2\|z\|^2 - |\langle v, z\rangle|^2}{(1+\|z\|^2)^2} \right\}$$
$$< -\frac{\|v\|^2}{(1+\|z\|^2)^2}$$

Thus the curvature is a negative definite $(1,1)$ form.

Def: Let the hyperplane bundle L be the inverse of L^{-1}, i.e. L is defined by the transition functions $\{\frac{\zeta^k}{\zeta^j}$ on $X_j \cap X_k\}$. i.e.

$$\Gamma(X, L) \ni \varphi \longleftrightarrow \{\varphi_j = \zeta^k/\zeta^j \, \varphi_k \text{ on } X_j \cap X_k\}$$

Similarly, $L^m = \{(\zeta^k/\zeta^j)^m \text{ on } X_j \cap X_k\}$.

L admits a natural metric $h_L(z) \triangleq \frac{1}{1+\|z\|^2}$, the Fubini-Study metric. The curvature of this metric is:

$$(F_L)_{\bar{k}j} = \left\{\frac{\delta_{jk}}{1+\|z\|^2} - \frac{z_k \bar{z}_j}{(1+\|z\|^2)^2}\right\}$$

which is strictly positive:

$$(F_L)_{\bar{k}j} v^j \bar{v}^k = \left\{\frac{\|v\|^2}{1+\|z\|^2} - \frac{\langle v, z\rangle\langle z, v\rangle}{(1+\|z\|^2)^2}\right\}$$
$$= \left\{ \frac{\|v\|^2 + \|v\|^2\|z\|^2 - |\langle v, z\rangle|^2}{(1+\|z\|^2)^2} \right\}$$
$$> \frac{\|v\|^2}{(1+\|z\|^2)^2}$$

The holomorphic sections of L^m can be identified with homogeneous polynomials $p(\zeta^0, \zeta^1, \cdots, \zeta^n)$ of order m. (Homogeneous means $p(\lambda\zeta^0, \lambda\zeta^1, \cdots \lambda\zeta^n) = \lambda^m p(\zeta^0, \zeta^1, \cdots, \zeta^n)$.) Given $p(\zeta)$, we may define, for instance on X_0, $p_0(z) = p(1, z^1, \cdots, z^n)$. Then, on $X_0 \cap X_n$:

$$p_0(1, z_1, \cdots, z_n) = p(1, \tfrac{\zeta^1}{\zeta^0}, \cdots, \tfrac{\zeta^n}{\zeta^0})$$
$$= (\tfrac{1}{\zeta^0})^m p(\zeta^0, \zeta^1, \cdots, \zeta^n)$$
$$= (\tfrac{\zeta^n}{\zeta^0})^m p(\tfrac{\zeta^0}{\zeta^n}, \tfrac{\zeta^1}{\zeta^n}, \cdots, 1)$$
$$= (\tfrac{\zeta^n}{\zeta^0})^m p_n(w^0, w^1, \cdots, 1)$$

In summary, we have the following set-up: $L \longrightarrow \mathbb{CP}^n$ admits a metric h_L with strictly positive curvature F_L. Then we may pick F_L to be a metric for $\mathbb{CP}^n$ since after all it's just a $(1,1)$-form. Thus in the future, the Fubini-

Study metric will mean 2 things:

1). The metric h_L on L

2). Curvature F_L of (L, h_L), regarded as a metric $g_{\bar{k}j} = (F_L)_{\bar{k}j}$ on $\mathbb{CP}^n$.

Exercise: Compute $g^{\ell\bar{k}}$, $\det g_{\bar{k}j}$ and $R_{\bar{k}j}$ $(= (n+1) g_{\bar{k}j})$.

Lemma. (Linear algebra). If $u \in \mathbb{C}^n$ is a unit vector, $\lambda \in \mathbb{C} \setminus \{0,1\}$, then

$$(\lambda \mathrm{Id} - u\bar{u}^t)^{-1} = \lambda^{-1}\mathrm{Id} + ((\lambda-1)^{-1} - \lambda^{-1}) u\bar{u}^t.$$

Pf: $(\lambda \mathrm{Id} - u\bar{u}^t)(\lambda^{-1}\mathrm{Id} + ((\lambda-1)^{-1} - \lambda^{-1}) u\bar{u}^t)$

$= \mathrm{Id} + \lambda((\lambda-1)^{-1} - \lambda^{-1}) u\bar{u}^t - \lambda^{-1}u\bar{u}^t - ((\lambda-1)^{-1} - \lambda^{-1}) u\bar{u}^t)$

$= \mathrm{Id} + \frac{1}{\lambda-1} u\bar{u}^t - \lambda^{-1}u\bar{u}^t - \frac{1}{\lambda-1} u\bar{u}^t + \lambda^{-1}u\bar{u}^t$

$= \mathrm{Id}$ $\square$

Using this lemma, we can calculate $(g_{\bar{k}j})^{-1}$:

$$(g_{\bar{k}j}) = \left(\frac{\delta_{jk}}{1+\|z\|^2} - \frac{z_k\bar{z}_j}{(1+\|z\|^2)^2}\right) = \frac{\|z\|^2}{(1+\|z\|^2)^2}\left(\frac{1+\|z\|^2}{\|z\|^2}\mathrm{Id} - \frac{z\bar{z}^t}{\|z\|^2}\right)$$

$$\Rightarrow (g_{\bar{k}j})^{-1} = \frac{(1+\|z\|^2)^2}{\|z\|^2}\left(\frac{\|z\|^2}{1+\|z\|}\mathrm{Id} + \left(\|z\|^2 - \frac{\|z\|^2}{1+\|z\|^2}\right) z\bar{z}^t\right)$$

$$= (1+\|z\|^2)\mathrm{Id} + (1+\|z\|^2)\|z\|^2 z\bar{z}^t$$

$$\Rightarrow g^{\ell\bar{k}} = (1+\|z\|^2)\delta_{\ell k} + (1+\|z\|^2)\|z\|^2 z^{\ell}\bar{z}^{k}$$

To calculate the determinant, we use:

Lemma. (Linear algebra). If $u \in \mathbb{C}^n$ is a unit vector, $\lambda \in \mathbb{C} \setminus \{0,1\}$, then

$$\det(\lambda \mathrm{Id} - u\bar{u}^t) = \lambda^{n-1}(\lambda - 1)$$

Pf: Complete u to an orthonormal basis $\{u, u_1, \dots, u_{n-1}\}$, then

$$\begin{cases} (\lambda \mathrm{Id} - u\bar{u}^t) u = (\lambda-1)u, \\ (\lambda \mathrm{Id} - u\bar{u}^t) u_i = \lambda u_i \end{cases}$$

$\Rightarrow \{u, u_1 \cdots u_{n-1}\}$ diagonalizes $\lambda\mathrm{Id} - u\bar{u}^t \Rightarrow \det(\lambda \mathrm{Id} - u\bar{u}^t) = \lambda^{n-1}(\lambda-1)$. $\square$

It follows $\det(g_{\bar{k}j}) = \frac{\|z\|^{2n}}{(1+\|z\|^2)^{2n}} \cdot \frac{(1+\|z\|^2)^{n-1}}{\|z\|^{2(n-1)}}\left(\frac{1+\|z\|^2}{\|z\|^2} - 1\right) = \frac{1}{(1+\|z\|^2)^{n+1}}$.

Finally, the Ricci curvature:

Lemma. If $A(t)$ is an $n\times n$ matrix valued function, $A(0)$ invertible, then:

$$(\log\det A(t))'(0) = \operatorname{tr} A^{-1}(0) A'(0).$$

Pf: Over $\mathbb{C}$, let $J = B^{-1}AB$ be the Jordan canonical form, λ_i the eigen-values.

Then $\log\det A(t) = \log\det J = \sum \log\lambda_i$

$$\begin{aligned}
\Rightarrow (\log\det A(t))'(0) &= \sum \lambda_i^{-1}\lambda_i'(0) \\
&= \operatorname{tr} J^{-1} J'(0) \\
&= \operatorname{tr}(B^{-1}AB)^{-1}(-B^{-1}B'B^{-1}AB + B^{-1}A'B + B^{-1}AB') \\
&= \operatorname{tr}(-B^{-1}A^{-1}B'B^{-1}AB + B^{-1}A^{-1}BB^{-1}A'B + B^{-1}A^{-1}BB^{-1}AB') \\
&= \operatorname{tr}(-B'B^{-1} + A^{-1}A' + B^{-1}B') \\
&= \operatorname{tr}(A^{-1}A')
\end{aligned}$$

$\square$

Now:
$$\begin{aligned}
R_{\bar{k}j} = R_{\bar{k}j}{}^{\ell}{}_{\ell} &= -\partial_{\bar{k}}(g^{\ell\bar{q}}\partial_j g_{\bar{q}\ell}) \\
&= -\partial_{\bar{k}}\partial_j \log\det g_{\bar{q}\ell} \\
&= -\partial_{\bar{k}}\partial_j(-(n+1))\log(1+\|z\|^2) \\
&= (n+1)\left(\frac{\delta_{jk}}{1+\|z\|^2} - \frac{\bar{z}^j z^k}{(1+\|z\|^2)^2}\right) \\
&= (n+1) g_{\bar{k}j}
\end{aligned}$$

Exercise: Our metric $g_{\bar{k}j} = \partial_{\bar{k}}\partial_j \log h$ is always Kähler since $\partial_\ell g_{\bar{k}j} = \partial_j g_{\bar{k}\ell}$. Now show that if X is Kähler, Y is a complex submanifold, then Y is Kähler. In particular any complex submanifold of $\mathbb{CP}^n$ is Kähler.

Pf: The question is local, thus we may assume that Y is locally the zero set of some holomorphic functions $f_{r+1},\cdots,f_n$, with linearly independent differentials.

Complete $f_{r+1},\cdots,f_n$ to a local coordinate chart $\{z_1,\cdots,z_n\}$, say, $z_{r+1}=f_{r+1},\cdots,z_n=f_n$. Then at $p\in Y$, $T^{1,0}_p M = \mathbb{C}\langle\partial_1,\cdots,\partial_r\rangle$, and $\{g_{\bar{k}j}\}|_Y$ satisfies $\partial_\ell g_{\bar{k}j}|_Y = \partial_j g_{\bar{k}\ell}|_Y$. The result follows.

§6. Kodaira Vanishing Theorem

Road Map:

Our first goal is the Kodaira Embedding Thm:

(**Kodaira Embedding Thm**) Let $L \to X$ be a positive line bundle over a compact complex manifold. Then for m large enough, the map

$$\begin{aligned} X &\longrightarrow \mathbb{CP}^{N_m} \\ z &\longmapsto [S_0(z), \cdots, S_{N_m}(z)] \end{aligned}$$

is an embedding. Here $\{S_\alpha(z),\ \alpha=0,\cdots,N_m\}$ is a basis of the space $H^0(X, L^m)$ and $\dim H^0(X, L^m) = N_m + 1$

Main Ingredients of the proof.

Need: many holomorphic sections of L^m

- Vanishing thms: If E is a holomorphic bundle, when is $\ker \square_{E\otimes\Lambda^{p,q}} = ?$

$$\cdots \underset{\bar\partial^\dagger}{\overset{\bar\partial}{\rightleftarrows}} E\otimes\Lambda^{p,q-1} \underset{\bar\partial^\dagger}{\overset{\bar\partial}{\rightleftarrows}} E\otimes\Lambda^{p,q} \underset{\bar\partial^\dagger}{\overset{\bar\partial}{\rightleftarrows}} E\otimes\Lambda^{p,q+1} \underset{\bar\partial^\dagger}{\overset{\bar\partial}{\rightleftarrows}} \cdots$$

$$\square = \bar\partial\bar\partial^\dagger + \bar\partial^\dagger\bar\partial$$

Observation: Naively, $\ker \square|_{E\otimes\Lambda^{p,q}}$ depends on metrics because $\bar\partial^\dagger$ does. It's however only dependent on the complex structure:

- Hodge decomposition thm.

$$\ker \square|_{E\otimes\Lambda^{p,q}} \equiv H_{\bar\partial}^{p,q}(X, E) \quad : \text{Dolbeault cohomology}.$$

which depends only on the complex structure.

- Sheaf cohomology: $H_{\bar\partial}^{p,q}(X,E) = H^q(X, \Omega^p(E))$, where $\Omega^p(E)$ is the sheaf of E-valued $(p,0)$ forms.

Advantage of sheaf cohomology:

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$

$$\Rightarrow \quad 0 \longrightarrow H^0(X,\mathcal{F}) \longrightarrow H^0(X,\mathcal{G}) \longrightarrow H^0(X,\mathcal{H}) \longrightarrow H^1(X,\mathcal{F}) \longrightarrow \cdots\cdots$$

Then once we know $H^1(X,\mathcal{F})$ vanishes and $H^0(X,\mathcal{H})$ is big enough, it would imply $H^0(X,\mathcal{G})$ is big enough.

Bochner-Kodaira Formulas.

- Characteristic feature: In geometry, there are many "Laplacians". A Bocher-Kodaira formula is a formula of the type:

$$\Delta = \hat{\Delta} + \text{Curvature terms.}$$

where Δ, $\hat{\Delta}$ are different Laplacians.

Base case: $E \longrightarrow X$: holomorphic line bundle. $H_{\bar{\alpha}\beta}$: metric on E. $g_{\bar{k}j}$: metric on X. (Hermitian metric $\Rightarrow$ $\overline{H_{\bar{\alpha}\beta}} = H_{\bar{\beta}\alpha}$). We have:

$$\cdots \xrightarrow{\bar{\partial}} E\otimes\Lambda^{p,q-1} \xrightarrow{\bar{\partial}} E\otimes\Lambda^{p,q} \xrightarrow{\bar{\partial}} E\otimes\Lambda^{p,q+1} \xrightarrow{\bar{\partial}} \cdots$$

In presence of metrics $H_{\bar{\alpha}\beta}$, $g_{\bar{k}j}$, there is a norm on $\Gamma(X, E\otimes\Lambda^{p,q})$, given by:

$\forall \varphi \in \Gamma(X, E\otimes\Lambda^{p,q})$, $\varphi = \frac{1}{p!q!}\sum \varphi^{\alpha}_{\bar{J}I}\, dz^I d\bar{z}^J$, where

$$\begin{cases} dz^I = dz^{i_1}\wedge\cdots\wedge dz^{i_p}, \\ d\bar{z}^J = d\bar{z}^{j_1}\wedge\cdots\wedge d\bar{z}^{j_q} \\ \varphi^{\alpha}_{\bar{J}I} = \varphi^{\alpha}_{\bar{j}_q\cdots\bar{j}_1 i_p\cdots i_1} \end{cases}$$

$$\|\varphi\|^2 \triangleq \frac{1}{p!q!}\int_X \varphi^{\alpha}_{\bar{J}I}\,\overline{\varphi^{\beta}_{\bar{K}L}}\, H_{\bar{\beta}\alpha}\, g^{K\bar{J}} g^{I\bar{L}} \cdot \frac{\omega^n}{n!}$$

$$= \frac{1}{p!\,q!}\int_X \varphi^{\alpha}_{\bar{j}_q\cdots\bar{j}_1 i_p\cdots i_1}\,\overline{\varphi^{\beta}_{\bar{k}_q\cdots\bar{k}_1 l_p\cdots l_1}}\, H_{\bar{\beta}\alpha}\, g^{k_1\bar{j}_1}\cdots g^{k_q\bar{j}_q}\, g^{i_p\bar{l}_p}\cdots g^{i_1\bar{l}_1}\, \frac{\omega^n}{n!}$$

Here ω denotes the symplectic form $\omega = \frac{i}{2} g_{\bar{k}j}\, dz^j\wedge d\bar{z}^k$, and $\frac{\omega^n}{n!}$ is the volume form on X: $\frac{\omega^n}{n!} = \det(g_{\bar{k}j})\prod_{j=1}^{n}(dz^j\wedge d\bar{z}^j)$

Note that we can also define the inner product similarly.

Now, we define the formal adjoint $\bar{\partial}^{\dagger}$ by:

$$\langle\bar{\partial}\varphi, \psi\rangle = \langle\varphi, \bar{\partial}^{\dagger}\psi\rangle$$

for any $\varphi \in \Gamma(X, E\otimes\Lambda^{p,q})$, $\psi \in \Gamma(X, E\otimes\Lambda^{p,q+1})$, with compact support. Then, there is a natural Laplacian $\Box = \bar{\partial}\bar{\partial}^{\dagger} + \bar{\partial}^{\dagger}\bar{\partial}$ on $\Gamma(X, E\otimes\Lambda^{p,q})$. However, there is another natural Laplacian (metric):

$$\Gamma(X, E\otimes\Lambda^{p,q}) \ni \varphi \longmapsto -g^{j\bar{k}}\nabla_j\nabla_{\bar{k}}\varphi^{\alpha}_{\bar{J}K} \in \Gamma(X, E\otimes\Lambda^{p,q}).$$

Question: Compare "$\Box$" and "$-g^{j\bar{k}}\nabla_j\nabla_{\bar{k}}$".

A simple example: $\Box$ on $\Gamma(\mathbb{C}^n, \Lambda^{0,1})$, with flat metric $g_{\bar{k}j} = \delta_{kj}$. ($E$ trivial)

We compute $\bar\partial$ on $\Lambda^{0,0}$ and $\Lambda^{0,1}$. Take $f \in \Lambda^{0,0}$, $\varphi \in \Lambda^{0,1}$.

$$\bar\partial f = \partial_{\bar j} f \, d\bar z^j$$

$$\bar\partial\varphi = \bar\partial(\varphi_{\bar j} d\bar z^j) = \bar\partial\varphi_{\bar j} d\bar z^j = \partial_{\bar k}\varphi_{\bar j}\, d\bar z^k \wedge d\bar z^j = \tfrac{1}{2}(\partial_{\bar k}\varphi_{\bar j} - \partial_{\bar j}\varphi_{\bar k}) d\bar z^k \wedge d\bar z^j$$

i.e. $(\bar\partial\varphi)_{\bar j \bar k} = \partial_{\bar k}\varphi_{\bar j} - \partial_{\bar j}\varphi_{\bar k}$ in components.

Next we compute $\bar\partial^+$ on $\Lambda^{0,2}$ and $\Lambda^{0,1}$, by the defining equation:

$$\langle \bar\partial\varphi, \psi \rangle = \langle \varphi, \bar\partial^+\psi \rangle, \ \forall \varphi \in \Gamma_c(\mathbb{C}^n, \Lambda^{0,1}), \ \psi \in \Gamma(\mathbb{C}^n, \Lambda^{0,2})$$

$$\text{l.h.s.} = \tfrac{1}{2}\int_X (\bar\partial\varphi)_{\bar j\bar k} \overline{\psi_{\bar\ell\bar m}}\, g^{\ell\bar j} g^{m\bar k} \cdot vol$$

$$= \int_X (\partial_{\bar k}\varphi_{\bar j}) \overline{\psi_{\bar\ell\bar m}}\, g^{\ell\bar j} g^{m\bar k} \cdot vol$$

(note that $\partial_{\bar k}\varphi_{\bar j} - \partial_{\bar j}\varphi_{\bar k}$ anti-sym in $\bar k, \bar j$ but $\overline{\psi_{\bar\ell\bar m}}\, g^{\ell\bar j} g^{m\bar k}$ is also anti-sym in $\bar k, \bar j$, giving 2)

$$= -\int_X \varphi_{\bar j}\, \overline{\nabla_k \psi_{\bar\ell\bar m}\, g^{k\bar m}} \cdot vol$$

The last step using integration by parts and the reason we use ∇_k will be clarified in detail later.

Thus, on $\Lambda^{0,2}$, $(\bar\partial^+\psi)_{\bar\ell} = -\nabla_k \psi_{\bar\ell\bar m}\, g^{k\bar m} = -\nabla_k \psi_{\bar\ell\bar k}$ (flat metric).

On $\Lambda^{0,1}$, take $\varphi \in \Gamma_c(\mathbb{C}^n, \Lambda^{0,0})$, $\psi \in \Gamma(\mathbb{C}^n, \Lambda^{0,1})$:

$$\text{l.h.s.} = \int_X (\partial_{\bar j}\varphi) \overline{\psi_{\bar\ell}}\, g^{\ell\bar j} \cdot vol$$

$$= \int_X \varphi\, \overline{(-\nabla_j \psi_{\bar\ell}\, g^{j\bar\ell})} \cdot vol.$$

Thus on $\Lambda^{0,1}$, $\bar\partial^+\psi = -\nabla_j \psi_{\bar\ell}\, g^{j\bar\ell} = -\nabla_\ell \psi_{\bar\ell} = -\partial_\ell \psi_{\bar\ell}$

Hence on $\Lambda^{0,1}$, the Laplacian $\Box$ is given by ($\varphi = \varphi_{\bar j} d\bar z^j$)

$$(\Box\varphi)_{\bar j} = \{\bar\partial\bar\partial^+\varphi\}_{\bar j} + \{\bar\partial^+\bar\partial\varphi\}_{\bar j}$$

$$= \partial_{\bar j}(-\partial_\ell\varphi_{\bar\ell}) + (-\partial_k(\bar\partial\varphi)_{\bar j\bar k})$$

$$= -\partial_{\bar j}\partial_\ell\varphi_{\bar\ell} - (\partial_k(\partial_{\bar k}\varphi_{\bar j} - \partial_{\bar j}\varphi_{\bar k}))$$

$$= -\partial_{\bar j}\partial_\ell\varphi_{\bar\ell} + \partial_k\partial_{\bar j}\varphi_{\bar k} - \partial_k\partial_{\bar k}\varphi_{\bar j}$$

$$= -\partial_k\partial_{\bar k}\varphi_{\bar j}$$

Note that in the last step, we used the fact that on flat spaces, $\partial_{\bar j}\partial_k = \partial_k\partial_{\bar j}$ (or more precisely, $\nabla_{\bar j}\nabla_k - \nabla_k\nabla_{\bar j} = 0$). In general, this results in curvature terms. In this case, two Laplacians agree.

General case: $\Box = \bar\partial\bar\partial^+ + \bar\partial^+\bar\partial$ on $E \otimes \Lambda^{p,q}$.

(1). Computation of $\bar\partial$ on $E \otimes \Lambda^{p,q}$.

$$\bar\partial\left(\frac{1}{p!q!}\sum \varphi^{\alpha}_{\bar J I}\, dz^I\wedge d\bar z^J\right) = \frac{1}{p!q!}\sum \partial_{\bar k}\varphi^{\alpha}_{\bar J I}\, d\bar z^k\wedge dz^I\wedge d\bar z^J$$

Here we need some explaination: this means without anti-symmetrization,

$$(\bar\partial\varphi)^{\alpha}_{\bar J I\bar k} = (q+1)\,\partial_{\bar k}\varphi^{\alpha}_{\bar J I}.$$

However, with antisymmetrization, we can also write:

$$(\bar\partial\varphi)^{\alpha}_{\bar J I\bar k} = (\bar\partial\varphi)_{\bar j_q\cdots\bar j_1 I\bar k}$$

$$= \partial_{\bar k}\varphi_{\bar j_q\cdots\bar j_1 I} - \partial_{\bar j_1}\varphi_{\bar j_q\cdots\bar j_2\bar k I} - \partial_{\bar j_2}\varphi_{\bar j_q\cdots\bar k\bar j_1 I} - \cdots - \partial_{\bar j_q}\varphi_{\bar k\bar j_{q-1}\cdots\bar j_1 I}.$$

Note also that, even $E\otimes\Lambda^{p,q}$ is no longer a holomorphic bundle, $\bar\partial$ makes sense since the process of antisymmetrizing kills higher order differentiations, just as in de Rham case.

(2). Computation of $\bar\partial^{\dagger}$ on $E\otimes\Lambda^{p,q}$

Again, take $\varphi\in\Gamma(X,E\otimes\Lambda^{p,q})$ $\psi\in\Gamma(X,E\otimes\Lambda^{p,q+1})$. By definition, we have:

$$\langle\bar\partial\varphi,\psi\rangle = \langle\varphi,\bar\partial^{\dagger}\psi\rangle.$$

$$\text{l.h.s.} = \frac{1}{p!(q+1)!}\int_X (\bar\partial\varphi)^{\alpha}_{\bar J I\bar k}\,\overline{\psi^{\beta}_{\bar K M\bar\ell}}\, H_{\bar\beta\alpha}\, g^{K\bar J} g^{I\bar M} g^{\ell\bar k}\,\frac{\omega^n}{n!}$$

Observation:

(a). We can write $(\bar\partial\varphi)^{\alpha}_{\bar J I\bar k}$ in terms of genuine covariant derivative.

e.g. $\varphi\in\Gamma(X,E\otimes\Lambda^{p,1})$, $(\bar\partial\varphi)^{\alpha}_{\bar j I\bar k} = \partial_{\bar k}\varphi^{\alpha}_{\bar j I} - \partial_{\bar j}\varphi^{\alpha}_{\bar k I}$

$$= \nabla_{\bar k}\varphi^{\alpha}_{\bar j I} - \Gamma^{\bar m}_{\bar k\bar j}\varphi^{\alpha}_{\bar m I} - (\nabla_{\bar j}\varphi^{\alpha}_{\bar k I} - \Gamma^{\bar m}_{\bar j\bar k}\varphi_{\bar m I})$$

(Here recall that on a (1,0)-form $\nabla_k\varphi_j = \partial_k\varphi_j - \Gamma^m_{kj}\varphi_m \Rightarrow \nabla_{\bar k}\varphi_{\bar j} = \partial_{\bar k}\varphi_{\bar j} - \Gamma^{\bar m}_{\bar k\bar j}\varphi_{\bar m}$).

i.e. $(\bar\partial\varphi)^{\alpha}_{\bar j I\bar k} = (\nabla_{\bar k}\varphi^{\alpha}_{\bar j I} - \nabla_{\bar j}\varphi^{\alpha}_{\bar k I}) - (\Gamma^{\bar m}_{\bar k\bar j} - \Gamma^{\bar m}_{\bar j\bar k})\varphi^{\alpha}_{\bar m I}$.

The last term, being the difference of two connections, is a tensor, called the torsion tensor.

(b). $(g_{\bar k j})$ Kähler $\Leftrightarrow$ Torsion tensor $=0$

Recall that, $\Gamma^m_{kj} = g^{m\bar p}\partial_k g_{\bar p j}$, then $\Gamma^m_{kj} - \Gamma^m_{jk} = g^{m\bar p}(\partial_k g_{\bar p j} - \partial_j g_{\bar p k})$. Since $(g^{m\bar p})$ is invertible, the last term vanishes iff $\partial_k g_{\bar p j} - \partial_j g_{\bar p k} = 0$, iff $(g_{\bar k j})$ Kähler.

- Henceforth, we will assume that $(g_{\bar k j})$ is Kähler.

Now we have:

$$\frac{1}{p!(q+1)!}\int_X (\bar\partial\varphi)^{\alpha}_{\bar J I\bar k}\,\overline{\psi^{\beta}_{\bar K M\bar\ell}}\, H_{\bar\beta\alpha}\, g^{K\bar J} g^{I\bar M} g^{\ell\bar k}\,\frac{\omega^n}{n!}$$

$$= \frac{1}{p!q!}\int_X \nabla_{\bar k}\varphi^{\alpha}_{\bar J I}\,\overline{\psi^{\beta}_{\bar K M\bar\ell}}\, H_{\bar\beta\alpha}\, g^{K\bar J} g^{I\bar M} g^{\ell\bar k}\,\frac{\omega^n}{n!}$$

Here we introduce the following trick:

Lemma. (Ricci). $(\nabla_{\bar{k}} \varphi^{\alpha}_{\bar{J}I}) g^{K\bar{J}} = \nabla_{\bar{k}} (g^{K\bar{J}} \varphi^{\alpha}_{\bar{J}I})$.

Pf: The Chern connection preserves both the complex structure and metric structure:

$$\begin{aligned} \nabla_{\bar{k}} g_{i\bar{j}} &= \partial_{\bar{k}} g_{i\bar{j}} - \Gamma^{\bar{\ell}}_{\bar{k}\bar{i}} g_{\ell\bar{j}} \\ &= \partial_{\bar{k}} g_{i\bar{j}} - g^{p\bar{\ell}} \partial_{\bar{k}} g_{\bar{i}p} \cdot g_{\ell\bar{j}} \\ &= \partial_{\bar{k}} g_{\bar{i}j} - \partial_{\bar{k}} g_{\bar{i}j} \\ &= 0. \end{aligned}$$

□

Notice that $g^{K\bar{J}} \varphi^{\alpha}_{\bar{J}I}$ is a section of a holomorphic bundle, thus:

$$\begin{aligned} &\frac{1}{p!q!} \int_X \nabla_{\bar{k}} \varphi^{\alpha}_{\bar{J}I} \overline{\psi^{\beta}_{\bar{K}M\bar{\ell}}} H_{\bar{\beta}\alpha} g^{K\bar{J}} g^{I\bar{M}} g^{\ell\bar{k}} \frac{\omega^n}{n!} \\ = &\frac{1}{p!q!} \int_X \nabla_{\bar{k}} (g^{K\bar{J}} \varphi^{\alpha}_{\bar{J}I}) \overline{\psi^{\beta}_{\bar{K}M\bar{\ell}} g^{M\bar{I}} g^{k\bar{\ell}} H_{\bar{\alpha}\beta}} \cdot \frac{\omega^n}{n!} \\ = &\frac{1}{p!q!} \int_X \partial_{\bar{k}} (g^{K\bar{J}} \varphi^{\alpha}_{\bar{J}I}) W^{I\bar{k}}_{K\alpha} \frac{\omega^n}{n!} \\ = -&\frac{1}{p!q!} \int_X (g^{K\bar{J}} \varphi^{\alpha}_{\bar{J}I}) \partial_{\bar{k}} (W^{I\bar{k}}_{K\alpha} \frac{\omega^n}{n!}). \end{aligned}$$

where we denote for short $\overline{\psi^{\beta}_{\bar{K}M\bar{\ell}} g^{M\bar{I}} g^{k\bar{\ell}} H_{\bar{\alpha}\beta}} = W^{I\bar{k}}_{K\alpha}$; the last step used integration by parts.

Now, locally, $\omega^n = \det(g_{\bar{q}p}) \prod_{i=1}^{n} dz^i \wedge d\bar{z}^i$. thus

$$\begin{aligned} \partial_{\bar{k}} (W^{\bar{k}} \det g_{\bar{q}p}) &= \partial_{\bar{k}} W^{\bar{k}} \cdot \det g_{\bar{q}p} + W^{\bar{k}} \partial_{\bar{k}} \det g_{\bar{q}p} \\ &= \partial_{\bar{k}} W^{\bar{k}} \cdot \det g_{\bar{q}p} + \det g_{\bar{q}p} \cdot W^{\bar{k}} \partial_{\bar{k}} \log \det g_{\bar{q}p} \\ &= (\det g_{\bar{q}p}) (\partial_{\bar{k}} W^{\bar{k}} + W^{\bar{k}} g^{p\bar{q}} \partial_{\bar{k}} g_{\bar{q}p}) \\ &= (\det g_{\bar{q}p}) \overline{(\partial_k W^k + g^{q\bar{p}} \partial_k g_{\bar{p}q} \cdot W^k)} \end{aligned}$$

where in the 3rd step, we used that $(\log \det A)' = tr(A')$.

Recall also that $\Gamma^{\ell}_{qk} = g^{\ell\bar{p}} \partial_q g_{\bar{p}k} \Rightarrow g^{q\bar{p}} \partial_k g_{\bar{p}q} = \Gamma^{q}_{kq}$. Thus the above equation becomes:

$$\begin{aligned} \partial_{\bar{k}} (W^{\bar{k}} \cdot \det g_{\bar{q}p}) &= \det g_{\bar{q}p} \overline{(\partial_k W^k + \Gamma^{q}_{kq} W^k)} \\ &= \det g_{\bar{q}p} \overline{(\partial_k W^k + \Gamma^{q}_{qk} W^k)} \quad \text{(by the Kähler condition)} \\ &= \det g_{\bar{q}p} \overline{\nabla_k W^k} \\ &= \det g_{\bar{q}p} \nabla_{\bar{k}} W^{\bar{k}} \end{aligned}$$

Summing up, we obtain:

$$-\frac{1}{p!q!}\int_X (g^{k\bar{j}}\varphi^{\alpha}_{\bar{J}I})(\partial_{\bar{k}} W^{I\bar{k}}_{k\alpha}\cdot\frac{\omega^n}{n!})$$
$$= -\frac{1}{p!q!}\int_X g^{k\bar{j}}\varphi^{\alpha}_{\bar{J}I}(\nabla_{\bar{k}} W^{I\bar{k}}_{k\alpha})\frac{\omega^n}{n!}$$
$$= \frac{1}{p!q!}\int_X g^{k\bar{j}}\varphi^{\alpha}_{\bar{J}I}\overline{(-g^{k\bar{\ell}}\nabla_k\psi^{\beta}_{\bar{K}M\bar{\ell}})}\,H_{\bar{\beta}\alpha}\,g^{I\bar{M}}\cdot\frac{\omega^n}{n!}$$

Hence by definition:
$$(\bar{\partial}^{\dagger}\psi)^{\beta}_{\bar{K}M} = -g^{k\bar{\ell}}\nabla_k\psi^{\beta}_{\bar{K}M\bar{\ell}}.$$

Now we can derive the formula for the Laplacian $\Box$: $\forall\varphi\in\Gamma(X, E\otimes\Lambda^{p,q})$

$$\begin{aligned}(\bar{\partial}^{\dagger}\bar{\partial}\varphi)_{\bar{J}I} &= -g^{\ell\bar{m}}\nabla_\ell(\bar{\partial}\varphi)^{\alpha}_{\bar{J}I\bar{m}}\\
&= -g^{\ell\bar{m}}\nabla_\ell(\nabla_{\bar{m}}\varphi^{\alpha}_{\bar{j}_q\cdots\bar{j}_1 I} - \nabla_{\bar{j}_1}\varphi^{\alpha}_{\bar{j}_q\cdots\bar{j}_2\bar{m}I} - \cdots - \nabla_{\bar{j}_q}\varphi^{\alpha}_{\bar{m}\bar{j}_{q-1}\cdots\bar{j}_1 I})\\
&= -g^{\ell\bar{m}}\nabla_\ell\nabla_{\bar{m}}\varphi^{\alpha}_{\bar{j}_q\cdots\bar{j}_1 I} + g^{\ell\bar{m}}(\nabla_\ell\nabla_{\bar{j}_1}\varphi^{\alpha}_{\bar{j}_q\cdots\bar{j}_2\bar{m}I} + \cdots + \nabla_\ell\nabla_{\bar{j}_q}\varphi^{\alpha}_{\bar{m}\bar{j}_{q-1}\cdots\bar{j}_1 I})\end{aligned}$$

$$\begin{aligned}(\bar{\partial}\bar{\partial}^{\dagger}\varphi)_{\bar{J}I} &= (\bar{\partial}\bar{\partial}^{\dagger}\varphi)^{\alpha}_{\bar{j}_q\cdots\bar{j}_1 I}\\
&= (-1)^p(\bar{\partial}\bar{\partial}^{\dagger}\varphi)^{\alpha}_{\bar{j}_q\cdots\bar{j}_2 I\bar{j}_1}\\
&= (-1)^p(\nabla_{\bar{j}_1}(\bar{\partial}^{\dagger}\varphi)^{\alpha}_{\bar{j}_q\cdots\bar{j}_2 I} - \nabla_{\bar{j}_2}(\bar{\partial}^{\dagger}\varphi)^{\alpha}_{\bar{j}_q\cdots\bar{j}_3\bar{j}_1 I} - \cdots - \nabla_{\bar{j}_q}(\bar{\partial}^{\dagger}\varphi)^{\alpha}_{\bar{j}_1\bar{j}_{q-1}\cdots\bar{j}_2 I})\\
&= (-1)^p(\nabla_{\bar{j}_1}(-g^{\ell\bar{m}}\nabla_\ell\varphi^{\alpha}_{\bar{j}_q\cdots\bar{j}_2 I\bar{m}}) + \nabla_{\bar{j}_2}(g^{\ell\bar{m}}\nabla_\ell\varphi^{\alpha}_{\bar{j}_q\cdots\bar{j}_3\bar{j}_1 I\bar{m}}) + \cdots + \nabla_{\bar{j}_q}(g^{\ell\bar{m}}\nabla_\ell\varphi^{\alpha}_{\bar{j}_1\bar{j}_{q-1}\cdots\bar{j}_2 I\bar{m}}))\\
&= -g^{\ell\bar{m}}(\nabla_{\bar{j}_1}\nabla_\ell\varphi^{\alpha}_{\bar{j}_q\cdots\bar{j}_2\bar{m}I} - \nabla_{\bar{j}_2}\nabla_\ell\varphi^{\alpha}_{\bar{j}_q\cdots\bar{j}_3\bar{j}_1\bar{m}I} - \cdots - \nabla_{\bar{j}_q}\nabla_\ell\varphi^{\alpha}_{\bar{j}_1\bar{j}_{q-1}\cdots\bar{j}_2\bar{m}I})\end{aligned}$$

$$\begin{aligned}\Rightarrow (\Box\varphi)^{\alpha}_{\bar{J}I} &= \{\bar{\partial}^{\dagger}(\bar{\partial}\varphi)\}^{\alpha}_{\bar{J}I} + \{\bar{\partial}(\bar{\partial}^{\dagger}\varphi)\}^{\alpha}_{\bar{J}I}\\
&= -g^{\ell\bar{m}}\nabla_\ell\nabla_{\bar{m}}\varphi^{\alpha}_{\bar{J}I} + g^{\ell\bar{m}}\{[\nabla_\ell, \nabla_{\bar{j}_1}]\varphi^{\alpha}_{\bar{j}_q\cdots\bar{j}_2\bar{m}I} + \cdots + [\nabla_\ell, \nabla_{\bar{j}_q}]\varphi^{\alpha}_{\bar{m}\bar{j}_{q-1}\cdots\bar{j}_1 I}\}\end{aligned}$$

Note that the commutators can be replaced by curvature terms, for instance:

$$\begin{aligned}[\nabla_\ell, \nabla_{\bar{j}_1}]\varphi^{\alpha}_{\bar{j}_q\cdots\bar{j}_2\bar{m}I} &= F_{\bar{j}_1\ell}{}^{\alpha}{}_{\beta}\varphi^{\beta}_{\bar{j}_q\cdots\bar{j}_2\bar{m}I}\\
&\quad + R_{\bar{j}_1\ell\bar{j}_q}{}^{\bar{k}}\varphi^{\alpha}_{\bar{k}\bar{j}_{q-1}\cdots\bar{j}_2\bar{m}I} + \cdots + R_{\bar{j}_1\ell\bar{j}_2}{}^{\bar{k}}\varphi^{\alpha}_{\bar{j}_q\cdots\bar{j}_3\bar{k}\bar{m}I} + R_{\bar{j}_1\ell\bar{m}}{}^{\bar{k}}\varphi^{\alpha}_{\bar{j}_q\cdots\bar{j}_2\bar{k}I}\\
&\quad - R_{\bar{j}_1\ell}{}^{k}{}_{i_p}\varphi^{\alpha}_{\bar{j}_q\cdots\bar{j}_2\bar{m}k i_{p-1}\cdots i_1} - \cdots - R_{\bar{j}_1\ell}{}^{k}{}_{i_1}\varphi^{\alpha}_{\bar{j}_q\cdots\bar{j}_2\bar{m}I i_p\cdots i_2 k}.\end{aligned}$$

Thm. (Kodaira Vanishing Theorem, Version 1)

Let E be a holomorphic line bundle over a compact Kähler manifold X. Let h be a metric on E, $F_{\bar{k}j}$ its curvature. Assume:

$$F_{\bar{k}j} + R_{\bar{k}j} \geq \varepsilon\cdot g_{\bar{k}j} \qquad (*)$$

for some constant $\varepsilon > 0$. Then $\ker\Box|_{E\otimes\Lambda^{0,1}} = 0$.

Pf: Apply Bochner–Kodaira formula to this case:

$$(\Box\varphi)_{\bar j} = -g^{l\bar m}\nabla_l\nabla_{\bar m}\varphi_{\bar j} + g^{l\bar m}[\nabla_l, \nabla_{\bar j}]\varphi_{\bar m}$$
$$= -g^{l\bar m}\nabla_l\nabla_{\bar m}\varphi_{\bar j} + g^{l\bar m}\{F_{\bar j l}\varphi_{\bar m} + R_{\bar j l\bar m}{}^{\bar k}\varphi_{\bar k}\}$$

Under the Kähler condition: $R_{\bar j l\bar m}{}^{\bar k} = R_{\bar m l\bar j}{}^{\bar k}$

$$(\Box\varphi)_{\bar j} = -g^{l\bar m}\nabla_l\nabla_{\bar m}\varphi_{\bar j} + g^{l\bar m}F_{\bar j l}\varphi_{\bar m} + g^{l\bar m}R_{\bar m l\bar j}{}^{\bar k}\varphi_{\bar k}$$
$$= -g^{l\bar m}\nabla_l\nabla_{\bar m}\varphi_{\bar j} + F_{\bar j}{}^{\bar m}\varphi_{\bar m} + R_{\bar j}{}^{\bar m}\varphi_{\bar m}$$

Pair this with φ:

$$\langle\varphi, \Box\varphi\rangle = \int_X (\Box\varphi)_{\bar j}\overline{\varphi_{\bar k}}\, h\, g^{k\bar j}\frac{\omega^n}{n!}$$
$$= -\int_X g^{l\bar m}\nabla_l\nabla_{\bar m}\varphi_{\bar j}\,\overline{\varphi_{\bar k}}\, g^{k\bar j} h\frac{\omega^n}{n!} + \int_X (F_{\bar j}{}^{\bar m}\varphi_{\bar m} + R_{\bar j}{}^{\bar m}\varphi_{\bar m})\overline{\varphi_{\bar k}}\cdot h g^{k\bar j}\frac{\omega^n}{n!}$$

Now $\quad -\int_X g^{l\bar m}\nabla_l\nabla_{\bar m}\varphi_{\bar j}\,\overline{\varphi_{\bar k}}\, g^{k\bar j} h\frac{\omega^n}{n!}$

$$= -\int_X \nabla_l(g^{l\bar m}\nabla_{\bar m}\varphi_{\bar j})\,\overline{\varphi_{\bar k}}\, g^{k\bar j} h\frac{\omega^n}{n!}$$
$$= \int_X \nabla_{\bar m}\varphi_{\bar j}\cdot\overline{\nabla_{\bar l}\varphi_{\bar k}}\, g^{l\bar m}g^{k\bar j}h\frac{\omega^n}{n!}$$
$$= \|\nabla_{\bar m}\varphi_{\bar j}\|^2$$

Moreover, $\quad \int_X (F_{\bar j}{}^{\bar m}\varphi_{\bar m} + R_{\bar j}{}^{\bar m}\varphi_{\bar m})\overline{\varphi_{\bar k}}\cdot h g^{k\bar j}\frac{\omega^n}{n!}$

$$= \int_X (F_{\bar j l} + R_{\bar j l})g^{l\bar m}\varphi_{\bar m}\,\overline{\varphi_{\bar k}}\, g^{k\bar j} h\frac{\omega^n}{n!}$$
$$\geq \int_X \varepsilon\cdot g_{\bar j l}g^{l\bar m}\varphi_{\bar m}\overline{\varphi_{\bar k}}\, g^{k\bar j}h\frac{\omega^n}{n!}$$
$$= \varepsilon\int_X \varphi_{\bar j}\,\overline{\varphi_{\bar k}}\, g^{k\bar j}\frac{\omega^n}{n!}$$
$$= \varepsilon\|\varphi\|^2$$

$$\Longrightarrow \langle\Box\varphi, \varphi\rangle = \|\nabla_{\bar m}\varphi_{\bar j}\|^2 + \varepsilon\|\varphi\|^2 \geq \varepsilon\|\varphi\|^2.$$

If $\Box\varphi = 0$, then $\varepsilon\|\varphi\|^2 = 0 \Rightarrow \varphi = 0$. $\square$

Rmk: The above estimate, together with Hodge decomposition thm (to be proven in what follows), gives:

(a). (Hodge) $\Rightarrow H^1_{\bar\partial}(X, E) = 0$, i.e. $\forall f \in C^\infty(X, E\otimes\Lambda^{0,1})$, $\bar\partial f = 0 \Rightarrow f = \bar\partial u$ for some $u \in C^\infty(X, E)$.

(b). Furthermore, we have $\|u\|^2 \leq \frac{1}{\varepsilon}\|f\|^2$.

- The bounds are independent of h as long as $(*)$ condition is satisfied. More precisely, fix a metric $\hat h$ on φ, with curvature form $\hat F_{\bar k j}$. Then:

Thm. $\forall \varphi \in C^\infty(M,\mathbb{R})$, $h = e^{-\varphi}\hat{h}$. Assume:

$$\partial_j \partial_{\bar{k}} \varphi + F_{\bar{k}j} + R_{\bar{k}j} \geq \varepsilon g_{\bar{k}j}$$

Then $\exists u$, with $\bar{\partial} u = f$, with:

$$\int_X |u|^2 e^{-\varphi} \leq \frac{1}{\varepsilon} \int |f|^2 e^{-\varphi}$$

$$\text{or} \quad \|u\|^2 \leq \frac{1}{\varepsilon} \|f\|^2$$

□

This is a very important theme in current research.

Thm. (Kodaira Vanishing Theorem, Version 2)

Let $E \longrightarrow X$ be a positive holomorphic line bundle over $(X, g_{\bar{k}j})$, compact Kähler manifold, i.e. $\exists$ h a metric on E with $F_{\bar{k}j} = -\partial_j \partial_{\bar{k}} \log h > 0$. Then $\exists$ $m_0 > 0$, s.t. $\forall$ $m \geq m_0$, we have:

$$\operatorname{Ker} \square \big|_{E^m \otimes \Lambda^{0,1}} = 0$$

Pf: Bochner-Kodaira formula for $E^m \otimes \Lambda^{0,1}$ is:

$$\square \varphi_{\bar{j}} = -g^{\ell\bar{p}} \nabla_\ell \nabla_{\bar{p}} \varphi_{\bar{j}} + (m F_{\bar{j}\ell} + R_{\bar{j}\ell}) \varphi^\ell$$

Thus when $m \geq m_0 >> 0$, we have:

$$\langle \square \varphi, \varphi \rangle \geq \varepsilon \|\varphi\|^2$$

and the thm. follows. □

Kodaira-Akizuki-Nakano Formulas.

Previously, $\square$ is compared with the metric Laplacian. However, on Kähler manifolds we have another Laplacian of ∂: $\bar{\square} \triangleq \partial\partial^\dagger + \partial^\dagger\partial$, so we would like to compare $\square$ and $\bar{\square}$. To do this, we shall use the following operator Λ:

$$\Lambda: \quad \Gamma(X, E \otimes \Lambda^{p+1,q+1}) \ni \Phi \longmapsto (\Lambda\Phi)_{\bar{J}I} = g^{\ell\bar{m}} \Phi_{\bar{m}\ell\bar{J}I} \in \Gamma(X, E \otimes \Lambda^{p,q}).$$

Then we have the identities:

$$\begin{cases} [\partial, \Lambda] = \bar{\partial}^\dagger \\ [\bar{\partial}, \Lambda] = -\partial^\dagger \end{cases}$$

Having this, we can prove:

Thm (KAN)

$$\square = \bar{\square} + [F, \Lambda]$$

Pf: $$\square - \bar{\square} = \bar\partial\bar\partial^\dagger + \bar\partial^\dagger\bar\partial - \partial\partial^\dagger - \partial^\dagger\partial$$
$$= \bar\partial[\partial,\Lambda] + [\partial,\Lambda]\bar\partial + \partial[\bar\partial,\Lambda] + [\bar\partial,\Lambda]\partial$$
$$= \bar\partial\partial\Lambda - \cancel{\bar\partial\Lambda\partial} + \cancel{\partial\Lambda\bar\partial} - \Lambda\partial\bar\partial + \partial\bar\partial\Lambda - \cancel{\partial\Lambda\bar\partial} + \cancel{\bar\partial\Lambda\partial} - \Lambda\bar\partial\partial$$
$$= (\bar\partial\partial + \partial\bar\partial)\Lambda - \Lambda(\partial\bar\partial + \bar\partial\partial)$$
$$= [(\bar\partial\partial + \partial\bar\partial), \Lambda]$$

Recall that, for any $\varphi \in \Gamma(X, E\otimes\Lambda^{p,q})$
$$d_A^2\varphi = F\wedge\varphi$$
and
$$d_A^2\varphi = (\partial+\bar\partial)^2\varphi = (\partial\bar\partial + \bar\partial\partial)\varphi$$
$\Rightarrow$ $$\square - \bar\square = [F, \Lambda].$$ $\square$

Proof of the identities:

- $[\partial, \Lambda] = \bar\partial^\dagger$

$[\partial,\Lambda]\varphi = \partial\Lambda\varphi - \Lambda\partial\varphi.$

$$\{\partial\Lambda\varphi\}_{\bar L Mj} = \nabla_j(\Lambda\varphi)_{\bar L m_p\cdots m_1} - \nabla_{m_1}(\Lambda\varphi)_{\bar L m_p\cdots m_2 j} - \cdots - \nabla_{m_p}(\Lambda\varphi)_{\bar L j m_{p-1}\cdots m_1}$$
$$= \nabla_j(g^{a\bar b}\varphi_{\bar b a \bar L m_p\cdots m_1}) - \nabla_{m_1}(g^{a\bar b}\varphi_{\bar b a\bar L m_p\cdots m_2 j} - \cdots - \nabla_{m_p}(g^{a\bar b}\varphi_{\bar b a\bar L j m_{p-1}\cdots m_1}).$$
$$\{\Lambda\partial\varphi\}_{\bar L Mj} = g^{a\bar b}(\partial\varphi)_{\bar b a\bar L Mj}$$
$$= g^{a\bar b}(\nabla_j\varphi_{\bar b a\bar L M} - \nabla_{m_1}\varphi_{\bar b a\bar L m_p\cdots m_2 j} - \cdots - \nabla_{m_p}\varphi_{\bar b a\bar L j m_{p-1}\cdots m_1} - \nabla_a\varphi_{\bar b j\bar L M})$$

Substracting, using Ricci's lemma, we obtain:
$$\{[\partial,\Lambda]\varphi\}_{\bar L Mj} = g^{a\bar b}\nabla_a\varphi_{\bar b j\bar L M}$$
$$= g^{a\bar b}\nabla_a\varphi_{\bar L M\bar b j}$$
$$= -g^{a\bar b}\nabla_a\varphi_{\bar L Mj\bar b}$$
$$= (\bar\partial^\dagger\varphi)_{\bar L Mj}$$
(Here recall that $(\bar\partial^\dagger\psi)^\beta_{\bar K M} = -g^{k\bar\ell}\nabla_k\psi^\beta_{\bar K M\bar\ell}$.)

To show the second identity, we first compute $\partial^\dagger$. Given $\varphi\in\Gamma(X, E\otimes\Lambda^{p,q})$ $\psi\in\Gamma(X, E\otimes\Lambda^{p+1,q})$, by definition, we have $\langle\partial\varphi,\psi\rangle = \langle\varphi, \partial^\dagger\psi\rangle$, i.e.
$$\langle\varphi, \partial^\dagger\psi\rangle = \frac{1}{(p+1)!q!}\int_X (\partial\varphi)_{\bar L Mj}\overline{\psi^\beta_{\bar P Qr}}\, g^{P\bar L}g^{M\bar Q}g^{j\bar r}H_{\alpha\bar\beta}\frac{\omega^n}{n!}$$
$$= \frac{1}{p!\,q!}\int_X \nabla_j\varphi^\alpha_{\bar L M}\overline{\psi^\beta_{\bar P Qr}}\, g^{P\bar L}g^{M\bar Q}g^{j\bar r}H_{\alpha\bar\beta}\frac{\omega^n}{n!}$$

$$= \frac{1}{p!\,q!} \int_X \nabla_j \varphi^{\alpha}_{\bar{L}M} \, \overline{\psi^{\beta}_{\bar{P}Qr} g^{r\bar{j}}} \, g^{P\bar{L}} g^{M\bar{Q}} H_{\alpha\bar{\beta}} \frac{\omega^n}{n!}$$
$$= -\frac{1}{p!\,q!} \int_X \varphi^{\alpha}_{\bar{L}M} \, \overline{\nabla_{\bar{j}}(g^{r\bar{j}} \psi^{\beta}_{\bar{P}Qr})} \, g^{P\bar{L}} g^{M\bar{Q}} H_{\alpha\bar{\beta}} \frac{\omega^n}{n!}$$
$$\Rightarrow (\partial^{\dagger}\psi)^{\beta}_{\bar{P}Q} = -\nabla_{\bar{j}} g^{r\bar{j}} \psi^{\beta}_{\bar{P}Qr}$$

- $[\bar{\partial}, \Lambda] = -\partial^{\dagger}$

Take $\varphi \in \Gamma(X, \Lambda^{p,q} \otimes E)$, then $\Lambda\varphi \in \Gamma(X, \Lambda^{p-1,q-1} \otimes E)$, $\bar{\partial}\Lambda\varphi \in \Gamma(X, \Lambda^{p-1,q} \otimes E)$ and $\bar{\partial}\varphi \in \Gamma(X, \Lambda^{p,q+1} \otimes E)$, $\Lambda\bar{\partial}\varphi \in \Gamma(X, \Lambda^{p-1,q} \otimes E)$

$$\{\bar{\partial}\Lambda\varphi\}_{\bar{J}I\bar{j}} = \nabla_{\bar{j}} (\Lambda\varphi)_{\bar{J}I} - \nabla_{\bar{j}_1} (\Lambda\varphi)_{\bar{j}\bar{j}_{q-1}\cdots\bar{j}_2 I} - \cdots - \nabla_{\bar{j}_{q-1}} (\Lambda\varphi)_{\bar{j}\bar{j}_{q-2}\cdots\bar{j}_1 I}$$
$$= \nabla_{\bar{j}} g^{a\bar{b}} \varphi_{\bar{b}a\bar{J}I} - \nabla_{\bar{j}_1} g^{a\bar{b}} \varphi_{\bar{b}a\bar{j}_{q-1}\cdots\bar{j}_2\bar{j}I} - \cdots - \nabla_{\bar{j}_{q-1}} g^{a\bar{b}} \varphi_{\bar{b}a\bar{j}\bar{j}_{q-2}\cdots\bar{j}_1 I}$$
$$\{\Lambda\bar{\partial}\varphi\}_{\bar{J}I\bar{j}} = g^{a\bar{b}} (\bar{\partial}\varphi)_{\bar{b}a\bar{J}I\bar{j}}$$
$$= g^{a\bar{b}} (\nabla_{\bar{j}} \varphi_{\bar{b}a\bar{J}I} - \nabla_{\bar{b}} \varphi_{\bar{j}a\bar{J}I} - \nabla_{\bar{j}_1} \varphi_{\bar{b}a\bar{j}_{q-1}\cdots\bar{j}_2\bar{j}I} - \cdots - \nabla_{\bar{j}_{q-1}} \varphi_{\bar{b}a\bar{j}\bar{j}_{q-2}\cdots\bar{j}_1 I})$$

Substracting gives:

$$([\bar{\partial}, \Lambda]\varphi)_{\bar{J}I\bar{j}} = g^{a\bar{b}} \nabla_{\bar{b}} \varphi_{\bar{j}a\bar{J}I}$$
$$= g^{a\bar{b}} \nabla_{\bar{b}} \varphi_{\bar{J}I\bar{j}a}$$
$$= (-\partial^{\dagger}\varphi)_{\bar{J}I\bar{j}} \qquad \square$$

Rmk: The signs of these identities can be easily checked as follows.

For $\varphi \in \Gamma(X, \Lambda^{0,1})$, $\varphi = \varphi_{\bar{j}} d\bar{z}^j$, then

$$[\partial, \Lambda]\varphi = (\partial\Lambda - \Lambda\partial)\varphi = -\Lambda\partial\varphi$$

Now, $\partial\varphi = \nabla_k \varphi_{\bar{j}} dz^k \wedge d\bar{z}^j \Rightarrow (\partial\varphi)_{\bar{j}k} = \nabla_k \varphi_{\bar{j}}$. Thus

$$[\partial, \Lambda]\varphi = -g^{k\bar{j}} (\partial\varphi)_{\bar{j}k} = -g^{k\bar{j}} (\nabla_k \varphi_{\bar{j}}) = \bar{\partial}^{\dagger}\varphi$$

Similarly if we take $\psi \in \Gamma(X, \Lambda^{1,0})$, $\psi = \psi_k dz^k$, then

$$[\bar{\partial}, \Lambda]\psi = (\bar{\partial}\Lambda - \Lambda\bar{\partial})\psi = -\Lambda\bar{\partial}\psi.$$

where $\bar{\partial}\psi = \nabla_{\bar{j}} \psi_k d\bar{z}^j \wedge dz^k$, or $(\bar{\partial}\psi)_{\bar{j}k} = -\nabla_{\bar{j}} \psi_k$. Hence

$$[\bar{\partial}, \Lambda]\psi = -\Lambda\bar{\partial}\psi = -g^{k\bar{j}} (\bar{\partial}\psi)_{\bar{j}k} = g^{k\bar{j}} \nabla_{\bar{j}} \psi_k = -\partial^{\dagger}\psi_k.$$

Lemma. Let $E \longrightarrow X$ be a line bundle with metric h and curvature $F_{\bar{k}j}$. At each $z \in X$, let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the eigenvalues of $F_{\bar{k}j}$ w.r.t. $g_{\bar{k}j}$. Then:

$$\langle [F,\Lambda]\psi,\psi\rangle_z = \sum_{\bar K J}\Big(\sum_{a\in J}\lambda_a + \sum_{\bar b\in \bar K}\lambda_b - \sum_{c=1}^{n}\lambda_c\Big)|\psi_{\bar K J}|^2$$

where we take a coordinate system s.t. $dz^1,\cdots,dz^n$ are orthonormal at z.

Rmk: Recall that since $F_{\bar k j}$ and $g_{\bar k j}$ are both Hermitian, $T^{\ell}{}_j \triangleq g^{\ell\bar k}F_{\bar k j}$ is then a self-adjoint endomorphism of the holomorphic tangent space at z:

$$\langle Ta, b\rangle_g = \langle a, Tb\rangle_g$$

Indeed, $g_{\bar k j}T^j{}_\ell a^\ell \overline{b^k} = g_{\bar k j}g^{j\bar s}F_{\bar s \ell}a^\ell\overline{b^k} = F_{\bar k j}a^j\overline{b^k}$,

and $g_{\bar s j}a^j\overline{T^s{}_k b^k} = g_{\bar s j}a^j\overline{g^{s\bar r}F_{\bar r k}b^k} = g_{\bar s j}g^{r\bar s}F_{\bar k r}a^j\overline{b^k} = F_{\bar k j}a^j\overline{b^k}$.

Thus we may talk about the eigenvalues of T: $\lambda_1\le\lambda_2\le\cdots\le\lambda_n$.

Pf: We compute:

$$\begin{aligned}
(\Lambda F\psi)_{\bar K J\bar r s} &= g^{n\bar m}(F\psi)_{\bar m n\bar K J\bar r s}\\
&= g^{n\bar m}(F\psi)_{\bar K J\bar m n\bar r s}\\
&= g^{n\bar m}\big(F_{\bar r s}\psi_{\bar K J\bar m n} + \text{anti-symmetric terms of } ((\bar r,\bar K,\bar m),(s,J,n))\big)\\
&= g^{n\bar m}\Big(\underbrace{F_{\bar r s}\psi_{\bar K J\bar m n}}_{\textcircled{1}}\\
&\qquad -\underbrace{\sum_a F_{\bar k_a s}\psi_{\bar k_q\cdots\bar k_{a+1}\bar r\bar k_{a-1}\cdots\bar k_1 J\bar m n}}_{\textcircled{2}} - F_{\bar m s}\psi_{\bar K J\bar r n}\\
&\qquad -\underbrace{\sum_b F_{\bar r j_b}\psi_{\bar K j_p\cdots j_{b+1}s j_{b-1}\cdots j_1\bar m n}}_{\textcircled{3}} - F_{\bar r n}\psi_{\bar K J\bar m s}\\
&\qquad +\underbrace{\sum_a\sum_b F_{\bar k_a j_b}\psi_{\bar k_q\cdots\bar r\cdots\bar k_1 j_p\cdots s\cdots j_1\bar m n}}_{\textcircled{4}} + \sum_a F_{\bar k_a n}\psi_{\bar k_q\cdots\bar r\cdots\bar k_1 J\bar m s}\\
&\qquad +\sum_b F_{\bar m j_b}\psi_{\bar K j_p\cdots s\cdots j_1\bar r n} + F_{\bar m n}\psi_{\bar K J\bar r s}\Big)
\end{aligned}$$

Next,

$$\begin{aligned}
(F\Lambda\psi)_{\bar K J\bar r s} &= F_{\bar r s}g^{n\bar m}\psi_{\bar m n\bar K J} + \text{anti-symmetric terms in } ((\bar r,\bar K),(s,J))\\
&= \underbrace{F_{\bar r s}g^{n\bar m}\psi_{\bar m n\bar K J}}_{\textcircled{1}}\\
&\qquad -\underbrace{\sum_a F_{\bar k_a s}g^{n\bar m}\psi_{\bar m n\bar k_q\cdots\bar r\cdots\bar k_1 J}}_{\textcircled{2}} - \underbrace{\sum_b F_{\bar r j_b}g^{n\bar m}\psi_{\bar m n\bar K j_p\cdots s\cdots j_1}}_{\textcircled{3}}\\
&\qquad +\underbrace{\sum_a\sum_b F_{\bar k_a j_b}g^{n\bar m}\psi_{\bar m n\bar k_q\cdots\bar r\cdots\bar k_1 j_p\cdots s\cdots j_1}}_{\textcircled{4}}
\end{aligned}$$

Substracting gives (notice the underlined cancelation relation)

$$([\Lambda, F]\psi)_{\bar{K}J\bar{r}s} = -g^{n\bar{m}} F_{\bar{m}s} \psi_{\bar{K}J\bar{r}n} - g^{n\bar{m}} F_{\bar{r}n} \psi_{\bar{K}J\bar{m}s}$$
$$+ \sum_a g^{n\bar{m}} F_{\bar{k}_a n} \psi_{\bar{k}_1 \cdots \bar{r} \cdots \bar{k}_q J \bar{m} s} + \sum_b g^{n\bar{m}} F_{\bar{m} j_b} \psi_{\bar{K} j_1 \cdots s \cdots j_p \bar{r} n} + g^{n\bar{m}} F_{\bar{m}n} \psi_{\bar{K}J\bar{r}s}$$

Now in a local coordinate system where $g_{\bar{m}n} = \delta_{mn}$, $F_{\bar{k}j} = \lambda_j \delta_{kj}$, we have:

$$([\Lambda, F]\psi)_{\bar{K}J\bar{r}s} = -\lambda_s \psi_{\bar{K}J\bar{r}s} - \lambda_r \psi_{\bar{K}J\bar{r}s} + \sum_a \lambda_{k_a} \psi_{\bar{k}_1 \cdots \bar{r} \cdots \bar{k}_q J \bar{k}_a s}$$
$$+ \sum_b \lambda_{j_b} \psi_{\bar{K} j_1 \cdots s \cdots j_p \bar{r} j_b} + \sum_{c=1}^{n} \lambda_c \psi_{\bar{K}J\bar{r}s}$$
$$= -\lambda_s \psi_{\bar{K}J\bar{r}s} - \lambda_r \psi_{\bar{K}J\bar{r}s} - \sum_a \lambda_{k_a} \psi_{\bar{K}J\bar{r}s} - \sum_b \lambda_{j_b} \psi_{\bar{K}J\bar{r}s} + \sum_{c=1}^{n} \lambda_c \psi_{\bar{K}J\bar{r}s}.$$
$$= \Big(-\sum_{a \in K, r} \lambda_a - \sum_{b \in J, s} \lambda_b + \sum_{c=1}^{n} \lambda_c\Big) \psi_{\bar{K}J\bar{r}s}. \qquad \square$$

As an immediate corollary, we obtain:

Thm (KAN)

Let $E \longrightarrow X$ be a positive line bundle, with metric h and curvature

$$F = -\partial_j \bar{\partial}_{\bar{k}} \log h > 0$$

Choose the curvature form of E as metric on X, i.e. $g_{\bar{k}j} = F_{\bar{k}j}$ (which implies all $\lambda_a = 1$, $a = 1, \cdots, n$). Then:

$$\langle [F, \Lambda]\psi, \psi \rangle = (p+q-n) \|\psi\|^2$$

Consequently,

$$\operatorname{Ker} \Box|_{E \otimes \Lambda^{p,q}} = 0 \quad \text{for } p+q > n.$$

In particular,

$$\operatorname{Ker} \Box|_{E \otimes \Lambda^{n,q}} = 0 \quad \text{for any } q > 1. \qquad \square$$

Observations:

1. Together with Hodge thm, this again gives solvability with L^2-bounds $\bar{\partial} u = f$.
2. There are versions of all these where X is not compact but complete and pseudo-convex. (Demailly)
3. There are versions for X bounded pseudo-convex domains in $\mathbb{C}^n$ (not complete) (Hörmander. L^2-estimate)

§7. Hodge Decomposition Theorem

Let $E \longrightarrow X$ be a holomorphic vector bundle, $H_{\bar{\alpha}\beta}$ a metric on E, $g_{\bar{k}j}$ a metric on X (not necessarily Kähler. Define $\square = \bar{\partial}\bar{\partial}^{+} + \bar{\partial}^{+}\bar{\partial}$ on $C^{\infty}(X, E\otimes\Lambda^{p,q})$.

The main result of this chapter is the following:

Thm. (Hodge, Kodaira)

Let $L^2(X, E\otimes\Lambda^{p,q})$ denote the space of L^2 (p,q)-forms valued in E:

(a). $\exists$ an orthonormal basis $\{\psi_l\}$ of $L^2(X, E\otimes\Lambda^{p,q})$, $\psi_l \in C^{\infty}(X, E\otimes\Lambda^{p,q})$ and

$$\square\psi_l = \lambda_l\psi_l$$

for each $l \in \mathbb{N}$.

(b) $\exists$ an operator $G: L^2 \longrightarrow L^2$, bounded, self-adjoint operator so that

$$G\bar{\partial} = \bar{\partial}G$$

$$G\bar{\partial}^{+} = \bar{\partial}^{+}G$$

$$\square G = G\square = Id - \pi$$

where $\pi: L^2 \longrightarrow L^2$ is the orthogonal projection of L^2 inner product space onto $Ker\,\square = \{\varphi \in C^{\infty}(X, E\otimes\Lambda^{p,q}) \mid \square\varphi = 0\}$.

(c). For each λ, the eigenspace $\{\varphi \in C^{\infty}(X, E\otimes\Lambda^{p,q}) \mid \square\varphi = \lambda\varphi\}$ is finite dimensional and $\lambda_m \geq c\cdot m^{\delta}$ for some $\delta > 0$.

Cor. 1. $\forall \varphi \in C^{\infty}(X, E\otimes\Lambda^{p,q})$, φ can be written as $(Id = \square G + \pi)$:

$$\varphi = \pi\varphi + \square G\varphi = \pi\varphi + \bar{\partial}(\bar{\partial}^{+}G\varphi) + \bar{\partial}^{+}(\bar{\partial}G\varphi)$$

i.e. $C^{\infty}(X, E\otimes\Lambda^{p,q}) = Ker\,\square \oplus Range\,\bar{\partial} \oplus Range\,\bar{\partial}^{+}$, where $\oplus$ means that the summands are mutually orthogonal to each other. $\square$

Def. (Dolbeaut cohomology). For the complex:

$$\cdots \xrightarrow{\bar{\partial}} \Gamma(X, E\otimes\Lambda^{p,q-1}) \xrightarrow{\bar{\partial}} \Gamma(X, E\otimes\Lambda^{p,q}) \xrightarrow{\bar{\partial}} \Gamma(X, E\otimes\Lambda^{p,q+1}) \xrightarrow{\bar{\partial}} \cdots$$

it's Dolbeaut cohomology is defined as:

$$H^{p,q}_{\bar{\partial}}(X, E) \triangleq (ker\,\bar{\partial}|_{E\otimes\Lambda^{p,q}}) / (Im\,\bar{\partial}|_{E\otimes\Lambda^{p,q-1}})$$

i.e. $\forall \varphi, \varphi' \in Ker\,\bar{\partial}|_{E\otimes\Lambda^{p,q}}$, $\varphi \sim \varphi' \Leftrightarrow \varphi - \varphi' = \bar{\partial}\psi$ for some $\psi \in \Gamma(X, E\otimes\Lambda^{p,q-1})$.

Cor 2. If $\varphi \in C^\infty(X, E\otimes\Lambda^{p,q})$ and $\bar\partial\varphi = 0$, then

$$\varphi = \pi\varphi + \bar\partial\bar\partial^\dagger G\varphi$$

i.e. any Dolbeault cohomology class $[\varphi]$ can be represented by a harmonic representative $\pi\varphi \in \mathrm{Ker}\,\Box|_{E\otimes\Lambda^{p,q}}$. $\Box$

Rmk: $\mathrm{Ker}\,\Box|_{E\otimes\Lambda^{p,q}}$ depends aparently on metrics while $H^{p,q}_{\bar\partial}$ is independent of metric. Thus with the vanishing thms proven previously, we can obtain information (vanishing) of $H^{p,q}_{\bar\partial}$.

Sobolev Spaces $H_{(s)}(X, E\otimes\Lambda^{p,q})$

X: compact. We define the Sobolev norm $\|\cdot\|_{(s)}$ on $C^\infty(X, E\otimes\Lambda^{p,q})$.

$$\|\varphi\|^2_{(s)} \triangleq \sum_{k\le s}\int_X \nabla_{a_1\cdots a_k}\varphi^\alpha\,\overline{\nabla_{b_1\cdots b_k}\varphi^\beta}\, g^{A\bar B} H_{\bar\beta\alpha}\frac{\omega^n}{n!}.$$

where a_i, b_j range in $\{1,\cdots,n,\bar 1,\cdots,\bar n\}$. $H_{\bar\beta\alpha}$ is the metric on $E\otimes\Lambda^{p,q}$.

For e.g.

$$\|\varphi\|^2_{(0)} = \int_X \varphi^\alpha\overline{\varphi^\beta} H_{\bar\beta\alpha}\frac{\omega^n}{n!} = \|\varphi\|^2_{L^2}$$

$$\|\varphi\|^2_{(1)} = \|\varphi\|^2_{(0)} + \int_X \nabla_j\varphi^\alpha\,\overline{\nabla_k\varphi^\beta}\, g^{j\bar k}H_{\bar\beta\alpha}\frac{\omega^n}{n!} + \int_X \nabla_{\bar j}\varphi^\alpha\,\overline{\nabla_{\bar k}\varphi^\beta}\, g^{k\bar j}H_{\bar\beta\alpha}\frac{\omega^n}{n!}.$$

$$\begin{aligned}\|\varphi\|^2_{(2)} = \|\varphi\|^2_{(1)} &+ \int_X \nabla_j\nabla_k\varphi^\alpha\,\overline{\nabla_l\nabla_m\varphi^\beta}\, g^{j\bar l}g^{k\bar m}H_{\bar\beta\alpha}\frac{\omega^n}{n!}\\ &+ \int_X \nabla_{\bar j}\nabla_{\bar k}\varphi^\alpha\,\overline{\nabla_{\bar l}\nabla_{\bar m}\varphi^\beta}\, g^{l\bar j}g^{m\bar k}H_{\bar\beta\alpha}\frac{\omega^n}{n!}\\ &+ \int_X \nabla_j\nabla_{\bar k}\varphi^\alpha\,\overline{\nabla_l\nabla_{\bar m}\varphi^\beta}\, g^{j\bar l}g^{m\bar k}H_{\bar\beta\alpha}\frac{\omega^n}{n!}\\ &+ \int_X \nabla_{\bar j}\nabla_k\varphi^\alpha\,\overline{\nabla_{\bar l}\nabla_m\varphi^\beta}\, g^{l\bar j}g^{k\bar m}H_{\bar\beta\alpha}\frac{\omega^n}{n!} \qquad \text{etc.}\end{aligned}$$

Def. $H_{(s)}(X, E\otimes\Lambda^{p,q}) = \overline{C^\infty(X, E\otimes\Lambda^{p,q})}$ w/r. t. $\|\cdot\|_{(s)}$.

(= Space of Cauchy sequences in $C^\infty(X, E\otimes\Lambda^{p,q})$, Cauchy w/r. t. $\|\ \|_{(s)}$).

i.e. $\varphi\in H_{(s)}$, $\varphi = \{\varphi_l\}_{l\in\mathbb N}$. $\|\varphi_l - \varphi_m\|_{(s)} \longrightarrow 0$, as $l, m\longrightarrow\infty$.

By def. of $\|\cdot\|_{(s)}$, $\varphi_l \longrightarrow \varphi$ in $H_{(s)}$ for some $\varphi\in L^2$, thus we may think of $H_{(s)}$ as a subspace of L^2.

Observation: $\|\varphi_\ell - \varphi_m\|_{(s)} \longrightarrow 0 \Leftrightarrow \|\nabla^k\varphi_\ell - \nabla^k\varphi_m\|_{L^2} \longrightarrow 0$ for all $k \le s$, i.e. $\nabla^k\varphi_\ell \longrightarrow \psi_k$ in L^2, for some $\psi_k \in L^2$, $\forall k \le s$. Thus $\varphi \in H_{(s)}$ gives rise to a sequence $\{\varphi_\ell : \varphi_\ell \in C^\infty(X, E \otimes \Lambda^{p,q})\}$, $\varphi_\ell \longrightarrow \varphi$ in L^2, and $\nabla^k\varphi_\ell$ converges in L^2 (to some ψ_k) for all $k \le s$.

Def. The k-th derivative of φ (in the sense of distributions) is defined by

$$\nabla^k\varphi \triangleq \lim_\ell \nabla^k\varphi_\ell = \psi_k \in L^2$$

for all $k \le s$.

Formal adjointness

We know that, $\forall\, \Phi, \Psi \in C^\infty$,

$$\langle \Box\Phi, \Psi \rangle = \langle \Phi, \Box\Psi \rangle,$$

since $\Box = \bar\partial\bar\partial^+ + \bar\partial^+\bar\partial$ and

$$\begin{aligned}\langle \Box\Phi, \Psi \rangle &= \langle \bar\partial\bar\partial^+\Phi, \Psi \rangle + \langle \bar\partial^+\bar\partial\Phi, \Psi \rangle \\ &= \langle \bar\partial^+\Phi, \bar\partial^+\Psi \rangle + \langle \bar\partial\Phi, \bar\partial\Psi \rangle\end{aligned}$$

This remains true if $\varphi\ \psi \in H_{(2)}$

$$\langle \Box\varphi, \psi \rangle = \langle \varphi, \Box\psi \rangle$$

where the inner product taken in L^2.

Pf: Take $\varphi_\ell \longrightarrow \varphi$, $\psi_\ell \longrightarrow \psi$. Then $\Box\varphi_\ell$ and $\Box\psi_\ell$ are smooth

$$\langle \Box\varphi, \psi \rangle = \lim_{\ell\to\infty} \langle \Box\varphi_\ell, \psi_\ell \rangle = \lim_{\ell\to\infty} \langle \varphi_\ell, \Box\psi_\ell \rangle = \langle \varphi, \psi \rangle.$$

Basic facts about Sobolev spaces

(a). The operator $\Box$ satisfies, $\forall\, \Phi \in C^\infty(X, E \otimes \Lambda^{p,q})$

$$\|\Box\Phi\|_{(s)} \le C_s \|\Phi\|_{(s+2)}$$

Thus $\Box$ extends uniquely to a linear bounded operator $\Box : H_{(s+2)} \longrightarrow H_{(s)}$.

Indeed, $\varphi \in H_{(s+2)}$, write $\varphi = \lim_\ell \varphi_\ell$ w/r.t. $\|\cdot\|_{(s+2)}$, then

$$\|\Box\varphi_\ell - \Box\varphi_m\|_{(s)} \le C_s \|\varphi_\ell - \varphi_m\|_{(s+2)} \longrightarrow 0 \quad \text{as } \ell, m \longrightarrow 0.$$

Hence $\{\Box\varphi_\ell\}$ is Cauchy w/r.t. $\|\cdot\|_{(s)}$, thus defines an element of $H_{(s)}$.

(b). Sobolev lemma.

Let $n_{\mathbb{R}}$ be the real dimension of X. If $s > \frac{n_{\mathbb{R}}}{2}$, then $\exists\, C_s$, s.t. $\forall\, \Phi \in C^\infty$,

$$\|\Phi\|_{C^0} \leq C_s \|\Phi\|_{(s)}$$

where $\|\Phi\|_{C^0}$ is the sup norm:

$$\|\Phi\|_{C^0} = \sup_X (\Phi^{\alpha}_{\bar{K}J} \overline{\Phi^{\beta}_{\bar{L}H}} H_{\bar{\beta}\alpha} g^{J\bar{H}} g^{L\bar{K}})$$

Two corollaries are immediate:

Cor 1. If $\varphi \in H_{(s)}$ for $s > \frac{n_{\mathbb{R}}}{2}$, then φ is continuous.

Pf: Let $\varphi_\ell \longrightarrow \varphi$ in L^2, $\varphi_\ell \in C^\infty$, $\|\varphi_\ell - \varphi_m\|_{(s)} \longrightarrow 0$ as $\ell, m \longrightarrow \infty$. By Sobolev lemma, $\|\varphi_\ell - \varphi_m\|_{C^0} \leq C_s \|\varphi_\ell - \varphi_m\|_{(s)} \longrightarrow 0$ as $\ell, m \longrightarrow \infty$. i.e. $\{\varphi_\ell\}$ is uniformly Cauchy and thus converges to a continuous function $\hat{\varphi}$. But $\varphi_\ell \longrightarrow \tilde{\varphi}$ in L^2 for some $\tilde{\varphi}$. By uniqueness of limit, $\varphi = \tilde{\varphi}$, and thus φ is continuous. $\square$

Cor 2. If $s > \frac{n_{\mathbb{R}}}{2} + m$, then $\varphi \in H_{(s)} \Rightarrow \varphi \in C^m$. Thus $\bigcap_{s>0} H_{(s)} = C^\infty$ $\square$

(c). Rellich's lemma.

X: compact. $s < t$. Let $\{\varphi_j \in H_{(t)}\}$ be a sequence s.t. $\|\varphi_j\|_{(t)} \leq 1$. Then $\exists$ a subsequence $\{\varphi_{j_k}\}$, and $\varphi \in H_{(s)}$, and $\varphi_{j_k} \longrightarrow \varphi$ in $H_{(s)}$.

Linear elliptic PDE

(a). Let $\square$ be the Laplacian, then $\forall s$, $\exists\, C_s$ s.t. $\forall\, \varphi \in C^\infty$,

$$\|\varphi\|_{(s+2)} \leq C_s (\|\square\varphi\|_{(s)} + \|\varphi\|_{(s+1)})$$

This depends crucially on the ellipticity of Laplacian, and is called "A Priori estimate". Similar as above, this translates directly to $H_{(s+2)}$.

(b). Regularity Thm.

Let $\Omega \subseteq X$ be an open set and $u, f \in L^2$ s.t. $\square u = f$ on Ω in the sense of distributions, (i.e. by def. $\langle u, \square\varphi \rangle = \langle f, \varphi \rangle$, $\forall\, \varphi \in C^\infty_0(\Omega)$) and $\square u = f$ in the standard sense.

Construction of Green's operator G

(a). Let $\mathrm{Ker}\,\Box = \{\varphi \in H_{(2)} ;\ \Box\varphi = 0\}$. Then $\mathrm{Ker}\,\Box$ is finite dimensional

Pf: If $\{\varphi_j\}$ were an infinite orthonormal basis of $\mathrm{Ker}\,\Box$ in $H_{(2)}$. By a priori estimate, we have, for $\varphi_j - \varphi_\ell$ $(j \neq \ell)$

$$\|\varphi_j - \varphi_\ell\|_{(2)} \leq C \cdot (\|\Box(\varphi_j - \varphi_\ell)\|_{(0)} + \|\varphi_j - \varphi_\ell\|_{1})$$
$$= C \cdot \|\varphi_j - \varphi_\ell\|_{(1)}$$

since $\varphi_j, \varphi_\ell \in \mathrm{Ker}\,\Box$. But $\|\varphi_j\|_{(2)} = 1$, and thus by Rellich's lemma, we may assume $\|\varphi_j - \varphi_\ell\|_{(1)} \longrightarrow 0$, by passing to a subsequence. This tells us that

$$\sqrt{2} = \|\varphi_j - \varphi_\ell\|_{(2)} \longrightarrow 0 \quad \text{as } j, \ell \longrightarrow \infty.$$

Contradiction. $\Box$

Similarly, we can show that each eigenspace $\mathrm{Ker}(\Box - \lambda)$ is finite dimensional, since we have

$$\|\varphi\|_{(s+2)} \leq C \cdot (\|\Box\varphi\|_{(s)} + \|\varphi\|_{(s+1)})$$
$$\leq C \cdot (\|\Box\varphi - \lambda\varphi\|_{(s)} + |\lambda|\,\|\varphi\|_{(s)} + \|\varphi\|_{(s+1)})$$

and the above proof applies.

Note that by regularity thm, $\mathrm{Ker}\,\Box$ actually consists of smooth functions.

(b). Define the range of $\Box$ by:

$$R(\Box) \triangleq \{\varphi \in H_{(0)},\ \varphi = \Box\psi \text{ for some } \psi \in H_{(2)}\}$$

We have the key fact that:

- $R(\Box)$ is closed

Observations: In general, $W \subseteq H$, H: Hilbert space and W a subspace, we have

$$H = \overline{W} \oplus W^{\perp}$$

Then since the range is closed, we can write

$$L^2 = R(\Box) \oplus R(\Box)^{\perp}$$

i.e. $\forall \varphi \in L^2$, $\varphi = \Box\psi + \varphi_0$, with $\varphi_0 \in R(\Box)^{\perp}$. We may further choose $\psi \perp \mathrm{Ker}\,\Box$ in the L^2-sense. (One may worry that throwing away from ψ its component in $\mathrm{Ker}\,\Box$ may not be well-defined. But there is no trouble since $\mathrm{Ker}\,\Box$ is finite dimensional thus closed, and in fact consists of smooth functions).

We have shown that $R(\square)$ is closed in §4. We reproduce the proof here for the sake of completeness:

Claim: a priori estimate $\Rightarrow \exists\, C>0$, s.t. $\forall \varphi \in C^\infty$, $\varphi \perp \operatorname{Ker}\square$, we have

$$\|\varphi\|_{(2)} \le C \cdot \|\square\varphi\|_{(0)} \quad (AP')$$

Then, $\forall\, \Phi \in R(\square)$, and $\Phi_n = \square\varphi_n$, $\varphi_n \perp \operatorname{Ker}\square$, and $\Phi_n \longrightarrow \Phi$ in $H_{(2)}$. Now, by (AP') $\|\varphi_n - \varphi_m\|_{(2)} \leqslant C \cdot \|\square\varphi_n - \square\varphi_m\|_{(0)} = C\|\Phi_n - \Phi_m\|_{(0)} \longrightarrow 0 \Rightarrow \{\varphi_n\}$ converges in H_2, say, $\varphi_n \longrightarrow \varphi$. Then $\square\varphi = \lim_n \square\varphi_n = \Phi$.

Pf of claim:

Otherwise, $\forall\, n \in \mathbb{N}$, $\exists\, \varphi_n \in \Gamma(X,L)$, $\varphi_n \perp \operatorname{Ker}\square$ with $\|\varphi_n\|_{(2)} \geqslant n\,\|\square\varphi_n\|_{(0)}$. Define: $\psi_n \triangleq \varphi_n / \|\varphi_n\|_{(2)}$. Then $\|\square\psi_n\|_{(0)} \leqslant \frac{1}{n}$, $\|\psi_n\|_{(2)} = 1$.

By a priori estimate, $\|\psi_n - \psi_m\|_{(2)} \leqslant D\cdot(\|\square\psi_n - \square\psi_m\|_{(0)} + \|\psi_n - \psi_m\|_{(1)})$ for some $D>0$. Rellich's lemma $\Rightarrow \exists$ subsequence, which we may assume to be $\{\psi_n\}$ to start with, s.t. $\|\psi_n - \psi_m\|_{(1)} \longrightarrow 0$ as $n,m \to \infty$. $\Rightarrow \psi_n \to \psi$ in $H_{(2)}$, with $\|\psi\|_{(2)} = \lim \|\psi_n\|_{(2)} = 1$. and $\square\psi = \lim_n \square\psi_n = 0 \Rightarrow \psi \in \operatorname{Ker}\square \subseteq H_{(2)}$. On the other hand, $\psi_n \in (\operatorname{Ker}\square)^\perp \Rightarrow \psi \in (\operatorname{Ker}\square)^\perp \subseteq H_{(2)}$. It follows that $\psi = 0$, contradiction to $\|\psi\|_{(2)} = 1$.

Define Green's operator $G: L^2 \longrightarrow H_{(2)}$, $\varphi \longmapsto \psi$.
Then by def., we have

$$\square G\varphi = \square\psi = \varphi - \varphi_0 = (\mathrm{Id} - \pi_{R(\square)^\perp})\varphi$$

(C). We can identify $R(\square)^\perp = \operatorname{Ker}(\square)$ (in the standard sense, $\operatorname{Ker}\square$ on C^∞)
Indeed, $\forall f \in R(\square)^\perp$, i.e. $\langle f, \square\psi\rangle = 0$, $\forall \psi \in H_{(2)}$. In particular, take $\psi \in C^\infty$, $\Rightarrow \square f = 0$ in the sense of distributions. By elliptic regularity thm, f is smooth and $\square f = 0$ in the usual sense. Thus $f \in \operatorname{Ker}\square$.

Conversely, suppose $f \in \operatorname{Ker}\square$, consider $\psi \in H_{(2)}$. Now $f \in C^\infty \subseteq H_{(2)}$
$\Rightarrow$
$$\langle f, \square\psi\rangle_{L^2} = \langle \square f, \psi\rangle_{L^2} = 0$$
$\Rightarrow f \in R(\square)^\perp$.

In summary, $\forall \varphi \in L^2$, $\exists!\, \psi \in H_{(2)}$, $\varphi_0 \in \operatorname{Ker}\square$ s.t. $\varphi = \square\psi + \varphi_0$, and we defined

$G\varphi = \psi$, $\psi \perp \mathrm{Ker}\,\Box$. Then:

$$\Box G\varphi = \Box\psi = \varphi - \varphi_0 = \varphi - \pi\varphi = (\mathrm{Id} - \pi)\varphi.$$

where $\pi: L^2 \longrightarrow \mathrm{Ker}\,\Box$ is the orthogonal projection. By composing $H_{(2)} \hookrightarrow L^2$, we obtain $G: L^2 \longrightarrow L^2$.

Green's operator

Observation: The a priori estimate implies the following estimate. (actually AP'):

$$\|G\varphi\|_{(2)} \le C \cdot \|\varphi\|_{(0)} \qquad (AP'')$$

Indeed, in the above notation, $G\varphi = \psi$, and $\Box\psi = \varphi - \varphi_0$, $\psi \perp \mathrm{Ker}\,\Box$. By AP'

$$\|G\varphi\|_{(2)} = \|\psi\|_{(2)} \le C\|\Box\psi\|_{(0)} = C\|\varphi - \varphi_0\|_{(0)} \le C\|\varphi\|.$$

Def. An operator $G: H \longrightarrow H$, where H is a Hilbert space, is said to be compact if:

(i). G is bounded: $\|G\varphi\| \le C \cdot \|\varphi\|$, $\forall \varphi \in H$

(ii). $\forall \{\varphi_i\}$ a bounded sequence in H. $\{G\varphi_i\}$ contains a convergent subsequence (pre-compactness).

Claim: The Green's operator G is compact.

Pf: Let $\{\varphi_j\} \in L^2$, $\|\varphi_j\|_{(0)} \le 1$. By the estimate (AP'')

$$\|G\varphi_j\|_{(2)} \le C \cdot \|\varphi_j\|_{(0)} \le C.$$

Rellich's lemma $\Rightarrow$ $\forall\, t<2$, say, $t=0$, $\{G\varphi_j\}$ contains a convergent subsequence w.r.t. $\|\cdot\|_{(0)}$ norm.

Claim: G is self-adjoint and positive.

Pf: $\forall\, \varphi, \tilde{\varphi} \in L^2$, then $\varphi = \Box\psi + \varphi_0$, $\tilde{\varphi} = \Box\tilde{\psi} + \tilde{\varphi}_0$, and $\psi, \tilde{\psi} \in H_{(2)}$, $\psi, \tilde{\psi} \perp \mathrm{Ker}\,\Box$ w.r.t. the L^2-norm. Now:

$$\langle G\varphi, \tilde{\varphi}\rangle_{L^2} = \langle \psi, \Box\tilde{\psi} + \tilde{\varphi}_0\rangle_{L^2} = \langle \psi, \Box\tilde{\psi}\rangle_{L^2}$$

Similarly: $$\langle \varphi, G\tilde{\varphi}\rangle_{L^2} = \langle \Box\psi + \varphi_0, \tilde{\psi}\rangle_{L^2} = \langle \Box\psi, \tilde{\psi}\rangle_{L^2}$$

Hence $\langle G\varphi, \tilde{\varphi}\rangle_{L^2} = \langle \varphi, G\tilde{\varphi}\rangle_{L^2}$. Taking $\tilde{\varphi} = \varphi \Rightarrow \langle G\varphi, \varphi\rangle_{L^2} = \langle \psi, \Box\psi\rangle_{L^2} \ge 0$.

Claim: $G\bar\partial = \bar\partial G$, $G\bar\partial^{\dagger} = \bar\partial^{\dagger} G$

Pf: Since $L^2 = \operatorname{Ker}\square \oplus R(\square)$, it suffices to check $G\bar\partial = \bar\partial G$ on each factor.

Let $\varphi \in \operatorname{Ker}\square$, i.e. $(\bar\partial^{\dagger}\bar\partial + \bar\partial\bar\partial^{\dagger})\varphi = 0 \Leftrightarrow \bar\partial^{\dagger}\varphi = 0$, $\bar\partial\varphi = 0$. Then $G\bar\partial\varphi = 0$. On the other hand, $\bar\partial G\varphi = 0$ since $G\varphi = 0$.

Next, let $\varphi \in R(\square)^{\perp}$, $\varphi = \square\psi$. Then $\bar\partial G\varphi = \bar\partial\psi$. On the other hand,

$$\begin{aligned} G(\bar\partial\varphi) &= G(\bar\partial(\bar\partial\bar\partial^{\dagger} + \bar\partial^{\dagger}\bar\partial)\psi) \\ &= G(\bar\partial\bar\partial^{\dagger}\bar\partial\psi) \\ &= G((\bar\partial\bar\partial^{\dagger} + \bar\partial^{\dagger}\bar\partial)\bar\partial\psi) \\ &= G\square\bar\partial\psi \\ &= (\mathrm{Id} - \pi)\bar\partial\psi. \end{aligned}$$

Thus it suffices to show that $\pi\bar\partial\psi = 0$. $\forall \eta \in C^{\infty}$, we have

$$\langle \pi\bar\partial\psi, \eta\rangle = \langle \bar\partial\psi, \pi\eta\rangle = \langle \psi, \bar\partial^{\dagger}\pi\eta\rangle.$$

But since $\pi\eta \in \operatorname{Ker}\square \Rightarrow \bar\partial^{\dagger}\pi\eta = \bar\partial\pi\eta = 0 \Rightarrow \pi\bar\partial\psi = 0$.

Taking adjoint of $\bar\partial G = G\bar\partial$ gives $G\bar\partial^{\dagger} = \bar\partial^{\dagger} G$, since G is self-adjoint.

The above claims finish part b) of Hodge-Kodaira thm.

A bit functional analysis: Spectrum of compact self-adjoint operators.

Lemma. Let $G: H \longrightarrow H$ be a compact, non-negative, self-adjoint operator on a Hilbert space H. Then:

(i). The upper bound $\mu = \sup_{\|u\|=1} \langle Gu, u\rangle$ is an eigenvalue, i.e. $\exists\, 0 \neq v \in H$ with $Gv = \mu v$. If $\mu = 0$, $G \equiv 0$.

(ii). Each eigen-space $\{v \mid Gv = \lambda v\}$ is finite dimensional.

(iii). The only accumulation points of the eigenvalues $\lambda > 0$ of G is $\lambda = 0$

(iv). The span of (i.e. the space of finite linear combinations of) eigenspaces with positive eigenvalues is dense in $\operatorname{Range}(G)$ ($= R(G)$ for short).

(v). G admits an orthonormal basis of eigen-vectors.

Spectrum of G

Pf: Elementary observation: Let A be a self-adjoint, non-negative, bounded operator. Then

$$\|Au\| \leq \langle Au, u\rangle^{\frac{1}{2}} \|A\|^{\frac{1}{2}}$$

In fact, if we define an inner product $[u, v] \triangleq \langle Au, v\rangle$, which is non-negative, possiblely degenerate inner product. Thus by Cauchy-Schwartz:

$$\begin{aligned}
\|Au\|^2 = \langle Au, Au\rangle = [u, Au] &\leq [u,u]^{\frac{1}{2}} [Au, Au]^{\frac{1}{2}} \\
&= \langle Au, u\rangle^{\frac{1}{2}} \langle A(Au), Au\rangle^{\frac{1}{2}} \\
&\leq \langle Au, u\rangle^{\frac{1}{2}} \|AAu\|^{\frac{1}{2}} \|Au\|^{\frac{1}{2}} \\
&\leq \langle Au, u\rangle^{\frac{1}{2}} \|A\|^{\frac{1}{2}} \|Au\|
\end{aligned}$$

$$\Rightarrow \|Au\| \leq \langle Au, u\rangle^{\frac{1}{2}} \|A\|^{\frac{1}{2}}$$

(i). Let $A \triangleq \mu \mathrm{Id} - G$, non-negative, self-adjoint and bounded. The above inequality:

$$\|(\mu \mathrm{Id} - G)u\| \leq C \cdot \langle (\mu \mathrm{Id} - G)u, u\rangle^{\frac{1}{2}}$$

Let $\{u_j\}$ satisfy $\langle Gu_j, u_j\rangle \longrightarrow \mu$, and $\|u_j\| = 1$. Thus

$$\|(\mu \mathrm{Id} - G)u_j\| \longrightarrow 0 \quad \text{when } j \longrightarrow 0.$$

$$\Rightarrow (\mu u_j - Gu_j) \longrightarrow 0 \quad \text{as vectors in } H.$$

Since G is compact and $\{u_j\}$ is bounded, by passing to a subsequence if necessary, we may assume that $Gu_j \longrightarrow v \in H$.

If $\mu \neq 0$, $u_j = \frac{1}{\mu} Gu_j \longrightarrow \frac{1}{\mu} v$ and $Gv = \lim_j \mu G u_j = \mu v$.

If $\mu = 0$, $A \equiv 0 \Rightarrow G \equiv 0$ and the statement is trivially true.

(ii). Let $\{u_j\}$ be an orthonormal basis of the λ-eigenspace. Since $\lambda > 0$, $u_j = \frac{1}{\lambda} Gu_j$. If the λ-eigenspace were infinite dimensional, since G is compact, by going to a subsequence if necessary, we may assume that Gu_j converges. This implies $\lambda\sqrt{2} = \lambda \|u_j - u_k\| = \|Gu_j - Gu_k\| \longrightarrow 0$ $(j, k \longrightarrow \infty)$, contradiction.

(iii). Let $\lambda_j > 0$ be eigenvalues and $\lambda_j \longrightarrow \lambda$. If $\lambda > 0$, we may choose as above a sequence u_j, $Gu_j = \lambda_j u_j$ and $\{Gu_j\}$ converges, i.e. $\|Gu_j - Gu_{j+1}\| \longrightarrow 0$ when

$j \longrightarrow \infty$. But

$$\|Gu_j - Gu_{j+1}\| = \|\lambda_j u_j - \lambda_{j+1} u_{j+1}\| \longrightarrow \sqrt{2}\lambda , \quad j \longrightarrow \infty$$

contradiction.

(iv). Define $\lambda_1 = \sup_{\|u\|=1} \langle Gu, u\rangle$, and assume that $\lambda_1 > 0$ ($G \not\equiv 0$, by (a)). Then $\exists\, u_1$ eigenvector : $Gu_1 = \lambda_1 u_1$. Let $H_1 = (\mathbb{C}\{u_1\})^{\perp}$, then $G|_{H_1} : H_1 \longrightarrow H_1$. Indeed, $\forall\, v \in H_1$, $\langle Gv, u_1\rangle = \langle v, Gu_1\rangle = \langle v, \lambda_1 u_1\rangle = 0$. Define $\lambda_2 = \sup_{\|u\|=1, u\in H_1} \langle Gu, u\rangle$, and by a) again. we find $u_2 \in H_1$, $Gu_2 = \lambda_2 u_2 \ldots$

Inductively, at the k-th stage, we can find $Gu_1 = \lambda_1 u_1, \cdots, Gu_k = \lambda_k u_k$ and define $H_k = \mathbb{C}\{u_1, \cdots, u_k\}^{\perp}$, then $G|_{H_k} : H_k \longrightarrow H_k$.

Claim: $\|G|_{H_k}\| \le \lambda_{k+1}$, where $\lambda_{k+1} = \sup_{\|u\|=1, u\in H_k} \langle Gu, u\rangle$.

Recall from elementary observation that:

$$\|G|_{H_k}\| = \sup_{\|u\|=1, u\in H_k} \|Gu\| \le \sup_{\|u\|=1, u\in H_k} \langle Gu, u\rangle^{\frac{1}{2}} \|G|_{H_k}\|^{\frac{1}{2}}$$
$$= \lambda_{k+1}^{\frac{1}{2}} \|G|_{H_k}\|^{\frac{1}{2}}$$
$$\Rightarrow \quad \|G|_{H_k}\| \le \lambda_{k+1}$$

By (iii). $\lambda_{k+1} \longrightarrow 0$. Let π_k be the orthogonal projection onto $\mathbb{C}\{u_1, \cdots, u_k\}$. Then $G\pi_k = \pi_k G$: $\forall\, u \in H$,

$$\begin{aligned} G\pi_k u &= G\Big(\sum_{j=1}^{k} \langle u_j, u\rangle u_j\Big) \\ &= \sum_{j=1}^{k} \langle u_j, u\rangle \lambda_j u_j \\ &= \sum_{j=1}^{k} \langle Gu_j, u\rangle u_j \\ &= \sum_{j=1}^{k} \langle u_j, Gu\rangle u_j \\ &= \pi_k Gu. \end{aligned}$$

Now let $u = Gv \in R(G)$, $u - \pi_k u = Gv - \pi_k Gv = G(v - \pi_k v)$. Note that $v - \pi_k v \in H_k \Rightarrow \|u - \pi_k u\| = \|G(v - \pi_k v)\| \le \lambda_{k+1} \|v - \pi_k v\| \le \lambda_{k+1} \|v\| \longrightarrow 0$ as $k \longrightarrow \infty$. Thus $\mathbb{C}\{u_1, \cdots, u_k, \cdots\}$ is dense in $R(G)$.

(v). Since $\mathbb{C}\{u_1, \cdots, u_k, \cdots\}$ is dense in $R(G) \Rightarrow$ the span is dense in $\overline{R(G)}$. Moreover, since G is bounded and self-adjoint, $R(G)^{\perp} = \mathrm{Ker}G$. ($\because$ $\forall\, u, v \in H$, $\langle Gu, v\rangle = \langle u, Gv\rangle \Rightarrow Gv = 0$ iff $v \perp$ all Gu's). Now $H = \overline{R(G)} \oplus \mathrm{Ker}G$. Take an orthonormal basis for $\operatorname{Ker} G$ and adjoint it to the basis $\{u_j\}$ obtained above, then we have an orthonormal basis of eigenvectors. $\square$

Now it follows from our construction that L^2 admits a basis of eigenvectors $\{\varphi_\lambda\}$ of G, $G\varphi_\lambda = \lambda\varphi_\lambda$. Next we shall prove the growth of these eigenvalues.

Growth of eigenvalues.

Let $\{\varphi_\lambda(z)\}$ be an orthonormal basis of eigenfunctions of $\square$. Given $\Lambda > 0$, define the spectral function of $\square$:

$$e_\Lambda(\omega, z) \triangleq \sum_{\lambda \le \Lambda} \varphi_\lambda(\omega) \overline{\varphi_\lambda(z)}.$$

More precisely, $\varphi_\lambda(\omega) = \{\varphi^\alpha_{\lambda \bar{J} I}(\omega)\}$ and

$$e_\Lambda(\omega, z) = \frac{1}{p!q!} \sum_{\lambda \le \Lambda} \varphi^\alpha_{\lambda \bar{J} I}(\omega) \overline{\varphi^\beta_{\lambda \bar{K} L}(z)}\, g^{I\bar{L}}(\omega) g^{K\bar{J}}(\omega)\, H_{\bar{\beta}\alpha}(\omega)$$

Note that $\varphi^\beta_{\lambda \bar{K} L}(z) \in E_z \otimes \Lambda^{p,q}_z$, and $\varphi^\alpha_{\lambda \bar{J} I}\, g^{I\bar{L}} g^{K\bar{J}} H_{\bar{\beta}\alpha}(\omega) \in (\overline{E_\omega \otimes \Lambda^{p,q}_\omega})^*$

$\Rightarrow$ $e_\Lambda(\omega, z) \in (\overline{E_\omega \otimes \Lambda^{p,q}_\omega})^* \otimes (\overline{E_z \otimes \Lambda^{p,q}_z})$ and $e_\lambda(z, z)$ is a scalar.

We shall use the spectral function to study the growth of λ's.

Observation 1: $\int_X e_\Lambda(z, z) \frac{\omega^n}{n!} = \sum_{\lambda \le \Lambda} \|\varphi_\lambda\|^2_{L^2} = \#\{\lambda \le \Lambda\}$

Observation 2: To show the growth $\lambda_m \ge D \cdot m^\delta$, it suffices to show that $\exists\, C$ $p > 0$, independent of z, so that:

$$|e_\Lambda(z, z)| \le C \cdot \Lambda^p \quad (*).$$

Indeed, this would imply by integrating over M:

$$\#\{\lambda \le \Lambda\} \le D \cdot \Lambda^p \quad \text{for some } D.$$

Reorder λ's so that $\lambda_1 \le \lambda_2 \le \cdots$. Then the eigenvalues satisfy:

$$\lambda_{[D\Lambda^p + 1]} \ge \Lambda$$

Set $N = [D \cdot \Lambda^p + 1] \le D' \Lambda^p \Rightarrow \lambda_N \ge C \cdot N^{\frac{1}{p}}$.

Proof of $(*)$.

Strategy: By the Sobolev lemma, the desired estimate should follow from estimates for $\|\cdot\|_{(s)}$, where $s>n$. $n=\dim_{\mathbb{C}} X$.

Recall the a priori estimate:

$$\|\varphi\|_{(s+2)} \le C\cdot(\|\Box\varphi\|_{(s)} + \|\varphi\|_{(s+1)})$$

which is equivalent to:

$$\|\varphi\|^2_{(s+2)} \le C'\cdot(\|\Box\varphi\|^2_{(s)} + \|\varphi\|^2_{(s+1)})$$

We shall now derive a technical improvement of this estimate:

$$\|\varphi\|^2_{(s+2)} \le C\cdot(\|\Box\varphi\|^2_{(s)} + \|\varphi\|^2_{(s)}).$$

C depending on s only. We do this, for instance, for $s=0$.

$$AP: \quad \|\varphi\|^2_{(2)} \le C\cdot(\|\Box\varphi\|^2_{(0)} + \|\varphi\|^2_{(1)})$$

It suffices to bound $\|\varphi\|^2_{(1)}$ by $\|\Box\varphi\|^2_{(0)}$ and $\|\varphi\|^2_{(0)}$. But by def.

$$\|\varphi\|^2_{(1)} = \|\varphi\|^2_{(0)} + \int_X \nabla_j\varphi\,\overline{\nabla_k\varphi}\,g^{j\bar{k}}\frac{\omega^n}{n!} + \cdots$$

Using integration by parts, we have

$$\int_X \nabla_j\varphi\,\overline{\nabla_{\bar{k}}\varphi}\,g^{j\bar{k}}\frac{\omega^n}{n!} = -\int_X \varphi\,\overline{g^{\bar{k}j}\nabla_j\nabla_{\bar{k}}\varphi}\,\frac{\omega^n}{n!}$$

Note that by Bochner-Kodaira type formulas, $-g^{\bar{k}j}\nabla_j\nabla_{\bar{k}}$ differs from $\Box$ only by tensor terms (with torsion occuring in non-Kähler cases), which are not differential operators (or diff. op. of order 0), and these terms can be bounded by $C\cdot\|\varphi\|^2_{(0)}$. Thus the result follows.

Now from $\|\varphi\|^2_{(2)} \le C\cdot(\|\Box\varphi\|^2_{(0)} + \|\varphi\|^2_{(0)})$

$$\begin{aligned}
\Rightarrow \quad \|\varphi\|^2_{(4)} &\le C\cdot(\|\Box\varphi\|^2_{(2)} + \|\varphi\|^2_{(2)}) \\
&\le C'(\|\Box^2\varphi\|^2_{(0)} + \|\Box\varphi\|^2_{(0)} + \|\varphi\|^2_{(0)})
\end{aligned}$$

$\Rightarrow \quad \cdots$

$\Rightarrow \quad \|\varphi\|^2_{(2k)} \le C\left(\sum_{j=1}^{k}\|\Box^j\varphi\|^2 + \|\varphi\|^2_{(0)}\right)$ for any $\varphi\in C^\infty$.

Suppose $\varphi\in H^\Lambda \triangleq \mathrm{Span}\{u_1,\cdots,u_\lambda\}$, $\lambda\le\Lambda$. Then $\|\Box\varphi\|\le\Lambda\|\varphi\|$. Indeed, we have

$$\begin{aligned}
&\varphi = \textstyle\sum_{\lambda\le\Lambda} \langle u_\lambda,\varphi\rangle u_\lambda \\
\Rightarrow\ &\Box\varphi = \textstyle\sum_{\lambda\le\Lambda} \lambda\langle u_\lambda,\varphi\rangle u_\lambda \\
\Rightarrow\ &\|\Box\varphi\|^2 = \textstyle\sum \lambda^2 |\langle u_\lambda,\varphi\rangle|^2 \le \Lambda^2 \sum |\langle u_\lambda,\varphi\rangle|^2 = \Lambda^2\|\varphi\|^2.
\end{aligned}$$

Now by induction, we have:

$$\|\Box^{\ell}\varphi\|^2 \leq \Lambda^{2\ell}\|\varphi\|^2$$

Thus if $\varphi \in H^{\Lambda}$, then:

$$\|\varphi\|^2_{(2k)} \leq C\cdot(\Lambda^{2k}+1)\|\varphi\|^2_{(0)}$$

or equivalently:

$$\|\varphi\|_{(2k)} \leq C_k(\Lambda^{k}+1)\|\varphi\|^2_{(0)}$$

Now by Sobolev's inequality, if $2k>n$, we have:

$$\|\varphi\|_{C^0} \leq \|\varphi\|_{(2k)} \leq C_k(\Lambda^k+1)\|\varphi\|_{(0)}$$

for any $\varphi \in H^{\Lambda}$. Now $\forall f \in L^2$, consider the orthogonal projection of f onto H^{Λ}:

$$\pi_\Lambda(f)(z) = \langle f(w), e_\Lambda(w,z)\rangle_{L^2},$$

integrating w.r.t. w. Since now $\pi_\Lambda(f) \in H^\Lambda$, this implies that:

$$\|\pi_\Lambda(f)\|_{C^0} \leq C_k(\Lambda^k+1)\|\pi_\Lambda(f)\|_{(0)} \leq C_k(\Lambda^k+1)\|f\|_{(0)}$$

In particular:

$$\begin{aligned}\|e_\Lambda(\cdot,z)\|_{L^2} &= \sup_{f\neq 0}\frac{|\langle f, e_\Lambda(\cdot,z)\rangle|}{\|f\|_{(0)}}\\ &\leq C_k(\Lambda^k+1) \qquad (**)\end{aligned}$$

On the other hand, for a fixed z_0:

$$\|e_\Lambda(\cdot,z_0)\|^2_{L^2} = \langle e_\Lambda(\cdot,z_0), e_\Lambda(\cdot,z_0)\rangle = e_\Lambda(z_0,z_0)$$

Thus $(**)$ implies that

$$|e_\Lambda(z_0,z_0)| \leq C_k^2(\Lambda^k+1)^2$$

for any $z_0 \in X$. Thus $(*)$ is proved.

Remaining steps

(1). A priori inequality: $\forall \varphi \in C_0^\infty$,

$$\|\varphi\|^2_{(s+2)} \leq C(\|\Box\varphi\|^2_{(s)} + \|\varphi\|^2_{(s+1)})$$

(2). Prove the basic property of Sobolev spaces:

- Sobolev's lemma
- Rellich's lemma
- Equivalence between different definitions.

(3). Regularity thms.

Elliptic linear PDE's on $\mathbb{R}^n$.

Let $\mathbb{R}^n = \{(x_1, \cdots, x_n)\}$ and $D^\alpha \triangleq (\frac{1}{i}\frac{\partial}{\partial x_1})^{\alpha_1} \cdots (\frac{1}{i}\frac{\partial}{\partial x_n})^{\alpha_n}$, where $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$, and $|\alpha| = \sum \alpha_i$. In this notation, the formal Taylor series of a function $u(x)$ is:

$$u(x) = \sum_\alpha \frac{1}{\alpha!} D^\alpha u(0) (ix)^\alpha$$

where $\alpha! \triangleq \alpha_1! \cdots \alpha_n!$ and $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.

Def: (1) L is a linear partial differential operator of order m if

$$(Lu)(x) = \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha u(x)$$

(2). The symbol $\sigma_L(x, \xi)$ of L is the following function

$$\sigma_L(x, \xi) = \sum_{|\alpha| = m} a_\alpha(x) \xi^\alpha$$

for any $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^n$.

(3). L is said to be elliptic at x_0 if $|\sigma_L(x_0, \xi)| \ge C \cdot |\xi|^m$. It's elliptic if it's elliptic at all points.

Rmk: $|\sigma_L(x_0, \xi)| \ge C \cdot |\xi|^m$ is equivalent to the requirement that $\sigma_L(x_0, \xi) \ne 0$, $\forall \xi \ne 0$. Indeed "$\Rightarrow$" is easy to see. Conversely, $|\sigma_L(x, \cdot)|$ achieves minimal value C on the (compact) unit sphere. Thus $\forall \xi$,

$$\begin{aligned} |\sigma_L(x, \xi)| &= |\sigma_L(x, \tfrac{\xi}{|\xi|}) |\xi|^m| \quad (\sigma_L(x, \xi) \text{ homogeneous of degree } m). \\ &= |\xi|^m |\sigma_L(x, \tfrac{\xi}{|\xi|})| \\ &\ge C \cdot |\xi|^m \end{aligned}$$

E.g. $\Box$ is elliptic of order 2. Recall from Bochner-Kodaira formula:

$$\begin{aligned} \Box \varphi^\alpha_{\bar{J}I} &= -g^{j\bar{k}} \nabla_j \nabla_{\bar{k}} \varphi^\alpha_{\bar{J}I} + \text{Curvature terms} \\ &= -g^{j\bar{k}} (\partial_j + \Gamma_j)(\partial_{\bar{k}} + \Gamma_k) \varphi^\alpha_{\bar{J}I} \\ &= -g^{j\bar{k}} \partial_j \partial_{\bar{k}} \varphi^\alpha_{\bar{J}I} + \text{lower degree terms}. \end{aligned}$$

Thus the symbol of $\Box$ is given by:

$$\sigma_\Box(z, \xi) = -g^{j\bar{k}}(z) \xi_j \bar{\xi}_k = -|\xi|^2$$

for any $\xi \in \Lambda^{1,0}$. Thus $\Box$ is elliptic of order 2. Actually the symbol of $\Box$ is a well-defined function on $\Lambda^1 M$.

Sobolev spaces in $\mathbb{R}^n$.

Def: (1). $\forall\ \varphi \in C_0^\infty(\mathbb{R}^n)$. $\|\varphi\|_{(k)}^2 \triangleq \sum_{|\alpha|\le k} \|D^\alpha \varphi\|^2$

(2). $H_{(k)}(\mathbb{R}^n) \triangleq \overline{C_0(\mathbb{R}^n)}$ w.r.t. the norm $\|\cdot\|_{(k)}$.

Basic properties about Fourier transform.

Let $\mathcal{S}$ be the space of Schwartz functions. i.e. $\mathcal{S} \triangleq \{\varphi \in C^\infty(\mathbb{R}^n)$ s.t. $\forall\ \alpha, N$. $(1+|x|^2)^N |D^\alpha \varphi(x)| \le C_{N,\alpha}$ for all $x \in \mathbb{R}^n\}$ (rapidly decaying condition.) The space contains $C_0^\infty(\mathbb{R}^n)$ and functions like $e^{-\frac{|x|^2}{2}}$ etc.

Define the Fourier transform:

$$\varphi(x) \longrightarrow \hat{\varphi}(\xi) \triangleq \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} \varphi(x)\,dx$$

Then:

(1). $\forall\ \varphi \in \mathcal{S}$. $\hat{\varphi} \in \mathcal{S}$. The inverse is given by Fourier inversion formula.

$$\varphi(x) = \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} \hat{\varphi}(\xi)\,d\xi.$$

(2). Plancheral formula: $\|\varphi\|_{L^2} = \|\hat{\varphi}\|_{L^2}$.

(3). $(D^\alpha \varphi)\hat{}\,(\xi) = \xi^\alpha \hat{\varphi}(\xi)$.

These properties imply that there is a simple characterization of Sobolev norms: $\forall\ \varphi \in \mathcal{S}(\mathbb{R}^n)$:

$$\begin{aligned} \|\varphi\|_{(k)}^2 &= \sum_{|\alpha|\le k} \|D^\alpha \varphi\|_{L^2}^2 \\ &= \sum_{|\alpha|\le k} \|(D^\alpha\varphi)\hat{}\,\|_{L^2}^2 \\ &= \sum_{|\alpha|\le k} \|\xi^\alpha \hat{\varphi}(\xi)\|_{L^2}^2 \\ &= \sum_{|\alpha|\le k} \int_{\mathbb{R}^n} |\xi^\alpha|^2 |\hat{\varphi}(\xi)|^2 d\xi. \end{aligned}$$

Observe that $\exists\ c, C > 0$ constants s.t.

$$c(1+|\xi|^2)^k \le \sum_{\alpha \le k} |\xi^\alpha|^2 \le C\cdot(1+|\xi|^2)^k.$$

Hence:

$$c\int_{\mathbb{R}^n} (1+|\xi|^2)^k |\hat{\varphi}(\xi)|^2 d\xi \le \|\varphi\|_{(k)}^2 \le C\cdot \int_{\mathbb{R}^n} (1+|\xi|^2)^k |\hat{\varphi}(\xi)|^2 d\xi$$

and the norm defined by $\int_{\mathbb{R}^n} (1+|\xi|^2)^k |\hat{\varphi}(\xi)|^2 d\xi$ is equivalent to $\|\cdot\|_{(k)}$.

Proof of the a priori estimate:

Step 1. Let $L_m = \sum_{|\alpha|=m} C_\alpha D^\alpha$ be an elliptic differential operator, homogeneous of degree m and with constant coefficients C_α. Then $\forall\ \varphi \in C_0^\infty(\mathbb{R}^n)$:

$$\|\varphi\|_{(m+s)}^2 \le C \cdot (\|L_m\varphi\|_{(s)}^2 + \|\varphi\|_{(m+s-1)}^2)$$

Pf: We just show $s=0$. $m>0$ is similar.

$$\|L_m\varphi\|_{(m+s)}^2 = \int_{\mathbb{R}^n} |(L_m\varphi)^\wedge(\xi)|^2 d\xi$$
$$= \int_{\mathbb{R}^n} |\sum_{|\alpha|\le m} C_\alpha \xi^\alpha \hat{\varphi}(\xi)|^2 d\xi$$
$$= \int_{\mathbb{R}^n} |\sigma_{L_m}(\xi)\hat{\varphi}(\xi)|^2 d\xi$$
$$= \int_{\mathbb{R}^n} |\sigma_{L_m}(\xi)|^2 |\hat{\varphi}(\xi)|^2 d\xi$$

Now by def., $|\sigma_{L_m}(\xi)|^2 \ge C|\xi|^{2m}$:

$$\Rightarrow \|L_m\varphi\|_{(0)}^2 \ge C \cdot \int_{\mathbb{R}^n} |\xi|^{2m} |\hat{\varphi}(\xi)|^2 d\xi$$
$$\Rightarrow \|L_m\varphi\|_{(0)}^2 + \|\varphi\|_{(0)}^2 \ge C \cdot \int_{\mathbb{R}^n} (|\xi|^m+1)^2 |\hat{\varphi}(\xi)|^2 d\xi$$
$$\ge C \cdot \|\varphi\|_{(m)}^2.$$

Step 2. Let z_0 be any given point in $\mathbb{R}^n$, L any elliptic linear differential operator of order m. Then $\exists\ V_{z_0} \ni z_0$, $\overline{V}_{z_0}$ compact, s.t.

$$\|\varphi\|_{(m)} \le C(\|L\varphi\|_{(0)} + \|\varphi\|_{(m-1)})$$

for any $\varphi \in C_0^\infty(V_{z_0})$.

Pf: Define L_m by $L_m\varphi = \sum_{|\alpha|=m} a_\alpha(z_0) D^\alpha \varphi(z)$. Then

$$\|L\varphi\|_{(0)} \ge \|L_m\varphi\|_{(0)} - \|(L-L_m)\varphi\|_{(0)}$$

and $\|(L-L_m)\varphi\|_{(0)} = \sum_{|\alpha|=m} \|(a_\alpha(z)-a_\alpha(z_0))D^\alpha\varphi\|_{(0)} + \sum_{|\alpha|\le m-1} \|a_\alpha(z) D^\alpha \varphi(z)\|$

Claim: $\forall \varepsilon>0$, $\exists\ V_{x_0} \ni x_0$ open nhd, $\overline{V}_{x_0}$ compact so that $\forall\ \varphi \in C_0^\infty(V_{x_0})$

$$\|(L-L_m)\varphi\|_{(0)} \le \varepsilon \cdot \|\varphi\|_{(m)} + C'\|\varphi\|_{(m-1)}$$

Then assuming this claim, $\forall \varphi \in C_0^\infty(V_{x_0})$,

$$\|\varphi\|_{(m)} \le C \cdot (\|L_m\varphi\|_{(0)} + \|\varphi\|_{(m-1)})$$
$$\le C \cdot (\|(L-L_m)\varphi\|_{(0)} + \|L\varphi\|_{(0)} + \|\varphi\|_{(m-1)})$$
$$\le C \cdot \varepsilon \|\varphi\|_{(m)} + C \cdot C'\|\varphi\|_{(m-1)} + C\|L\varphi\|_{(0)} + C\|\varphi\|_{(m-1)}$$

Taking $C \cdot \varepsilon = \frac{1}{2}$, we obtain:

$$\|\varphi\|_{(m)} \le 2C\|L\varphi\|_{(0)} + 2C(C'+1)\|\varphi\|_{(m-1)}$$
$$\le C''(\|L\varphi\|_{(0)} + \|\varphi\|_{(m-1)}).$$

Now we check the claim:

$$(L-L_m)\varphi = \sum_{|\alpha|\le m-1} a_\alpha(x) D^\alpha \varphi + \sum_{|\alpha|=m} (a_\alpha(x) - a_\alpha(x_0)) D^\alpha \varphi$$

For $|\alpha| \le m-1$ terms,

$$\| a_\alpha(x) D^\alpha \varphi \|_{(0)}^2 = \int_{\mathbb{R}^n} |a_\alpha(x) D^\alpha \varphi(x)|^2 dx$$

Thus on any open subset of $\mathbb{R}^n$ with compact closure. $|a_\alpha(x)|$ is bounded. and

$$\| a_\alpha(x) D^\alpha \varphi \|_{(0)}^2 \le C \cdot \|\varphi\|_{(m-1)}$$

Now choose V_{x_0} so that $\sup_{x\in V_{x_0}} |a_\alpha(x) - a_\alpha(x_0)|^2 < \varepsilon$. Then $\forall \varphi \in C_0^\infty(V_{x_0})$, $|\alpha| = m$ we have:

$$\begin{aligned} \|(a_\alpha(x) - a_\alpha(x_0)) D^\alpha \varphi\|_{(0)}^2 &= \int_{\mathbb{R}^n} |(a_\alpha(x) - a_\alpha(x_0)) D^\alpha \varphi|^2 dx \\ &= \int_{V_{x_0}} |(a_\alpha(x) - a_\alpha(x_0)) D^\alpha \varphi|^2 dx \\ &\le \varepsilon \int_{V_{x_0}} |D^\alpha \varphi|^2 dx \\ &= \varepsilon \int_{\mathbb{R}^n} |D^\alpha \varphi|^2 dx \\ &\le \varepsilon \|\varphi\|_{(m)} \end{aligned}$$

Now, to prove $\|\varphi\|_{(m+s)} \le C(\|L\varphi\|_{(s)} + \|\varphi\|_{(s+m-1)})$, we apply similar techniques as above. Observe that $\forall s \in \mathbb{N}$, we have:

$$\|(L-L_m)\varphi\|_{(s)} \le \| \sum_{|\alpha|\le m-1} (a_\alpha(x) D^\alpha \varphi) \|_{(s)} + \| \sum_{|\alpha|=m-1} (a_\alpha(x) - a_\alpha(x_0)) D^\alpha \varphi \|_{(s)}$$

For $|\alpha| \le m-1$ terms:

$$\| a_\alpha(x) D^\alpha \varphi \|_{(s)}^2 = \sum_{|\beta|\le s} \| D^\beta (a_\alpha(x) D^\alpha \varphi) \|_{(0)}^2$$

and fix each β.

$$D^\beta \left(\sum_{|\alpha|\le m-1} a_\alpha(x) D^\alpha \varphi \right) = \sum_{|\gamma|\le m+s-1} (b_\gamma(x) D^\gamma \varphi)$$

Thus

$$\| \sum_{|\alpha|\le m-1} a_\alpha(x) D^\alpha \varphi \|_{(s)} \le C \cdot \|\varphi\|_{(m+s-1)}.$$

For $|\alpha| = m-1$ terms,

$$\begin{aligned} \| (a_\alpha(x) - a_\alpha(x_0)) D_\alpha \varphi \|_{(s)}^2 &= \sum_{|\beta|\le s} \| D^\beta (a_\alpha(x) - a_\alpha(x_0)) D^\alpha \varphi \|_{(0)}^2 \\ &= \sum_{|\beta|\le s} \| (a_\alpha(x) - a_\alpha(x_0)) D^{\beta+\alpha} \varphi + \sum_{\gamma \le |\beta|+|\alpha|-1} b_\gamma(x) D^\gamma \varphi \|_{(0)}^2 \\ &\le \sum_{|\beta|\le s} \| (a_\alpha(x) - a_\alpha(x_0)) D^{\beta+\alpha} \varphi \|^2 + \sum_{\beta\le s} \| \sum_{\gamma \le s+m-1} b_\gamma D^\gamma \varphi \|_{(0)}^2 \\ &\le \varepsilon \cdot \|\varphi\|_{(m+s)} + C \|\varphi\|_{(m+s-1)}. \end{aligned}$$

Rmk: Even for s non-integers or negative numbers, we may define the Sobolev norm by:

$$\|\varphi\|_{(s)}^2 \triangleq \int_{\mathbb{R}^n} (1+|\xi|^2)^s |\hat{\varphi}(\xi)|^2 d\xi$$

The same a priori estimate holds for all $s \in \mathbb{R}$. The proof requires some more difficult arguments.

Step 3. Fix a compact set $K \subseteq \mathbb{R}^n$, $\mathring{K} \neq \phi$, we prove the AP estimate for any $\varphi \in C_0^\infty(\mathring{K})$.

Clearly, $K \subseteq \bigcup_{\omega \in K} V_\omega$, where V_ω is constructed as in step 2. Extract a finite subcover, $K \subseteq \bigcup_{j=1}^n V_{x_j}$, and take a partition of unity subordinate to V_{x_j}:

$$0 \le \rho_j \in C_0^\infty(V_{x_j}), \quad \sum_{j=1}^N \rho_j \equiv 1.$$

Then: $\forall \varphi \in C_0^\infty(K)$, $\varphi = \sum_{j=1}^N \rho_j \varphi$ with $\rho_j \varphi \in C_0^\infty(V_{x_j})$,

$$\begin{aligned}\|\varphi\|_{(m)} &\le \sum_{j=1}^N \|\rho_j \varphi\|_{(m)} \\ &\le C \cdot \sum_{j=1}^N (\|L(\rho_j\varphi)\|_{(0)} + \|\rho_j\varphi\|_{(m-1)})\end{aligned}$$

Obseravation 1: $\forall s \in \mathbb{N}$. $\|\rho_j\varphi\|_{(s)} \le C \cdot \|\varphi\|_{(s)}$:

$$\begin{aligned}\|\rho\varphi\|_{(s)}^2 &= \sum_{|\alpha|\le s} \int |D^\alpha(\rho\varphi)|^2 dx \\ &= \sum_{|\gamma|\le s} \int |b_\gamma D^\gamma \varphi|^2 dx \\ &\le \sum_{|\gamma|\le s} C \cdot \int |D^\gamma \varphi|^2 dx \\ &\le C \cdot \|\varphi\|_{(s)}^2\end{aligned}$$

Observation 2: $\|L(\rho_j\varphi) - \rho_j L\varphi\|_{(0)} \le C \cdot \|\varphi\|_{(m-1)}$:

In fact

$$\begin{aligned}L(\rho\varphi) - \rho L\varphi &= \sum_\alpha a_\alpha D^\alpha(\rho\varphi) - \sum \rho a_\alpha D^\alpha \varphi \\ &= \sum_{\beta+\gamma=\alpha} a_\alpha D^\beta \rho D^\gamma \varphi - \sum \rho a_\alpha D^\alpha \varphi \\ &= \sum_{\beta+\gamma=\alpha,\ |\gamma|<|\alpha|\le m} a_\alpha D^\beta \rho\, D^\gamma \varphi\end{aligned}$$

$$\Rightarrow \|L(\rho\varphi) - \rho L\varphi\|_{(0)} \le C \cdot \|\varphi\|_{(m-1)}$$

Hence

$$\begin{aligned}\|\varphi\|_{(m)} &\le C \cdot \sum_{j=1}^N (\|L(\rho_j\varphi)\|_{(0)} + \|\rho_j\varphi\|_{(m-1)}) \\ &\le C \cdot \sum_{j=1}^N (\|\rho_j L\varphi\|_{(0)} + \|L(\rho_j\varphi) - \rho_j L\varphi\|_{(0)} + C'\|\varphi\|_{(m-1)}) \\ &\le C \cdot \sum_{j=1}^N (\rho_j \|L\varphi\|_{(0)} + C \cdot C''\|\varphi\|_{(m-1)} + C \cdot C'\|\varphi\|_{(m-1)}) \\ &\le C(\|L\varphi\|_{(0)} + \|\varphi\|_{(m-1)})\end{aligned}$$

$\square$

Rmk: This step is also true for all $s \in \mathbb{R}$, by using some integral inequality.

Proof of Sobolev lemma:

If $s > \frac{n}{2}$, then $\forall \varphi \in C_0^\infty(\mathbb{R}^n)$

$$\|\varphi\|_{C^0} \leq C \cdot \|\varphi\|_{(s)}$$

Pf: By the Fourier inversion formula:

$$\varphi(x) = \int e^{i\langle x, \xi\rangle} \hat{\varphi}(\xi)\, d\xi$$

and thus

$$|\varphi(x)| \leq \int |\hat{\varphi}(\xi)|\, d\xi$$
$$\leq \left(\int (1+|\xi|^2)^s |\hat{\varphi}(\xi)|^2 d\xi\right)^{\frac{1}{2}} \cdot \left(\int (1+|\xi|^2)^{-s} d\xi\right)^{\frac{1}{2}} \quad \text{(Cauchy-Schwartz)}$$

Now if $s > \frac{n}{2}$, $\int_{\mathbb{R}^n} (1+|\xi|^2)^{-s} d\xi < \infty$ and thus

$$|\varphi(x)| \leq C \cdot \|\varphi\|_{(s)}$$
$$\Rightarrow \|\varphi\|_{C^0} \leq C \cdot \|\varphi\|_{(s)}. \qquad \square$$

Similarly, we can show that $\forall \varphi \in C_0^\infty(\mathbb{R}^n)$, if $s > \frac{n}{2} + k$,

$$\|\varphi\|_{C^k} \leq C \cdot \|\varphi\|_{(s)}$$

In fact, $\forall k \in \mathbb{N}$, $|\alpha| = k$, and $s > \frac{n}{2} + k$:

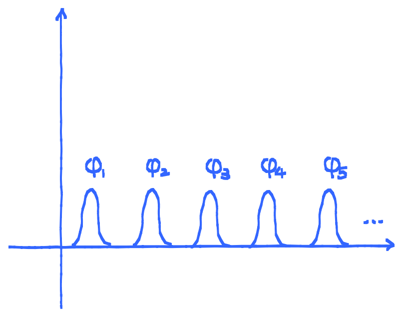
$$|D^\alpha \varphi(x)| \leq \int |\xi^\alpha| |\hat{\varphi}(\xi)|\, d\xi$$
$$\leq \int (1+|\xi|^2)^{\frac{|\alpha|}{2}} |\hat{\varphi}(\xi)|\, d\xi$$
$$\leq \left(\int (1+|\xi|^2)^s |\hat{\varphi}(\xi)|^2 d\xi\right)^{\frac{1}{2}} \left(\int (1+|\xi|^2)^{-s+|\alpha|} d\xi\right)^{\frac{1}{2}}$$
$$\leq C \cdot \|\varphi\|_{(s)}$$

and the result follows.

Proof of Rellich's lemma

Recall that if $0 \leq s < t$, and $\{\varphi_j \in H_{(t)} \mid \|\varphi_j\|_{(t)} \leq 1,\ \operatorname{supp}\varphi_j \subseteq K \overset{\text{cpt}}{\subseteq} \mathbb{R}^n\}$. Then Rellich's lemma $\Rightarrow \exists$ subsequence $\{\varphi_{j_\ell}\}$ converging w/r.t. $\|\cdot\|_{(s)}$.

The lemma is clearly wrong without the assumption that K is a fixed compact subset of $\mathbb{R}^n$. For instance:



The translated bump functions $\|\varphi_j\|_{(t)} \le C$, but has no convergent subsequence at all.

Pf:
$$\|\varphi_j - \varphi_k\|_{(s)}^2 = \int_{\mathbb{R}^n} (1+|\xi|^2)^s \, |\hat{\varphi}_j(\xi) - \hat{\varphi}_k(\xi)|^2 d\xi$$
$$= \int_{\mathbb{R}^n} (1+|\xi|^2)^{-(t-s)} (1+|\xi|^2)^t \, |\hat{\varphi}_j(\xi) - \hat{\varphi}_k(\xi)|^2 d\xi$$
$$= \int_{|\xi|\le R} + \int_{|\xi|>R}$$

To extract a convergent subsequence, we need the above two integrals to converge.
Now $\forall \xi \in \mathbb{R}^n$,
$$\hat{\varphi}(\xi) = \int e^{-i\langle x,\xi\rangle} \varphi(x)\, dx$$
We will show that $\{\hat{\varphi}_j(\xi)\}$ is an equicontinuous family, and then the second integral will converge once $R \gg 0$, and the first integral will converge by passing to a uniformly convergent subsequence on the compact set $\{|\xi| \le R\}$.

Step 1: $\{\hat{\varphi}_j(\xi)\}$ is equi-continuous.

(a). $\{\hat{\varphi}_j(\xi)\}$ is uniformly bounded.
$$|\hat{\varphi}_j(\xi)| \le \int_{\mathbb{R}^n} |\varphi_j(x)| dx = \int_K |\varphi_j(x)|\, dx$$
$$\le \left(\int_K |\varphi_j(x)|^2 dx\right)^{\frac{1}{2}} \cdot \left(\int_K 1 \cdot dx\right)^{\frac{1}{2}}$$
$$\le \mathrm{Vol}(K)^{\frac{1}{2}} \cdot \|\varphi_j\|_{(t)}$$
$$\le C$$

(b). $\{\frac{\partial}{\partial \xi_j} \hat{\varphi}_j(\xi)\}$ is uniformly bounded.
$$\left|\frac{\partial}{\partial \xi_j} \hat{\varphi}_j(\xi)\right| \le \int_{\mathbb{R}^n} |\, i x_j e^{-i\langle x,\xi\rangle} \varphi_j(x)\,| dx$$
$$= \int_K |x_j| \, |\varphi_j(x)|\, dx$$
$$\le \left(\int_K |\varphi_j(x)|^2 dx\right)^{\frac{1}{2}} \left(\int_K |x_j|^2 dx\right)^{\frac{1}{2}}$$
$$\le C \cdot \|\varphi_j\|_{(t)}$$
$$\le C$$

This implies equi-continuity: By the mean value thm:

$$|\hat{\varphi}_j(\xi) - \hat{\varphi}_j(\eta)| \le \max_{[\xi,\eta]} |\nabla \hat{\varphi}_j| \cdot |\xi - \eta|$$
$$\le C \cdot |\xi - \eta|$$

Hence by Arzela-Ascoli thm. $\forall$ compact subset of $\mathbb{R}^n$, $\exists$ subsequence $\{\hat{\varphi}_{jk}(\xi)\}$ converging uniformly on this compact set.

Step 2. $\forall \varepsilon > 0$, $\exists R \gg 0$ s.t. $(1+|R|^2)^{-(t-s)} \le \frac{\varepsilon}{8}$. By step 1, we have a subsequence $\{\varphi_{j\ell}\}$ s.t. $\{\hat{\varphi}_{j\ell}(\xi)\}$ converges uniformly on $\{|\xi| \le R\}$. Now

$$\int_{|\xi|>R} (1+|\xi|^2)^{-(t-s)} (1+|\xi|^2)^t |\hat{\varphi}_{j\ell}(\xi) - \hat{\varphi}_{jk}(\xi)|^2 d\xi$$
$$\le 2 \int_{|\xi|>R} \frac{\varepsilon}{4} (1+|\xi|^2)^t (|\hat{\varphi}_{j\ell}(\xi)|^2 + |\hat{\varphi}_{jk}(\xi)|^2) d\xi$$
$$\le \frac{\varepsilon}{4} \int_{\mathbb{R}^n} (1+|\xi|^2)^t (|\hat{\varphi}_{j\ell}(\xi)|^2 + |\hat{\varphi}_{jk}(\xi)|^2) d\xi$$
$$= \frac{\varepsilon}{4} (\|\varphi_{j\ell}\|_{(t)} + \|\varphi_{jk}\|_{(t)})$$
$$\le \frac{\varepsilon}{2}$$

Next, by equi-continuity, we may choose $\mathbb{N} \ni N \gg 0$ so that whenever $\ell, k \ge N$,

$$\sup_{|\xi| \le R} |\hat{\varphi}_{j\ell}(\xi) - \hat{\varphi}_{jk}(\xi)|^2 \le (1+R^2)^{-s} \cdot (vol(|\xi| \le R))^{-1} \cdot \frac{\varepsilon}{2}$$

Then

$$\int_{|\xi| \le R} (1+|\xi|^2)^s |\hat{\varphi}_{j\ell}(\xi) - \hat{\varphi}_{jk}(\xi)|^2 d\xi \le \int_{|\xi| \le R} (1+|\xi|^2)^s \sup_\xi |\hat{\varphi}_{j\ell}(\xi) - \hat{\varphi}_{jk}(\xi)|^2 d\xi$$
$$\le \int_{|\xi| \le R} (1+R^2)^s \cdot (1+R^2)^{-s} (vol)^{-1} \frac{\varepsilon}{2} d\xi$$
$$\le \frac{\varepsilon}{2}.$$

Summing up, we obtain the desired:

$$\|\varphi_{j\ell} - \varphi_{jk}\|^2 \le \varepsilon \qquad (\ell, k \ge N).$$ □

Reducing compact manifolds to $\mathbb{R}^n$.

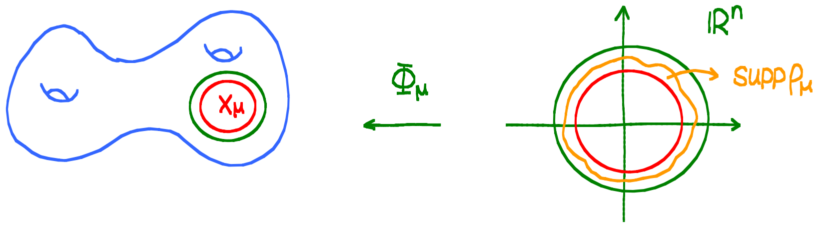
Recall that on a compact manifold X, $s \in \mathbb{N}$,

$$\|\varphi\|_{(s)} \triangleq \sum_{|\alpha| \le s} \int |\nabla^\alpha \varphi|^2 \sqrt{g}\, dx$$

We may also introduce another norm as follows. Write $X = \bigcup_{\mu=1}^n X_\mu$, where X_μ corresponds to the unit ball in $\mathbb{R}^n$, and $\{\rho_\mu\}$ a partition of unity subordinate to this cover. Define:

$$\|\varphi\|'_{(s)} \triangleq \sum_{\mu=1}^N \|\rho_\mu \varphi\|^2_{(s, \mathbb{R}^n)}$$

where the r.h.s. uses the usual Sobolev norm on $\mathbb{R}^n$.



Claim: These two norms are equivalent, i.e. $\exists\, c, C > 0$ s.t.

$$c\cdot\|\varphi\|'_{(s)} \le \|\varphi\|_{(s)} \le C\cdot\|\varphi\|'_{(s)}$$

This follows from the next two simple observations:

(1). For each X_μ, we have: $\forall\, \varphi\in C_0^\infty(X_\mu)$

$$\|\varphi\|_{(s)} \sim \|\varphi\|_{(s,\mathbb{R}^n)}$$

(2). For either norm, we always have,

$\forall\, a\in C^\infty(X)$, $\|a\varphi\|_{(s)} \le C_a\|\varphi\|_{(s)}$

$\forall\, b\in C_0^\infty(\mathbb{R}^n)$, $\|b\varphi\|_{(s,\mathbb{R}^n)} \le C_b\|\varphi\|_{(s,\mathbb{R}^n)}$

(2) is easy. We check (1):

$(s=0)$: $\|\varphi\|^2_{(0)} = \int_X |\varphi|^2\sqrt{g}\,dx = \int_{\Phi_\mu^{-1}(X_\mu)} |\varphi\circ\Phi_\mu(y)|^2\sqrt{g(y)}\,dy$

$\|\varphi\|^2_{(0,\mathbb{R}^n)} = \int_{\Phi_\mu^{-1}(X_\mu)} |\varphi\circ\Phi_\mu(y)|^2\,dy$

Note that $g=\det g_{ij}>0$, and on the compact set $\Phi_\mu^{-1}(X_\mu)\subseteq\mathbb{R}^n$ is bounded from below and above. Hence $\|\varphi\|^2_{(0)} \sim \|\varphi\|^2_{(0,\mathbb{R}^n)}$

$(s=1)$: $\|\varphi\|^2_{(1)} = \|\varphi\|^2_{(0)} + \int_X g^{ij}\,\nabla_i\varphi\,\nabla_j\varphi\,\sqrt{g}\,dx$

$\|\varphi\|^2_{(1,\mathbb{R}^n)} = \|\varphi\|^2_{(0,\mathbb{R}^n)} + \int_{\mathbb{R}^n}\sum_i|\partial_i(\varphi\circ\Phi_\mu(y))|^2\,dy$

From simple linear algebra, we know that if h_{ij}, $\tilde h_{ij}$ are two positive definite bilinear forms on $\mathbb{R}^n$, then they are equivalent: $\lambda h(u,u)\le\tilde h(u,u)\le\Lambda h(u,u)$, $\forall\, u\in\mathbb{R}^n$. ($\lambda,\Lambda$ can be taken to be the minimal and maximal eigenvalues of the self-adjoint operator $h^{ip}\tilde h_{pj}$. Now over a compact set $\Phi_\mu^{-1}(X_\mu)$, take $h_{ij}=g^{ij}$, $\tilde h_{ij}=\delta^{ij}$, then $\exists\, c, C$ s.t $0<c\le\lambda(x)\le\Lambda(x)\le C$, $\forall\, x\in\Phi_\mu^{-1}(X_\mu)$, hence

$$c\int_{\mathbb{R}^n} g^{ij}\,\nabla_i\varphi\,\nabla_j\varphi\,\sqrt{g}\,dx \le \int_{\mathbb{R}^n}\sum_i|\nabla_i\varphi|^2dx \le C\cdot\int_{\mathbb{R}^n} g^{ij}\,\nabla_i\varphi\,\nabla_j\varphi\,\sqrt{g}\,dx$$

Now note that $\nabla_i\varphi = \partial_i\varphi + A_i\varphi \Rightarrow |\nabla_i\varphi|^2\le|\partial_i\varphi|^2+|A_i\varphi|^2$, but $|A_i|$ is bounded on the compact set $\Phi_\mu^{-1}(X_\mu)$. Similarly $|\partial_i\varphi|^2\le|\nabla_i\varphi|^2+|A_i\varphi|^2$. Thus by adding $\|\varphi\|^2_{(0)}$, $\|\varphi\|^2_{(0,\mathbb{R}^n)}$, these norms are again equal.

The general case follows from similar discussions. Now we can prove the equivalence of $\|\cdot\|_{(s)}$ and $\|\cdot\|'_{(s)}$.

By triangle inequality,

$$\|\varphi\|_{(s)} \le \sum_{\mu=1}^N \|\rho_\mu\varphi\|^2_{(s)} \le C\cdot \sum_{\mu=1}^N \|\rho_\mu\varphi\|^2_{(s,\mathbb{R}^n)}$$

On the other hand, each term $\mu=1,\cdots,N$

$$\|\rho_\mu\varphi\circ\Phi_\mu(y)\|^2_{(s,\mathbb{R}^n)} \le C'\cdot\|\rho_\mu\varphi\|^2_{(s)} \le C'_\mu\|\varphi\|^2_{(s)}$$

Summing up, the claim follows.

Using this equivalence of norms, we may transform a priori estimate, Rellich's lemma, Sobolev lemma from $\mathbb{R}^n$ to compact manifolds. For instance, we prove a priori estimate for compact manifolds.

$\forall\varphi\in C^\infty(X)$, write $\varphi=\sum_\mu\rho_\mu\varphi$. Now

$$\begin{aligned}\|\varphi\|^2_{(s+m)} &\le \sum_\mu\|\rho_\mu\varphi\|^2_{(s+m)}\\ &\le C\cdot\sum_\mu\|(\rho_\mu\varphi)\circ\Phi_\mu\|^2_{(s+m,\mathbb{R}^n)}\\ &\le C'\sum_\mu\left(\|L((\rho_\mu\varphi)\circ\Phi_\mu)\|^2_{(s,\mathbb{R}^n)}+\|(\rho_\mu\varphi)\circ\Phi_\mu\|^2_{(s+m-1,\mathbb{R}^n)}\right)\end{aligned}$$

Observation: $(\rho_\mu\varphi)\circ\Phi_\mu=(\rho_\mu\circ\Phi_\mu)\cdot(\varphi\circ\Phi_\mu)$ and $\rho_\mu\circ\Phi_\mu\in C_0^\infty(\mathbb{R}^n)$. Then:

$$\begin{aligned}\|L((\rho_\mu\varphi)\circ\Phi_\mu)\|^2_{(s,\mathbb{R}^n)} &\le \|(\rho_\mu\circ\Phi_\mu)\circ L(\varphi\circ\Phi_\mu)\|^2_{(s,\mathbb{R}^n)}\\ &\quad+\|\sum_{|\nu|\le m-1}((\tilde\rho_\mu\circ\Phi_\mu)\cdot D^\nu\varphi)\|_{(s,\mathbb{R}^n)}\\ &\le C\cdot\big(\|(\rho_\mu\circ\Phi_\mu)\circ L(\varphi\circ\Phi_\mu)\|^2_{(s,\mathbb{R}^n)}\\ &\quad+\|(\rho_\mu\varphi)\circ\Phi_\mu\|_{(s+m-1,\mathbb{R}^n)}\big).\end{aligned}$$

where $\tilde\rho_\mu$ are order ≥ 1 derivatives of ρ_μ, coming from derivatives of L by Leibnitz rule. (we may also require $|\partial^\nu\rho_\mu|\le\rho_\mu$) Thus:

$$\begin{aligned}\|\varphi\|^2_{(s+m)} &\le C'\sum_\mu\left(\|\rho_\mu\circ\Phi_\mu\cdot L(\varphi\circ\Phi_\mu)\|^2_{(s,\mathbb{R}^n)}+\|(\rho_\mu\varphi)\circ\Phi_\mu\|^2_{(s+m-1,\mathbb{R}^n)}\right)\\ &\le C'\left(\|L\varphi\|^2_{(s)}+\|\varphi\|^2_{(s+m-1)}\right)\end{aligned}$$

The transformation of Rellich's lemma and Sobolev lemma from $\mathbb{R}^n$ to X is similar. $\square$

Elliptic regularity

Basic technique: Method of difference quotients.

Assume $u \in H_{(s)}(\mathbb{R}^n)$. $\forall h \neq 0$, let

$$\Delta_h^i u \triangleq \frac{1}{h}(u(x_1,\cdots,x_i+h,\cdots,x_n) - u(x_1,\cdots,x_n))$$

Then : $\sup_h\left(\sum_{i=1}^n \|\Delta_h^i u\|_{(s)}\right) < \infty \iff u \in H_{(s+1)}(\mathbb{R}^n)$

Pf: "$\Leftarrow$" $\|\Delta_h^i u\|_{(s)}^2 = \int (1+|\xi|^2)^s |(\Delta_h^i u)^\wedge(\xi)|^2 d\xi$

We compute: $(\Delta_h^1 u)(\xi) = \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} \frac{1}{h}(u(x_1+h,\cdots,x_n) - u(x_1,\cdots,x_n))\,dx$

$$= \frac{1}{h}(e^{ih\xi_1}-1)\,\hat{u}(\xi)$$

Note that $\left|\frac{1}{h}(e^{ih\xi_1}-1)\right| = \left|\xi_1 \int_0^1 e^{ith\xi_1} dt\right| \le |\xi_1|$

$$\Rightarrow \|\Delta_h^1 u\|_{(s)}^2 \le \int (1+|\xi|^2)^s |\xi_1|^2 |\hat{u}(\xi)|^2 d\xi$$

$$\le \int (1+|\xi|^2)^{s+1} |\hat{u}(\xi)|^2 d\xi$$

$$= \|u\|_{(s+1)}^2$$

"$\Rightarrow$" $\int (1+|\xi|^2)^s |\xi_j|^2 |\hat{u}(\xi)|^2 d\xi = \int \lim_{h\to 0} (1+|\xi|^2)^s \left|\frac{e^{ih\xi_j}-1}{h}\right|^2 |\hat{u}(\xi)|^2 d\xi$

By Fatou's lemma ($f_j \ge 0$, then $\int \liminf f_j\, d\mu \le \liminf \int f_j\, d\mu$)

$\int \lim_{h\to 0} (1+|\xi|^2)^s \left|\frac{e^{ih\xi_j}-1}{h}\right|^2 |\hat{u}(\xi)|^2 d\xi \le \lim_{h\to 0} \|\Delta_h^j u\|_{(s)}^2 \le$ const. independent of h. $\square$

Manipulations with Δ_h^j.

(1). $\Delta_h^j\left(\frac{\partial u}{\partial x_k}\right) = \frac{\partial}{\partial x_k}(\Delta_h^j u)$

Indeed, $\Delta_h\left(\frac{\partial u}{\partial x_k}\right) = \frac{1}{h}\left(\frac{\partial u}{\partial x_k}(x+h) - \frac{\partial u}{\partial x_k}\right) = \frac{\partial}{\partial x_k}\left(\frac{1}{h}(u(x+h) - u(x))\right)$.

(2). Let $a \in C_0^\infty(\mathbb{R}^n)$, then:

$$\|\Delta_h^j(au) - a\Delta_h^j(u)\|_{(s)} \le C_a \|u\|_{(s)}$$

where C_a is a constant independent of u and h.

Note that, since Δ_h behaves like a differential, we trivially have

$$\|\Delta_h(au)\|_{(s)} \le C_a' \|u\|_{(s+1)}.$$

Pf: $\Delta_h(au)(x) = \frac{1}{h}(a(x+h)u(x+h) - a(x)u(x))$

$$= \frac{1}{h}(a(x+h)u(x+h) - a(x)u(x+h) + a(x)u(x+h) - a(x)u(x))$$

$$= u(x+h)\,\Delta_h(a)(x) + a(x)\,\Delta_h(u)(x)$$

$\Rightarrow \Delta_h^j(au) - a\Delta_h^j(u) = \Delta_h(a)\,u(x+h)$.

Observation 1: $\|u(x+h)\|_{(s)} = \|u(x)\|_{(s)}$

Observation 2: $\Delta_h(a)$ is smooth, and is bounded independently of h.

In fact: $a(x_1+h, x_2, \cdots) - a(x) = \int_0^1 \frac{d}{dt} a(x_1+t\cdot h, x_2, \cdots)\, dt$
$= \int_0^1 \frac{\partial a}{\partial x_1}(x_1+th, x_2, \cdots)\, h\, dt$
$= h \int_0^1 \frac{\partial a}{\partial x_1}(x_1+th, x_2, \cdots)\, dt$

$\Rightarrow \quad \Delta_h(a) = \int_0^1 \frac{\partial a}{\partial x_1}(x_1+th, x_2, \cdots)\, dt$

$\Rightarrow \quad \|\Delta_h(a)\|_{(s)} \le \int_0^1 \|\frac{\partial a}{\partial x_1}\|_{(s)}\, dt = C_a$

The result follows.

(3). More generally, if $L = \sum a_\alpha(x) D^\alpha$, $a_\alpha(x) \in C_0^\infty(\mathbb{R}^n)$, similar proof shows that:

$$\|\Delta_h(Lu) - L(\Delta_h u)\|_{(s)} \le C \cdot \|u\|_{(s+m)}$$

Note that our elliptic operator doesn't satisfy $a_\alpha(x) \in C_0^\infty(\mathbb{R}^n)$, but we will introduce a cut off trick to deal with this.

Proof of regularity thm.

Note that if $u \in H_{(s)}(\mathbb{R}^n)$, we have two ways of defining Lu. One way is to regard $L: H_{(s)} \longrightarrow H_{(s-m)}$, and Lu is its image in $H_{(s-m)}$. i.e. $\exists\, u_j \in C_0^\infty(\mathbb{R}^n)$, $u_j \longrightarrow u$ w/r.t. $\|\cdot\|_{(s)}$, and $Lu = \lim Lu_j$ w/r.t. $\|\cdot\|_{(s-m)}$.

The second way is in the sense of distributions. We say that $Lu = g$ for some function g if

$$\int_{\mathbb{R}^n} u\,\Big(\sum_{|\alpha|\le m}(-1)^{|\alpha|} D^\alpha\Big)(a_\alpha \cdot \varphi) = \int_{\mathbb{R}^n} g\varphi,$$

$\forall\, \varphi \in C_0^\infty(\Omega)$. In case $u \in C^\infty(\Omega)$, the l.h.s. becomes $\int_{\mathbb{R}^n}(\sum_{|\alpha|\le m} a_\alpha D^\alpha u)\varphi$, by integration by parts.

Observation: If $Lu = f$ in the first sense and $Lu = g$ in the second sense, then $f = g$.

Indeed, take $u_j \longrightarrow u$ in $H_{(s)}$, $u_j \in C_0^\infty(\mathbb{R}^n)$. Denote the operator $\sum_{|\alpha|\le m}(-1)^{|\alpha|} D^\alpha(a_\alpha \cdot)$ by L^t, we have, $\forall\, \varphi \in C_0^\infty(\mathbb{R}^n)$,

$$\Big|\int_{\mathbb{R}^n}(u_j - u_k) L^t\varphi\, dx\Big| = \Big|\int_{\mathbb{R}^n}(u_j-u_k)\hat{}\cdot (L^t\varphi)\hat{}\, d\xi\Big|$$
$$\le \Big|\int_{\mathbb{R}^n} |u_j - u_k|\hat{}\,(1+|\xi|^2)^s d\xi\Big|\,\Big|\int_{\mathbb{R}^n} |L^t\varphi|\hat{}\,(1+|\xi|^2)^{-s} d\xi\Big|$$
$$= \|u_j - u_k\|_{(s)} \|L^t\varphi\|_{(-s)} \longrightarrow 0 \quad (j, k \longrightarrow \infty),$$

since $L^t\varphi$ is compactly supported and thus $\|L^t\varphi\|_{(-s)} < \infty$. Letting $j \longrightarrow \infty$, and integrating by parts for u_j, we have:

$$\left| \int_{\mathbb{R}^n} f\varphi - \int_{\mathbb{R}^n} u_k L^t \varphi \right| \longrightarrow 0 \quad \text{as} \quad k \longrightarrow \infty,$$

which means $\lim_{k\to\infty} u_k = f$ in the sense of distributions.

Next, we introduce local Sobolev space: Given $\Omega \subseteq \mathbb{R}^n$, $H^{loc}_{(s)}(\Omega) \triangleq \{\varphi \mid \forall \chi \in C_0^\infty(\Omega), \chi\varphi \in H_{(s)}(\mathbb{R}^n)\}$.

Observation: $\bigcap_s H^{loc}_{(s)}(\Omega) = C^\infty(\Omega)$.

Claim: $Lu = f$ in Ω, $f \in C^\infty(\Omega)$, and $u \in H^{loc}_{(s+m)}(\Omega)$, then $u \in H^{loc}_{(s+m+1)}(\Omega)$.

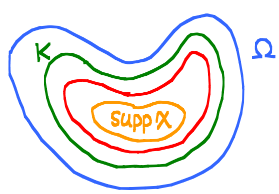
Then by the above observation, $u \in C^\infty(\Omega)$.

Recall the AP estimate. $\forall K \subseteq \Omega$ compact subset ($\mathring{K} \neq \phi$), $\forall \varphi \in C_0^\infty(K)$

$$\|\varphi\|_{(s+m)} \leq C(\|L\varphi\|_{(s)} + \|\varphi\|_{(s+m-1)})$$

Let $H^{comp}_{(s+m)}(K) \triangleq \overline{C_0^\infty(K)}$ w/r. t. $\|\cdot\|_{(s+m)}$ norm, then $H^{comp}_{(s+m)}(K) \subseteq H^{loc}_{(s+m)}(\Omega)$.

Want: $\forall \chi \in C_0^\infty(\Omega)$, $\chi u \in H_{(s+1)}(\mathbb{R}^n)$. Since $\operatorname{Supp}\chi \subseteq \Omega$, we may pick inclusion of open neighborhoods:



We wish to show that $\sup_h \|\Delta_h(\chi u)\|_{(s+m)} < \infty$ as in difference quotient.

Now we have:

$$\|\Delta_h(\chi u)\|_{(s+m)} \leq C(\|L(\Delta_h(\chi u))\|_{(s)} + \|\Delta_h(\chi u)\|_{(s+m-1)})$$

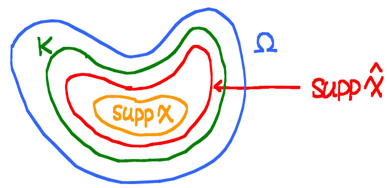
But

$$\|L(\Delta_h(\chi u)) - \Delta_h(L(\chi u))\|_{(s)} \leq C \cdot \|\chi u\|_{(s+m)} < \infty$$

and

$$\|\Delta_h(L\chi u) - \Delta_h(\chi L u)\|_{(s)} \leq \|L\chi u - \chi L u\|_{(s+1)}$$
$$\leq C \cdot \|u\|_{(s+m)} < \infty \qquad \square$$

A small technical remark: In using difference quotient, we need L to have its coefficients compactly supported. This could not happen for elliptic operators! Thus, we need to pick another $\hat{\chi} \in C_0^\infty(\Omega)$, $\hat{\chi} \equiv 1$ on $\operatorname{supp}\chi$ to do a cut-off



and we may so pick $\hat{\chi}$ that the support of $\hat{\chi}L$ after small h shifts still remain inside K, and on suppχ we can apply our AP estimate.

§8. Geometry of Subbundles

Def: Let $E \longrightarrow X$ be a holomorphic vector bundle. A subbundle $E' \longrightarrow X$ of E is holomorphic vector bundle s.t. E'_z is a subspace of E_z, $\forall z \in X$.

In terms of transition functions, $X = \cup X_\mu$, $E \longleftrightarrow \{t_{\mu\nu}(z)\}$, $E' \longleftrightarrow \{t'_{\mu\nu}(z)\}$, then being a subbundle means:

$$t_{\mu\nu}(z) = \begin{pmatrix} t'_{\mu\nu}(z) & b_{\mu\nu} \\ 0 & t''_{\mu\nu} \end{pmatrix}$$

where $t'_{\mu\nu}(z)$ is an $r' \times r'$ matrix, $t''_{\mu\nu}$ an $(r-r') \times (r-r')$ matrix, $b_{\mu\nu}$ an $r' \times (n-r')$ matrix.

Given $E \longrightarrow X$ a holomorphic vector bundle, $\{H_{\bar\alpha\beta}\}$ a metric on E, then we have a unitary Chern connection on E:

$$\begin{cases} \nabla_{\bar j} \varphi^\alpha = \partial_{\bar j} \varphi \\ \nabla_j \varphi^\alpha = H^{\alpha\bar\gamma} \partial_j (H_{\bar\gamma\beta} \varphi^\beta) \end{cases}$$

Let E' be a subbundle (with $\operatorname{rank} E' = r' < r = \operatorname{rank} E$), we have two ways of constructing a connection on E': (In the following, let $\mu \in \{1, \cdots, n, \bar 1, \cdots, \bar n\}$

(1). $\{H_{\bar\alpha\beta}\}$ restricts to a metric on E', with the requirement to be a Chern connection, it defines a unique connection.

(2). $\forall \varphi \in \Gamma(X, E') \subseteq \Gamma(X, E)$, we can first take $\nabla_\mu \varphi \in \Gamma(X, E)$, and take the orthogonal projection from E to E':

$$\nabla'_\mu \varphi = \pi'(\nabla_\mu \varphi)$$

Note that for any scalar function,

$$\begin{aligned} \nabla'_\mu (f\varphi) &= \pi'(\partial_\mu(f)\varphi) + f \nabla_\mu(\varphi) \\ &= \partial_\mu(f)\varphi + f \nabla'_\mu \varphi \end{aligned}$$

and thus defines a connection on E'. This agrees with the first construction since $\forall \varphi, \varphi_1, \varphi_2 \in \Gamma(X, E')$:

$$\nabla'_{\bar j} \varphi = \pi'(\partial_{\bar j} \varphi) = \partial_{\bar j} \varphi \quad (\text{since } E' \text{ is holomorphic})$$

$$\begin{aligned} \partial_\mu \langle \varphi_1, \varphi_2 \rangle &= \langle \nabla_\mu \varphi_1, \varphi_2 \rangle + \langle \varphi_1, \nabla_{\bar\mu} \varphi_2 \rangle \\ &= \langle \pi' \nabla_\mu \varphi_1, \varphi_2 \rangle + \langle \varphi_1, \pi' \nabla_{\bar\mu} \varphi_2 \rangle \end{aligned}$$

$$= \langle \nabla'_\mu \varphi_1, \varphi_2 \rangle + \langle \varphi_1, \nabla'_{\bar\mu} \varphi_2 \rangle$$

Def. $\forall \varphi \in \Gamma(X, E')$, $B_\mu \varphi \triangleq \nabla_\mu \varphi - \nabla'_\mu \varphi = \pi''(\nabla_\mu \varphi)$, where π'' is the projection onto the orthogonal complement of E'' of E'. (As real bundles $E \cong E' \oplus E''$).

Observation :
$$\begin{aligned} B_\mu(f\varphi) &= \nabla_\mu(f\varphi) - \nabla'_\mu(f\varphi) \\ &= \partial_\mu(f)\cdot\varphi + f\cdot\nabla_\mu\varphi - \partial_\mu(f)\cdot\varphi - f\cdot\nabla'_\mu\varphi \\ &= f(\nabla_\mu\varphi - \nabla'_\mu\varphi) \\ &= f\cdot B_\mu\varphi. \end{aligned}$$

i.e. $B_\mu \in \Gamma(X, \mathrm{Hom}(E', E''))$. Equivalently, if $B = B_\mu dz^\mu$, then $B \in \Gamma(X, \Lambda^1 \otimes \mathrm{Hom}(E', E''))$. Moreover, $B_{\bar j}\varphi = \partial_{\bar j}\varphi - \partial_{\bar j}\varphi = 0 \Rightarrow B \in \Gamma(X, \Lambda^{1,0} \otimes \mathrm{Hom}(E', E''))$. B is called the second fundamental form of E'

Connection on E''.

E'' is a smooth subbundle of E, but it's not necessarily holomorphic. We can define a connection on E'' by, $\forall \psi \in \Gamma(X, E'')$:

$$\nabla_\mu \psi = \pi''(\nabla_\mu \psi)$$

Again this is a unitary connection, but in general, it's not a Chern connection since E'' is not necessarily holomorphic. Also set:

$$C_\mu \psi \triangleq \nabla_\mu \psi - \nabla''_\mu \psi$$

Claim: $B_\mu^\dagger = -C_{\bar\mu}$.

Pf: $\forall \varphi \in \Gamma(X, E')$, $\psi \in \Gamma(X, E'')$, $\langle \varphi, \psi \rangle = 0$

$$\begin{aligned} \Rightarrow \quad 0 = \partial_\mu(\langle \varphi, \psi \rangle) &= \langle \nabla_\mu \varphi, \psi \rangle + \langle \varphi, \nabla_{\bar\mu} \psi \rangle \\ &= \langle \pi'' \nabla_\mu \varphi, \psi \rangle + \langle \varphi, \pi' \nabla_{\bar\mu} \psi \rangle \\ &= \langle B_\mu \varphi, \psi \rangle + \langle \varphi, C_{\bar\mu} \psi \rangle \end{aligned}$$

Hence $C \triangleq C_\mu dz^\mu \in \Gamma(X, \Lambda^{0,1} \otimes \mathrm{Hom}(E'', E'))$. It also follows that $B \equiv 0 \Leftrightarrow C \equiv 0$, which happens only when E'' is holomorphic as well.

Curvatures of E, E', E''.

Let $\Phi \in \Gamma(X,E)$, write $\Phi = \varphi + \psi$, with $\varphi \in \Gamma(X,E')$, $\psi \in \Gamma(X,E'')$. Then

$$\begin{aligned}\nabla_\mu \Phi &= \nabla_\mu(\varphi+\psi)\\ &= \nabla'_\mu \varphi + B\varphi + \nabla''_\mu \psi + C_\mu \psi\\ &= \begin{pmatrix} \nabla'_\mu & C_\mu \\ B_\mu & \nabla''_\mu \end{pmatrix}\begin{pmatrix}\varphi\\ \psi\end{pmatrix}\end{aligned}$$

Write

$$d_\nabla \Phi = \begin{pmatrix} d_{\nabla'} & C \\ B & d_{\nabla''}\end{pmatrix}\begin{pmatrix}\varphi\\ \psi\end{pmatrix},$$

then the curvature:

$$\begin{aligned}d_\nabla^2 \Phi &= F\Phi\\ &= \begin{pmatrix} d^2_{\nabla'} + C\wedge B & d_{\nabla'}C + C d_{\nabla''} \\ B d_{\nabla'} + d_{\nabla''}B & d^2_{\nabla''} + B\wedge C\end{pmatrix}\begin{pmatrix}\varphi\\ \psi\end{pmatrix}\end{aligned}$$

Explicitly, recall that $F = F_{\bar{k}j}\, dz^j \wedge d\bar{z}^k$, $B = B_j dz^j$ and $C = C_{\bar{k}} d\bar{z}^k$. In our notation, $(C\wedge B)_{\bar{k}j} = C_{\bar{k}} d\bar{z}^k \wedge B_j dz^j = -C_{\bar{k}} B_j\, dz^j \wedge d\bar{z}^k$.

$$\Rightarrow F_{\bar{k}j}\big|_{Hom(E',E')} = F'_{\bar{k}j} - C_{\bar{k}} B_j = F'_{\bar{k}j} + B^{\dagger}_{\bar{k}} B_j.$$

Thus

$$\begin{aligned}\langle F_{\bar{k}j}\varphi, \varphi\rangle &= \langle F'_{\bar{k}j}\varphi, \varphi\rangle + \langle B^{\dagger}_{\bar{k}} B_j \varphi, \varphi\rangle\\ &= \langle F'_{\bar{k}j}\varphi, \varphi\rangle + \langle B_j\varphi, B_k\varphi\rangle\end{aligned}$$

Since $\langle B_j\varphi, B_k\varphi\rangle$ is always positive definite, $\{F_{\bar{k}j}\}$ is more "positive" than $\{F'_{\bar{k}j}\}$. The equation $d^2_\nabla \Phi$ can be cleaned up to be:

$$F = \begin{pmatrix} F' + B^{\dagger}\wedge B, & -d_\nabla B^{\dagger} \\ d_\nabla B & , \; F'' + B\wedge B^{\dagger}\end{pmatrix}$$

Necessary condition for Hermitian-Einstein metrics.

Let $E \longrightarrow X$ be a holomorphic vector bundle on (X,ω) a compact Kähler manifold. $\omega = \frac{i}{2} g_{\bar{k}j}\, dz^j \wedge d\bar{z}^k$ the Kähler form.

Question: When does there exist $H_{\bar{\alpha}\beta}$ on E, so that $\Lambda F = \mu\, Id$, where

Λ is the Hodge operator and μ is a constant. i.e

$$(g^{j\bar{k}} F_{\bar{k}j})^{\alpha}{}_{\beta} = \mu \delta^{\alpha}{}_{\beta} \qquad (H\text{-}E).$$

As $F_{\bar{k}j} = -\partial_{\bar{k}}(H^{-1}\partial_j H)$, this equation is non-linear in H.

Def: (Degree of E). Given $E \longrightarrow (X, \omega)$,

$$\deg E \triangleq \frac{i}{2}\int_X \mathrm{Tr}(F) \wedge \frac{\omega^{n-1}}{(n-1)!}$$

Note that if $F = F_{\bar{k}j}{}^{\alpha}{}_{\beta}\, dz^j \wedge d\bar{z}^k$, then $\mathrm{Tr}(F) = F_{\bar{k}j}{}^{\alpha}{}_{\alpha}\, dz^j \wedge d\bar{z}^k$. It's a closed form whose cohomology class is $C_1(E)$. Thus the degree doesn't depend on the the metric $\{H_{\bar{\alpha}\beta}\}$ or $\{g_{\bar{k}j}\}$ but only E and $[\omega]$.

Def. (Slope of E). $\quad \mathrm{Slope}(E) \triangleq \deg E / \mathrm{rank} E$

Key observation (Kobayashi, Lübke)

Thm. If $E \longrightarrow (X, \omega)$ admits a Hermitian-Einstein metric, then $\forall E'$ which is a holomorphic subbundle of E,

$$\mathrm{slope}(E') \leq \mathrm{slope}(E).$$

Equality holds iff $E = E' \oplus E''$ with E'' holomorphic.

Def. A holomorphic vector bundle E is said to be stable in the sense of Mumford – Takemoto if $\forall E'$ holomorphic subbundle, $\mathrm{slope}(E') \leq \mathrm{slope}(E)$.

Pf of thm.

Let $E' \subseteq E$ be a holomorphic subbundle. If a Hermitian-Einstein metric exists, we can use it to compute slope E.

$$\deg E = \frac{i}{2}\int_X (\mathrm{Tr} F) \wedge \frac{\omega^{n-1}}{(n-1)!}$$

where $F = F_{\bar{k}j}{}^{\alpha}{}_{\beta}\, dz^j \wedge d\bar{z}^k$. To calculate $\mathrm{Tr} F \wedge \frac{\omega^{n-1}}{(n-1)!}$, we may do it pointwise. Write $\omega = \frac{i}{2}(dz_1 \wedge d\bar{z}_1 + \cdots + dz_n \wedge d\bar{z}_n)$, then

$$\frac{\omega^{n-1}}{(n-1)!} = \left(\frac{i}{2}\right)^{n-1} \left(\sum_{j=1}^{n} dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge \widehat{(dz_j \wedge d\bar{z}_j)} \wedge \cdots \wedge dz_n \wedge d\bar{z}_n \right)$$

and thus:

$$F\wedge\frac{\omega^{n-1}}{(n-1)!}=\sum_{j=1}^{n}F_{j\bar{j}}{}^{\alpha}{}_{\beta}\,dz_j\wedge d\bar{z}_j\wedge dz_1\wedge d\bar{z}_1\wedge\cdots\wedge\widehat{(dz_j\wedge d\bar{z}_j)}\wedge\cdots\wedge dz_n\wedge d\bar{z}_n$$
$$=g^{j\bar{k}}F_{\bar{k}j}{}^{\alpha}{}_{\beta}\frac{\omega^n}{n!}$$

(Generally for any 2-form valued tensor T: $T\wedge\frac{\omega^{n-1}}{(n-1)!}=g^{j\bar{k}}T_{\bar{k}j}\frac{\omega^n}{n!}$).

Hence:

$$\begin{aligned}\mathrm{Tr}(F)\wedge\frac{\omega^{n-1}}{(n-1)!}&=\mathrm{Tr}(\Lambda F)\frac{\omega^n}{n!}\\&=\mathrm{Tr}(\mu\mathrm{Id})\frac{\omega^n}{n!}\\&=\mu\cdot\mathrm{rank}E\,\frac{\omega^n}{n!}\end{aligned}$$

and

$$\int_X\mathrm{Tr}(F)\wedge\frac{\omega^{n-1}}{(n-1)!}=\mu\cdot\mathrm{rank}E\cdot\mathrm{vol}(X)$$

On the other hand, recall that we have:

$$F_{\bar{k}j}|_{E'}=F'_{\bar{k}j}+B^{+}_{\bar{k}}B_j$$

$$\begin{aligned}\Rightarrow\qquad \deg E'&=\int_X\mathrm{Tr}(F_{\bar{k}j}|_{E'}-B^{+}_{\bar{k}}B_j)\frac{\omega^{n-1}}{(n-1)!}\\&=\int_X\mathrm{Tr}(g^{j\bar{k}}(F_{\bar{k}j}|_{E'}-B^{+}_{\bar{k}}B_j))\frac{\omega^n}{n!}\\&=\int_X\mathrm{Tr}(\Lambda F|_{E'})\frac{\omega^n}{n!}-\int_X\mathrm{Tr}(g^{j\bar{k}}\bar{B}_{\bar{k}}B_j)\frac{\omega^n}{n!}\\&=\mu\cdot\mathrm{rank}E'\cdot\mathrm{vol}(X)-\|B\|^2_{L^2}\\&\le\mu\cdot\mathrm{rank}E'\cdot\mathrm{vol}(X)\end{aligned}$$

$$\Rightarrow\qquad \mathrm{slope}(E')\le\mu\cdot\mathrm{vol}(X)=\mathrm{slope}(E)$$

Note that "=" holds iff $B\equiv0$. i.e. iff E splits as direct sum of holomorphic vector bundles: $E\cong E'\oplus E''$. □

Another example of a global aspect in canonical metrics: The Yang-Mills equation.

Recall that from electromagnetism, we are interested in:

- The functional: $A=A_\mu dx^\mu\longmapsto I(A)\triangleq\int_{\mathbb{R}^{3,1}}|F_{\mu\nu}|^2$, $F\triangleq dA$.
- The critical points of I, i.e. those connections $A_\mu dx^\mu$ satisfying
$$\frac{\delta I}{\delta A_\mu}=0$$

We are interested in the following generalizations.

$E\longrightarrow(X,\omega)$: holomorphic vector bundle over a compact Kähler manifold.

$A = A_j dz^j$: a connection on E. Consider the Yang-Mills functional:

$$I(A) \triangleq \int_X g^{l\bar{k}} g^{j\bar{m}} F_{\bar{k}j}{}^{\alpha}{}_{\beta} \overline{F_{\bar{l}m}{}^{\gamma}{}_{\delta}} H_{\bar{\gamma}\alpha} H^{\beta\bar{\delta}} \frac{\omega^n}{n!}$$

Introduce the Hodge $*$ operator, $* : \Lambda^{p,q} \longrightarrow \Lambda^{n-p,n-q}$, with the following defining property: $\varphi \wedge * \overline{\psi} = \langle \varphi, \psi \rangle \frac{\omega^n}{n!}$. In this notation, we have:

$$I(A) = \int_X \mathrm{Tr}(F \wedge * F)$$

In case $\dim X = 2$, $* : \Lambda^{1,1} \longrightarrow \Lambda^{1,1}$, and $*^2 = id_{\Lambda^{1,1}}$. Consider the eigenspace orthogonal decomposition $\Lambda^{1,1} \cong \Lambda^{1,1}_+ \oplus \Lambda^{1,1}_-$, $F = F_+ + F_-$ with $*F_+ = F_+$ and $*F_- = -F_-$. Notice that:

$$\begin{aligned} I(A) &= \int_X \mathrm{Tr}((F_+ + F_-) \wedge *(F_+ + F_-)) \\ &= \int_X \mathrm{Tr}(F_+ \wedge * F_+ + F_+ \wedge * F_- + F_- \wedge * F_+ + F_- \wedge * F_-) \\ &= \| F_+ \|_{L^2} + \| F_- \|_{L^2} \end{aligned}$$

On the other hand, observe that:

$$\begin{aligned} C_2(E) &= \int_X \mathrm{Tr}(F \wedge F) \\ &= \int_X \mathrm{Tr}((F_+ + F_-) \wedge (*F_+ - *F_-)) \\ &= \| F_+ \|_{L^2} - \| F_- \|_{L^2} \end{aligned}$$

is topological. $\Rightarrow$ $I(A) \geq C_2(E) = $ const. Thus if we can find a connection with $F_- = 0$, $I(A)$ achieves minimum. The equation $F_- = 0$ itself is easier than the non-linear YM equation.

§9. Kähler Manifolds

Recall that, for a complex manifold X and $E \longrightarrow X$ holomorphic vector bundle we have:

$$\bar{\partial} : \Gamma(X, E\otimes\Lambda^{p,q}) \longrightarrow \Gamma(X, E\otimes\Lambda^{p,q+1})$$
$$\phi \quad \mapsto \quad \bar{\partial}\phi$$

$\forall\ \varphi, \psi \in \Gamma_c(X, E\otimes\Lambda^{p,q})$, we have

$$\langle \varphi, \psi \rangle = \frac{1}{p!q!}\int_X \varphi^{\alpha}_{\bar{J}I}\overline{\psi^{\beta}_{\bar{K}L}}\, g^{K\bar{J}} g^{I\bar{L}} H_{\bar{\beta}\alpha} \cdot vol.$$

is an L^2-inner product. To define this we only needed:

(1). A hermitian metric on E and on TX.

(2). A volume element

Recall that if (X, g) is Kähler, then:

$$(\bar{\partial}^{\dagger}\psi)^{\beta}_{\bar{K}L} = -g^{k\bar{\ell}}\nabla_k \psi^{\beta}_{\bar{K}L\bar{\ell}}$$

Rmk: Kähler condition enters as we used integration by parts. Furthermore, ∇ is the connection on $E\otimes\Lambda^{p,q}$, which is the Levi-Civita and Chern connection.

Kähler condition

Let (X, J), complex manifold, J: almost complex structure, g: Riemannian metric, compatible with J:

$$g(Ju, Jv) = g(u, v).$$

i.e. it's a Hermitian metric. Let $\omega(u, v) = g(Ju, v)$, ω is alternating. Note that g induces the following structure:

(1). ∇_{LC}: Levi-Civita connection on $T^{\mathbb{C}}X = TX\otimes_{\mathbb{R}}\mathbb{C} = T^{1,0}X \oplus T^{0,1}X$.

(2). A hermitian metric on $T^{1,0}$ and thus the Chern connection ∇_{CH}.

Thm. The following are equivalent:

(1). (X, J, g) is Kähler

(2). $\nabla_{LC} J = 0$

(3). $\nabla_{LC}\,\omega = 0$

(4). $\nabla_{LC} = \nabla_{Ch}$

(5). $\forall p\in X,\ \exists U\ni p$ s.t. $z(p)=0$ and $g_{\bar{k}j}=\delta_{kj}+O(|z|^2)$

Generalized Jacobi's identity.

Let $A\in\Gamma(X,\Lambda^a\otimes End(E))$, $B\in\Gamma(X,\Lambda^b\otimes End E)$, we may define their generalized Lie bracket $[A,B]\in\Gamma(X,\Lambda^{a+b}\otimes End(E))$:

$$[A,B]\triangleq AB-(-1)^{ab}BA$$

We have Jacobi's identity:

$$ad_{[A,B]}=[ad_A,ad_B]$$

i.e. $\forall C\in\Gamma(X,\Lambda^c\otimes End(E))$:

$$[[A,B],C]=[A[B,C]]-(-1)^{ab}[B,[A,C]].$$

Recall that we defined

$$L:\Gamma(X,E\otimes\Lambda^{p,q})\longrightarrow\Gamma(X,E\otimes\Lambda^{p+1,q+1}),\ \psi\longmapsto\omega\wedge\psi$$

and its adjoint:

$$\Lambda:\Gamma(X,E\otimes\Lambda^{p,q})\longrightarrow\Gamma(X,E\otimes\Lambda^{p-1,q-1})$$

Thm. (Hodge identities) (X,J,g) Kähler and $(E,H)/X$ a hermitian holomorphic vector bundle. Then:

(1). $[\bar\partial^\dagger,L]=\sqrt{-1}\,\partial$ (2). $[\partial^\dagger,L]=-\sqrt{-1}\,\bar\partial$

(3). $[\Lambda,\bar\partial]=-\sqrt{-1}\,\partial^\dagger$ (4). $[\Lambda,\partial]=\sqrt{-1}\,\bar\partial^\dagger$

where $\partial=\nabla^{1,0}_{CH}$ on E.

Pf: We have shown this in local coordinates before. Here is the proof in the notation introduced above.

Note that $(1)^\dagger\Rightarrow(3)$ and $(2)^\dagger\Rightarrow(4)$, and $\overline{(1)}\Rightarrow(2)$. Thus it suffices to show (1).

Since $\bar\partial\psi=d\bar z^\beta\wedge\nabla_{\bar\beta}\psi$, we have $\bar\partial^\dagger\psi=-\iota(dz^\alpha)\nabla_\alpha\psi$, where $\iota(dz^\alpha)$ acts on forms by:

(i). On 1-forms, $\iota(dz^\alpha)(\theta)=\langle\theta,dz^\alpha\rangle_g$, the Euclidean inner product. Thus

$$\iota(dz^\alpha)(dz^\beta)=0,\quad \iota(dz^\alpha)(d\bar z^\beta)=g^{\alpha\bar\beta}.$$

(ii). $\iota(dz^\alpha)$ acts as a derivation: For $\varphi\wedge\psi$,

$$\iota(dz^\alpha)(\varphi\wedge\psi)=(\iota(dz^\alpha)\varphi)\wedge\psi+(-1)^{|\varphi|}\varphi\wedge(\iota(dz^\alpha)\psi).$$

Then:

$$\begin{aligned}\bar\partial^\dagger L\psi&=\bar\partial^\dagger(\omega\wedge\psi)\\&=-\iota(dz^\alpha)\nabla_\alpha(\omega\wedge\psi)\\&=-\iota(dz^\alpha)\,\omega\wedge\nabla_\alpha\psi\\&=-(\iota(dz^\alpha)\omega)\wedge\nabla_\alpha\psi-\omega\wedge(\iota(dz^\alpha)\nabla_\alpha\psi)\\&=-(\iota(dz^\alpha)\omega)\wedge\nabla_\alpha\psi+L\bar\partial^\dagger\psi\end{aligned}$$

But $\omega=\sqrt{-1}g_{\bar\beta\gamma}dz^\gamma\wedge d\bar z^\beta$

$$\begin{aligned}\Rightarrow \iota(dz^\alpha)(\omega)&=\sqrt{-1}g_{\bar\beta\gamma}\,\iota(dz^\alpha)(dz^\gamma)\wedge d\bar z^\beta-\sqrt{-1}g_{\bar\beta\gamma}dz^\gamma\wedge\iota(dz^\alpha)(d\bar z^\beta)\\&=-\sqrt{-1}g_{\bar\beta\gamma}\,g^{\alpha\bar\beta}dz^\gamma\end{aligned}$$

$$\begin{aligned}\Rightarrow -\iota(dz^\alpha)(\omega)\wedge\nabla_\alpha\psi&=\sqrt{-1}g_{\bar\beta\gamma}g^{\alpha\bar\beta}dz^\gamma\wedge\nabla_\alpha\psi\\&=\sqrt{-1}\,dz^\alpha\wedge\nabla_\alpha\psi\\&=\sqrt{-1}\,\partial\psi\end{aligned}$$

$\square$

Applications

(1). (KAN identity) $\square_{\bar\partial}-\square_\partial=[\sqrt{-1}F,\Lambda]$.

This is proven before. In particular, if $E=\mathcal{O}_X$ is trivial, $F=0$.

$$\square_\partial=\square_{\bar\partial}$$

Thm. For Kähler manifolds, $\Delta_d=2\square_\partial=2\square_{\bar\partial}$.

Pf: Recall that
$$\begin{aligned}\Delta_d&=dd^\dagger+d^\dagger d\\&=[d,d^\dagger]\qquad(d\text{ has degree }1,\ d^\dagger\text{ degree }-1)\\&=[\partial+\bar\partial,\ \partial^\dagger+\bar\partial^\dagger]\\&=[\partial,\partial^\dagger]+[\bar\partial,\bar\partial^\dagger]+[\bar\partial,\partial^\dagger]+[\partial,\bar\partial^\dagger]\\&=\square_\partial+\square_{\bar\partial}+[\bar\partial,\partial^\dagger]+[\partial,\bar\partial^\dagger]\end{aligned}$$

Thus it suffices to show that $[\partial,\bar\partial^\dagger]=[\bar\partial,\partial^\dagger]=0$.

$$\begin{aligned}[\partial,\bar\partial^\dagger]&=[-\sqrt{-1}[\bar\partial^\dagger,L],\bar\partial^\dagger]\\&=-\sqrt{-1}\left([\bar\partial^\dagger,[L,\bar\partial^\dagger]]-(-1)^{1\cdot 2}[L,[\bar\partial^\dagger,\bar\partial^\dagger]]\right)\end{aligned}$$

$$= -\sqrt{-1}\,([[L,\partial^\dagger],\bar\partial^\dagger] - [L, 2\bar\partial^\dagger\bar\partial^\dagger])$$
$$= \sqrt{-1}\,([[\bar\partial^\dagger, L],\bar\partial^\dagger])$$
$$= -[\partial,\bar\partial^\dagger].$$

$\Rightarrow [\partial,\bar\partial^\dagger] = 0$. Similarly for $[\bar\partial,\partial^\dagger]$. $\square$

Cor. Δ_d preserves (p,q)-types, i.e. $\psi \in \Gamma(X,\Lambda^{p,q}) \Rightarrow \Delta_d\psi \in \Gamma(X,\Lambda^{p,q})$. $\square$

Thm. (Hodge decomposition) X : compact Kähler manifold. then:

$$H^r(X) = \oplus_{p+q=r} H^{p,q}(X)$$

Pf: Take a Kähler metric on X. By Hodge's thm: $H^r(X) \cong \mathcal{H}^r(X)$, the space of harmonic d-forms. $\forall \psi \in \mathcal{H}^r(X)$, we may decompose it into its (p,q)-type:

$$\psi = \Sigma_{p+q=r}\, \psi^{p,q}$$

and

$$0 = \Delta_d\psi = \Sigma_{p+q=r}\, (\Delta_d\psi)^{p,q}$$
$$= \Sigma_{p+q=r}\, \Delta_d\psi^{p,q}$$

$\Rightarrow \Delta_d\psi^{p,q} = 2\,\Box_{\bar\partial}\psi^{p,q} = 0 \Rightarrow \psi^{p,q} \in \mathcal{H}^{p,q}(X)$. $\square$

Cor. If X is a compact Kähler manifold. then $h^r(X)$ is even for r odd.
Pf: Indeed by the thm, $H^r(X) \cong H^{r,0}(X) \oplus H^{r-1,1}(X) \oplus \cdots \oplus H^{1,r-1}(X) \oplus H^{0,r}(X)$.
Note that $\overline{(\cdot)}: H^{p,q}(X) \xrightarrow{\cong} H^{q,p}(X)$ is a complex conjugate linear isomorphism of vector spaces $\Rightarrow h^r(X) = 2h^{r,0} + 2h^{r-1,1} + \cdots$ is even. $\square$

E.g. Hopf surface : Non-Kähler complex manifold.
$S^3 \times \mathbb{R} \cong \mathbb{R}^4 \setminus \{0\} \cong \mathbb{C}^2 \setminus \{0\}$ is a complex manifold. Define $\mathbb{Z} \curvearrowright S^3 \times \mathbb{R}$ by
$(\vec{v}, \lambda) \mapsto (e^\lambda \cdot \vec{v})$

$$n \cdot (\vec{v}, \lambda) = (\vec{v}, \lambda + n)$$

(or on $\mathbb{C}^2 \setminus \{0\}$, n acts by multiplication by e^n). $\mathbb{Z}$ preserves the complex structure on $\mathbb{C}^2 \setminus \{0\}$. Thus $S^3 \times \mathbb{R}/\mathbb{Z} \cong S^3 \times S^1$ is a complex manifold.

But it's not Kähler since $h^1(X) = 1$ is odd.

(ii) ($\partial\bar\partial$ - lemma) If $\varphi \in \Omega^{p,q}$ is d-exact, i.e. $\varphi = d\psi$ for some ψ, then $\exists \eta \in \Omega^{p-1,q-1}$ s.t. $\varphi = \partial\bar\partial\eta$.

Rmk: The converse is easy: $\varphi = \partial\bar\partial\eta = (\partial+\bar\partial)\bar\partial\eta = d\bar\partial\eta$.

Pf of lemma.

φ is d-exact $\Rightarrow$ $[\varphi] = 0$ in $H^r(X)$, $r = p+q$. Let

$$\Gamma(X, \Lambda^{p,q}) = \mathcal{H}^{p,q}(X) \oplus \operatorname{Im}\bar\partial \oplus \operatorname{Im}\bar\partial^\dagger$$
$$\Rightarrow \varphi = \Box_{\bar\partial} G_{\bar\partial}\varphi = (\bar\partial\bar\partial^\dagger + \bar\partial^\dagger\bar\partial) G_{\bar\partial}\varphi.$$

Since $[\bar\partial, G_{\bar\partial}] = 0$,

$$\Rightarrow \varphi = \bar\partial\bar\partial^\dagger G_{\bar\partial}\varphi + \bar\partial^\dagger G_{\bar\partial}\bar\partial\varphi$$
$$= \bar\partial\bar\partial^\dagger G_{\bar\partial}\varphi$$

Let $\eta = \bar\partial^\dagger G_{\bar\partial}\varphi \in \Gamma(X, \Lambda^{p,q-1})$, then

$$\begin{aligned} \partial\eta &= \partial\bar\partial^\dagger G_{\bar\partial}\varphi \\ &= \bar\partial^\dagger\partial G_{\bar\partial}\varphi \quad ([\partial, \bar\partial^\dagger] = 0) \\ &= \bar\partial^\dagger\partial G_{\partial}\varphi \quad (G_{\bar\partial} = G_{\partial}) \\ &= \bar\partial^\dagger G_{\partial}\partial\varphi \\ &= 0 \end{aligned}$$

$\Rightarrow \eta$ is ∂-closed $\Rightarrow \eta = \pi_\partial\eta + \Box_\partial G_\partial\eta \quad (\pi_\partial\eta \in \mathcal{H}^{p,q}_{\partial} = \mathcal{H}^{p,q}_{\bar\partial})$

$$\Rightarrow \varphi = \bar\partial(\pi_\partial\eta + (\partial\partial^\dagger + \partial^\dagger\partial)G_\partial\eta)$$
$$= \bar\partial\partial\partial^\dagger G_\partial\eta. \qquad \square$$

§9. The Calabi Conjecture

X: compact complex manifold.

Question: Let $\omega_0 = \frac{i}{2} g^0_{\bar{k}j} dz^j \wedge d\bar{z}^k$ be a Kähler form on X. Let $T_{\bar{k}j}$ be a $(1,1)$ form in $C_1(X)$. Is there a Kähler form ω' in the same class as ω_0 satisfying

$$R_{\bar{k}j}(\omega) = T_{\bar{k}j} \; ?$$

Here $C_1(X) \triangleq C_1(T^{1,0}X)$.

Calabi conjectured that the answer is yes, and such an ω is unique within the class of ω_0.

Thm. (Yau) The Calabi conjecture is true.

Cor. Let X be a compact complex manifold with $C_1(X)=0$. Then if ω_0 is any Kähler class, there is a unique $\omega \in [\omega_0]$ with

$$R_{\bar{k}j}(\omega) = 0$$

Reduction to a Monge-Ampère equation

$\omega \in [\omega_0]$ means that $\omega - \omega_0 = d\Phi$ is exact. By the $\partial\bar{\partial}$-lemma, we have $\omega - \omega_0 = \frac{i}{2}\partial\bar{\partial}\varphi$ where $\varphi \in C^\infty(X, \mathbb{R})$. Thus we are looking for with:

$$\omega \triangleq \omega_0 + \frac{i}{2}\partial\bar{\partial}\varphi > 0 \quad (\omega_0\text{-pluri-subharmonic condition})$$

and

$$R_{\bar{k}j}(\omega) = T_{\bar{k}j} \qquad (*)$$

The equation $(*)$ may be rewritten as:

$$-\partial_j \partial_{\bar{k}} \log \omega^n = T_{\bar{k}j} \qquad (*)'$$

Observe that ω^n is not a scalar function on X, but if we change variables, we get $\omega^n(z) = \tilde{\omega}^n(w) \left|\frac{\partial w}{\partial z}\right|^2$, and $\log\omega^n = \log\tilde{\omega}^n + \log(\frac{\partial w}{\partial z}) + \log(\overline{\frac{\partial w}{\partial z}})$. Thus

$$\begin{aligned} -\partial\bar{\partial}\log\omega^n &= -\partial\bar{\partial}\log\tilde{\omega}^n - \partial\bar{\partial}\log(\tfrac{\partial w}{\partial z}) - \partial\bar{\partial}(\log(\overline{\tfrac{\partial w}{\partial z}})) \\ &= -\partial\bar{\partial}\log\tilde{\omega}^n \end{aligned}$$

is a well-defined $(1,1)$-form.

To write the l.h.s. of $(*)'$ in terms of scalars, note that

$$-\partial_j\partial_{\bar{k}}\log\omega^n = -\partial_j\partial_{\bar{k}}\log\omega_0^n - \partial_j\partial_{\bar{k}}\log\frac{\omega^n}{\omega_0^n}$$
$$= R_{\bar{k}j}(\omega_0) - \partial_j\partial_{\bar{k}}\log\frac{\omega^n}{\omega_0^n}$$

where $\frac{\omega^n}{\omega_0^n}$ is now a scalar function. Then $(*)'$ is equivalent to

$$-\partial_j\partial_{\bar{k}}\log\left(\frac{\omega^n}{\omega_0^n}\right) = T_{\bar{k}j} - R_{\bar{k}j}(\omega_0)$$
$$= \partial_j\partial_{\bar{k}}F$$

In other words,

$$-\partial_j\partial_{\bar{k}}\left(\log\frac{\omega^n}{\omega_0^n} - F\right) = 0$$

where F is a given smooth function on X (by $\partial\bar{\partial}$-lemma again).

Claim: $\log\frac{\omega^n}{\omega_0^n} - F \equiv c$.

Pf: $\partial_j\partial_{\bar{k}}h = 0 \Rightarrow \int_X \partial_j\partial_{\bar{k}}h\cdot\bar{h}\,g^{j\bar{k}}\frac{\omega^n}{n!} = 0$

$$\Rightarrow\ 0 = \int_X \partial_{\bar{k}}h\,\overline{\partial_{\bar{j}}h}\,g^{j\bar{k}}\frac{\omega^n}{n!}$$
$$= \|\bar{\partial}h\|_{L^2}^2$$

Thus h is globally holomorphic and thus a constant $\square$

Changing F into $F+c$, we are reduced to solve:

$$\omega^n = e^F\omega_0^n$$

More explicitly:

$$\det(g^0_{\bar{k}j} + \partial_j\partial_{\bar{k}}\varphi) = e^F\det(g^0_{\bar{k}j})$$

where F is a global smooth function determined up to a constant. This is the Monge-Ampère equation. Note that the constant can be fixed as follows:

$$\int_X\omega^n = \int_X\left(\omega_0 + \tfrac{i}{2}\partial\bar{\partial}\varphi\right)^n$$
$$= \int_X \omega_0^n + n\,\omega_0^{n-1}\wedge\left(\tfrac{i}{2}\partial\bar{\partial}\varphi\right) + \cdots + \left(\tfrac{i}{2}\partial\bar{\partial}\varphi\right)^n$$
$$= \int_X\omega_0^n$$

since $\partial\bar{\partial}\varphi = d(\bar{\partial}\varphi)$ is exact and ω_0 is closed. Thus $\omega^n = e^F\omega_0^n$ gives

$$\int_X e^F\omega_0^n = \int_X\omega^n = \int_X\omega_0^n$$

and this condition uniquely determines F in the Monge-Ampère equation.

Method of Continuity.

Introduce the following family of equations, for the unknown φ_t $(0 \le t \le 1)$:

$$\det(g^{0}_{\bar{k}j} + \partial_j \partial_{\bar{k}} \varphi_t) = e^{tF - C_t} \det(g^{0}_{\bar{k}j}) \qquad (*)_t$$

where C_t is so chosen that

$$\int_X e^{tF - C_t} \omega_0^n = \int_X \omega_0^n$$

$$\text{i.e.} \quad C_t = \log \frac{\int_X e^{tF} \omega_0^n}{\int_X \omega_0^n}$$

Define $I = \{ t \in [0,1] \mid (*)_t$ admits a smooth solution satisfying the ω_0-pluri-subharmonic condition $g^{0}_{\bar{k}j} + \partial_j \partial_{\bar{k}} \varphi_t > 0 \}$

We shall show that:

(a). $I \neq \phi$ (easy, since $0 \in I$)

(b). I is open

(c). I is closed

Then it will follow that $I = [0,1]$ and $(*)$ admits a smooth solution.

Proof of (b). — Implicit function thm. for Banach spaces.

Rewrite $(*)_t$ as:

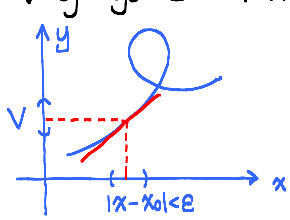
$$\log \frac{\det(g^{0}_{\bar{k}j} + \partial_j \partial_{\bar{k}} \varphi)}{\det(g^{0}_{\bar{k}j})} - (tF - C_t) = 0 \qquad (*)_t$$

We want to show that if $(*)_t$ admits a solution φ_{t_0} at t_0, then $\exists\, \varepsilon > 0$ s.t. $\forall t$: $|t - t_0| < \varepsilon$, $(*)_t$ admits a C^∞, ω_0-pluri-subharmonic solution φ_t.

Recall the standard implicit function thm on $\mathbb{R}^2$: let $E(x,y)$ be a C^2 function on $\mathbb{R}^2$ and assume that:

$$\begin{cases} E(x_0, y_0) = 0 \\ \frac{\partial E}{\partial y}(x_0, y_0) \neq 0 \end{cases}$$

Then $\exists\, \varepsilon > 0$ and a nhd V of y_0 s.t. $\forall x$: $|x - x_0| < \varepsilon$, $\exists!\, y \in V$ s.t. $E(x,y) = 0$.



Thm. (Implicit function thm for Banach spaces)

Let B_i, $i=1,2,3$ be Banach spaces, and $E: B_1 \times B_2 \longrightarrow B_3$ be a C^1-continuous function. Assume that $E(x_0, y_0) = 0$ for some $x_0 \in B_1$ and $y_0 \in B_2$ $\frac{\partial E}{\partial y}(x_0, y_0)$ is an invertible operator from B_2 to B_3, with bounded inverse. Then $\exists$ nhd V of x_0, $\tilde{V}$ of y_0 so that $\forall x \in V$, $\exists! \, y \in \tilde{V}$ s.t. $E(x,y) = 0$.

We need to set up the proof of (b) so as to apply the implicit function thm. Let

$$E(t, \varphi) \triangleq \log \frac{\det(g^0_{\bar{k}j} + \partial_j \partial_{\bar{k}} \varphi)}{\det(g^0_{\bar{k}j})} - (tF - C_t)$$

where $E: \mathbb{R} \times B_2 \longrightarrow B_3$ for appropriate Banach spaces B_2, B_3.

Def. (Schauder spaces) On $\mathbb{R}^n$, fix $0 < \alpha < 1$.

$$\|\varphi\|_{C^{0,\alpha}} \triangleq \sup |\varphi| + \sup_{x \neq y} \frac{|\varphi(x) - \varphi(y)|}{|x-y|^\alpha}$$

$C^{0,\alpha}(\mathbb{R}^n) \triangleq \{\varphi \mid \|\varphi\|_{C^{0,\alpha}} < \infty\}$. Observe that $\varphi, \psi \in C^{0,\alpha} \Rightarrow \varphi \cdot \psi \in C^{0,\alpha}$, $e^\varphi \in C^{0,\alpha}$.
More generally, $C^{k,\alpha}(\mathbb{R}^n) \triangleq \{\varphi \mid \partial^\gamma \varphi \in C^{0,\alpha}(\mathbb{R}^n), \forall |\gamma| \leq k\}$.

On a compact manifold, similar spaces $C^{k,\alpha}(X)$ can be defined via taking a partition of unity.

Next, observe that the solution to $(*)_t$ is not unique : φ_t satisfies $(*)_t$ $\Rightarrow$ so does $\varphi_t +$ const. Introduce:

$$C^{k,\alpha}_0(X) \triangleq \{\varphi \in C^{k,\alpha}(X) \mid \int_X \varphi \omega_0^n = 0\}$$

Then we claim that $E(t, \varphi): \mathbb{R} \times C^{2,\alpha}_0 \longrightarrow C^{0,\alpha}_0$. Indeed:

$$\int_X \left(\frac{\omega^n}{\omega_0^n} - e^{tF - C_t}\right) \omega_0^n = \int_X \omega^n - \int_X e^{tF - C_t} \omega_0^n = 0$$

since C_t is so chosen.

Verifying the hypotheses of the IFT.

Given $f: B_2 \longrightarrow B_3$, we say that f is differentiable at $y_0 \in B_2$ if $\exists$ a bounded linear map $L: B_2 \longrightarrow B_3$ (bounded: $\exists C > 0$ s.t. $\forall y \in B_2$, $\|Ly\|_{B_3} \leq C \|y\|_{B_2}$).

s.t. $$\lim_{y\to y_0} \frac{\| f(y) - f(y_0) - L_{y_0}(y-y_0)\|_{B_3}}{\| y - y_0\|_{B_2}} = 0$$

$Hom(B_2, B_3) = \{$ bounded linear operators sending B_2 into $B_3\}$ is a Banach space, with norm defined by:

$$\| T \| \triangleq \sup_{\substack{y\in B_2,\\ y\neq 0}} \frac{\| Ty\|_{B_3}}{\| y\|_{B_2}}$$

$E \in C^1(\Omega)$ if $y \mapsto L_y$ is defined for all $y \in \Omega$ and is continuous as a mapping $\Omega \longrightarrow Hom(B_2, B_3)$.

Thus now in our case, we want to show that

(1). $E \in C^1$

(2). Compute "$\frac{\partial E}{\partial \varphi}$" $\in Hom(C_0^{2,\alpha}, C_0^{0,\alpha})$ (boundedness)

(3). "$\frac{\partial E}{\partial \varphi}$" has a bounded inverse.

Computation of "$\frac{\partial E}{\partial \varphi}$":

By variation: imaging that $\varphi \longmapsto \varphi + \delta\varphi$.

$$\delta(E) = \delta\left(\frac{\det(\overset{\circ}{g}_{\bar{k}j} + \partial_j \partial_{\bar{k}} \varphi)}{\det \overset{\circ}{g}_{\bar{k}j}} - e^{tF - C_t}\right)$$

$$= \frac{1}{\det \overset{\circ}{g}_{\bar{k}j}} \delta(\det(\overset{\circ}{g}_{\bar{k}j} + \partial_j \partial_{\bar{k}} \varphi))$$

Observe that if G is an invertible matrix:

$$\delta(\det G) = ((\det G)^{-1} \delta(\det G)) \cdot \det G$$
$$= \delta(\ln \det G) \cdot \det G$$
$$= Tr(G^{-1} \delta G) \det G$$

Hence:

$$\delta E = \frac{1}{\det(\overset{\circ}{g}_{\bar{k}j})} \det(g_{\bar{k}j}) \cdot g^{j\bar{k}} \delta g_{\bar{k}j}$$
$$= \frac{1}{\det(\overset{\circ}{g}_{\bar{k}j})} \det(g_{\bar{k}j}) \cdot g^{j\bar{k}} \partial_j \partial_{\bar{k}} \delta\varphi$$
$$= \frac{1}{\det(\overset{\circ}{g}_{\bar{k}j})} \det g_{\bar{k}j} \cdot \Delta_g \delta\varphi.$$

and

$$L: \delta\varphi \longmapsto \frac{\det g_{\bar{k}j}}{\det \overset{\circ}{g}_{\bar{k}j}} \Delta_g \delta\varphi.$$

Claim: L_φ admits a bounded inverse, i.e. $\forall\, \mu \in C_0^{0,\alpha}$, $\exists !\, \nu \in C_0^{2,\alpha}$ s.t.

$$L\nu = \mu$$

Furthermore,

$$\|\nu\|_{C^{2,\alpha}} \le C \|\mu\|_{C^{0,\alpha}}$$

for some constant $C > 0$. This follows from the following:

Basic fact from linear analysis:

Let (X, g_{ij}) be any compact Riemannian manifold, then the equation

$$\Delta_g \nu = \mu$$

admits a solution iff

$$\int_X \mu \sqrt{g} = 0 .$$

The solution ν is unique if require the normalization condition:

$$\int_X \nu \sqrt{g} = 0$$

Now in our case. $L_\varphi \nu = \mu \Longleftrightarrow \Delta_g \nu = \frac{\omega_0^n}{\omega^n} \mu$. This is solvable since $\mu \in C_0^{0,\alpha}$, and the solution is unique if we specify

$$\int_X \nu\, \omega_0^n = 0$$

The similar a priori estimates hold for Schauder spaces, which implies the boundedness of the inverse of L. (the Green's operator)

Proof of (c): I is closed.

We want to show that. $t_j \in I$, $t_j \longrightarrow T$, $\Longrightarrow T \in I$. i.e. φ_{t_j} admits a solution of $(*)_{t_j}$, and $t_j \longrightarrow T$, then $\exists\, \varphi$ a solution of $(*)_T$ satisfying the Monge-Ampère equation and the pluri-subharmonic condition.

To start, we would wish to show that a subsequence, renamed still by $\{\varphi_j\}$, converges in $C^{3,\alpha}$ (or all $C^{m,\alpha}$). If so, and the limit φ satisfies the M-A equation, we see that the determinant of $g^0_{\bar{k}j} + \partial_j \partial_{\bar{k}} \varphi$ is fixed pointwise. If we can further show that the eigenvalues of $g^0_{\bar{k}j} + \partial_j \partial_{\bar{k}} \varphi$ is bounded from above, they will also be bounded from below, whence $g^0_{\bar{k}j} + \partial_j \partial_{\bar{k}} \varphi > 0$.

To achieve this, it suffices to show that, $\forall m$,

$$\|\varphi_{t_j}\|_{C^m} \le A_m. \qquad (AP)_m,$$

since $\|\varphi_{t_j}\|_{C^1} + \|\varphi_{t_j}\|_{C^0} \le const \Rightarrow \{\varphi_{t_j}\}$ is equicontinuous. (The standard trick as used in Hodge theory: $|\varphi_{t_i}(x) - \varphi_{t_i}(y)| \le \sup\|\nabla\varphi\|\cdot|x-y|$.) Then by Arezela-Ascoli thm. there would be a convergent subsequence.

Schauder theory

$(AP)_m$, $0 \le m \le 3 \Rightarrow (AP)_m$ for all m. (In fact, $0 \le m \le 2+\varepsilon$ will do. This is because a priori estimate is valid only for $\varepsilon > 0$: $\|\varphi\|_{C^{2,\varepsilon}}$). The 4 cases

$m=3$ (Calabi)

$m=2$ (Aubin-Yau, Pogorelov)

$m=1$

$m=0$ (This is the hardest part, done by Yau using Moser's iteration).

Furthermore, $(m=2) + (m=0) \Rightarrow (m=1)$.

10 years ago, Kolodziej proved a stronger estimate:

Thm. (Kolodziej).

$$\det(\overset{\circ}{g}_{\bar{k}j} + \partial_j\partial_{\bar{k}}\varphi) = f \det \overset{\circ}{g}_{\bar{k}j}$$

with $f>0$. Then $\forall\, p>1$, $\exists\, \alpha>0$, so that whenever $\|f\|_{L^p} < \infty$, we have

$$\|\varphi\|_{C^\alpha} < Const$$

Proof of the C^0-estimate

Moser iteration in the simplest case.

Consider $\Delta u = f$, where Δ is the Laplacian on a compact manifold (X, g_{ij}). Assume that u is normalized: $\int_X u\sqrt{g}\,dx = 0$. We want to estimate for $\|u\|_{C^0}$.

(a).

$$\begin{aligned}\int_X u\cdot f\sqrt{g}\,dx &= \int_X u\cdot\Delta u\sqrt{g}\,dx\\ &= \int_X u\cdot\frac{1}{\sqrt{g}}(\partial_j(\sqrt{g}\,g^{ij}\partial_i u))\sqrt{g}\,dx\\ &= -\int_X(\partial_j u\cdot\partial_i u)g^{ij}\sqrt{g}\,dx\\ &= -\int_X\|\nabla u\|^2\sqrt{g}\,dx\end{aligned}$$

$$\Longrightarrow \qquad \|\nabla u\|_{L^2}^2 \le \left|\int_X u\cdot f\sqrt{g}\,dx\right| \le \|u\|_{L^2}\|f\|_{L^2}$$

Recall Poincaré's inequality:

$$\|u\|_{L^2}^2 \le C\cdot\left(\|\nabla u\|_{L^2}^2 + \left(\int_X u\sqrt{g}\right)^2\right)$$

Combined with above it gives

$$\|u\|_{L^2}^2 \le C\cdot\|u\|_{L^2}\|f\|_{L^2}$$

$$\Longrightarrow \|u\|_{L^2} \le C\cdot\|f\|_{L^2}$$

(b). Moser iteration: Let $\beta \equiv \frac{n}{n-2}\ (>1)$, $n = \dim_{\mathbb{R}} X$. Then $\forall\, p \ge 2$,

$$\max(1, \|u\|_{L^{p\beta}}) \le (C\cdot p)^{\frac{1}{p}}\max(1, \|u\|_{L^p})$$

where C is a constant independent of p.

Assuming this, we can bound $\|u\|_{L^\infty}\ (= \|u\|_{C^0})$ in the following manner: Start with p (thinking of $p=2$ as the starting point from (a).).

$$\log\max(1, \|u\|_{L^{\beta p}}) \le \frac{1}{p}\log(C\cdot p) + \log\max(1, \|u\|_{L^p})$$

Iterate with p replaced by $p\beta$:

$$\begin{aligned}\log\max(1, \|u\|_{L^{\beta^2 p}}) &\le \frac{1}{p\beta}\log(C\cdot p\beta) + \log\max(1, \|u\|_{L^{\beta p}})\\ &\le \frac{1}{p\beta}\log(C\cdot p\beta) + \frac{1}{p}\log(C\cdot p) + \log\max(1, \|u\|_{L^p})\end{aligned}$$

After k-steps, we get:

$$\log\max(1, \|u\|_{L^{\beta^k p}}) \le \underbrace{\sum_{\ell=1}^{k-1}\frac{1}{p\beta^\ell}\log(C\cdot p\beta^\ell)}_{\text{converges when } k\to\infty} + \log\max(1, \|u\|_{L^p})$$

Hence in the limit,

$$\log\max(1, \|u\|_{L^\infty}) \le A + \log\max(1, \|u\|_{L^p})$$

(c). Proof of the Moser iteration step

Recall from calculus: What's the anti-derivative of $|x|^\alpha$?

$$(x\cdot|x|^\alpha)' = (\alpha+1)|x|^\alpha$$

Then if $\Delta u = f$,

$$u\cdot|u|^\alpha\Delta u = u|u|^\alpha\cdot f$$

$$\Longrightarrow \qquad \int_X u|u|^\alpha\Delta u\sqrt{g}\,dx = \int_X u|u|^\alpha f\sqrt{g}\,dx$$

Integrating by parts, this gives:

$$\int_X u|u|^\alpha f\sqrt{g}\,dx = -\int_X \nabla(u|u|^\alpha)\cdot\nabla u\sqrt{g}\,dx$$
$$= -\int_X (\alpha+1)|u|^\alpha \|\nabla u\|^2\sqrt{g}\,dx$$
$$= -(\alpha+1)\int_X |u^{\frac{\alpha}{2}}\|\nabla u\||^2\sqrt{g}\,dx$$
$$= -\frac{(\alpha+1)}{(1+\frac{\alpha}{2})^2}\int_X \|\nabla(u\cdot|u|^{\frac{\alpha}{2}})\|^2\sqrt{g}\,dx$$

$\Longrightarrow$

$$-\frac{(\alpha+1)}{(1+\frac{\alpha}{2})^2}\int_X \|\nabla(u\cdot|u|^{\frac{\alpha}{2}})\|^2\sqrt{g}\,dx = \int_X u|u|^\alpha f\sqrt{g}\,dx$$

Similar as using Poincaré's inequality above, we need the Sobolev inequality:

$$\|v\|^2_{L^{2\beta}} \leq C\cdot(\|\nabla v\|^2_{L^2} + \|v\|^2_{L^2})$$

In particular, apply Sobolev's inequality to $v = u\cdot|u|^{\frac{\alpha}{2}}$, we have:

$$\|u|u|^{\frac{\alpha}{2}}\|^2_{L^{2\beta}} \leq C\cdot(\|\nabla(u\cdot|u|^{\frac{\alpha}{2}})\|^2_{L^2} + \|u\cdot|u|^{\frac{\alpha}{2}}\|^2_{L^2})$$

i.e.

$$\left(\int_X |u|^{(1+\frac{\alpha}{2})2\beta}\sqrt{g}\,dx\right)^{\frac{1}{\beta}} \leq C\cdot\left(\frac{(1+\frac{\alpha}{2})^2}{(\alpha+1)}\int_X |u|^{\alpha+1} f\sqrt{g}\,dx + \int_X |u|^{\alpha+2}\sqrt{g}\,dx\right)$$

Set $p = 2+\alpha$. Then

$$\left(\int_X |u|^{\beta p}\sqrt{g}\,dx\right)^{\frac{1}{\beta}} \leq C\cdot\left(p\cdot\left|\int_X |u|^{\alpha+1} f\sqrt{g}\,dx\right| + \left|\int_X |u|^p\sqrt{g}\,dx\right|\right)$$

Note that

$$\left|\int_X |u|^{\alpha+1} f\sqrt{g}\,dx\right| \leq \|f\|_{C^0}\int_X |u|^{\alpha+1}\sqrt{g}\,dx$$
$$\leq \|f\|_{C^0}\left(\int_X |u|^{\alpha+2}\sqrt{g}\,dx\right)^{\frac{\alpha+1}{\alpha+2}}\left(\int_X 1\cdot\sqrt{g}\,dx\right)^{\frac{1}{\alpha+2}}$$
$$\leq C'\cdot(\|u\|^p_{L^p})^{\frac{\alpha+1}{\alpha+2}}.$$

(Jordan-Hölder $\int |fg| \leq (\int |f|^p)^{\frac{1}{p}}(\int |g|^q)^{\frac{1}{q}}$)

Combined, since $\frac{\alpha+1}{\alpha+2} < 1$, there are two cases: if $\|u\|_{L^p} \leq 1$, $C'\cdot(\|u\|^p_{L^p})^{\frac{\alpha+1}{\alpha+2}} \leq C'\cdot 1$; otherwise, $C'(\|u\|^p_{L^p})^{\frac{\alpha+1}{\alpha+2}} \leq C'\cdot\|u\|^p_{L^p} \Longrightarrow$

$$\|u\|^p_{L^{\beta p}} \leq C' p\max(1, \|u\|^p_{L^p}).$$

Moser iteration for the Monge-Ampère equation.

Notation: $\omega = \frac{i}{2} g_{\bar{k}j}\,dz^j\wedge d\bar{z}^k$. $\omega_\varphi = \frac{i}{2}(g_{\bar{k}j} + \partial_{\bar{k}}\partial_j\varphi)\,dz^j\wedge d\bar{z}^k$. Then the Monge-Ampère equation becomes:

$$\omega_\varphi^n = e^f\omega^n \quad , (f = (tF - c_t))$$

Similar as the simple case above, we want to control $\|\nabla v\|^2_{L^2}$. The trick is the following:

$$e^f\omega^n - \omega^n = \omega_\varphi^n - \omega^n$$

$$= (\omega_\varphi - \omega)(\omega_\varphi^{n-1} + \omega_\varphi^{n-2}\wedge\omega + \cdots + \omega^{n-1})$$
$$= \tfrac{i}{2}\partial\bar\partial\varphi\,(\omega_\varphi^{n-1} + \cdots + \omega^{n-1}).$$

As in the Laplacian case, we integrate by parts.

$$\int_X \varphi|\varphi|^\alpha (e^f - 1)\omega^n = \int_X \varphi|\varphi|^\alpha (\tfrac{i}{2}\partial\bar\partial\varphi)\wedge(\omega_\varphi^{n-1} + \cdots + \omega^{n-1})$$
$$= \int_X \varphi|\varphi|^\alpha (\tfrac{i}{2} d\bar\partial\varphi)\wedge(\omega_\varphi^{n-1} + \cdots + \omega^{n-1})$$
$$= -\int_X d(\varphi|\varphi|^\alpha)(\tfrac{i}{2}\bar\partial\varphi)\wedge(\omega_\varphi^{n-1} + \cdots + \omega^{n-1})$$
$$= -\int_X (\alpha+1)|\varphi|^\alpha \cdot \tfrac{i}{2}\partial\varphi\wedge\bar\partial\varphi\wedge(\omega_\varphi^{n-1} + \cdots + \omega^{n-1})$$
$$= -\frac{(\alpha+1)}{(1+\frac{\alpha}{2})^2}\int_X \partial(\varphi|\varphi|^{\frac{\alpha}{2}})\wedge\bar\partial(\varphi|\varphi|^{\frac{\alpha}{2}})\wedge(\omega_\varphi^{n-1} + \cdots + \omega^{n-1})$$

Now we make two observations:

(1). $\int_X \partial\psi\wedge\bar\partial\psi\wedge\omega^{n-1} = \|\nabla\psi\|_{L^2}^2$.

This follows, since for any (1,1)-form $T_{\bar k j}\, dz^j\wedge d\bar z^k$

$$\int_X T_{\bar k j}\, dz^j\wedge d\bar z^k\wedge\omega^{n-1} = \int_X g^{j\bar k}\, T_{\bar k j}\,\omega^n$$

Thus,

$$\int_X \partial\psi\wedge\bar\partial\psi\wedge\omega^{n-1} = \int_X g^{j\bar k}\,\partial_{\bar k}\psi\,\partial_j\psi\cdot\omega^n$$
$$= \|\nabla\psi\|_{L^2}^2.$$

In particular,

$$\frac{\alpha+1}{(1+\frac{\alpha}{2})^2}\int_X i(\partial(\varphi|\varphi|^{\frac{\alpha}{2}})\wedge\bar\partial(\varphi|\varphi|^{\frac{\alpha}{2}})\wedge\omega^{n-1} = \frac{(\alpha+1)}{(1+\frac{\alpha}{2})^2}\int_X |\nabla(\varphi|\varphi|^{\alpha})|^2\omega^n$$

(2). The remaining terms $\int i\,\partial(\varphi|\varphi|^{\frac{\alpha}{2}})\wedge\bar\partial(\varphi|\varphi|^{\frac{\alpha}{2}})\,\omega_\varphi^k\wedge\omega^{n-k-1}$ are all non-negative. In fact, the integrand is non-negative pointwise. More precisely, we say a (k,k)-form is positive if w/r.t. some $(1,0)$ frame $\{e_i\}$, it can be written as a positive linear combination of terms of the form:

$$(i e_{i_1}\wedge\bar e_{i_1})\wedge\cdots\cdots\wedge(i e_{i_k}\wedge\bar e_{i_k}).$$

Note that this agrees with the standard definition of positivity for (1,1) forms. Moreover, if ψ, ψ' are positive $\Longrightarrow$ so is $\psi\wedge\psi'$. Hence

$$\underbrace{i(\partial u\wedge\bar\partial\bar u)}_{\geq 0}\wedge\underbrace{\omega_\varphi^k}_{\geq 0}\wedge\underbrace{\omega^{n-k-1}}_{\geq 0} \geq 0$$

In fact, as an (n,n)-form, it's a positive multiple of the volume form.

$$\psi\prod_{i=1}^n(\tfrac{i}{2}dz^i\wedge d\bar z^i), \quad \text{with } \psi\geq 0.$$

Combining these, we obtain the desired bound for $\|\nabla(\varphi\cdot|\varphi|^{\frac{\alpha}{2}})\|_{L^2}$. Applying Moser's iteration as above and we get the C^0 bound. This is the major contribution of Yau.

Proof of the C^2- estimate.

In fact we shall just control something weaker:

$$(\omega+\tfrac{i}{2}\partial\bar{\partial}\varphi)^n = e^f\omega^n \quad (*)$$

Then $\exists\, C>0$, depending only on f, ω and $\operatorname{osc}(\varphi)$ ($\triangleq \sup\varphi - \inf\varphi$, which can be treated now as a known quantity, by the C^0-estimate above), s.t.

$$\|\Delta\varphi\|_{C^0} \le C$$

The C^2-estimate follows from this and a priori estimate (similar as done in Hodge theory).

Idea of proof (Aubin, Yau, Pogorelov)

Apply the maximum principal to the expression $\log(n+\Delta\varphi) - A\varphi$. More precisely, we claim that $\exists\, A, C_1, C_2 > 0$ s.t.

$$(*) \Longrightarrow \Delta'(\log(n+\Delta\varphi) - A\varphi) \ge C_1\cdot\Big(\sum_{j=1}^{n}\frac{1}{\lambda_j}\Big) - C_2 \quad (**)$$

at the maximum point of $\log(n+\Delta\varphi) - A\varphi$. Here λ_i's are the eigenvalues of ω_φ w/r.t. ω, Δ' is the Laplacian of $g'_{\bar{k}j} = g_{\bar{k}j} + \partial_j\partial_{\bar{k}}\varphi$.

Assuming this claim, then since p is a maximum,

$$0 \ge C_1\Big(\textstyle\sum_{j=1}^{n}\frac{1}{\lambda_j}\Big) - C_2$$

($\Delta'(\log(n+\Delta\varphi) - A\varphi)$ is the trace of the Hessian of $\log(n+\Delta\varphi) - A\varphi$, which is a negative semi-definite matrix since p is a maximum $\Rightarrow$ the trace ≤ 0).

$$\Longrightarrow \quad \frac{1}{\lambda_j} \le C_3, \quad \forall\, j \quad (\text{or } \lambda_j \text{ bounded from below})$$

However, by $(*)$, $\prod_{j=1}^{n}\lambda_j = e^{f(p)}$ is bounded, hence λ_j's are bounded from below and above at p. i.e. $\exists\, C_4 \ge C_5$ s.t.

$$C_5 \le \lambda_j \le C_4$$

Now consider a general point. Since p is a maximum,

$$\log(n+\Delta\varphi(z)) - A\varphi(z) \le \log(n+\Delta\varphi(p)) - A\varphi(p)$$

$$\Rightarrow \log(n+\Delta\varphi(z)) \le \log(n+\Delta\varphi(p)) + A(\varphi(z)-\varphi(p))$$
$$= \log(\mathrm{Tr}\,\omega') + A(\varphi(z)-\varphi(p))$$
$$\le C_6 + A\cdot \mathrm{osc}(\varphi)$$
$$\Rightarrow (n+\Delta\varphi(z)) \le e^{C_6 + A\cdot \mathrm{osc}(\varphi)} \le C_7.$$

Proof of (**):

The trick here is to use the endomorphism $h^{\ell}{}_m = g^{\ell\bar{p}} g'_{\bar{p}m}$ to calculate Δh. We begin with:

$$\mathrm{Tr}\,h = h^{\ell}{}_{\ell}$$
$$= g^{\ell\bar{p}}(g_{\bar{p}\ell} + \partial_\ell \partial_{\bar{p}}\varphi)$$
$$= n + \Delta\varphi$$

and compute:

$$\Delta'(\mathrm{Tr}\,h) = (g')^{j\bar{k}} \partial_j \partial_{\bar{k}} (\mathrm{Tr}\,h)$$
$$= (g')^{j\bar{k}} \partial_{\bar{k}} \partial_j (\mathrm{Tr}\,h)$$
$$= (g')^{j\bar{k}} \mathrm{Tr}(\nabla'_{\bar{k}} ((\nabla'_j h) h^{-1} \cdot h))$$
$$= (g')^{j\bar{k}} \mathrm{Tr}\{ (\nabla'_{\bar{k}} (\nabla'_j h) h^{-1}) \cdot h) + (\nabla'_j h)\cdot h^{-1} \cdot \nabla'_{\bar{k}} h\}$$

where ∇' is the covariant derivative with respect to $g'_{\bar{k}j}$. Now recall that $R_{\bar{k}j} = -\partial_{\bar{k}}((g)^{-1}\partial_j g)$: curvature form in matrix notation. Remember that $h = g^{-1}g' \in \Gamma(X, \mathrm{End}(T^{1,0}X))$ (a holomorphic bundle with induced metrics). Then:

$$R_{\bar{k}j} = -\partial_{\bar{k}}(g^{-1}\partial_j g)$$
$$= -\partial_{\bar{k}}(h g'^{-1} \partial_j (g' h^{-1}))$$
$$= -\partial_{\bar{k}}\{ (h g'^{-1} \partial_j g' \cdot h^{-1} + h g'^{-1} g' \partial_j (h^{-1})\}$$
$$= -\partial_{\bar{k}}\{ (h g'^{-1} \partial_j g' \cdot h^{-1} + h \partial_j (h^{-1})\}$$

Recall that covariant differential of an endomorphism T takes the form:

$$\nabla_j T = \partial_j T + A_j T - T A_j$$

where $A_j = g^{-1}\partial_j g$. Thus $\nabla'_j(h^{-1}) = \partial_j(h^{-1}) + g'^{-1}\partial_j g' \cdot h^{-1} - h^{-1}(g'^{-1}\partial_j g') \Rightarrow$ $h\nabla'_j(h^{-1}) = h\partial_j(h^{-1}) + hg'^{-1}\partial_j g' \cdot h^{-1} - (g'^{-1}\partial_j g')$. Plugging in we get:

$$
\begin{aligned}
R_{\bar{k}j} &= -\partial_{\bar{k}}\left(h\nabla_j' h^{-1} + (g'^{-1}\partial_j g')\right) \\
&= -\partial_{\bar{k}}(g'^{-1}\partial_j g') + \partial_{\bar{k}}(\nabla_j' h \cdot h^{-1}) \\
&= R'_{\bar{k}j} + \partial_{\bar{k}}(\nabla_j' h \cdot h^{-1}) \\
&= R'_{\bar{k}j} + \nabla'_{\bar{k}}(\nabla_j' h \cdot h^{-1})
\end{aligned}
$$

Hence we obtain:

$$
\Delta'(\mathrm{Tr}h) = g'^{j\bar{k}}\,\mathrm{Tr}((R_{\bar{k}j} - R'_{\bar{k}j})\cdot h) + g'^{j\bar{k}}\,\mathrm{Tr}((\nabla_j' h \cdot h^{-1})(\nabla'_{\bar{k}} h))
$$

Then:

$$
\begin{aligned}
\Delta' \log(\mathrm{Tr}h) &= g'^{j\bar{k}}\partial_j\partial_{\bar{k}}(\log \mathrm{Tr}h) \\
&= g'^{j\bar{k}}\partial_j \frac{\partial_{\bar{k}}(\mathrm{Tr}h)}{\mathrm{Tr}h} \\
&= \frac{\Delta'(\mathrm{Tr}h)}{\mathrm{Tr}h} - g'^{j\bar{k}}\frac{(\partial_{\bar{k}}\mathrm{Tr}h)(\partial_j \mathrm{Tr}h)}{(\mathrm{Tr}h)^2} \\
&= \frac{\Delta'(\mathrm{Tr}h)}{\mathrm{Tr}h} - \frac{\|\nabla' \mathrm{Tr}h\|^2}{(\mathrm{Tr}h)^2}
\end{aligned}
$$

Plugging in $\Delta'(\mathrm{Tr}h)$:

$$
\Delta' \log(\mathrm{Tr}h) = \frac{g'^{j\bar{k}}\,\mathrm{Tr}((R_{\bar{k}j} - R'_{\bar{k}j})\cdot h)}{\mathrm{Tr}h} + \left\{ \frac{(g'^{j\bar{k}})\,\mathrm{Tr}((\nabla_j' h\cdot h^{-1})\cdot(\nabla'_{\bar{k}} h))}{\mathrm{Tr}h} - \frac{(g'^{j\bar{k}})\partial_j \mathrm{Tr}h\cdot\partial_{\bar{k}}\mathrm{Tr}h}{(\mathrm{Tr}h)^2} \right\}
$$

Basic observations:

1). Yau's inequality: the term in the bracket $\{\cdot\} \geq 0$ (to be proved)

2). The $(g')^{j\bar{k}}\,\mathrm{Tr}(R'_{\bar{k}j}\cdot h)$ can be simplified as follows:

$$
\begin{aligned}
(g')^{j\bar{k}}\,\mathrm{Tr}(R'_{\bar{k}j}\cdot h) &= (g')^{j\bar{k}} R'_{\bar{k}j}{}^{\ell}{}_{m} h^{m}{}_{\ell} \\
&= (Ric')^{\ell}{}_{m} h^{m}{}_{\ell} \\
&= (Ric')^{\ell}{}_{m} g^{m\bar{p}} g'_{\bar{p}\ell} \\
&= (Ric')_{\bar{p}m}\, g^{m\bar{p}} \qquad (\bullet)
\end{aligned}
$$

However, we know that $(Ric)_{\bar{k}j} = -\partial_j\partial_{\bar{k}}\log\det(g_{\bar{q}p})$. Thus

$$
(Ric')_{\bar{p}m} = -\partial_m\partial_{\bar{p}}\log\det(g'_{\bar{q}p})
$$

and by the Monge-Ampère equation

$$
\det(g'_{\bar{q}p}) = \det(g_{\bar{q}p})\cdot e^{tF - c_t}.
$$

we have:

$$(Ric')_{\bar p m} = -\partial_m\partial_{\bar p}\log\det(g_{\bar q p}) - \partial_m\partial_{\bar p}(tF - C_t)$$
$$= \underbrace{(Ric)_{\bar p m} - \partial_m\partial_{\bar p}\, tF}$$

(unlike Ric', this quantity is prescribed)

Sustitute this into (•), we get:

$$(g')^{j\bar k}\, Tr(R'_{\bar k j}\cdot h) = g^{m\bar p}((Ric)_{\bar p m} - \partial_m\partial_{\bar p}(tF))$$
$$= R - \Delta(tF)$$

where R denotes the scalar curvature of $(g_{\bar k j})$.

3). We estimate

$$(g')^{j\bar k}\, Tr(R_{\bar k j}\cdot h) = (g')^{j\bar k}(R_{\bar k j}{}^{\ell}{}_m\, h^m{}_\ell) \qquad (\text{Recall that } h = g^{-1}g' \Rightarrow (g')^{-1} = h^{-1}g^{-1})$$
$$= (h^{-1})^j{}_r\, g^{r\bar k}\, R_{\bar k j}{}^{\ell}{}_m\, h^m{}_\ell$$
$$= (h^{-1})^j{}_r\, R^r{}_j{}^{\ell}{}_m h^m{}_\ell$$

Since $R^r{}_j{}^{\ell}{}_m$ is the curvature tensor of a known metric (a.k.a. bi-sectional curvature), it can be bounded from below by $\min_{i,j}|R^i{}_j{}^i{}_j| \triangleq B$

$$(g')^{j\bar k}\, Tr(R_{\bar k j}\cdot h) \geq -B\, Tr(h)\, Tr(h^{-1})$$

(This can be proved, for instance, by diagonalizing h, then

$$l.h.s. = \lambda_j^{-1} R^j{}_j{}^{\ell}{}_\ell \lambda_\ell \quad)$$

From these observations, we have

$$\Delta' Tr(\log h) \geq -B(Tr h^{-1}) - \frac{1}{Tr h}(R - \Delta(tF))$$
$$\geq -B(Tr h^{-1}) - C_3 (Tr h)^{-1}$$
$$= -C_4\, Tr h^{-1}$$

(the last inequality follows by diagonalizing h (positive endomorphism), and

$$(Tr h)^{-1} = (\lambda_1 + \cdots + \lambda_n)^{-1} \leq \lambda_i^{-1} \leq Tr(h^{-1}) \quad).$$

This is close to the desired inequality, but the constant has wrong sign. That's why we need to substract $A\Delta'\varphi$:

$$\Delta'\varphi = (g')^{j\bar k}\partial_j\partial_{\bar k}\varphi$$
$$= (g')^{j\bar k}((g')_{\bar k j} - g_{\bar k j})$$
$$= n - Tr(h^{-1}) \qquad (h^{-1} = g'^{-1}\cdot g)$$

Altogether, choose $A \gg 0$, we have:

$$\Delta'(\log \mathrm{Tr}h - A\varphi) \geq -C_4 \mathrm{Tr}h^{-1} - A(n - \mathrm{Tr}h^{-1})$$
$$= (A - C_4)\mathrm{Tr}h^{-1} - A\cdot n$$
$$\geq C_1 \mathrm{Tr}h^{-1} - C_2,$$

as claimed in (**).

Proof of Yau's inequality.

This is a tensorial inequality, and thus it suffices to work in a coordinate system so that at z:

$$\begin{cases} g_{\bar{k}j}(z) = \delta_{\bar{k}j}, \\ g'_{\bar{k}j}(z) = \lambda_j \delta_{\bar{k}j} \quad (\lambda_j = 1 + \varphi_{\bar{j}j}), \\ \nabla'_p = \partial_p \ \text{ at } z. \end{cases}$$

Thus we want to show that:

$$(g'^{j\bar{k}})(\partial_j \mathrm{Tr}h)(\partial_{\bar{k}} \mathrm{Tr}h) \leq \mathrm{Tr}h \cdot \{(g')^{p\bar{q}} \mathrm{Tr}(\nabla'_p h \cdot h^{-1} \cdot \nabla'_{\bar{q}} h)\}$$

i.e.

$$\sum_{p,\bar{q}} \frac{\delta^{p\bar{q}}}{(1+\varphi_{\bar{p}p})} (\partial_p(\textstyle\sum_i (1+\varphi_{\bar{i}i}))\, \partial_{\bar{q}}(\sum_j (1+\varphi_{\bar{j}j})) = \sum_p \frac{1}{(1+\varphi_{\bar{p}p})} \sum_{i,j} (\partial_p \varphi_{\bar{i}i})(\partial_{\bar{p}} \varphi_{\bar{j}j})$$

$$\leq \{\textstyle\sum_i (1+\varphi_{\bar{i}i})\} \cdot \{\sum_{p,\bar{q}} \frac{\delta^{p\bar{q}}}{(1+\varphi_{\bar{p}p})} \sum_j (\partial_p (1+\varphi_{\bar{j}j})) \frac{1}{1+\varphi_{\bar{j}j}} \partial_{\bar{q}}(1+\varphi_{\bar{j}j})\}$$

$$= \{\textstyle\sum_i (1+\varphi_{\bar{i}i})\} \cdot \{\sum_p \frac{1}{(1+\varphi_{\bar{p}p})} \cdot \sum_j \frac{\partial_p \varphi_{\bar{j}j} \cdot \partial_{\bar{p}} \varphi_{\bar{j}j}}{(1+\varphi_{\bar{j}j})}\}$$

This is just the Cauchy-Schwartz inequality. Indeed, write:

$$\sum_p \frac{1}{(1+\varphi_{\bar{p}p})} \sum_{i,j} (\partial_p \varphi_{\bar{i}i})(\partial_{\bar{p}} \varphi_{\bar{j}j}) = \sum_{i,j} \{\sum_p (\frac{\partial_p \varphi_{\bar{i}i}}{(1+\varphi_{\bar{p}p})^{\frac{1}{2}}})(\frac{\partial_{\bar{p}} \varphi_{\bar{j}j}}{(1+\varphi_{\bar{p}p})^{\frac{1}{2}}})\}$$

$$\leq \sum_{i,j} (\sum_p \frac{|\partial_p \varphi_{\bar{i}i}|^2}{(1+\varphi_{\bar{p}p})})^{\frac{1}{2}} (\sum_p \frac{|\partial_{\bar{p}} \varphi_{\bar{j}j}|^2}{(1+\varphi_{\bar{p}p})})^{\frac{1}{2}} \quad \text{(Cauchy-Schwartz in } p, \text{ for fixed } i,j)$$

$$= (\sum_i (\sum_p \frac{|\partial_p \varphi_{\bar{i}i}|^2}{(1+\varphi_{\bar{p}p})})^{\frac{1}{2}}) \cdot (\sum_j (\sum_p \frac{|\partial_p \varphi_{\bar{j}j}|^2}{(1+\varphi_{\bar{p}p})})^{\frac{1}{2}})$$

$$= (\sum_i (\sum_p \frac{|\partial_p \varphi_{\bar{i}i}|^2}{(1+\varphi_{\bar{p}p})})^{\frac{1}{2}})^2$$

$$= \left(\sum_i (1+\varphi_{\bar{i}i})^{\frac{1}{2}} \cdot (\sum_p \frac{|\partial_p \varphi_{\bar{i}i}|^2}{(1+\varphi_{\bar{i}i})(1+\varphi_{\bar{p}p})})^{\frac{1}{2}}\right)^2$$

$$\leq \{\sum_i (1+\varphi_{\bar{i}i})\} \cdot \{\sum_{i,p} \frac{|\partial_p \varphi_{\bar{i}i}|^2}{(1+\varphi_{\bar{i}i})(1+\varphi_{\bar{p}p})}\} \quad \text{(Cauchy-Schwartz in } i)$$

Proof of the C^3-estimate

We shall prove the bound: $S \leq Const$, where $S = |\nabla' \overline{\nabla'} \nabla' \varphi|^2$. More precisely, $S = (g')^{j\bar m} (g')^{p\bar k} \varphi_{j\bar k \ell} \overline{\varphi_{m\bar p q}} (g')^{\ell \bar q}$, with $\varphi_{j\bar k \ell} = \nabla'_\ell \varphi_{\bar k j}$ w/r.t. the metric $g'_{\bar k j} = g_{\bar k j} + \partial_j \partial_{\bar k} \varphi$. The proof is due to Cartan, Nirenberg and Yau.

Claim: $\exists$ constants $A_1, C_5, C_6 > 0$ s.t.

$$\Delta'(S - A_1 \Delta\varphi) \geq C_5 S - C_6 \qquad (***)$$

Assuming this claim, we may apply the maximum principal again. At a point p, where $S - A_1\Delta\varphi$ attains its maximum, similar as before, we have

$$\Delta'(S - A_1\Delta\varphi)(p) \leq 0$$

$$\Rightarrow \qquad S(p) \leq C_7$$

Then for a general $z \in M$,

$$S(z) - A_1 \Delta\varphi(z) \leq S(p) - A_1 \Delta\varphi(p)$$

$$\begin{aligned} \Rightarrow \qquad S(z) &\leq S(p) + A_1(\Delta\varphi(z) - \Delta\varphi(p)) \\ &\leq C_7 + A_1 \cdot 2 \cdot \|\Delta\varphi\|_{C^0} \\ &\leq C_8 \end{aligned}$$

Proof of (***).

For this, observe that: $S = |\nabla' h \cdot h^{-1}|^2_{g'}$, where $h^j{}_k = g^{j\bar p} g'_{\bar p k}$. Indeed

$$\begin{aligned} |\nabla' h \cdot h^{-1}|^2_{g'} &= \nabla'_j h^a{}_b (h^{-1})^b{}_c \, \overline{\nabla'_k h^d{}_e (h^{-1})^e{}_f} \, (g')^{j\bar k} (g')_{\bar d a} (g')^{c\bar f} \\ &= (\nabla'_j g^{a\bar u}) g'_{\bar u b} \, g'^{b\bar v} g_{\bar v c} \, \overline{(\nabla'_k g^{d\bar s}) g'_{\bar s e} \cdot g'^{e\bar r} g_{\bar r f}} \cdot (g')^{j\bar k} (g')_{\bar d a} (g')^{c\bar f} \\ &= (\nabla'_j g^{a\bar u} \cdot g_{\bar u c}) \cdot \overline{(\nabla'_k g^{d\bar r}) g_{\bar r f}} \cdot (g')^{j\bar k} (g')_{\bar d a} (g')^{c\bar f} \\ &= (-g^{a\bar m} \nabla'_j (g_{\bar m n}) g^{n\bar u} g_{\bar u c}) \cdot \overline{(-g^{d\bar s} \nabla'_k (g_{\bar s t}) g^{t\bar r} \cdot g_{\bar r f})} \cdot (g')^{j\bar k} \cdot (g')_{\bar d a} \cdot (g')^{c\bar f} \\ &= (g^{a\bar m} \nabla'_j (\partial_{\bar m} \partial_c \varphi) \cdot \overline{g^{d\bar s} \nabla'_k (\partial_{\bar s} \partial_f \varphi)} \cdot (g')^{j\bar k} \cdot (g')_{\bar d a} \cdot (g')^{c\bar f} \\ &= \end{aligned}$$

Now we compute $\Delta' S$:

$$\Delta' S = \Delta' |\nabla' h \cdot h^{-1}|^2_{g'}$$
$$= \Delta' \langle \nabla' h \cdot h^{-1}, \nabla' h \cdot h^{-1} \rangle$$

where $\Delta' = (g')^{j\bar{k}} \nabla_j \nabla_{\bar{k}}$. Then:

$$\Delta' S = (g')^{j\bar{k}} (\nabla_j \langle \nabla_{\bar{k}} \nabla' h \cdot h^{-1}, \nabla' h \cdot h^{-1} \rangle + \nabla_j \langle \nabla' h \cdot h^{-1}, \nabla_k \nabla' h \cdot h^{-1} \rangle$$
$$= (g')^{j\bar{k}} (\langle \nabla_j \nabla_{\bar{k}} \nabla' h \cdot h^{-1}, \nabla' h \cdot h^{-1} \rangle + \langle \nabla_{\bar{k}} \nabla' h \cdot h^{-1}, \nabla_{\bar{j}} \nabla' h \cdot h^{-1} \rangle$$
$$+ \langle \nabla_j \nabla' h \cdot h^{-1}, \nabla_k \nabla' h \cdot h^{-1} \rangle + \langle \nabla' h \cdot h^{-1}, \nabla_{\bar{j}} \nabla_k \nabla' h \cdot h^{-1} \rangle)$$
$$= \langle \Delta' \nabla' h \cdot h^{-1}, \nabla' h \cdot h^{-1} \rangle + \langle \nabla' h \cdot h^{-1}, \bar{\Delta}' (\nabla' h \cdot h^{-1}) \rangle$$
$$+ |\bar{\nabla}' (\nabla' h \cdot h^{-1})|^2 + |\nabla' (\nabla' h \cdot h^{-1})|^2$$

Next, note that

$$\Delta' (\nabla_j' h \cdot h^{-1}) = g'^{\ell\bar{k}} \nabla_\ell' \nabla_{\bar{k}}' (\nabla_j' h \cdot h^{-1})$$
$$= g'^{\ell\bar{k}} \nabla_\ell' (R_{\bar{k}j} - R'_{\bar{k}j}) \quad \text{(by an earlier computation in } C^2\text{-estimate)}$$
$$= g'^{\ell\bar{k}} (\nabla_\ell' R_{\bar{k}j} - \nabla_j' R'_{\bar{k}\ell}) \quad \text{(by the } 2^{nd}\text{- Bianchi's identity: } \nabla_j R_{\bar{k}\ell} = \nabla_\ell R_{\bar{k}j})$$
$$= g'^{\ell\bar{k}} \nabla_\ell' R_{\bar{k}j} - \nabla_j' (Ric')$$

Up to now, we haven't used the condition that $(g'_{\bar{k}j})$ solves the MA equation, which says that Ric' is known:

$$Ric'_{\bar{a}b} = -\partial_b \partial_{\bar{a}} \log\det(g'_{\bar{p}q})$$
$$= Ric_{\bar{a}b} - \partial_b \partial_{\bar{a}} F \cdot t$$

Thus $\nabla_j'(Ric')$ is bounded and so is $g'^{\ell\bar{k}} \nabla_\ell' R_{\bar{k}j}$. It follows that:

$$\| \langle \Delta' \nabla' h \cdot h^{-1}, \nabla' h \cdot h^{-1} \rangle \|_{g'} \le C_9 S + C_{10}$$

(There is some subtle details here: the norm here is taken w/r.t. $g'_{\bar{k}j}$, which is a metric to be solved. However, by the C^2-estimate, $\{g'_{\bar{k}j}\}$ and $\{g_{\bar{k}j}\}$ are uniformly equivalent:

$$\left.\begin{array}{l} Tr\, h = g^{j\bar{k}} g'_{\bar{k}j} = n + \Delta\varphi \\ \Delta\varphi \text{ bounded} \end{array}\right\} \Rightarrow \text{the sum of eigenvalues of } h \text{ is bounded;}$$

Moreover, $\prod \lambda_i$ is fixed $\Rightarrow$ Each λ_i is bounded. Thus h is a uniformly bounded endomorphism $\Rightarrow g' = gh$ is uniformly bounded.)

Hence:

$$\Delta' S \geq \langle \Delta' \nabla' h \cdot h^{-1}, \nabla' h \cdot h^{-1} \rangle + \langle \nabla' h \cdot h^{-1} \ \bar{\Delta}'(\nabla' h \cdot h^{-1}) \rangle$$
$$\geq -C_{11} S - C_{12}.$$

where we leave out the positive terms $|\bar{\nabla}'(\nabla' h \cdot h^{-1})|^2 + |\nabla'(\nabla' h \cdot h^{-1})|^2$, and $\bar{\Delta}'(\nabla' h \cdot h^{-1}) = \Delta' +$ Curvature terms of $(g_{i\bar{j}})$. The curvature terms are bounded again by the C^2-estimate.

Next, we compute $\Delta' \Delta \varphi$: (c.f. C^2-estimate)

$$\begin{aligned}
\Delta'(\Delta \varphi) &= \Delta'(\mathrm{Tr}\, h) \\
&= g'^{p\bar{q}} (\mathrm{Tr}(\nabla_{\bar{q}} (\nabla_p h \cdot h^{-1} \cdot h))) \\
&= g'^{p\bar{q}} (\mathrm{Tr}(R_{\bar{q}p} \cdot h - R'_{\bar{q}p} \cdot h) + \mathrm{Tr}(\nabla_p h \cdot h^{-1} \cdot \nabla_{\bar{q}} h)) \\
&= \underbrace{g'^{p\bar{q}} (\mathrm{Tr}(R_{\bar{q}p} \cdot h)) - (R - \Delta F_t)}_{\text{known}} + g'^{p\bar{q}} \mathrm{Tr}(\underbrace{\nabla_p h \cdot h^{-1} \cdot \nabla_{\bar{q}} h}_{\text{of order } S}).
\end{aligned}$$

By our C^2-estimate, $\{g_{p\bar{q}}\}$ and $\{g'_{p\bar{q}}\}$ are equivalent metrics, and the last term is of order S, we have:

$$\Delta'(\Delta \varphi) \geq -(C_{13} S + C_{14})$$

Combining these two estimates as we did for C^2-estimate, we obtain the claimed inequality (***).

A final remark. Why these estimates imply the desired convergence.

Differentiating the MA equation w/r.t. ∇_ℓ gives

$$(g')^{j\bar{k}} \nabla_\ell \nabla_j \nabla_{\bar{k}} \varphi = \text{known quantity}$$

$\Rightarrow \nabla_\ell \varphi$ satisfies the Laplacian equation with C^0 r.h.s. By the classical Schauder elliptic regularity, which is part of classical linear PDE theory, one gets that $\nabla_\ell \varphi$ is of class $C^{1,\nu}$, $\forall \nu \in (0,1)$. This in turn implies that φ is of class $C^{2,\nu}$. ⋯ (Black box). This gives the desired convergence and we may apply the Ascoli-Arezela thm.