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Leonhard Euler: His Life, the Man, and His Works*

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Abstract. On the occasion of the 300th anniversary (on April 15, 2007) of Euler's birth, an attempt is made to bring Euler's genius to the attention of a broad segment of the educated public. The three stations of his life—Basel, St. Petersburg, and Berlin—are sketched and the principal works identified in more or less chronological order. To convey a flavor of his work and its impact on modern science, a few of Euler's memorable contributions are selected and discussed in more detail. Remarks on Euler's personality, intellect, and craftsmanship round out the presentation.

Key words. Leonhard Euler, sketch of Euler's life, works, and personality

AMS subject classification. 01A50

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Seh ich die Werke der Meister an,
So sehe ich, was sie getan;
Betracht ich meine Siebensachen,
Seh ich, was ich hätt sollen machen.
–GOETHE, Weimar 1814/1815

1. Introduction.

It is a virtually impossible task to do justice, in a short span of time and space, to the great genius of Leonhard Euler. All we can do, in this lecture, is to bring across some glimpses of Euler's incredibly voluminous and diverse work, which today fills 74 massive volumes of the *Opera omnia* (with two more to come). Nine additional volumes of correspondence are planned and have already appeared in part, and about seven volumes of notebooks and diaries still await editing!

We begin in section 2 with a brief outline of Euler's life, going through the three stations of his life: Basel, St. Petersburg (twice), and Berlin. In section 3, we identify in more or less chronological order Euler's principal works and try to convey a flavor and some characteristic features of his work by describing in more detail a few of his many outstanding contributions. We conclude in section 4 with remarks on Euler's personality and intellect, as gained from testimonials of his contemporaries, and on the quality of his craft, and in section 5 with some bibliographic information for further reading.

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2. His Life.

2.1. Basel 1707–1727: Auspicious Beginnings. Leonhard Euler was born on April 15, 1707, the first child of Paulus Euler and Margaretha Brucker. Paulus Euler came from modest folk, mostly artisans, while Margaretha Brucker's ancestors include a number of well-known scholars in the classics. Euler's father at the time was a vicar at the church of St. Jakob, just outside the old city walls of Basel. Although a theologian, Paulus had interests in mathematics and took courses from the famous Jakob Bernoulli during the first two years of his study at the university. About a year and a half after Leonhard's birth, the family moved to Riehen, a suburb of Basel, where Paulus Euler assumed the position of Protestant minister at the local parish. He served in that capacity faithfully and devotedly for the rest of his life.

Fig. 1 *The parish residence and church in Riehen.*

The parish residence, as it looks today (Figure 1), seems comfortable enough, but at the time it had one floor less and only two rooms with heating. The living quarters it provided, therefore, were rather cramped, especially after the family increased by another child, Anna Maria, in 1708. Two more children, Maria Magdalena and Johann Heinrich, were to follow later on.

Leonhard received his first schooling in mathematics at home from his father. Around the age of eight he was sent to the Latin school in Basel and given room and board at his maternal grandmother's house. To compensate for the poor quality then prevailing at the school, Paulus Euler hired a private tutor for his son, a young theologian by the name of Johannes Burckhardt, himself an enthusiastic lover of mathematics. In October of 1720, at the age of thirteen (not unusual at the time), Leonhard enrolled at the University of Basel, first at the philosophical faculty, where he took the freshman courses on elementary mathematics given by Johann Bernoulli, the younger brother of the now deceased Jakob. The young Euler pursued his mathematical studies with such a zeal that he soon caught the attention of Bernoulli, who encouraged him to study more advanced books on his own and even offered him assistance at his house every Saturday afternoon. In 1723, Euler graduated with a master's degree and a public lecture (in Latin) comparing Descartes's system of natural philosophy with that of Newton.

Following the wishes of his parents, he then entered the theological faculty, devoting, however, most of his time to mathematics. Euler's father eventually had to concede, probably at the urging of Johann Bernoulli, that Leonhard was predestined

Fig. 2 *The old university of Basel and Johann* I *Bernoulli.*

to a career in mathematics rather than one in theology. This is how Euler himself recounts this early learning experience at the university in his brief autobiography of 1767 (here freely translated from German; see Fellmann [10, Engl. transl., pp. 1–7]):

> In 1720 I was admitted to the university as a public student, where I soon found the opportunity to become acquainted with the famous professor Johann Bernoulli, who made it a special pleasure for himself to help me along in the mathematical sciences. Private lessons, however, he categorically ruled out because of his busy schedule. However, he gave me a far more beneficial advice, which consisted in myself getting a hold of some of the more difficult mathematical books and working through them with great diligence, and should I encounter some objections or difficulties, he offered me free access to him every Saturday afternoon, and he was gracious enough to comment on the collected difficulties, which was done with such a desired advantage that, when he resolved one of my objections, ten others at once disappeared, which certainly is the best method of making happy progress in the mathematical sciences.

These personal meetings have become known, and famous, as the *privatissima*, and they continued well beyond his graduation. It was during these *privatissima* that Johann Bernoulli more and more began to admire the extraordinary mathematical talents of the young Euler.

Barely nineteen years old, Euler dared to compete with the greatest scientific minds of the time by responding to a prize question of the Paris Academy of Sciences with a memoir on the optimal placing of masts on a ship. He, who at that point in his life had never so much as seen a ship, did not win first prize, but still a respectable second. A year later, when the physics chair at the University of Basel became vacant, the young Euler, dauntlessly again, though with the full support of his mentor, Johann Bernoulli, competed for the position, but failed, undoubtedly because of his youth and lack of an extensive record of publications. In a sense, this was a blessing in disguise, because in this way he was free to accept a call to the Academy of Sciences in St. Petersburg, founded a few years earlier by the czar Peter I (the Great), where he was to find a much more promising arena in which to fully develop himself. The groundwork for this appointment had been laid by Johann Bernoulli and two of his sons, Niklaus II and Daniel I, both of whom were already active at the Academy.

2.2. St. Petersburg 1727–1741: Meteoric Rise to World Fame and Academic Advancement. Euler spent the winter of 1726 in Basel studying anatomy and physiology in preparation for his anticipated duties at the Academy. When he arrived in St. Petersburg and started his position as an adjunct of the Academy, it was soon determined, however, that he should devote himself entirely to the mathematical sciences. In addition, he was to participate in examinations for the cadet corps and act as a consultant to the Russian state in a variety of scientific and technological questions.

Fig. 3 *The Academy in St. Petersburg and Peter* I. *(Photograph of the Academy of Sciences courtesy of Andreas Verdun.)*

Euler adjusted easily and quickly to the new and sometimes harsh life in the northern part of Europe. Contrary to most other foreign members of the Academy he began immediately to study the Russian language and mastered it quickly, both in writing and speaking. For a while he shared a dwelling with Daniel Bernoulli, and he was also on friendly terms with Christian Goldbach, the permanent Secretary of the Academy and best known today for his—still open—conjecture in number theory. The extensive correspondence between Euler and Goldbach that ensued has become an important source for the history of science in the 18th century.

Euler's years at the Academy of St. Petersburg proved to be a period of extraordinary productivity and creativity. Many spectacular results achieved during this time (more on this later) brought him instant world fame and increased status and esteem within the Academy. A portrait of Euler stemming from this period is shown in Figure 4.

In January of 1734 Euler married Katharina Gsell, the daughter of a Swiss painter teaching at the Academy, and they moved into a house of their own. The marriage brought forth thirteen children, of whom, however, only five reached the age of adulthood. The first-born child, Johann Albrecht, was to become a mathematician himself and later in life was to serve Euler as one of his assistants.

Euler was not spared misfortunes. In 1735, he fell seriously ill and almost lost his life. To the great relief of all, he recovered, but suffered a repeat attack three years later of (probably) the same infectious disease. This time it cost him his right eye, which is clearly visible on all portraits of Euler from this time on (for example, the famous one in Figure 6, now hanging in the Basel Museum of Arts).

The political turmoil in Russia that followed the death of the czarina Anna Ivanovna induced Euler to seriously consider, and eventually decide, to leave St. Pe-

Fig. 4 *Euler, ca.* 1737.

tersburg. This all the more as he already had an invitation from the Prussian king Frederick II to come to Berlin and help establish an Academy of Sciences there. This is how Euler put it in his autobiography:

> ... in 1740, when His still gloriously reigning Royal Majesty [Frederick II] came to power in Prussia, I received a most gracious call to Berlin, which, after the illustrious Empress Anne had died and it began to look rather dismal in the regency that followed, I accepted without much hesitation ....

In June of 1741, Euler, together with his wife Katharina, the six-year-old Johann Albrecht, and the one-year-old toddler Karl, set out on the journey from St. Petersburg to Berlin.

2.3. Berlin 1741–1766: The Emergence of Epochal Treatises. Because of his preoccupation with the war campaign in Silesia, Frederick II took his time to set up the Academy. It was not until 1746 that the Academy finally took shape, with Pierre-Louis Moreau de Maupertuis its president and Euler the director of the Mathematics Class. In the interim, Euler did not remain idle; he completed some twenty memoirs, five major treatises (and another five during the remaining twenty years in Berlin), and composed over 200 letters!

Even though Euler was entrusted with manifold duties at the Academy—he had to oversee the Academy's observatory and botanical gardens, deal with personnel matters, attend to financial affairs, notably the sale of almanacs, which constituted the major source of income for the Academy, not to speak of a variety of technological and engineering projects—his mathematical productivity did not slow down. Nor was he overly distracted by an ugly priority dispute that erupted in the early 1750s over Euler's principle of least action, which was also claimed by Maupertuis and which the Swiss fellow mathematician and newly elected academician Johann Samuel König asserted to have already been formulated by Leibniz in a letter to the mathematician Jakob Hermann. König even came close to accusing Maupertuis of plagiarism. When challenged to produce the letter, he was unable to do so, and Euler was asked to investigate. Not sympathetic to Leibniz's philosophy, Euler sided with Maupertuis and in turn accused König of fraud. This all came to a boil when Voltaire, aligned with König, came forth with a scathing satire ridiculing Maupertuis and not sparing

Fig. 5 *The Berlin Academy and Frederick* II. *(Left panel reprinted with permission from the Archiv der Berlin-Brandenburgischen Akademie der Wissenschaften.)*

Fig. 6 *Euler,* 1753.

Euler either. So distraught was Maupertuis that he left Berlin soon thereafter, and Euler had to conduct the affairs of the Academy as *de facto*, if not *de jure*, president of the Academy.

By now, Euler was sufficiently well-off that he could purchase a country estate in Charlottenburg, in the western outskirts of Berlin, which was large enough to provide a comfortable home for his widowed mother (whom he had come to Berlin in 1750), his sister-in-law, and all the children. At just twenty years old, his first-born son, Johann Albrecht, was elected in 1754 to the Berlin Academy on the recommendation of Maupertuis. With a memoir on the perturbation of cometary orbits by planetary attraction he won in 1762 a prize of the Petersburg Academy, but had to share it with Alexis-Claude Clairaut. Euler's second son, Karl, went to study medicine in Halle, whereas the third, Christoph, became an officer in the military. His daughter Charlotte married into Dutch nobility, and her older sister Helene married a Russian officer later in 1777.

Euler's relation with Frederick II was not an easy one. In part, this was due to the marked difference in personality and philosophical inclination: Frederick—

proud, self-assured, worldly, a smooth and witty conversationalist, sympathetic to French enlightenment; Euler—modest, inconspicuous, down-to-earth, and a devout protestant. Another, perhaps more important, reason was Euler's resentment for never having been offered the presidency of the Berlin Academy. This resentment was only reinforced after Maupertuis's departure and Euler's subsequent efforts to keep the Academy afloat, when Frederick tried to interest Jean le Rond d'Alembert in the presidency. The latter indeed came to Berlin, but only to inform the king of his disinterest and to recommend Euler for the position instead. Still, Frederick not only ignored d'Alembert's advice, but ostentatiously declared himself the head of the Academy! This, together with many other royal rebuffs, finally led Euler to leave Berlin in 1766, in defiance of several obstacles put in his way by the king. He indeed already had a most cordial invitation from Empress Catherine II (the Great) to return to the Academy of St. Petersburg, which he accepted, and was given an absolutely triumphant welcome back.

Fig. 7 *The Euler house and Catherine* II. *(Left panel reprinted with permission from the Archiv der Berlin-Brandenburgischen Akademie der Wissenschaften.)*

2.4. St. Petersburg 1766–1783: The Glorious Final Stretch. Highly respected at the Academy and adored at Catherine's court, Euler now held a position of great prestige and influence that had been denied him in Berlin for so long. He in fact was the spiritual if not the appointed leader of the Academy. Unfortunately, however, there were setbacks on a personal level. A cataract in his left (good) eye, which already began to bother him in Berlin, now became increasingly worse, so that in 1771 Euler decided to undergo an operation. The operation, though successful, led to the formation of an abscess, which soon destroyed Euler's vision almost entirely. Later in the same year, his wooden house burned down during the great fire of St. Petersburg, and the almost blind Euler escaped from being burnt alive only by a heroic rescue by Peter Grimm, a workman from Basel. To ease the misfortune, the Empress granted funds to build a new house (the one shown in Figure 7 with the top floor having been added later). Another heavy blow hit Euler in 1773 when his wife Katharina Gsell died. Euler remarried three years later so as not to be dependent on his children.

In spite of all these fateful events, Euler remained mathematically as active as ever, if not more so. Indeed, about half of his scientific output was published, or originated, during this second St. Petersburg period, among which his two "best-sellers," *Letters to a German Princess* and *Algebra*. Naturally, he could not have done it without good secretarial and technical help, which he received from, among

Fig. 8 *Euler,* 1778.

others, Niklaus Fuss, a compatriot from Basel and future grandson-in-law of Euler, and Euler's own son, Johann Albrecht. The latter, by now secretary of the Academy, also acted as the protocolist of the Academy sessions, over which Euler, as the oldest member of the Academy, had to preside.

The high esteem in which Euler was held at the Academy and at court is touchingly revealed by a passage in the memoirs of the Countess Dashkova, a directress of the Academy appointed by the empress. She recounts the first time she accompanied the old Euler to one of the sessions of the Academy, probably Euler's last. Before the session started, a prominent professor and State Councilor as a matter of course claimed the chair of honor, next to the director's chair. The countess then turned to Euler and said: "Please be seated wherever you want; the seat you select will of course become the first of all."

Leonhard Euler died from a stroke on September 18, 1783 while playing with one of his grandchildren. Formulae that he had written down on two of his large slates describing the mathematics underlying the spectacular balloon flight undertaken on June 5, 1783, by the brothers Montgolfier in Paris were found on the day of his death. Worked out and prepared for publication by his son, Johann Albrecht, they became the last article of Euler; it appeared in the 1784 volume of the *Mémoires.* A stream of memoirs, however, all queued up at the presses of the Academy, were still to be published for nearly fifty years after Euler's death.

3. His Works. In the face of the enormous volume of Euler's writings, we content ourselves with briefly identifying his principal works, and then select, and describe in more detail, a few of Euler's prominent results in order to convey a flavor of his work and some of its characteristic features. The papers will be cited by their Eneström-Index numbers (E-numbers).

3.1. The Period in Basel. During the relatively short time of Euler's creative activity in Basel, he published two papers (E1, E3) in the *Acta Eruditorum* (Leipzig), one on isochronous curves, the other on so-called reciprocal curves, both influenced by Johann Bernoulli, and the work on the Paris Academy prize question (E4). The major work of this period is probably his *Dissertatio physica de sono* (E2), which he submitted in support of his application to the physics chair at the University of Basel and had printed in 1727 in Basel. In it, Euler discusses the nature and propagation of


Q. F. F. Q. S.
DISSERTATIO PHYSICA
DE SONO,
QUAM
ANNUENTE NUMINE DIVINO
JUSSU MAGNIFICI ET SAPIENTISSIMI PHILOSOPHORUM ORDINIS
PRO
VACANTE PROFESSIONE PHYSICA
Ad d. 18. Febr. A. MDCCXXVII.
In Auditorio Juridico hora 9.
Publico Eruditorum Examini subjicit
LEONHARDUS EULERUS
A. L. M.
Respondente
Præstantissimo Adolescente
ERNESTO LUDOVICO BURCARDO
Phil. Cand.
BASILEÆ,
Typis E. & J. R. THURNISIORUM, Fratrum.

Fig. 9 *Physical Dissertation on Sound,* 1727. *(Reprinted with permission from Birkhäuser Verlag.)*

sound, in particular the speed of sound, and also the generation of sound by musical instruments. Some of this work is preliminary and has been revisited by Euler in his *Tentamen* (cf. section 3.2.1) and, thirty years later, in several memoirs (E305–E307).

3.2. First St. Petersburg Period. In spite of the serious setbacks in health, Euler's creative output during this period is nothing short of astonishing. Major works on mechanics, music theory, and naval architecture are interspersed with some 70 memoirs on a great variety of topics that run from analysis and number theory to concrete problems in physics, mechanics, and astronomy. An account of the mathematical work during this period is given in Sandifer [22].


MECHANICA
SIVE
MOTVS
SCIENTIA
ANALYTICE
EXPOSITA
AVCTORE
LEONHARDO EVLERO
ACADEMIAE IMPER. SCIENTIARVM MEMBRO ET MATHESEOS SVBLIMIORIS PROFESSORE.
TOMVS I.
INSTAR SVPPLEMENTI AD COMMENTAR. ACAD. SCIENT. IMPER.
PETROPOLI
EX TYPOGRAPHIA ACADEMIAE SCIENTIARVM.
A. 1736.

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SIVE
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PETROPOLI
EX TYPOGRAPHIA ACADEMIAE SCIENTIARVM.
A. 1736.

Fig. 10 *Mechanics,* 1736. *(Reprinted with permission from Birkhäuser Verlag.)*

3.2.1. Major Works. The two-volume *Mechanica* (E15, E16) is the beginning of a far-reaching program, outlined by Euler in Vol. I, sect. 98, of composing a comprehensive treatment of all aspects of mechanics, including the mechanics of rigid, flexible, and elastic bodies, as well as fluid mechanics and celestial mechanics. The present work is restricted almost entirely to the dynamics of a point mass, to its free motion in Vol. I and constrained motion in Vol. II. In either case the motion may take place either in a vacuum or in a resisting medium. The novelty of the *Mechanica* consists in the systematic use of (the then new) differential and integral calculus, including differential equations, and in this sense it represents the first treatise on what is now called analytic (or rational) mechanics. It had won the praise of many leading scientists of the time, Johann Bernoulli among them, who said of the work that "it does honor to Euler's genius and acumen." Also Lagrange, who in 1788 wrote his own *Mécanique analytique*, acknowledges Euler's mechanics to be "the first great work where Analysis has been applied to the science of motion." Implementation and systematic treatment of the rest of Euler's program, never entirely completed, occupied him throughout much of his life.

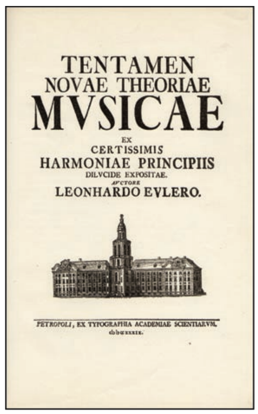


Fig. 11 *Tentamen,* 1739 (1731). *(Reprinted with permission from Birkhäuser Verlag.)*

It is evident from Euler's notebooks that he thought a great deal about music and musical composition while still in Basel and had plans to write a book on the subject. These plans matured only later in St. Petersburg and gave rise to the *Tentamen novae theoriae musicae* (E33), usually referred to as the *Tentamen*, published in 1739 but completed already in 1731. (An English translation was made available by Smith [27, pp. 21–347].) The work opens with a discussion of the nature of sound as vibrations of air particles, including the propagation of sound, the physiology of auditory perception, and the generation of sound by string and wind instruments. The core of the work, however, deals with a theory of pleasure that music can evoke, which Euler develops by assigning to a tone interval, a chord, or a succession of such, a numerical value—the "degree"—which is to measure the agreeableness, or pleasure, of the respective musical construct: the lower the degree, the more pleasure. This is done in the context of Euler's favorite diatonic-chromatic temperament, but a complete mathematical theory of temperaments, both antique and contemporary ones, is also given.

In trying to make music an exact science, Euler was not alone: Descartes and Mersenne did the same before him, as did d'Alembert and many others after him (cf. Bailhache [2] and Assayag, Feichtinger, and Rodrigues [1]). In 1766 and 1774, Euler returns to music in three memoirs (E314, E315, and E457).

Euler's two-volume *Scientia navalis* (E110, E111) is a second milestone in his development of rational mechanics. In it, he sets forth the principles of hydrostatics and develops a theory of equilibrium and oscillations about the equilibrium of three-

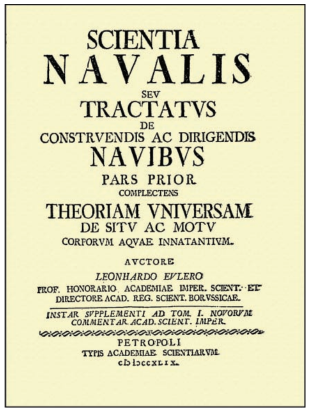


Fig. 12 *Naval Science,* 1749 (1740–1741).

dimensional bodies submerged in water. This already contains the beginnings of the mechanics of rigid bodies, which much later is to culminate in his *Theoria motus corporum solidorum seu rigidorum*, the third major treatise on mechanics (cf. section 3.3.1). The second volume applies the theory to ships, shipbuilding, and navigation.

3.2.2. Selecta Euleriana.

Selectio 1. The Basel Problem. This is the name that has become attached to the problem of determining the sum of the reciprocal squares,

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots\ . \tag{3.1}$$

In modern terminology, this is called the zeta function of 2, where more generally

$$\zeta(s)=1+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\cdots\ . \tag{3.2}$$

The problem had stumped the leading mathematicians of the time—Leibniz, Stirling, de Moivre, and all the Bernoullis—until Euler came along. Typically for Euler, he started, using his tremendous dexterity of calculation and his adroitness in speeding up slowly converging series, to calculate $\zeta(2)$ in E20 to seven decimal places (cf. Gautschi [13, sect. 2]). (Stirling, already in 1730, actually calculated the series to nine decimal places, but Euler did not yet know this.) The breakthrough came in 1735 (published as E41 in 1740) when he showed by a brilliant but daring procedure (using Newton's identities for polynomials of infinite degree!) that

$$\zeta(2)=\frac{\pi^2}{6}\,.$$

Spectacular as this achievement was, Euler went on to use the same method, with considerably more labor, to determine $\zeta(s)$ for all even $s=2n$ up to 12. He found $\zeta(2n)$ to be always a rational number multiplied by the $2n$th power of π. It was in connection with the Basel problem that Euler in 1732 discovered a general summation procedure, found independently by Maclaurin in 1738, and promptly used it to calculate $\zeta(2)$ to twenty decimal places (cf. Gautschi [13, sect. 5.1]). Eventually, Euler

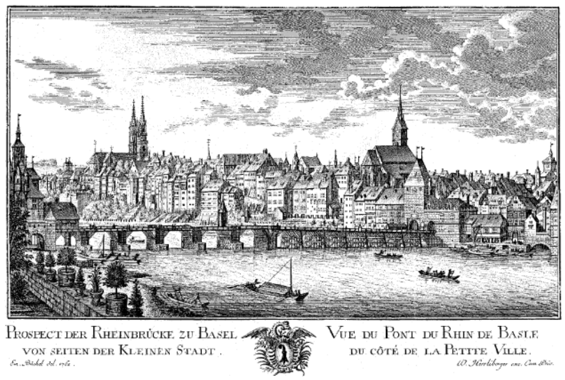


Fig. 13 *Basel, mid* 18*th century. (Reprinted with permission from the University Library of Berne, Central Library, Ryhiner Collection.)*

managed to place his approach on a more solid footing, using his own partial fraction expansion of the cotangent function, and he succeeded, in E130 (see also E212, Part II, Chap. 5, p. 324), to prove the general formula

$$\zeta(2n) = \frac{2^{2n-1}}{(2n)!}\,|B_{2n}|\pi^{2n}. \tag{3.3}$$

Here, B_{2n} are the Bernoulli numbers (introduced by Jakob Bernoulli in his *Ars conjectandi*), which Euler already encountered in his general summation formula.

Euler also tried odd values of s, but wrote in a letter to Johann Bernoulli that "the odd powers I cannot sum, and I don't believe that their sums depend on the quadrature of the circle [that is, on π]" (Fellmann [9, p. 84, footnote 56]). The problem in this case, as a matter of fact, is still open today. The Zürich historian Eduard Fueter once wrote that "where mathematical reason could not go any further, this for Euler was where the kingdom of God began." Could it be that here was an instance where Euler felt a brush with the kingdom of God?

Selectio 2. Prime Numbers and the Zeta Function. Let

$$\mathcal{P} = \{2, 3, 5, 7, 11, 13, 17, \ldots\}$$

be the set of all prime numbers, i.e., the integers > 1 that are divisible only by 1 and themselves. Euler's fascination with prime numbers started quite early and continued throughout his life, even though the rest of the mathematical world at the time (Lagrange excluded!) was rather indifferent to problems of this kind. An example of his profound insight into the theory of numbers is the discovery in 1737 (E72) of the fabulous product formula

$$\prod_{p\in\mathcal{P}} \frac{1}{1 - 1/p^s} = \zeta(s), \quad s > 1, \tag{3.4}$$

connecting prime numbers with the zeta function (3.2). How did he do it? Simply by starting with the zeta function and peeling away, layer after layer, all the terms whose integers in the denominators are divisible by a prime! Thus, from

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots,$$

dividing by 2^s and subtracting, one gets

$$\left(1 - \frac{1}{2^s}\right)\zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \cdots .$$

All the denominator integers divisible by 2 are gone. Doing the same with the next prime, 3, i.e., dividing the last equation by 3^s and subtracting, one gets

$$\left(1 - \frac{1}{2^s}\right)\left(1 - \frac{1}{3^s}\right)\zeta(s) = 1 + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{11^s} + \cdots ,$$

where all integers divisible by 3 are gone. After continuing in this way ad infinitum, everything will be gone except for the first term, 1,

$$\prod_{p\in\mathcal{P}} \left(1 - \frac{1}{p^s}\right)\zeta(s) = 1.$$

But this is the same as (3.4)!

The result provides a neat analytic proof of the fact (already known to the Greeks) that the number of primes is infinite. Indeed, since $\zeta(1)$—the harmonic series—diverges to ∞ (cf. Selectio 4), the product on the left of (3.4), if $s = 1$, cannot be finite.

The formula—the beginning of "analytic number theory"—in fact paved the way to important later developments in the distribution of primes.

Selectio 3. The Gamma Function. Following a correspondence in 1729 with Goldbach, Euler in E19 considers the problem of interpolating the sequence of factorials

$$n! = 1 \cdot 2 \cdot 3 \cdots n, \quad n = 1, 2, 3, \ldots, \tag{3.5}$$

at noninteger values of the argument. Euler quickly realized that this cannot be done algebraically, but requires "transcendentals," that is, calculus. He writes $n!$ as an infinite product,

$$\frac{1 \cdot 2^n}{1+n} \cdot \frac{2^{1-n} \cdot 3^n}{2+n} \cdot \frac{3^{1-n} \cdot 4^n}{3+n} \cdot \frac{4^{1-n} \cdot 5^n}{4+n} \cdots , \tag{3.6}$$

which formally, by multiplying out the numerators, can be seen to be the ratio of two infinite products, $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdots$ and $(n+1)(n+2)(n+3)\cdots$, which indeed reduces to (3.5). Now for $n = \frac{1}{2}$, Euler manages to manipulate the infinite product (3.6) into the square root of an infinite product for $\pi/4$ due to John Wallis; therefore, $\frac{1}{2}! = \frac{1}{2}\sqrt{\pi}$. This is why Euler knew that some kind of integration was necessary to solve the problem.

By a stroke of insight, Euler takes the integral $\int_0^1 x^e(1-x)^n \mathrm{d}x$—up to the factor $1/n!$, the n-times iterated integral of x^e, where e is an arbitrary number (not the basis of the natural logarithms!)—and finds the formula

$$(e+n+1)\int_0^1 x^e(1-x)^n \mathrm{d}x = \frac{n!}{(e+1)(e+2)\cdots(e+n)} . \tag{3.7}$$

He now lets $e = f/g$ be a fraction, so that

$$\frac{f+(n+1)g}{g^{n+1}} \int_0^1 x^{f/g}(1-x)^n \mathrm{d}x = \frac{n!}{(f+g)(f+2g)\cdots(f+ng)} .$$

If $f = 1$, $g = 0$, then on the right we have $n!$; on the left, we have to determine the limit as $f \to 1$, $g \to 0$, which Euler takes to be the desired interpolant, since it is meaningful also for noninteger n. Skillfully, as always, Euler carries out the limit by first changing variables, $x = t^{g/(f+g)}$, to obtain

$$\frac{f+(n+1)g}{f+g}\int_0^1 \left(\frac{1-t^{g/(f+g)}}{g}\right)^n \mathrm{d}t,$$

and then doing the limit as $g \to 0$ with $f = 1$ by the Bernoulli–l'Hôpital rule. The result is $\int_0^1 (-\ln t)^n \mathrm{d}t$. Here we can set $n = x$ to be any positive number, and thus we obtain $x! = \int_0^1 (-\ln t)^x \mathrm{d}t$, which today is written as

$$x! = \int_0^\infty \exp(-t)t^x \mathrm{d}t = \Gamma(x+1) \tag{3.8}$$

in terms of the gamma function Γ. It is easily verified that

$$\Gamma(x+1) = x\Gamma(x), \quad \Gamma(1) = 1, \tag{3.9}$$

so that indeed $\Gamma(n+1) = n!$ if n is an integer ≥ 0.

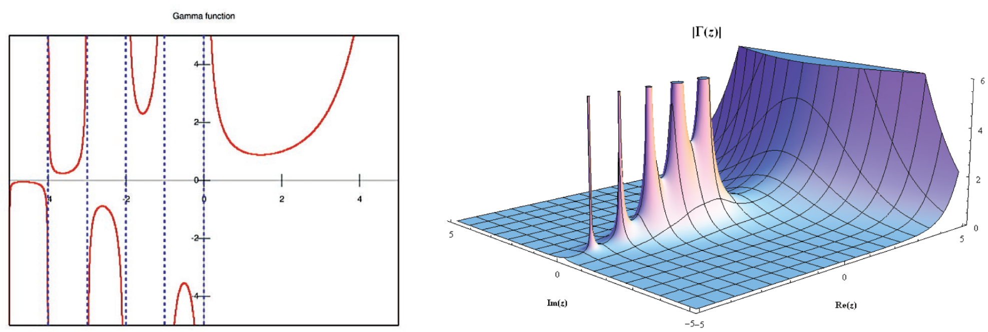


Fig. 14 *The gamma function; graph and contour map. (Per Wikipedia, permission is granted to copy, distribute and/or modify this document under the terms of the GNU Free Documentation License, Version* 1.2 *or any later version published by the Free Software Foundation; with no Invariant Sections, no Front-Cover Texts, and no Back-Cover Texts. Subject to disclaimers.)*

Euler's unfailing intuition in producing the gamma function had been vindicated early in the 20th century when it was shown independently by Harald Bohr and Johannes Mollerup that there is no other function on $(0, \infty)$ interpolating the factorials if, in addition to satisfying the difference equation (3.9), it is also required to be logarithmically convex. The gamma function indeed has become one of the most fundamental functions in analysis—real as well as complex.

The integral in (3.8) is often referred to as the second Eulerian integral, the first being

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t,$$

also called the beta function. The latter can be beautifully expressed in terms of the gamma function by

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

which is nothing but (3.7) for $e = x-1$, $n = y-1$.

For a recent historical essay on the gamma function, see Srinivasan [28].

Selectio 4. Euler's Constant. It is generally acknowledged that, aside from the imaginary unit $i = \sqrt{-1}$, the two most important constants in mathematics are $\pi = 3.1415\ldots$, the ratio of the circumference of a circle to its diameter, and $e = 2.7182\ldots$, the basis of the natural logarithms, sometimes named after Euler. They pop up everywhere, often quite unexpectedly. The 19th-century logician Auguste de Morgan said about π that "it comes on many occasions through the window and through the door, sometimes even down the chimney." The third most important constant is undoubtedly Euler's constant γ introduced by him in 1740 in E43. Of the three together—the "holy trinity," as they are sometimes called—the last one, γ, is the most mysterious one, since its arithmetic nature, in contrast to π and e, is still shrouded in obscurity. It is not even known whether γ is rational, even though most likely it is not; if it were, say, equal to p/q in reduced form, then high-precision continued fraction calculations of γ have shown that q would have to be larger than $10^{244,663}$ (Haible and Papanikolaou [14, p. 349]).

Euler's constant arises in connection with the harmonic series $\zeta(1) = 1+\frac{1}{2}+\frac{1}{3}+\cdots$ (so called because each of its terms is the harmonic mean of the two neighboring terms) and is defined as the limit

$$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}\cdots+\frac{1}{n}-\ln n\right) = 0.57721\ldots. \tag{3.10}$$

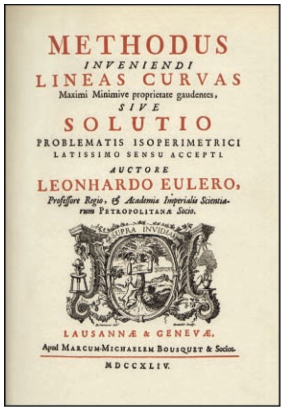
It has been known as early as the 14th century that the harmonic series diverges, but a rigorous proof of it is usually attributed to Jakob Bernoulli, who also mentioned another proof by his younger brother Johann, which, however, is not entirely satisfactory. At any rate, Euler, in defining his constant and showing it to be finite, puts in evidence not only the divergence of the harmonic series, but also its logarithmic rate of divergence. Beyond this, using his general summation formula (mentioned in Selectio 1), he computes γ to 16 correct decimal places (cf. Gautschi [13, sect. 5.2]), and to equally many decimals the sum of the first million terms of the harmonic series! Since later (in 1790) Lorenzo Mascheroni also considered Euler's constant, gave it the name γ, and computed it to 32 decimal places (of which, curiously, the 19th, 20th, and 21st are incorrect), the term "Euler–Mascheroni constant" is also in use. As of today, it appears that γ has been computed to 108 million decimal places, compared to over 2×10^{11} decimals for π and 50.1 billion for e.

An inspiring tale surrounding Euler's constant can be found in Havil [15], and a rather encyclopedic account in Krämer [18].

After all these spectacular achievements, the numerous other memoirs written on many different topics, and his responsibilities at the Academy, it is incredible that Euler still had the time and stamina to write a 300-page volume on elementary arithmetic for use in the St. Petersburg gymnasia. How fortunate were those St. Petersburg kids for having had Euler as their teacher!

3.3. Berlin. Next to some 280 memoirs, many quite important, and consultation on engineering and technology projects, this period saw the creation of a number of epochal scientific treatises and a highly successful and popular work on the philosophy of science.

3.3.1. Major Works. The brachistochrone problem—finding the path along which a mass point moves under the influence of gravity down from one point of a vertical plane to another (not vertically below) in the shortest amount of time—is an early



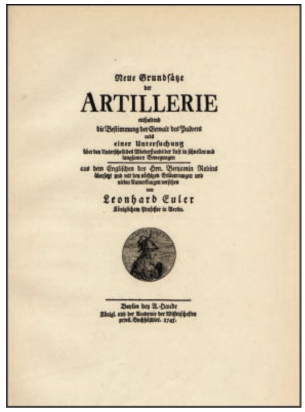


Fig. 15 *Calculus of Variations,* 1744, *and Artillerie,* 1745. *(Reprinted with permission from Birkhäuser Verlag.)*

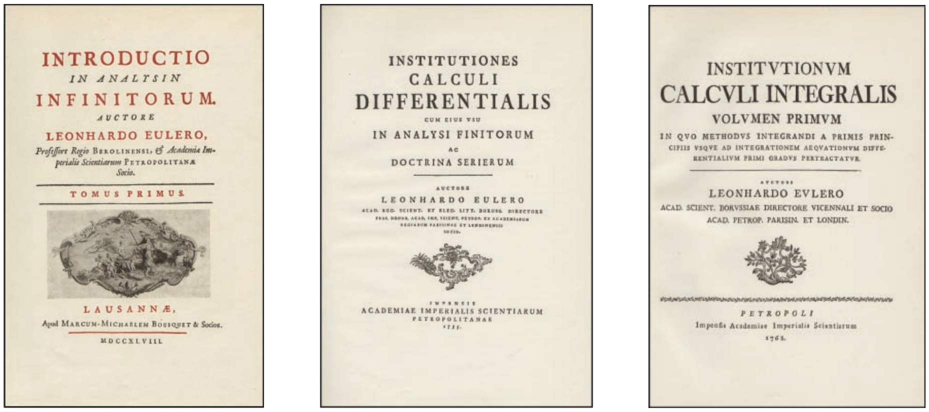
example of an optimization problem, posed by Johann Bernoulli, which seeks a function (or a curve) that renders optimal an analytic expression that depends on this function. In 1744 (E65), and later in 1766 (E296) adopting an improved approach of Lagrange, Euler vastly generalizes this problem, thereby creating an entirely new branch of mathematics, called (already by Euler) the "calculus of variations." He derives his famous Euler equation: a necessary condition in the form of a differential equation that any solution of the problem must satisfy. Typically for Euler, he illustrates this by many—some hundred!—examples, among them the principle of least action that caused so much turmoil in the mid-1700s (cf. section 2.3).

Two smaller treatises, one on planetary and cometary trajectories (E66) and another on optics (E88), appeared at about the same time (1744, resp., 1746). The latter is of historical interest insofar as it started the debate of Newton's particle versus Euler's own wave theory of light.

In deference to his master, king Frederick II, Euler translated an important work on ballistics by the Englishman Benjamin Robins, even though the latter had been unfairly critical of Euler's *Mechanica* of 1736. He added, however, so many commentaries and explanatory notes (also corrections!) that the resulting book—his *Artillerie* of 1745 (E77)—is about five times the size of the original. Niklaus Fuss in his 1783 *Eulogy* of Euler (cf. *Opera omnia*, Ser. I, Vol. 1, pp. xliii–xcv) remarks: "...the only revenge [Euler] took against his adversary because of the old injustice consists in having made [Robins's] work so famous as, without him, it would never have become."

The two-volume *Introductio in analysin infinitorum* of 1748 (E101, E102) together with the *Institutiones calculi differentialis* of 1755 (E212) and the three-volume *Institutiones calculi integralis* of 1768–1770 (E342, E366, E385)—a "magnificent trilogy" (Fellmann [9, sect. 4])—establishes analysis as an independent, autonomous discipline, and represents an important precursor of analysis as we know it today.

In the first volume of the *Introductio*, after a treatment of elementary functions, Euler summarizes his many discoveries in the areas of infinite series, infinite products, partition of numbers, and continued fractions. On several occasions, he uses the fundamental theorem of algebra, clearly states it, but does not prove it. He develops a clear concept of function—real- as well as complex-valued—and emphasizes the fundamental role played in analysis by the number e and the exponential and logarithm


INTRODUCTIO
IN ANALYSIN
INFINITORUM.
AUCTORE
LEONHARDO EULERO,
TOMUS PRIMUS.
LAUSANNÆ,
MDCCXLVIII.

INSTITUTIONES
CALCULI
DIFFERENTIALIS
IN ANALYSI FINITORUM
AC
DOCTRINA SERIERUM
AUCTORE
LEONHARDO EULERO
ACADEMIAE IMPERIALIS SCIENTIARUM
PETROPOLITANAE

INSTITVTIONVM
CALCVLI INTEGRALIS
VOLVMEN PRIMVM
AVCTORE
LEONHARDO EVLERO
PETROPOLI

Fig. 16 *Infinitesimal Analysis,* 1748, *and Differential and Integral Calculus,* 1755, 1763, 1773. *(Reprinted with permission from Birkhäuser Verlag.)*

functions. The second volume is devoted to analytic geometry: the theory of algebraic curves and surfaces.

Differential Calculus also has two parts, the first being devoted to the calculus of differences and differentials, the second to the theory of power series and summation formulae, with many examples given for each. Chapter 4 of the second part, incidentally, contains the first example, in print, of a Fourier series; cf. also p. 297 of the *Opera omnia*, Ser. I, Vol. 10. Another chapter deals with Newton's method, and improvements thereof, for solving nonlinear equations, and still another with criteria for algebraic equations to have only real roots.

The three-volume *Integral Calculus* is a huge foray into the realm of quadrature and differential equations. In the first volume, Euler treats the quadrature (i.e., indefinite integration) of elementary functions and techniques for reducing the solution of linear ordinary differential equations to quadratures. In the second volume, he presents, among other things, a detailed theory of the important linear second-order differential equations, and in the third volume a treatment, to the extent known at the time (mostly through Euler's own work), of linear partial differential equations. A fourth volume, published posthumously in 1794, contains supplements to the preceding volumes. Euler's method—a well-known approximate method for solving arbitrary first-order differential equations, and the more general Taylor series method, are embedded in Chapter 7 of the second section of Volume 1.

Euler's program for mechanics (cf. section 3.2.1) progressed steadily as he tackled the problem of developing a theory of the motion of solids. An important milestone in this effort was the memoir E177 in which was stated for the first time, in full generality, what today is called Newtonian mechanics. The great treatise *Theoria motus corporum solidorum seu rigidorum* (E289) which followed in 1765, also called the "Second *Mechanics*," represents a summary of Euler's mechanical work up to this time. In addition to an improved exposition of his earlier mechanics of mass points (cf. section 3.2.1), it now contains the differential equations (Euler's equations) of motion of a rigid body subject to external forces. Here, Euler introduces the original idea of employing two coordinate systems—one fixed, the other moving, attached to the body—and deriving differential equations for the angles between the respective

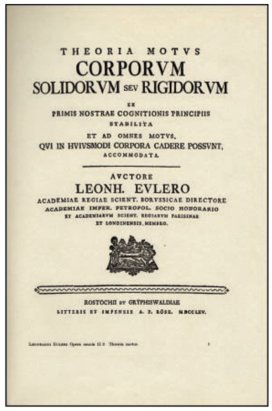
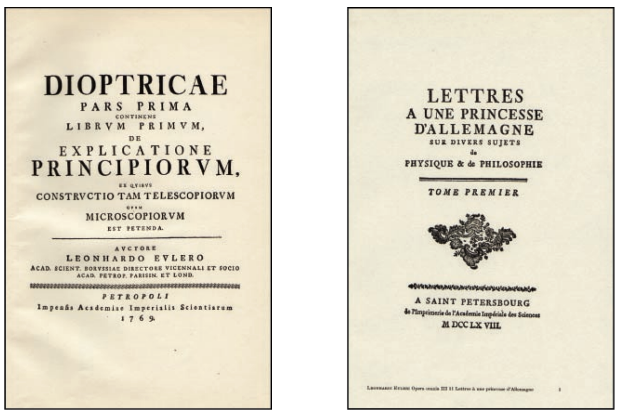

THEORIA MOTVS
CORPORVM
SOLIDORVM SEV RIGIDORVM
EX
PRIMIS NOSTRAE COGNITIONIS PRINCIPIIS
STABILITA
ET AD OMNES MOTVS,
QVI IN HVIVSMODI CORPORA CADERE POSSVNT,
ACCOMMODATA.
AVCTORE
LEONH. EVLERO

Fig. 17 *Theoria motus corporum,* 1765. *(Reprinted with permission from Birkhäuser Verlag.)*


DIOPTRICAE
PARS PRIMA
CONTINENS
LIBRVM PRIMVM,
DE
EXPLICATIONE
PRINCIPIORVM,
EX QVIBVS
CONSTRVCTIO TAM TELESCOPIORVM
QVAM
MICROSCOPIORVM
EST PETENDA.
AVCTORE
LEONHARDO EVLERO
PETROPOLI
Impenſis Academiae Imperialis Scientiarum
1769.

LETTRES
A UNE PRINCESSE
D'ALLEMAGNE
SUR DIVERS SUJETS
de
PHYSIQUE & de PHILOSOPHIE
TOME PREMIER
A SAINT PETERSBOURG
M DCC LX VIII.

Fig. 18 *Optics,* 1769–1771*, and Letters,* 1768, 1772 (1760–1762)*. (Reprinted with permission from Birkhäuser Verlag.)*

coordinate axes, now called the Euler angles. The intriguing motion of the spinning top is one of many examples worked out by Euler in detail.

Later, in 1776, Euler returns to mechanics again with his seminal work E479, where one finds the definitive formulation of the principles of linear and angular momentum.

Throughout his years in Berlin and beyond, Euler was deeply occupied with geometric optics. His memoirs and books on this topic, including the monumental three-volume *Dioptrics* (E367, E386, E404), written mostly while still in Berlin, fill no fewer than seven volumes in his *Opera omnia.* A central theme and motivation of this work was the improvement of optical instruments like telescopes and microscopes, notably ways of eliminating chromatic and spherical aberration through intricate systems of lenses and interspaced fluids.

Euler's philosophical views on science, religion, and ethics are expressed in over 200 letters written between 1760 and 1762 (in French) to a German princess and published later in 1768 and 1772 (E343, E344, E417). (For a recent edition of these letters, see Euler [8].) While Euler's role as a philosopher may be controversial (even his best friend Daniel Bernoulli advised him to better deal with "more sublime matters"), his *Letters*, written with extreme clarity and also accessible to people not trained in the sciences, "even to the gentle sex," as Fuss remarks in his *Eulogy*, became an instant success and were translated into all major languages.

3.3.2. Selecta Euleriana.

Selectio 5. The Königsberg Bridge Problem. The river Pregel, which flows through the Prussian city of Königsberg, divides the city into an island and three distinct land masses, one in the north, one in the east, and one in the south. There are altogether seven bridges, arranged as shown in green on the left of Figure 19, connecting the three land masses with each other and with the island. The problem is this: Can one take a stroll from one point in the city to another by traversing each bridge exactly once? In particular, can one return to the starting point in the same manner?

Evidently, this is a problem that cannot be dealt with by the traditional methods of analysis and algebra. It requires a new kind of analysis that deemphasizes metric properties in favor of positional properties. Euler solved the problem in 1735, published as E53 in 1741, by showing that such paths cannot exist. He does this by an ingenious process of abstraction, associating with the given land and bridge configuration (what today is called) a connected graph, i.e., a network of vertices and connecting edges, each vertex representing a piece of land and each edge a bridge connecting the respective pieces of land. In the problem at hand, there are four distinct pieces of land, hence four vertices, and they are connected with edges as shown on the right of Figure 19. It is obvious what is meant by a path along edges from one vertex to another. A closed path is called a circuit, and paths or circuits are (today) called Eulerian if each edge is traversed exactly once.

Euler recognized that in modern terminology a crucial concept here is the *degree* of a vertex, i.e., the number of edges emanating from it. If, in an arbitrary connected

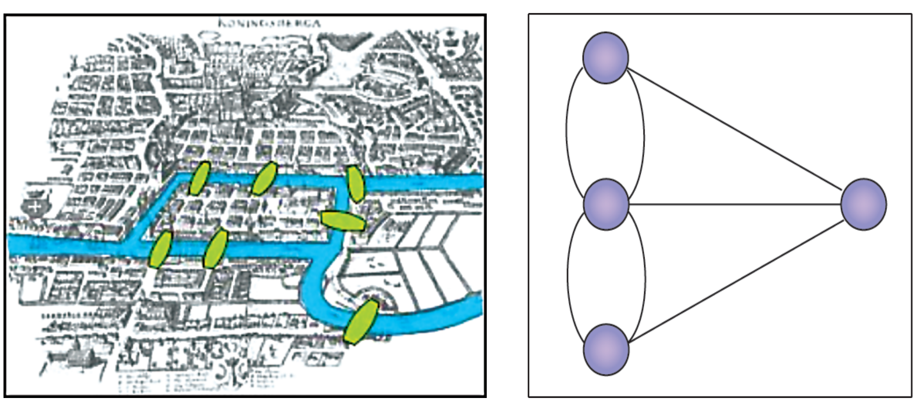


Fig. 19 *The Königsberg bridge problem. (Left image created by Bogdan Giuşcă, as displayed in the Wikipedia article "Leonhard Euler." Per Wikipedia, permission is granted to copy, distribute and/or modify this document under the terms of the GNU Free Documentation License, Version 1.2 or any later version published by the Free Software Foundation; with no Invariant Sections, no Front-Cover Texts, and no Back-Cover Texts. Subject to disclaimers.)*

graph, n denotes the number of vertices of odd degree, he in effect proves that (a) if $n = 0$, the graph has at least one Eulerian circuit, and he indicates how to find it; (b) if $n = 2$, it has at least one Eulerian path, but no circuit, and again he shows us how to find it; (c) if $n > 2$, it has neither. (The case $n = 1$ is impossible.) Since the Königsberg bridge graph has $n = 4$, we are in case (c), hence it is impossible to traverse the city in the manner required in the problem.

Here again, like in the calculus of variations, one can admire Euler's powerful drive and capacity of starting with a concrete example and deriving from it, by a process of sweeping generalization, the beginnings of a whole new theory, in the present case, the theory of graphs and topological networks.

Selectio 6. Euler's Buckling Formula (1744). In a first supplement to his *Methodus* (cf. Figure 15, left), Euler applies the calculus of variations to elasticity theory, specifically to the bending of a rod subject to an axial load. He derives the critical load under which the rod buckles. This load depends on the stiffness constant of the material, on the way the rod is supported at either end, and it is inversely proportional to the square of the length of the rod. A particular configuration of two rods loaded on top by a connecting bar (assumed to be of infinite stiffness) is shown in Figure 20, during the initial phase (left), and at the time of buckling (right). Here, the top end of the rods is slidably supported and the bottom end clamped. For a video, see http://epubs.siam.org/sam-bin/getfile/SIREV/articles/70271_02.avi.

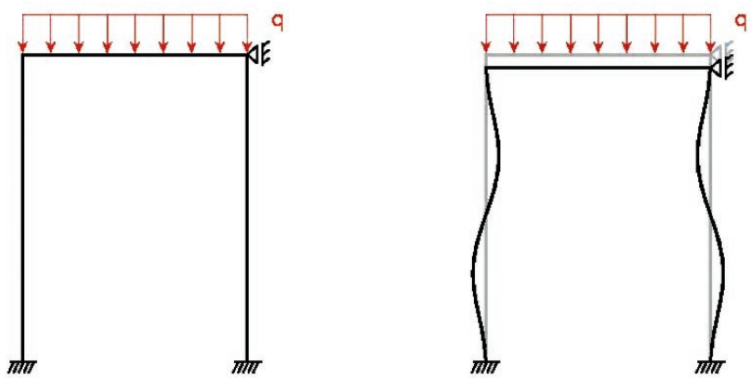


Fig. 20 *The buckling of a rod. (Images and video courtesy of Wolfgang Ehlers.)*

The critical load is the first elasto*static* eigenvalue of the problem. Euler also calculates the elasto*kinetic* eigenvalues, the eigenfrequencies of the rod's transversal oscillations, and the associated eigenfunctions, which determine the shapes of the deformed rod.

Selectio 7. Euler Flow. In a series of three memoirs, E225–E227, all published in 1757, and another three papers (E258, E396, E409), Euler gave his definitive treatment of continuum and fluid mechanics, the culmination of a number of earlier memoirs on the subject. It contains the celebrated Euler equations, expressing the conservation of mass, momentum, and energy. In two (three) dimensions, these constitute a system of four (five) nonlinear hyperbolic partial differential equations, which have to be solved, given appropriate initial and boundary conditions. Naturally, in Euler's time,

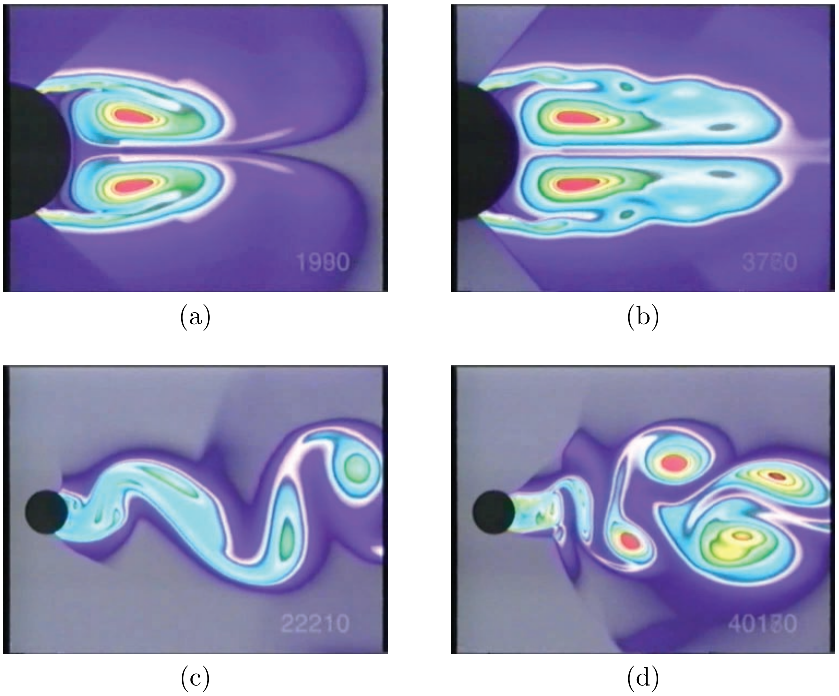


Fig. 21 *Transonic Euler flow at Mach .85 about a cylinder. (Images and video courtesy of Nicola Botta.)*

this was virtually impossible to do, except in very special cases, and indeed Euler in the introduction to E226 had to write that "if there remain any difficulties, they shall not be on the side of mechanics, but solely on the side of analysis: for this science has not yet been carried to the degree of perfection which would be necessary in order to develop analytic formulae which include the principles of the motion of fluids." Nowadays, however, the Euler equations are widely being used in computer simulation of fluids.

An example is the asymmetric flow of a compressible, inviscid fluid about a circular cylinder at transonic speed, calculated in 1995 by Botta [4]. Four color-coded snapshots of the two-dimensional flow (vorticity contour lines), as it develops behind the cylinder, are shown in Figure 21: (a) the onset of the flow, (b) a regimen of Kelvin–Helmholtz instability, (c) the flow after breakdown of symmetry, and (d) the formation of vortex pairs. (The scaling of (c) and (d) differs from that of (a) and (b).) For the complete Euler-flow video, see http://epubs.siam.org/sam-bin/getfile/SIREV/articles/70271_03.avi.

Selectio 8. Euler's Polyhedral Formula (1758). In a three-dimensional convex polyhedron (not necessarily regular), let V denote the number of vertices, E the number of edges, and F the number of faces. Thus, in the case of an octahedron (cf. Figure 22), one has $V = 6, E = 12$, and $F = 8$. Mentioned in 1750 in a letter to Goldbach, and later published in E231, Euler proves for the first time the extremely simple but stunning formula

$$V - E + F = 2. \tag{3.11}$$

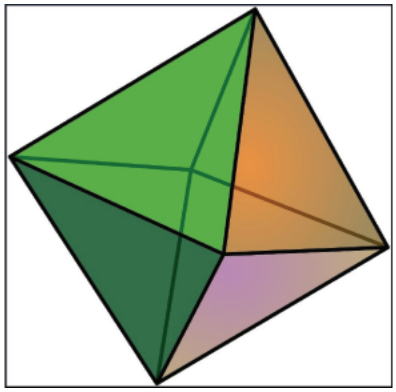


Fig. 22 *Octahedron. (From the Wikipedia article "Octahedron." Per Wikipedia, permission is granted to copy, distribute and/or modify this document under the terms of the GNU Free Documentation License, Version 1.2 or any later version published by the Free Software Foundation; with no Invariant Sections, no Front-Cover Texts, and no Back-Cover Texts. Subject to disclaimers.)*

The way he did it is to chop off triangular pyramids from the polyhedron, one after another, in such a manner that the sum on the left of (3.11) remains the same. Once he got it chopped down to a tetrahedron, that sum is easily seen to be 2. (For a critical and historical review of Euler's proof, see Francese and Richeson [11].) Descartes, some 100 years earlier, already knew, but did not prove, something close to the formula (3.11).

The expression on the left-hand side of (3.11) is an example of an Euler characteristics, a topological invariant for polyhedra. Euler characteristics have been defined for many other topological spaces and today still come up often in homological algebra.

The generalization to higher-dimensional polytopes leads to what is called Euler–Poincaré characteristics, where the pattern of alternating signs can be seen to come from the dimensionality of the respective facets, something already noted in 1852 by another Swiss mathematician, Ludwig Schläfli [25, sect. 32].

Selectio 9. Euler and *q*-Theory. The story here begins with a letter Euler wrote in 1734 to Daniel Bernoulli, in which he considered the (somewhat bizarre) problem of interpolating the common logarithm $\log x$ at the points $x_r = 10^r$, $r = 0, 1, 2, \ldots$. He essentially writes down Newton's interpolation series $S(x)$ (without mentioning Newton by name) and remarks that, when $x = 9$, the series converges quickly, but to a wrong value, $S(9) \neq \log 9$ (cf. Gautschi [12]). Rather than losing interest in the problem, Euler must have begun pondering the question about the nature of the limit function $S(x)$: what is it, if not the logarithm?

Almost twenty years later, in 1753, he returned to this problem in E190, now more generally for the logarithm to base $a > 1$, and studied the respective limit function $S(x; a)$ in great detail. Intuitively, he must have perceived its importance. Today we know (Koelink and Van Assche [17]) that it can be thought of as a q-analogue of the logarithm, where $q = 1/a$, and some of the identities derived by Euler (in part already contained in Vol. 1, Chap. 16 of his *Introductio*) are in fact special cases of the q-binomial theorem—a centerpiece of q-theory in combinatorial analysis and physics. Thus, Euler must be counted among the precursors of q-theory, which was only developed about 100 years later by Heinrich Eduard Heine.

Selectio 10. The Euler–Fermat Theorem and Cryptology. Let $\mathbb{N}$ be the set of positive integers, and $\varphi(n)$, $n \in \mathbb{N}$, Euler's totient function, that is, the number of integers $1, 2, 3, \ldots, n$ coprime to n. The Euler–Fermat theorem, published 1763 as E271, states that for any $a \in \mathbb{N}$ coprime to n,

$$a^{\varphi(n)} \equiv 1 \pmod{n}. \tag{3.12}$$

It generalizes the "little Fermat" theorem, which is the case $n = p$ a prime number, and therefore $\varphi(p) = p - 1$.

In cryptography, one is interested in the secure transmission of messages, whereby a message M is transmitted from a sender to the receiver in encrypted form: The sender encodes the message M into E, whereupon the receiver has to decode E back into M. It is convenient to think of M as a number in $\mathbb{N}$, for example, the number obtained by replacing each letter, character, and space in the text by its ASCII code. The encrypted message E is then $E = f(M)$, where $f : \mathbb{N} \to \mathbb{N}$ is some function on $\mathbb{N}$. The problem is to find a function f that can be computed by the general public but is extremely difficult to invert (i.e., to obtain M from E), unless one is in the possession of a secret key associated with the function f.

A solution to this problem is the now widely used RSA encryption scheme (named after its inventors R. Rivet, A. Shamir, and L. Adleman). To encode the message M, one selects two distinct (and very large) prime numbers p, q and defines a "modulus" $n = pq$ assumed to be larger than M. Then an integer e, $1 < e < \varphi(n)$, is chosen with e coprime to $\varphi(n)$. The numbers n, e form the "public key," i.e., they are known to the general public. The encoded message M is $E = f(M)$, where $f(M) \equiv M^e \pmod{n}$. The "private key" is n, d, where d is such that $de \equiv 1 \pmod{\varphi(n)}$. To compute d, one needs to know p and q, since $n = pq$, $\varphi(n) = (p-1)(q-1)$. The general public, however, knows only n, so must factor n into prime numbers to get a hold of p, q. If n is sufficiently large, say $n > 10^{300}$, this, today, is virtually impossible. The person who selected p and q, on the other hand, is in possession of d, and can decode the ciphertext E as follows,

$$E^d \equiv (M^e)^d \pmod{n} \equiv M^{ed} \pmod{n} \equiv M^{N\varphi(n)+1} \pmod{n}, \quad N \in \mathbb{N},$$

by the choice of d. Using now the Euler–Fermat theorem (3.12), with $a = M^N$ (almost certainly coprime to $n = pq$ or can be made so), one gets

$$E^d \equiv M a^{\varphi(n)} \pmod{n} \equiv M \pmod{n} = M,$$

since $M < n$. (It is true that M, e, n, and d are typically very large numbers so that the computations described may seem formidable. There are, however, efficient schemes to execute them; see, e.g., Silverman [26, Chaps. 16, 17].)

3.4. Second St. Petersburg Period. This may well be Euler's most productive period, with well over 400 published works to his credit, not only on each of the topics already mentioned, but also on geometry, probability theory and statistics, cartography, and even widow's pension funds and agriculture. In this enormous body of work there figure three treatises on algebra, lunar theory, and naval science, and what appear to be fragments of major treatises on number theory (E792), natural philosophy (E842), and dioptrics (E845).

3.4.1. Major Works. Soon into this second St. Petersburg period, another of Euler's "bestsellers" appeared: the *Vollständige Anleitung zur Algebra* (E387, E388),

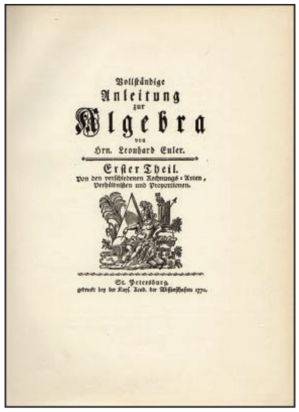

Vollständige
Anleitung
zur
Algebra
von
Hrn. Leonhard Euler.
Erster Theil.
Von den verschiedenen Rechnungs-Arten,
Verhältnissen und Proportionen.
St. Petersburg,
gedruckt bey der Kays. Acad. der Wissenschaften 1770.

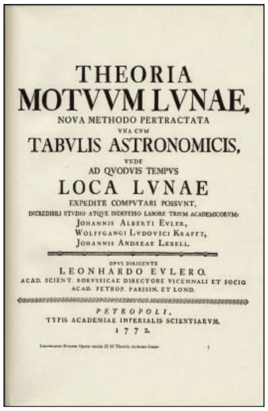
Fig. 23 *Algebra,* 1770. *(Reprinted with permission from Birkhäuser Verlag.)*

or *Algebra* for short. Even before publication of the German original, a translation into Russian came out, and translations into all major languages were soon to follow. (The French translation by Johann III Bernoulli includes a long supplement by Lagrange containing an exposé on the arithmetic theory of continued fractions and many addenda to the last section of the *Algebra* dealing with Diophantine equations.)

Euler wrote this 500-page work to introduce the absolute beginner into the realm of algebra. He dictated the work to a young man—a tailor's apprentice—whom he brought with him from Berlin, and who (according to the preface of the work) "was fairly good at computing, but beyond that did not have the slightest notion about mathematics .... As far as his intellect is concerned, he belonged among the mediocre minds." Nevertheless, it is said that, when the work was completed, he understood everything perfectly well and was able to solve algebraic problems posed to him with great ease.

It is indeed a delight to witness in this work Euler's magnificent didactic skill, to watch him progress in ever so small steps from the basic principles of arithmetic to algebraic (up to quartic) equations, and finally to the beautiful art of Diophantine analysis. Equally delightful is to see how the theory is illustrated by numerous well-chosen examples, many taken from everyday life.

The orbit of the moon, with all its irregularities, had long fascinated mathematicians like Clairaut and d'Alembert, as well as Euler, who already in 1753 published his *Theoria motus lunae* (E187), the "First Lunar Theory." The theory he developed there, while tentative, provided astronomers with formulae needed to prepare lunar tables, which in turn served seafaring nations for over a century with accurate navigational aids. Euler's definitive work on the subject, however, is his "Second Lunar Theory" (E418) of 1772, a monumental work dealing in a more effective way than before with the difficult three-body problem, i.e., the study of the motion of three bodies—in this case the sun, the earth, and the moon, thought of as point masses—moving under the influence of mutual gravitational forces. Already Newton is reputed to have said that "an exact solution of the three-body problem exceeds, if I am not mistaken, the power of any human mind." Today it is known, indeed, that an exact solution is not possible. Euler grapples with the problem by introducing appropriate variables, again choosing two coordinate systems—one fixed, the other moving—applying processes of successive approximation, and making use, when needed, of observational data.


THEORIA
MOTVVM LVNAE,
NOVA METHODO PERTRACTATA
VNA CVM
TABVLIS ASTRONOMICIS,
VNDE
AD QVODVIS TEMPVS
LOCA LVNAE
EXPEDITE COMPVTARI POSSVNT,
INCREDIBILI STVDIO ATQVE INDEFESSO LABORE TRIVM ACADEMICORVM:
JOHANNIS ALBERTI EVLER,
WOLFGANGI LVDOVICI KRAFFT,
JOHANNIS ANDREAE LEXELL.
OPVS DIRIGENTE
LEONHARDO EVLERO
ACAD. SCIENT. BORVSSICAE DIRECTORE VICENNALI ET SOCIO
ACAD. PETROP. PARISIN. ET LOND.
PETROPOLI,
TYPIS ACADEMIAE IMPERIALIS SCIENTIARVM.
1772.

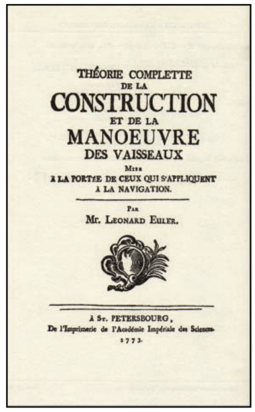

THÉORIE COMPLETTE
DE LA
CONSTRUCTION
ET DE LA
MANOEUVRE
DES VAISSEAUX
MISE
À LA PORTÉE DE CEUX QUI S'APPLIQUENT
À LA NAVIGATION.
PAR
MR. LEONARD EULER.
À ST. PETERSBOURG,
De l'Imprimerie de l'Académie Impériale des Sciences.
1773.

Fig. 24 *Second Lunar Theory,* 1772, *and Second Theory of Ships,* 1773. *(Reprinted with permission from Birkhäuser Verlag.)*

According to L. Courvoisier (cf. *Opera omnia*, Ser. II, Vol. 22, p. xxviii), "all later progress in celestial mechanics is based, more or less, on the ideas contained in the works of Euler, [and the later works of] Laplace and Lagrange."

The *Théorie complete de la construction et de la manœuvre des vaisseaux* (E426), also called the "Second Theory of Ships," is a work that treats the topic indicated in the title for people having no or little mathematical knowledge, in particular for the sailors themselves. Not surprisingly, given the level of presentation and the author's extraordinary didactic skill, the work proved to be very successful. The French maritime and finance minister (and famous economist) Anne Robert Jacques Turgot proposed to King Louis XVI that all students in marine schools (and also those in schools of artillery) be required to study Euler's relevant treatises. Very likely, Napoléon Bonaparte was one of those students. The king even paid Euler 1,000 rubles for the privilege of having the works reprinted, and czarina Catherine II, not wanting to be outdone by the king, doubled the amount and pitched in an additional 2,000 rubles!

3.4.2. Selecta Euleriana.

Selectio 11. Partition of Numbers. Euler's interest in the partition of numbers, i.e., in expressing an integer as a sum of integers from some given set, goes back to 1740 when Philippe Naudé the younger, of the Berlin Academy, in a letter to Euler asked in how many ways the integer 50 can be written as a sum of seven different positive integers. This gave rise to a series of memoirs, spanning a time interval of about 20 years, beginning with E158, published (with a delay of 10 years) in 1751, and ending with E394, published in 1770. In this work, Euler almost single-handedly created the theory of partition. A systematic exposition of part of this work can also be found in Volume 1, Chapter 16, of his *Introductio* (cf. section 3.3.1) and relevant correspondence with Niklaus I Bernoulli in the *Opera omnia*, Ser. IVA, Vol. 2, pp. 481–643, especially pp. 518, 537ff, 555ff.

Euler, as de Moivre before him (cf. Scharlau [24, p. 141f]), attacked problems of this type by a brilliant use of generating functions and formal power series. Thus, in the case of Naudé's inquiry, in Euler's hands this becomes the problem of finding the coefficient of z^7x^{50} in the expansion of $(1 + xz)(1 + x^2z)(1 + x^3z)(1 + x^4z)\cdots$, for which Euler finds the answer 522, "a most perfect solution of Naudé's problem," as he proudly wrote (at the end of section 19 of E158). In the context of "unrestricted partitions," Euler in the penultimate paragraph of E158 surprises us with the marvelous expansion

$$(1-x)(1-x^2)(1-x^3)(1-x^4)\cdots = \sum_{n=-\infty}^{\infty} (-1)^n x^{n(3n-1)/2},$$

which he conjectured as early as 1742 by numerical computation, and then labored on it for almost ten years to find a proof (in E244, a "masterpiece" according to C. G. J. Jacobi). He used (in E175) the expansion to obtain his astonishing recurrence relation for $s(n)$, the sum of divisors of n (including 1 and n), and (in E191) the reciprocal expansion to obtain a similar recurrence for the partition function $p(n)$, the number of ways n can be written as a sum of natural numbers. In E394, Euler considers the problem of how many ways any given number can be thrown by n ordinary dice. He shows that the answer is given by the appropriate coefficient in the expansion of $(x+x^2+x^3+x^4+x^5+x^6)^n$. Of course, Euler also solves the same problem for more general dice having an arbitrary number of sides, which may even differ from die to die.

Euler's magnificent work on partitions has not found much response among his contemporaries; it was only in the 20th century that his work was continued and significantly expanded by such mathematicians as Ramanujan, Hardy, and Rogers.

Selectio 12. Euler's Gear Transmission. In connection with the design of water turbines, Euler developed optimal profiles for teeth in cogwheels that transmit motion with a minimum of resistance and noise (E330, OII.17, pp. 196–219). These profiles involve segments of circular evolvents as shown in Figure 25. For the gear in action, see the video at http://epubs.siam.org/sam-bin/getfile/SIREV/articles/70271_04.avi.

The technical realization of this design took shape only later in what is called the involute gear. Euler not only is the inventor of this kind of gear, but he also anticipated the underlying geometric equations now usually called the Euler–Savary equations.

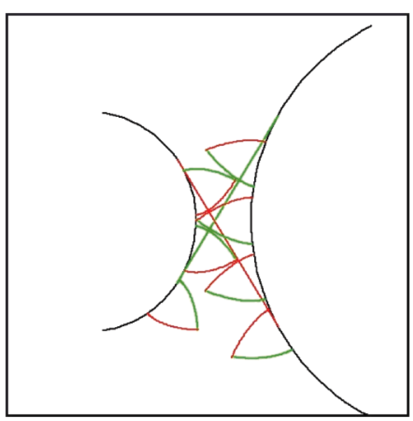


Fig. 25 *Euler gear,* 1767. *(Image and video courtesy of Bert Jüttler.)*

Selectio 13. Euler's Disk. In a number of memoirs (E257, E292, E336, E585) from the 20-year period 1761–1781, Euler analyzes the motion of a rigid body around a moving axis, including the effects of friction. An interesting example is the Euler disk, a circular (homogeneous) metal disk being spun on a clean smooth surface. At first, it will rotate around its vertical axis, but owing to friction, the axis is beginning to tilt and the disk to roll on a circular path. The more the axis is tilting, the wider the circular path and the higher the pitch of the whirring sound emitted by the point of contact of the disk with the surface. Thus, paradoxically, the speed of the motion seems to increase, judging from the rising pitch of the sound, although energy is being dissipated through friction. The disk, eventually, comes to an abrupt halt, flat on the surface.

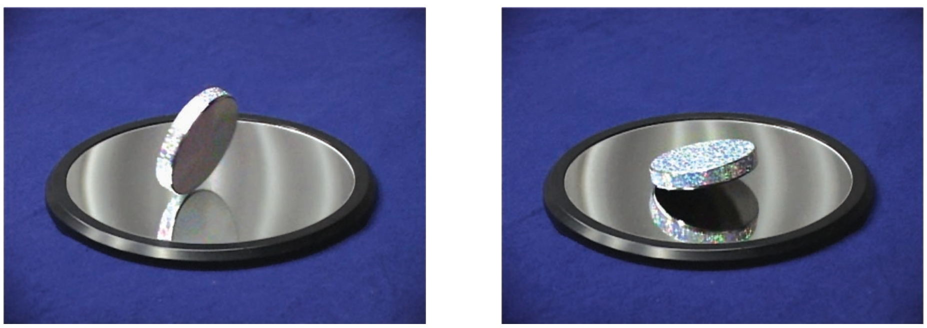


Fig. 26 *Euler disk. (Produced by Multimedia Services, ETH Zürich.)*

Two snapshots, one from the initial phase and the other from a later phase of the motion, are shown in Figure 26 on the left and right, respectively. For the complete Euler-disk video, see http://epubs.siam.org/sam-bin/getfile/SIREV/articles/70271_05.avi.

The key toward explaining the motion are Euler's equations, a set of differential equations involving the Euler angles and other parameters. The technical details of the motion, though, are still being analyzed today (cf., e.g., Le Saux, Leine, and Glocker [19] and the literature cited therein).

4. The Man.

4.1. Personality. From various testimonials of Euler's contemporaries, and also, of course, from Euler's extensive correspondence, one can form a fairly accurate picture of Euler's personality. A valuable source is the eulogy of Niklaus Fuss (*Opera omnia*, Ser. I, Vol. 1, pp. xliii–xcv), who during the last ten years of Euler's life had seen him regularly, almost on a daily basis, as one of his assistants. Also based on personal acquaintance is the eulogy of the marquis Nicolas de Condorcet (*Opera omnia*, Ser. III, Vol. 12, pp. 287–310), which, however, deals more with Euler's work. Euler comes across as a modest, inconspicuous, uncomplicated, yet cheerful and sociable person. He was down-to-earth and upright; "honesty and uncompromising rectitude, acknowledged Swiss national virtues, he possessed to a superior degree," writes Fuss. Euler never disavowed—in fact was proud of—his Swiss heritage. Fuss (who also originated from Basel) recalled that Euler "always retained the Basel dialect with all the peculiarities of its idiom. Often he amused himself to recall for me certain provincialisms and figures of speech, or mix into his parlance Basel expressions whose use and meaning I had long forgotten." He even made sure that he and his children retained the Basel civic rights.

Feelings of rancor, due to either priority issues or unfair criticism, were totally foreign to Euler. When Maclaurin, for example, discovered the well-known summation formula which Euler obtained six years earlier, Euler did not object, let alone complain, when for some time the formula was generally referred to as the "Maclaurin summation formula." It may even have pleased him that others hit upon the same fortunate idea. In due time, of course, the formula became justly known as the Euler–Maclaurin summation formula. Another example is Maupertuis's claim for the principle of least action (cf. section 2.3), which Euler had already enunciated before, much more clearly and exhaustively; yet Euler remained supportive of Maupertuis. Euler's forgiving way of reacting to Robins's criticism of the *Mechanica* has already been mentioned in section 3.3.1.

Sharing ideas with others and letting others take part in the process of discovery is another noble trait of Euler. A case in point is the way he put on hold his already extensive work on hydrodynamics, so that his friend Daniel Bernoulli, who was working on the same topic, could complete and publish his own *Hydrodynamics* first! It became a classic.

An important aspect of Euler's personality is his religiousness: By his upbringing in the Riehen parish environment, he was a devout protestant and even served as an elder in one of the protestant communities in Berlin. Indeed, he felt increasingly uncomfortable and frustrated in the company of so many "free-spirits"—as he and others called the followers of French enlightenment—that populated and began to dominate the Berlin Academy. He gave vent to his feelings in the (anonymously published) pamphlet *Rettung der göttlichen Offenbarung gegen die Einwürfe der Freygeister* (E92, *Opera omnia*, Ser. III, Vol. 12, pp. 267–286). This frustration may well have had something to do with his atypically harsh treatment of Johann Samuel König in the dispute about the Euler/Maupertuis principle of least action (cf. section 2.3). It may also have been one, and not the least, of the reasons why Euler left Berlin and returned to St. Petersburg.

4.2. Intellect. There are two outstanding qualities in Euler's intellect: a phenomenal memory, coupled with an unusual power of mental calculation, and an ease in concentrating on mental work irrespective of any hustle and bustle going on around him: "A child on the knees, a cat on his back, that's how he wrote his immortal works," recounts Dieudonné Thiébault, the French linguist and confidant of Frederick II. With regard to memory, the story is well known of Euler's ability, even at an advanced age, to recite by heart all the verses of Virgil's *Aeneid.* One of these, Euler says in a memoir, has given him the first ideas in solving a problem in mechanics. Niklaus Fuss also tells us that during a sleepless night, Euler mentally calculated the first six powers of all the numbers less than twenty (less than 100 in Condorcet's account), and several days later was able to recall the answers without hesitation. "Euler calculates as other people breathe," Condorcet wrote.

Equipped with such intellectual gifts, it is not surprising that Euler was extremely well read. In Fuss's words,

> he possessed to a high degree what commonly is called erudition; he had read the best writers of antique Rome; the older mathematical literature was very well known to him; he was well versed in the history of all times and all people. Even about medical and herbal remedies, and chemistry, he knew more than one could expect from a scholar who doesn't make these sciences a special subject of his study.

Many visitors who came to see Euler went away "with a mixture of astonishment and admiration. They could not understand how a man who during half a century seemed to have occupied himself solely with discoveries in the natural sciences and mathematics could retain so many facts that to him were useless and foreign to the subject of his researches."

4.3. Craftsmanship. Euler's writings have the marks of a superb expositor. He always strove for utmost clarity and simplicity, and he often revisited earlier work when he felt they were lacking in these qualities. Characteristically, he will proceed from very simple examples to ever more complicated ones before eventually revealing the underlying theory in its full splendor. Yet, in his quest for discovery, he could be fearless, even reckless, but owing to his secure instinct, he rarely went astray when his argumentation became hasty. He had an eye for what is essential and unifying. In mechanics, Gleb Konstantinovich Mikhailov [20, p. 67] writes, "Euler possessed a rare gift of systematizing and generalizing scientific ideas, which allowed him to present large parts of mechanics in a relatively definitive form." Euler was open and receptive to new ideas. In the words of André Weil [30, pp. 132–133],

> ... what at first is striking about Euler is his extraordinary quickness in catching hold of any suggestion, wherever it came from.... There is not one of these suggestions which in Euler's hands has not become the point of departure of an impressive series of researches.... Another thing, not less striking, is that Euler never abandons a research topic, once it has excited his curiosity; on the contrary, he returns to it, relentlessly, in order to deepen and broaden it on each revisit. Even if all problems related to such a topic seem to be resolved, he never ceases until the end of his life to find proofs that are "more natural," "simpler," "more direct."

4.4. Epilogue. In closing, let me cite the text (translated from German)—concise but to the point—that Otto Spiess had inscribed on a memorial plaque attached near the house in Riehen in which Euler grew up:

LEONHARD EULER
1707–1783
Mathematician, physicist, engineer,
astronomer and philosopher, spent his
youth in Riehen. He was a great scholar
and a kind man.

5. Further Reading. For readers interested in more details, we recommend the authoritative scientific (yet formula-free!) biography by Fellmann [10], the essays in the recent book by Henry [16], and several accounts on Euler and parts of his work that have recently appeared: Bogolyubov, Mikhailov, and Yushkevich [3], Bradley, D'Antonio, and Sandifer [5], Dunham [6], [7], Nahin [21], Sandifer [22], [23], and Varadarajan [29].

The web site of the U.S. Euler Archive,

http://www.math.dartmouth.edu/∼euler,

also provides detailed information about Euler's complete works, arranged by their E-numbers.

Sources and Acknowledgments. The sources for the videos posted here, with permission, are as follows. Video `buckle.avi`: Professor Wolfgang Ehlers, Institute of Applied Mechanics (CE), University of Stuttgart, Germany. Video `eulerflow.avi`: 2-*dimensional compressible inviscid flow about a circular cylinder—a computer simulation by Nicola Botta*, ©1993 Eidgenössische Technische Hochschule Zürich. Video `zahn.avi`: Professor Bert Jüttler, Institute of Applied Geometry, Johannes Kepler Universität, Linz, Austria. Video `eulerdisk.avi`: produced at the author's request by Olaf A. Schulte, Multimedia Services, ETH Zürich, Zürich, Switzerland, ©2007 Walter Gautschi.

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