Lesson 7: An Area Formula for Triangles

Student Outcomes

- Students prove the formula $\text{Area} = \frac{1}{2}bc\sin(A)$ for a triangle. They reason geometrically and numerically to find the areas of various triangles.

Lesson Notes

This lesson starts with triangles in the Cartesian plane. Students discover that the height of a triangle can be calculated using the sine function for rotations that correspond to angles between $0°$ and $180°$. They create convincing arguments as they calculate the areas of various triangles and eventually generalize their work to derive a formula for the area of any oblique triangle. The focus of this lesson is **G-SRT.D.9**, which calls for students to derive an area formula for a triangle. Students complete an Exploratory Challenge that leads them to this formula through a series of triangle area problems. In Geometry, students derived the formula $\text{Area} = \frac{1}{2}bc\sin(A)$ in Module 2 Lesson 31. However, in that lesson, the formula only applied to acute triangles because the trigonometry functions were only defined for acute angles.

This lesson begins by introducing the area formula and then making a connection to the definition of the sine function presented in Algebra II Module 2. In the Exploratory Challenge, students start with triangles drawn in the coordinate plane with one vertex located at the origin and then generalize specific examples to derive the formula. The standard calls for the construction of an auxiliary line through one vertex of a triangle that is perpendicular to the side opposite the vertex, which is included in Exercises 7 and 8. By defining the height of a triangle in terms of the sine function definition from Algebra II, we do not need to treat the case with an obtuse triangle as separate from an acute triangle when deriving the formula. You also may wish to review the Geometry Module 2 Lesson 31 to see how we proved the formula.

Prior to the lesson, consider building students' fluency with converting between radians and degrees and with the relationships within special triangles.

Students need a calculator for this lesson.

Scaffolding:

- This lesson has quite a bit of scaffolding built in for students working below grade level. Students begin with concrete examples that can be solved using $\text{Area} = \frac{1}{2}bh$ and make connections to the trigonometry functions via special triangle relationships. Additional scaffolds are provided by situating these triangles in a coordinate grid to reinforce measurement concepts.
- For students working above grade level, skip the first two exercises and begin with the Exploratory Challenge on Exercise 3. Students can more fully explore the last exercise in this lesson, which uses the area formula to develop a general formula for the area of a regular polygon.
- For early finishers of Exercises 1 and 2, have groups come up with a second way to determine the area of the triangles.

Classwork

Opening (2 minutes)

Lead a short discussion to activate prior knowledge about measuring the area and perimeter of geometric figures.

- How could you quickly estimate the area and circumference of the circle as well as the area and perimeter of the triangles shown in Exercise 1?
  - *To estimate the area, we could count the square units and estimate the partial square units since these figures are located in the coordinate plane. To estimate the perimeter of the triangles and the circumference of the circle, we could use a piece of string to measure the length around each figure and then see how many units long it is by comparing it to the number line shown on the coordinate grid.*
- What formulas could we use to precisely calculate the area and circumference of the circle and the area and perimeter of the triangle shown in Exercise 1? What dimensions are needed to use these formulas?
  - *The area formula is $A = \pi r^2$ where r is the radius of a circle. The circumference of a circle is $C = \pi d$ where d is the diameter of the circle. The area of a triangle is $A = \frac{1}{2} b \cdot h$ where b is the length of one side and h is the altitude of the triangle. We find the perimeter of a triangle by adding the measures of all the sides. We would need to know the lengths of three sides to find the perimeter.*

Exploratory Challenge/Exercises 1–6 (15 minutes): Triangles in Circles

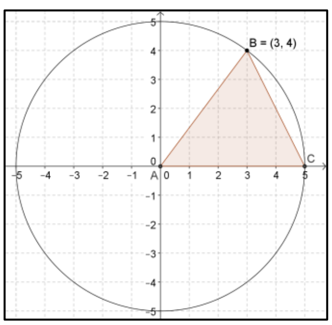
Organize your class into small groups, and have them work through the first few exercises in this Exploratory Challenge. These exercises are designed to scaffold students from concrete to more abstract examples to help them discover that they can determine an altitude of any triangle if they know an angle and a side adjacent to it using the sine function.

Exploratory Challenge/Exercises 1–6: Triangles in Circles

In this Exploratory Challenge, you will find the area of triangles with base along the positive x-axis and a third point on the graph of the circle $x^2 + y^2 = 25$.

1. **Find the area of each triangle shown below. Show work to support your answer.**

   a.

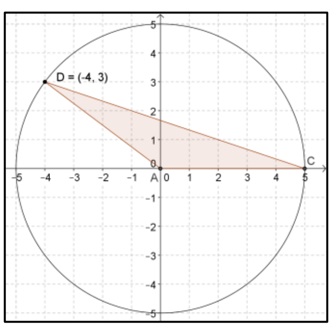


The base is given by the measure of $\overline{AC}$, which is 5 units. The height is the vertical distance from B to the horizontal axis, which is 4 units.

$\frac{1}{2}(5)(4) = 10$

The area is 10 square units.

   b.



The base is 5 units and the height is 3 units.

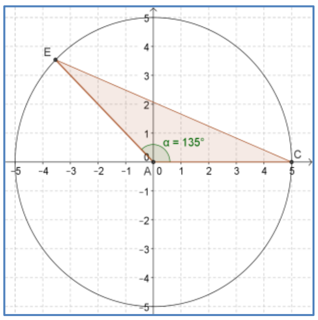
$\frac{1}{2}(5)(3) = 7.5$

The area is 7.5 square units.

If most of your students struggle on Exercise 1, consider pausing to review how to draw in the height of a triangle. Reinforce that the height is a segment from one vertex perpendicular to the opposite side (**G-SRT.D.9**). You may also discuss why it would make the most sense in these problems to use the measure of side $\overline{AC}$ for the base. Students need to use special right triangle relationships to answer the problems in Exercise 2. These ideas were most recently revisited in Lessons 1 and 2 of this module.

2. Find the area of each of the triangles shown below. Show work to support your answer.

a.

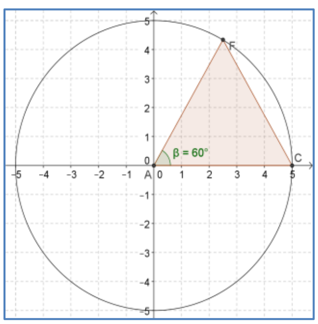


Draw a perpendicular line from E to the horizontal axis. The distance from E to the horizontal axis is a leg of a $45°$-$45°$-$90°$ right triangle whose sides are in a ratio $a: a: a\sqrt{2}$. Since $EA = 5$, the height of the triangle will be the solution to the equation $a\sqrt{2} = 5$. Thus, the height of triangle AEC is $\frac{5}{\sqrt{2}}$ or $\frac{5}{2}\sqrt{2}$ units.

$$\frac{1}{2}(5)\left(\frac{5}{2}\sqrt{2}\right) = \frac{25}{4}\sqrt{2}$$

The area is $\frac{25}{4}\sqrt{2}$ square units.

b.



Draw a perpendicular line from F to the horizontal axis. The height of the triangle is a leg of a $30°$-$60°$-$90°$ right triangle whose sides are the ratio $a: a\sqrt{3}: 2a$. Since $AF = 5$ units, the shorter leg will be $\frac{5}{2}$ units and the longer leg will be $\frac{5}{2}\sqrt{3}$ units. Thus, the height of the triangle is $\frac{5}{2}\sqrt{3}$ units.

$$\frac{1}{2}(5)\left(\frac{5}{2}\sqrt{3}\right) = \frac{25}{4}\sqrt{3}$$

The area is $\frac{25}{4}\sqrt{3}$ square units.

Before moving the class on to Exercise 3, have different groups of students present their solutions to the class. Focus the discussion on any different or unique approaches your students may have utilized. While the solutions above do not focus on using trigonometric ratios to determine the heights of the triangles, do not discount students that may have used this approach. Present solutions that use trigonometry last when you are discussing these problems.

The arguments presented in Exercise 3 part (a) focus on thinking about the circle shown as a dilation of the unit circle and applying the definitions of the sine and cosine functions covered in Algebra II Module 2. Some students may use the right trigonometry ratios they learned in Geometry Module 2 and may also specifically recall the formula they derived for area in Geometry Module 2 Lesson 31. The proof developed in Geometry relied on the definitions of the trigonometric functions that were defined for acute angles only. In this course, we extend use of the sine function to develop formulas for areas of triangles that include obtuse as well as acute triangles.

MP.3

3. Joni said that the area of triangle AFC in Exercise 2 part (b) can be found using the definition of the sine function.

a. What are the coordinates of point F in terms of the cosine and sine functions? Explain how you know.

The coordinates are the point on the unit circle given by $(\cos(\theta), \sin(\theta))$, where θ is the rotation of a ray from its initial position to its terminal position. This circle has a radius of 5 units so each point on this circle is a dilation by a factor of 5 of the points on the unit circle; thus, the coordinates are $(5\cos(\theta), 5\sin(\theta))$.

b. Explain why the y-coordinate of point F is equal to the height of the triangle.

If $\overline{AC}$ is the base of the triangle, then the height of the triangle is on a line perpendicular to the base that contains point F. The y-coordinate of any point in the Cartesian plane represents the distance from that point to the horizontal axis. Thus, the y-coordinate is equal to the height of this triangle.

c. Write the area of triangle AFC in terms of the sine function.

The height is the y-coordinate of a point on the circle of radius 5 units. This y-coordinate is $5\sin\left(\frac{\pi}{3}\right)$. Thus, the area in square units is given by

$$\begin{aligned}\text{Area} &= \frac{1}{2}(5)\left(5\sin\left(\frac{\pi}{3}\right)\right)\\ &= \frac{5}{2}\left(\frac{5\sqrt{3}}{2}\right)\\ &= \frac{25}{4}\sqrt{3}\end{aligned}$$

d. Does this method work for the area of triangle AEC?

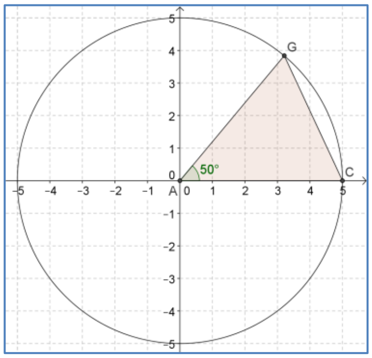
Yes. The angle $135°$ corresponds to a rotation of $\frac{3\pi}{4}$ radians. In square units, the area of triangle AEC is given by

$$\begin{aligned}\text{Area} &= \frac{1}{2}(5)\left(5\sin\left(\frac{3\pi}{4}\right)\right)\\ &= \frac{5}{2}\left(\frac{5\sqrt{2}}{2}\right)\\ &= \frac{25}{4}\sqrt{2}\end{aligned}$$

The next two exercises require students to use the sine function to determine the measurement of the height of the triangles.

4. Find the area of the following triangles.

a.

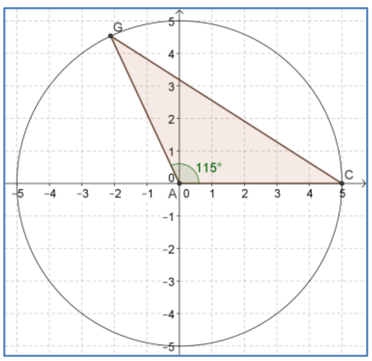


The base is 5 units, and if we draw a perpendicular line from point G to side $\overline{AC}$, then the distance from G to $\overline{AC}$ is given by $5\sin(50°)$.

$$\frac{1}{2}(5)(5\sin(50°)) \approx 9.58$$

The area is approximately 9.58 square units.

b.



Drawing a perpendicular line from point G to the horizontal axis gives a height for this triangle that is equal to $5\sin(115°)$.

$$\frac{1}{2}(5)\big(5\sin(115°)\big) = 11.33$$

The area is approximately 11.33 square units.

Monitor groups as they work on Exercise 4, and move them on to Exercise 5. Hold a brief discussion if needed before proceeding to Exercise 5.

- How are all these problems similar?
  - *They all have the same base along the horizontal axis. To determine the height of the triangle, we needed to draw an auxiliary line perpendicular to the base from the point on the circle that corresponded to the third vertex of the triangle. The y-coordinate of the point corresponded to the height of the triangle.*
- What did you have to do differently in Exercise 4 to determine the height of the triangles?
  - *The vertex on the circle did not correspond to easily recognized coordinates, nor could we use special right triangle ratios to determine the exact height. We had to use the sine function to determine the y-coordinate of the point that corresponded to the height of the triangle.*

By noting these similarities and differences, students should now be able to generalize their work on the first four exercises.

MP.2 & MP.8

5. **Write a formula that will give the area of any triangle with vertices located at $A(0,0)$, $C(5,0)$, and $B(x,y)$ a point on the graph of $x^2+y^2=25$ such that $y>0$.**

   $\text{Area} = \frac{25}{2}\sin(\theta)$, *where θ is the counterclockwise rotation of B from C about the origin.*

6. **For what value of θ will this triangle have maximum area? Explain your reasoning.**

   The triangle has maximum area when $\theta = \frac{\pi}{2}$ because the height is the greatest for this rotation since that is the rotation value when the height of the triangle is equal to the radius of the circle. For all other rotations, given that $y>0$, the height is less than the radius.

Discussion (5 minutes)

The approach presented in the previous six exercises depended on the triangles being positioned in a circle. This was done to best utilize the definition of the sine function, which is required to derive the area formula for a triangle that is the focus of this lesson. This discussion helps students to see how to apply the same reasoning to circles of any size and then to triangles where the two given sides of the triangle are not equal.

- How would your approach to finding the area change if the triangles were constructed in a circle with a different radius?
  - *The base would equal the radius and the height would be the radius multiplied by the sine of the angle.*
- All of the triangles in the previous examples were isosceles triangles. Explain why.
  - *Two sides were radii of the circle. All radii of a circle are congruent.*
- How would your approach to finding the area change if the triangles were scalene?
  - *The base would change, but we could still use the sine function to find a value for the height of the triangle if we knew two sides and the angle in between them.*

Exploratory Challenge/Exercises 7–10 (15 minutes): Triangles in Circles

In these exercises, students extend their thinking and begin to generalize a process for finding the height of a triangle using the sine function. As you debrief Exercise 7, be sure to emphasize that the height from D to point $\overline{AC}$ is still given by $4\sin(65°)$ because D can be thought of as a point on a circle with radius 4 units and center at the origin A. Changing the base to any other length will not affect the height of this triangle.

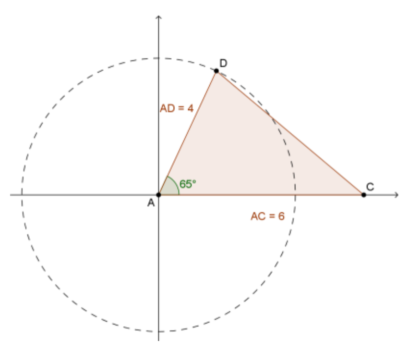
Exploratory Challenge/Exercises 7–10: Triangles in Circles

7. **Find the area of the following triangle.**

   The height is $4\sin(65°)$ units, and the base is 6 units.

   $\frac{1}{2}(6)(4\sin(65°)) \approx 10.88$

   The area is approximately 10.88 square units.



This exercise asks students to generalize what they did in Exercise 7. Ask students to compare the variable symbols that represent the parts of the triangle shown below and how they relate back to the numbers in the previous exercise.

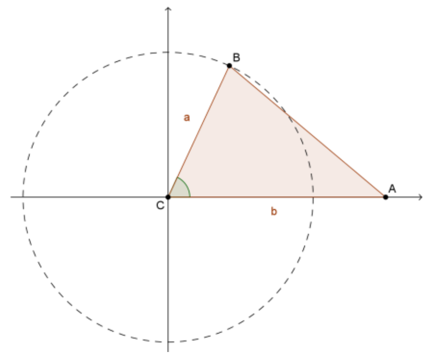
An oblique triangle is a triangle with no right angles. Students may need some introduction to this concept.

8. Prove that the area of any oblique triangle is given by the formula

$$\text{Area} = \frac{1}{2}ab\sin(C)$$

where a and b are adjacent sides of $\triangle ABC$ and C is the measure of the angle between them.

If we take B to be a point on a circle of radius a units centered at C, then the coordinates of B are given by $(a\cos(C), a\sin(C))$ where C is the measure of $\angle BCA$. Construct the height of $\triangle ABC$ from point B to side $\overline{AC}$.

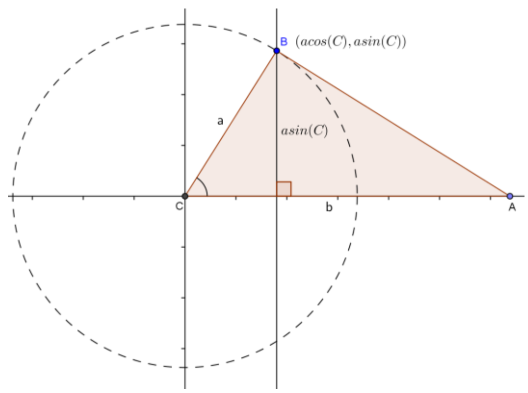


MP.3

Note the diagram has the auxiliary line drawn.

This height corresponds to the y-coordinate of B, which is $a\sin(C)$. The base is b units, and thus, the area is as follows:

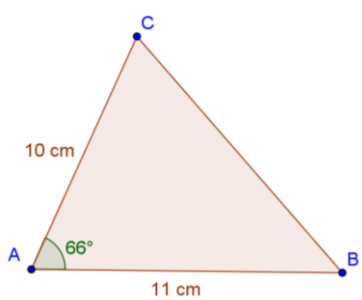
$$\begin{aligned}\text{Area} &= \frac{1}{2}(\text{base})(\text{height})\\ &= \frac{1}{2}(b)(a\sin(C))\\ &= \frac{1}{2}ab\sin(C).\end{aligned}$$



In the last exercises, students apply the formula that they just derived. If students are struggling to use the formula in Exercise 9 parts (b) and (c), remind them that these triangles could easily be rotated and translated to have the given angle correspond to the origin, which shows that the formula works regardless of the position of the triangle in a plane as long as two sides and the included angle are given.

9. Use the area formula from Exercise 8 to calculate the area of the following triangles.

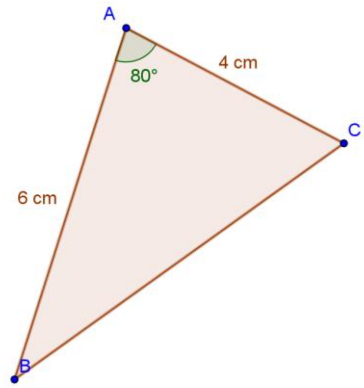
a.



$$\text{Area} = \frac{1}{2}(10)(11)\sin(66°)$$
$$\text{Area} = 55\sin(66°)$$
$$\text{Area} \approx 50.25$$

The area is $50.25\ \text{cm}^2$.

b.



$\text{Area} = \frac{1}{2}(6)(4)\sin(80°)$

$\text{Area} = 12\sin(80°)$

$\text{Area} \approx 11.82$

The area is 11.82 cm^2.

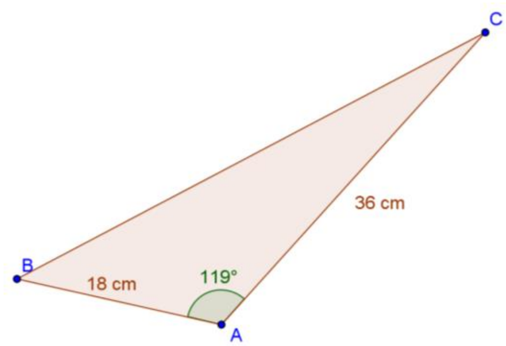
c. A quilter is making an applique design with triangular pieces like the one shown below. How much fabric is used in each piece?

$\text{Area} = \frac{1}{2}(18)(36)\sin(119°)$

$\text{Area} = 18^2\sin(119°)$

$\text{Area} \approx 283.38$

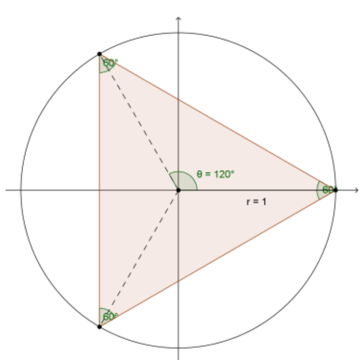
Each triangular piece is 283.38 cm^2 *of fabric.*



Exercise 10 can be used for early finishers or as an additional Problem Set exercise if time is running short. These problems are similar to the approach used by Archimedes to approximate the value of π in ancient times. Of course, he did not have the sine function at his disposal, but he did approximate the value of π by finding the area of regular polygons inscribed in a circle as the number of sides increased.

10. Calculate the area of the following regular polygons inscribed in a unit circle by dividing the polygon into congruent triangles where one of the triangles has a base along the positive x-axis.

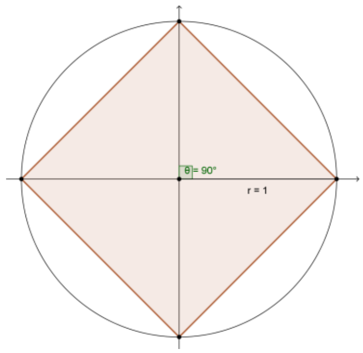
a.



$3\left(\frac{1}{2}\right)(1)(1 \cdot \sin(120°)) \approx 1.30$

The area is approximately 1.30 *square units.*

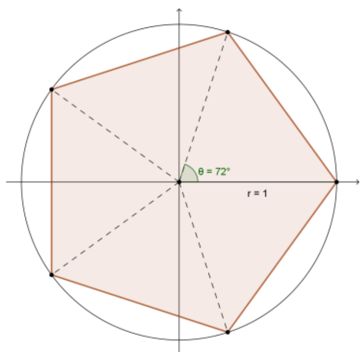
b.



$4\left(\frac{1}{2}\right)(1)(1 \cdot \sin(90°)) \approx 2$

The area is approximately 2 *square units.*

c.



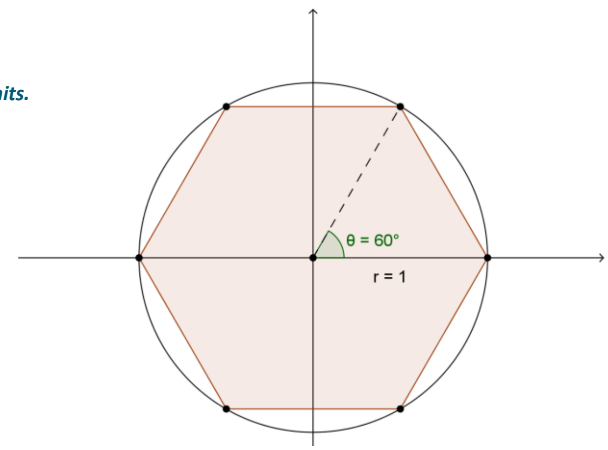
$5\left(\frac{1}{2}\right)(1)(1 \cdot \sin(72°)) \approx 2.38$

The area is approximately 2.38 *square units.*

d. **Sketch a regular hexagon inscribed in a unit circle with one vertex at $(1, 0)$, and find the area of this hexagon.**

$6\left(\frac{1}{2}\right)(1)(1 \cdot \sin(60°)) \approx 2.59$

The area is approximately 2.59 *square units.*



MP.8

e. **Write a formula that gives the area of a regular polygon with n sides inscribed in a unit circle if one vertex is at $(1, 0)$ and θ is the angle formed by the positive x-axis and the segment connecting the origin to the point on the polygon that lies in the first quadrant.**

$\text{Area} = \frac{n}{2}\sin\left(\frac{360°}{n}\right)$

f. **Use a calculator to explore the area of this regular polygon for large values of n. What does the area of this polygon appear to be approaching as the value of n increases?**

If we select $n = 10$, the area is 2.94 square units.

If we select $n = 30$, the area is 3.12 square units.

If we select $n = 100$, the area is $3.139\,525$ square units.

The area appears to be approaching π.

To further reinforce the solutions to part (f), ask students to calculate the area of the unit circle.

- What is the area of the unit circle?
  - *The area is π square units.*
- How does the area of a polygon inscribed in the circle compare to the area of the circle as the number of sides increases?
  - *As the number of sides increases, the polygon area would be getting closer to the circle's area.*

Another interesting connection can be made by graphing the related function and examining its end behavior. Students can confirm by graphing that the function, $f(x) = \frac{x}{2}\sin\left(\frac{2\pi}{x}\right)$ for positive integers x, appears to approach a horizontal asymptote of $y = \pi$ as the value of x increases.

Closing (4 minutes)

Use these questions as a quick check for understanding before students begin the Exit Ticket. You can encourage students to research other area formulas for oblique triangles as an extension to this lesson.

- Draw a triangle whose area can be calculated using the formula $\text{Area} = \frac{1}{2}ab\sin(C)$, and indicate on the triangle the parameters required to use the formula.
  - *Solutions will vary but should be a triangle with measurements provided for two sides and the angle formed by them.*
- Draw a triangle whose area CANNOT be calculated using this formula.
  - *Solutions will vary. One example would be an oblique triangle with three side measures given.*

Lesson Summary

The area of $\triangle ABC$ is given by the formula:

$$\text{Area} = \frac{1}{2}ab\sin(C)$$

where a and b are the lengths of two sides of the triangle and C is the measure of the angle between these sides.

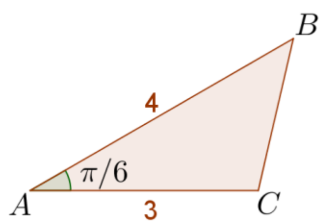
Exit Ticket (4 minutes)

Name ____ Date ____________________

Lesson 7: An Area Formula for Triangles

Exit Ticket

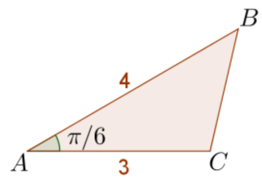
1. Find the area of $\triangle ABC$.



2. Explain why $\frac{1}{2}ab\sin(\theta)$ gives the area of a triangle with sides a and b and included angle θ.

Exit Ticket Sample Solutions

1. **Find the area of $\triangle ABC$.**

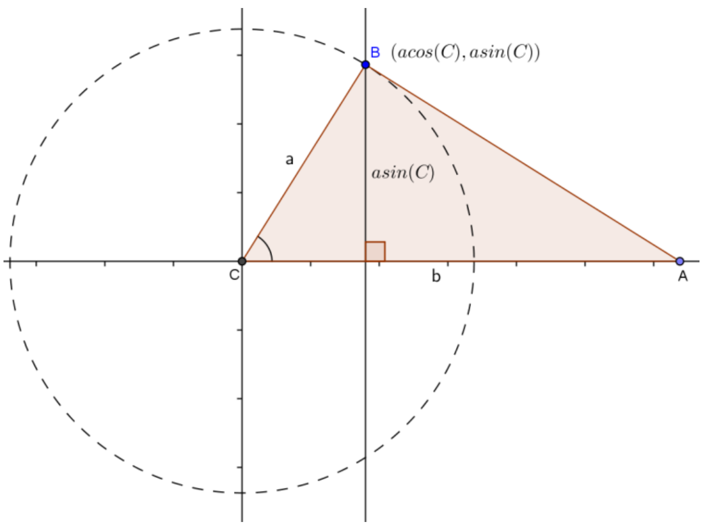


$\text{Area} = \frac{1}{2}(3)(4)\sin\left(\frac{\pi}{6}\right) = 3$

The area is 3 square units.

2. **Explain why $\frac{1}{2}ab\sin(\theta)$ gives the area of a triangle with sides a and b and included angle θ.**

In the diagram below, the height is the perpendicular line segment from point B to the base b. The length of this line segment is $a\sin(\theta)$, which is the y-coordinate of point B, a point on a circle of radius a with center at C as shown.

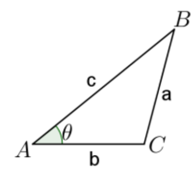


The area of a triangle is one-half the product of one side (the base) and the height to that side. If we let b be the base, then

$$\begin{aligned}\text{Area} &= \frac{1}{2}b \cdot a\sin(\theta)\\ &= \frac{1}{2}ab\sin(\theta).\end{aligned}$$

Problem Set Sample Solutions

1. Find the area of the triangle ABC shown to the right, with the following data:



a. $\theta = \frac{\pi}{6}$, $b = 3$, and $c = 6$.

$$\frac{1}{2}\left(b \cdot c \cdot \sin\left(\frac{\pi}{6}\right)\right) = \frac{1}{2}\left(18 \cdot \frac{1}{2}\right) = \frac{9}{2}$$

The area is $\frac{9}{2}$ *square units.*

b. $\theta = \frac{\pi}{3}$, $b = 4$, and $c = 8$.

$$\frac{1}{2}\left(b \cdot c \cdot \sin\left(\frac{\pi}{3}\right)\right) = \frac{1}{2}\left(32 \cdot \frac{\sqrt{3}}{2}\right) = 8\sqrt{3}$$

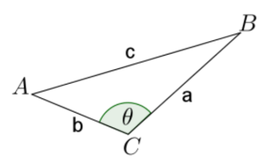
The area is $8\sqrt{3}$ *square units.*

c. $\theta = \frac{\pi}{4}$, $b = 5$, and $c = 10$.

$$\frac{1}{2}\left(b \cdot c \cdot \sin\left(\frac{\pi}{4}\right)\right) = \frac{1}{2}\left(50 \cdot \frac{\sqrt{2}}{2}\right) = \frac{25\sqrt{2}}{2}$$

The area is $\frac{25\sqrt{2}}{2}$ *square units.*

2. Find the area of the triangle ABC shown to the right, with the following data:



a. $\theta = \frac{3\pi}{4}$, $a = 6$, and $b = 4$.

$$\frac{1}{2}\left(a \cdot b \cdot \sin\left(\frac{\pi}{4}\right)\right) = \frac{1}{2}\left(24 \cdot \frac{\sqrt{2}}{2}\right) = 6\sqrt{2}$$

The area is $6\sqrt{2}$ *square units.*

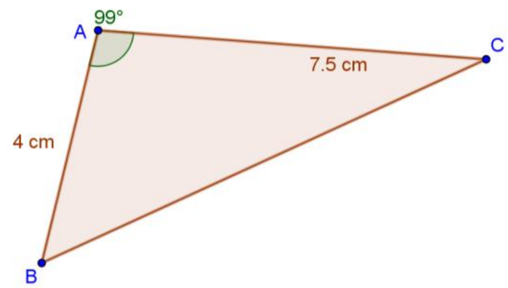
b. $\theta = \frac{5\pi}{6}$, $a = 4$, and $b = 3$.

$$\frac{1}{2}\left(a \cdot b \cdot \sin\left(\frac{\pi}{6}\right)\right) = \frac{1}{2}\left(12 \cdot \frac{1}{2}\right) = 3$$

The area is 3 *square units.*

3. **Find the area of each triangle shown below. State the area to the nearest tenth of a square centimeter.**

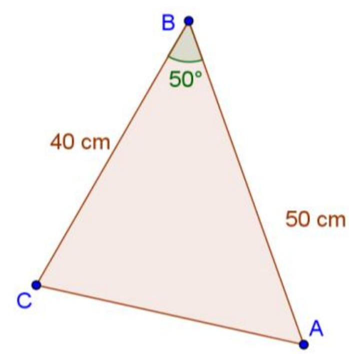
a.



$A = \frac{1}{2} \cdot 4 \cdot 7.5 \cdot \sin(99°) \approx 14.8$

The area is approximately 14.8 sq. cm.

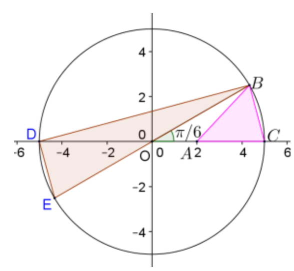
b.



$A = \frac{1}{2} \cdot 40 \cdot 50 \cdot \sin(50°) \approx 766.0$

The area is approximately 766 sq. cm.

4. **The diameter of the circle O in the figure shown to the right is $EB = 10$.**



a. Find the area of the triangle OBA.

$\frac{1}{2}\left(2 \cdot 5 \cdot \sin\left(\frac{\pi}{6}\right)\right) = \frac{1}{2}\left(2 \cdot 5 \cdot \frac{1}{2}\right) = \frac{5}{2}$

The area is $\frac{5}{2}$ *square units.*

b. Find the area of the triangle ABC.

$\frac{1}{2}(bh) = \frac{1}{2}\left(3 \cdot \frac{5}{2}\right) = \frac{15}{4}$

The area is $\frac{15}{4}$ *square units.*

c. **Find the area of the triangle DBO.**

$$\frac{1}{2}(bh) = \frac{1}{2}\left(5 \cdot \frac{5}{2}\right) = \frac{25}{4}$$

The area is $\frac{25}{4}$ square units.

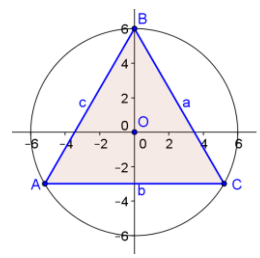
d. **Find the area of the triangle DBE.**

The area of triangle DBE is the sum of the areas of triangles DBO, OBA, and ABC.

$$\frac{25}{4} + \frac{15}{4} + \frac{5}{2} = \frac{50}{4}$$

The area of triangle DBE is $\frac{50}{4}$ square units.

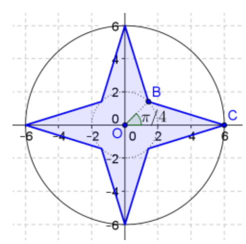
5. **Find the area of the equilateral triangle ABC inscribed in a circle with a radius of 6.**



$$3 \cdot \frac{1}{2}\left(6 \cdot 6 \cdot \sin\left(\frac{2\pi}{3}\right)\right) = \frac{54\sqrt{3}}{2}$$

The area is $\frac{54\sqrt{3}}{2}$ square units.

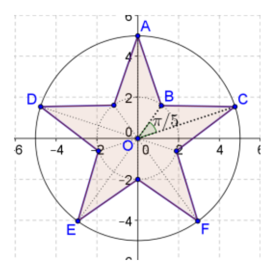
6. **Find the shaded area in the diagram below.**



$$8 \cdot \frac{1}{2}\left(6 \cdot 2 \cdot \sin\left(\frac{\pi}{4}\right)\right) = 24\sqrt{2}$$

The area is $24\sqrt{2}$ square units.

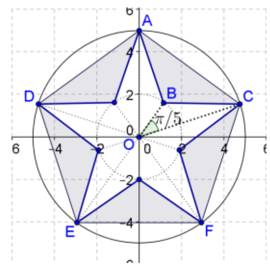
7. **Find the shaded area in the diagram below. The radius of the outer circle is 5; the length of the line segment OB is 2.**



$$10 \cdot \frac{1}{2}\left(5 \cdot 2 \cdot \sin\left(\frac{\pi}{5}\right)\right) \approx 29.389$$

The area is 29.289 square units.

8. **Find the shaded area in the diagram below. The radius of the outer circle is 5.**



$$5 \cdot \frac{1}{2}\left(5 \cdot 5 \cdot \sin\left(\frac{2\pi}{5}\right)\right) = \frac{125}{2}\sin\left(\frac{2\pi}{5}\right) = 59.441$$

The area of the pentagon is 59.441 square units. From Problem 7, we have the area of the star is 29.389. The shaded area is the area of the pentagon minus the area of the star.

The shaded area is 30.052 square units.

9. **Find the area of the regular hexagon inscribed in a circle if one vertex is at $(2, 0)$.**

$$6 \cdot \frac{1}{2}\left(2 \cdot 2 \cdot \sin\left(\frac{2\pi}{6}\right)\right) = 12 \cdot \sin\left(\frac{\pi}{3}\right) = 12 \cdot \frac{\sqrt{3}}{2} = 6\sqrt{3}$$

The area is $6\sqrt{3}$ square units.

10. **Find the area of the regular dodecagon inscribed in a circle if one vertex is at $(3, 0)$.**

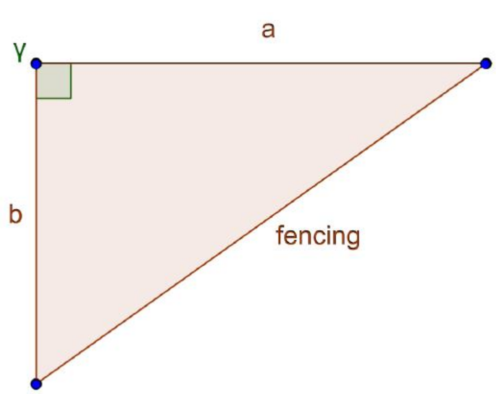
$$12 \cdot \frac{1}{2}\left(3 \cdot 3 \cdot \sin\left(\frac{2\pi}{12}\right)\right) = 54 \cdot \sin\left(\frac{\pi}{6}\right) = 54 \cdot \frac{1}{2} = 27$$

The area is 27 square units.

11. **A horse rancher wants to add on to existing fencing to create a triangular pasture for colts and fillies. She has $1{,}000$ feet of fence to construct the additional two sides of the pasture.**

   a. **What angle between the two new sides would produce the greatest area?**

   Our formula is $A = \frac{1}{2}ab\sin(\gamma)$. When a and b are constant, the equation is maximized when $\sin(\gamma)$ is maximized, which occurs at $\gamma = 90°$. Thus, the greatest area would be when the angle is a $90°$ angle.

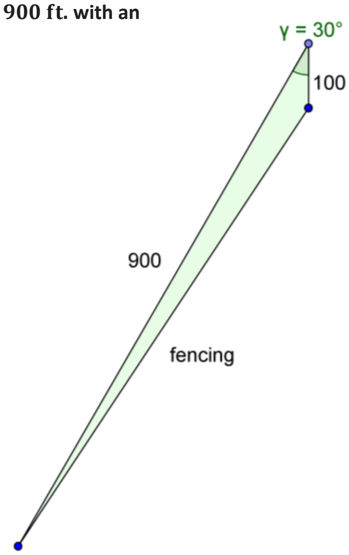


   b. **What is the area of her pasture if she decides to make two sides of 500 ft. each and uses the angle you found in part (a)?**

$$A = \frac{1}{2} \cdot 250\,000 \cdot 1 = 125\,000$$

   The area would be $125{,}000$ sq. ft.

c. **Due to property constraints, she ends up using sides of 100 ft. and 900 ft. with an angle of $30°$ between them. What is the area of the new pasture?**

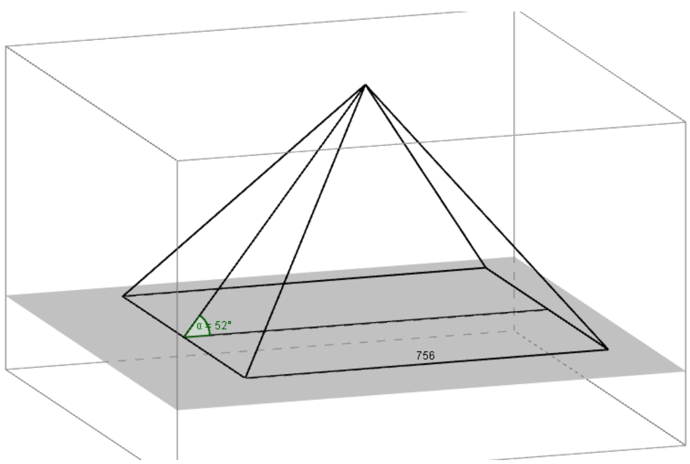


$$A = \frac{1}{2} \cdot 90000 \cdot \sin(30°)$$
$$= 45000 \cdot \frac{1}{2}$$
$$= 22500$$

The area would be $22,500$ *sq. ft.*

12. **An enthusiast of Egyptian history wants to make a life-size version of the Great Pyramid using modern building materials. The base of each side of the Great Pyramid was measured to be 756 ft. long, and the angle of elevation is about $52°$.**

a. **How much material will go into the creation of the sides of the structure (the triangular faces of the pyramid)?**



According to these measurements, each side of the Great Pyramid has a base of 756 ft. *The Great Pyramid's height is in the middle of the triangle, so we can construct a right triangle of side* $\frac{756}{2} = 378$ *and the height of the pyramid, and use the cosine function.*

$$\cos(52°) = \frac{378}{h}$$
$$h = \frac{378}{\cos(52°)}$$
$$h \approx 614$$

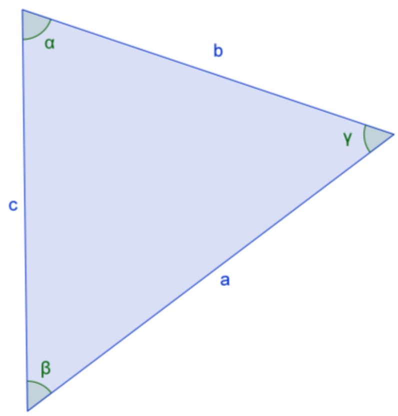
Area of one side: $\frac{1}{2} \cdot 614 \cdot 756 \approx 232\,092$. *For four sides, the area is* $928,368$ sq. ft.

b. **If the price of plywood for the sides is $\$0.75$ per square foot, what is the cost of just the plywood for the sides?**

The total price would be about $\$696,276$ for just the plywood going into the sides.

13. **Depending on which side you choose to be the "base," there are three possible ways to write the area of an oblique triangle, one being $A = \frac{1}{2}ab\sin(\gamma)$.**

a. **Write the other two possibilities using $\sin(\alpha)$ and $\sin(\beta)$.**



$\frac{1}{2}bc\sin(\alpha)$

and

$\frac{1}{2}ac\sin(\beta)$

b. **Are all three equal?**

Yes, since each expression is equal to the area of the triangle, they are all equal to each other.

c. **Find $\frac{2A}{abc}$ for all three possibilities.**

$$\frac{2A}{abc} = \frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c}$$

d. **Is the relationship you found in part (c) true for all triangles?**

There was nothing special about the triangle we picked, so it should be true for all triangles.