Journal of Process Control 24 (2014) 144–157

Contents lists available at ScienceDirect

Journal of Process Control

journal homepage: www.elsevier.com/locate/jprocont

A neutral system approach to H_∞ PD/PI controller design of processes with uncertain input delay

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ARTICLE INFO

Article history:
Received 4 January 2013
Received in revised form 6 January 2014
Accepted 6 January 2014
Available online 9 February 2014

Keywords:
H_∞ control
Neutral systems
Uncertain time-invariant delay
PD/PI control
Linear matrix inequality (LMI)

ABSTRACT

This paper presents a neutral system approach to the design of an H_∞ controller for input delay systems in presence of uncertain time-invariant delay. It is shown that when proportional derivative (PD) controller is applied to a time-delay system, the resulting closed loop system is generally a time-delay system of neutral type with delay term coefficients depending on the controller parameters. A descriptor model transformation is used to derive an advantageous bounded real lemma representation for the system. Furthermore, new delay-dependent sufficient conditions for the existence of an H_∞ PD and PI controller in presence of uncertain delay are derived in terms of matrix inequalities. Some case studies and numerical examples are given in order to illustrate the advantages of the proposed method.

1. Introduction

Design of the PID family controllers (P, PI, PD and PID) is still one of the attractive topics among the control research community due to their wide application in industrial systems [1–4]. However, studies show that these controllers may perform inferior to a simple manual tuning [5]. This is mostly because of intrinsic nonlinearity and plants aging behavior which cause time variation of physical parameters of the plants. Moreover, many processes are under input or output time-delay which produces a decrease in the system phase and also gives rise to a non-rational transfer function of the system, making them more difficult to analyze and control [6]. Bjorklund and Ljung [7] showed that none of the time-delay estimation methods could accurately estimate the actual delay even when the under study plant is linear time invariant. Design of a PID controller becomes more challenging, when the system under study includes uncertainty in time-delay. In this paper, we study a descriptor approach design of H_∞ PD/PI controller for a linear system with uncertain delay.

Although there are various techniques presented in the literature for designing PID controllers [8,9], the problem is still open especially for time-delay processes. Moreover, the number of PID design methods for multi-input multi-output (MIMO) is limited [10]. Recently, new method to compute the set of stabilizing PID controller parameters for arbitrary time-delay system is given [1]. Wang used a graphical approach to find the stabilizing region of PID controllers for high-order all pole plants with time-delay [3]. Stabilization of a class of delayed unstable processes by simple controllers was investigated by Lee et al. [4] which uses Nyquist method to determine the parameter ranges of stabilizing PID family controllers. A review on various PI/PID controller design methods for integrating and unstable system was presented in Ref. [11]. Moreover, many results have been appeared in the literature using PID family controllers in particular applications [12–14]. In addition to the above methods, Smith predictor structure can be considered as the first control method for single-input single-output (SISO) linear time-delay processes [15]. A predictive PI (PIP) controller with the same structure as Smith predictor is presented by Hägglund [16] which is suitable for processes with long dead-time. In order to improve the robustness of this controller, Normey-Rico et al. [17] proposed a modified Smith predictor using an additional filter to the structure of the PIP controller (FPIP). They also presented a unified approach for designing dead-time compensators for systems with time-delay on the base of FPIP in [18]. Extensions of this work for multi-input

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http://dx.doi.org/10.1016/j.jprocont.2014.01.003

multi-output (MIMO) square and MIMO non-square processes has been reported in Refs. [19] and [20], respectively. Furthermore, Liu et al. [21] and Lu et al. [22] presented some solutions for unstable systems with time delay using Smith predictor approach. Recently, a methodology was provided for control of unstable systems with long time-delay based on generalized predictor approach [23]. In a second paper, the same authors presented an extension of this method to multi-input-multi output systems in Ref. [24]. Moreover, a review on the design, analysis and tuning of dead time compensators for stable and unstable processes using smith predictor method has been obtained by Normey-Rico and Camacho [25].

For linear systems with uncertain input delay, Shariati et al. presented a synthesis of H_∞ PD/PI controller in Ref. [26]. They showed that applying a PD or PID controller to a linear system with input delay leads to a time-delay system of retarded or neutral type. In the neutral type which contains delays both in its state and its derivative of state [27], the resulting delay term coefficients depend on the controller gains. Up to the present, many papers have considered Lyapunov based stabilization and H_∞ control of neutral systems. In this line of research, robust H_∞ state feedback control of uncertain neutral system has been considered by Chen [28]. H_∞ state feedback controller is obtained by formulating an optimization problem through linear matrix inequality constraints. Moreover, H_∞ output feedback control of neutral systems has been the center of attention in some researches [29]. Observer-based H_∞ state feedback control for a class of uncertain neutral systems is considered by Lien [30]. In the time-varying delay case, Suplin et al. [31] proposed delay-dependent sufficient conditions for H_∞ control of a neutral system in presence of time-varying state delay. Recently, the stability problem of a class of linear switched systems with time-varying input delay is also addressed [32]. Furthermore, Liu et al. [33] presented some necessary and sufficient stability conditions for continuous-time positive systems with time-varying delay. A review on these results shows that few papers have focused on the special neutral time-delay systems with both delayed term coefficients depending on the control law parameters. Recently, this class of time-delay systems of neutral type was investigated using an H_∞ proportional-derivative state feedback and state-derivative feedback controller in Refs. [34] and [35], respectively. This line of research was followed in Ref. [26] designing an H_∞ PD/PI controller for linear systems with uncertain delay.

In the present work, a new bounded real lemma (BRL) is introduced and the design of an output feedback PD controller for a system with uncertain time-invariant input delay is addressed. This method can be easily used for designing a PI controller by augmenting an integrator to the system model. This point is absolutely attractive from the perspective of process control practitioners, especially noting that this method is applicable for both SISO and MIMO processes in simple conventional control structure. Moreover, the proposed methodology can be applied for a wide variety of process model formulations including stable/unstable, minimum/non-minimum phase and integrative systems. In this paper we introduce a new form of the Lyapunov–Krasovskii functional that contains two integral terms with the second derivative of $x(t)$ as:

$$\int_{t-\tau}^{t} \ddot{x}^T(\alpha) Z_2 \ddot{x}(\alpha)\, d\alpha \quad \text{and} \quad \int_{-\tau}^{0}\int_{t+\beta}^{t} \ddot{x}^T(\alpha) Z_2 \ddot{x}(\alpha)\, d\alpha\, d\beta$$

or

$$\int_{t-\tau}^{t} \dot{y}^T(\alpha) Z_2 \dot{y}(\alpha)\, d\alpha \quad \text{and} \quad \int_{-\tau}^{0}\int_{t+\beta}^{t} \dot{y}^T(\alpha) Z_2 \dot{y}(\alpha)\, d\alpha\, d\beta$$

with $y(t) = \dot{x}(t)$ in the descriptor form which plays an important role in the main result of the paper. This paper has a descriptor approach to the resulting closed-loop system of neutral type with delay term coefficients depending on the control law parameters, however, all of the results carried out for the neutral case are valid for the retarded type systems, as well. From the perspective of PID family controller design, in addition to providing a unified method for designing H_∞ PD/PI controller applicable for both SISO and MIMO linear systems with uncertain input delay, the presented descriptor approach expands the feasibility region of the resulting matrix inequality conditions and consequently, obtains better closed-loop performance compared to the results provided in Ref. [26].

This paper is organized as follows. Problem formulation and preliminary results are introduced in Section 2. In Section 3, a new bounded real lemma and H_∞ PD/PI controller design is given for uncertain input delay. Some illustrative case studies are provided in Section 4 to show the effectiveness of the proposed methods. Finally, the concluding remarks are given in Section 5.

2. Problem formulation and preliminary results

In this paper, we consider the following time-delay system with input delay:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t-\tau) + Fw(t) \\ z(t) &= C_z x(t) + D_z w(t) \\ y_p(t) &= C_y x(t) \end{aligned} \tag{1}$$

where $x(t)$ is the state, $u(t) \in \mathcal{R}^l$ is the system input, w is the disturbance input of system and belongs to the Sobolev space $\mathcal{W}^{1,2}(0, \infty, \mathcal{R}^p) \cap \mathcal{L}^2(0, \infty, \mathcal{R}^p)$ (this means that both $w(t)$ and $\dot{w}(t)$ belong to $\mathcal{L}^2(0, \infty)$), $y_p(t) \in \mathcal{R}^r$ is the measured output, τ is the uncertain time-invariant delay of the system and is assumed to satisfy $0 \le \tau < \bar{\tau}$ and $z(t) \in \mathcal{R}^q$ is the controlled system output. The matrices $A \in \mathcal{R}^{n\times n}$, $B \in \mathcal{R}^{n\times l}$, $F \in \mathcal{R}^{n\times p}$, $C_z \in \mathcal{R}^{q\times n}$, $C_y \in \mathcal{R}^{r\times n}$, $D_Z \in \mathcal{R}^{q\times p}$ are assumed to be known. Similar to the results recently presented for proportional-derivative state feedback [34], by applying a PD control law such as:

$$u = K_p y_p + K_d \dot{y}_p \tag{2}$$

to the time-delay system (1), the corresponding closed-loop system is generally governed by the following state space equations:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_h x(t-\tau) + A_d \dot{x}(t-\tau) + Fw(t) \\ z(t) &= C_z x(t) + D_z w(t) \end{aligned} \tag{3}$$

where, $A_h = BK_pC_y$ and $A_d = BK_dC_y$. If the term $A_d\dot{x}(t-\tau)$ is expressed by the state variables, the closed-loop system (3) transforms to a retarded type representation, otherwise, the above state space equation remains neutral with both A_h and A_d coefficients depending on the controller parameters K_p and K_d. In this paper we consider the latter case which is more challenging and involves both retarded and neutral type closed loop systems. Before proceeding further, let us state two lemmas which will be used in the main result of the paper.

Lemma 1. [36]: Assume $a(.) \in \mathcal{R}^{n_a}$, $b(.) \in \mathcal{R}^{n_b}$ and $N \in \mathcal{R}^{n_a \times n_b}$ are defined on the interval Ω, then for any matrices $X, Y, Z \in \mathcal{R}^{n_a \times n_b}$, the following inequality holds:

$$-2\int_{\Omega} a^T(\alpha) N b(\alpha)\, d\alpha \le \int_{\Omega} \begin{bmatrix} a(\alpha) \\ b(\alpha) \end{bmatrix}^T \begin{bmatrix} X & Y-N \\ Y^T-N^T & Z \end{bmatrix} \begin{bmatrix} a(\alpha) \\ b(\alpha) \end{bmatrix} d\alpha, \tag{4}$$

where

$$\begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix} > 0 \tag{5}$$

Lemma 2. [35]: Consider the neutral system (1) and define $d(t) = [mw^T(t)\,(1-m)\dot{w}^T(t)]^T$ where m is a real scalar value and $0 < m < 1$. If $\left\| T_{zd} \right\|_\infty < \gamma$, then the inequality $\left\| T_{zw} \right\|_\infty < \gamma$ is satisfied.

The above lemma can be proved using matrix H_∞ norm properties. The detailed proof is given in Ref. [35].

For a prescribed scalar $\gamma > 0$, we define the performance index

$$J(w) = \int_0^\infty (z^T z - \gamma^2 w^T w)\, d\tau \tag{6}$$

Furthermore, we represent (3) in the equivalent descriptor form [38]:

$$\begin{aligned} \dot{x}(t) &= y(t) \\ y(t) &= Ax(t) + A_h x(t-\tau) + A_d\dot{x}(t-\tau) + Fw(t) \\ z(t) &= C_z x(t) + D_z w(t) \end{aligned} \tag{7}$$

or

$$\begin{aligned} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} &= \begin{bmatrix} 0 & I \\ A & -I \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ A_h & A_d \end{bmatrix} \begin{bmatrix} x(t-\tau) \\ y(t-\tau) \end{bmatrix} + \begin{bmatrix} 0 \\ F \end{bmatrix} w(t) \\ z(t) &= C_z x(t) + D_z w(t) \end{aligned} \tag{8}$$

where, $A_h = BK_pC_y$, $A_d = BK_dC_y$.

3. Main results

Delay is appeared in the input of many industrial plants. Thus, Eq. (3) can be considered as a general closed-loop state space representation of these plants when they are under PD controller. In this section, a new delay-dependent bounded real lemma representation is derived for the special neutral system of (3). Furthermore, as we will explain later, we obtain matrix inequality conditions to find H_∞ PD controller parameters.

3.1. Bounded real lemma

The bounded real lemma (BRL) characterizes the H_∞ norm of a system in term of its state space representation. For a prescribed $\gamma > 0$, we consider the performance index (6). Moreover, throughout Lemma 3, the following assumption is considered to enable the application of Lyapunov method for stability of neutral systems [27]:

Assumption 1. Let the difference operator D: $C[-\bar{\tau}, 0]$, given by $Dx_t = x(t) - A_d x(t-\tau)$ be delay-independently stable with respect to all delays.

A sufficient condition for Assumption 1 is given by the inequality $\left\| A_d \right\| < 1$, where $\|\cdot\|$ denotes any matrix norm.

We use linear neutral system (3) directly to derive a new delay-dependent bounded real lemma representation in terms of some linear matrix inequalities. Hence we obtain the following lemma.

Lemma 3. Consider the system (3). For prescribed scalars $\gamma > 0$ and $0 < m < 1$, the cost function $J(w) < 0$ for all non-zero $w \in \mathcal{W}^{1,2}(0, \infty, \mathcal{R}^p) \cap \mathcal{L}^2(0, \infty, \mathcal{R}^p)$ and all delay $0 \le \tau < \bar{\tau}$, if there exist positive definite symmetric matrices $P_1, Z_1, Z_2, Q, R_1, R_2 \in \mathcal{R}^{n \times n}$ and matrices $P_2, P_3 \in \mathcal{R}^{n \times n}$, $X, Y \in \mathcal{R}^{2n \times 2n}$, $K_p, K_d \in \mathcal{R}^{l \times q}$ satisfying the following LMIs:

$$\begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}^T & \Sigma_{22} \end{bmatrix} < 0 \tag{9}$$

and

$$\begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix} > 0 \tag{10}$$

where

$$\Sigma_{11} = \begin{bmatrix} P^T\begin{bmatrix}0 & I\\ A & -I\end{bmatrix} + \begin{bmatrix}0 & A^T\\ I & -I\end{bmatrix}P + \bar{\tau}X + Y + Y^T & -Y + P^T\begin{bmatrix}0 & 0\\ A_h & A_d\end{bmatrix} & 0 & P^T\begin{bmatrix}0\\ F\end{bmatrix} & 0\\ * & \begin{bmatrix}-Q & 0\\ 0 & -R_1\end{bmatrix} & 0 & 0 & 0\\ * & * & -R_2 & 0 & 0\\ * & * & * & -m^2\gamma^2 I & 0\\ * & * & * & * & -(1-m)^2\gamma^2 I \end{bmatrix}$$

$$\Sigma_{12} = \begin{bmatrix} \begin{bmatrix}Q & 0\\ 0 & R_1\end{bmatrix} & \begin{pmatrix}0\\ \bar{\tau}Z_1\end{pmatrix} & \begin{pmatrix}0\\ A^T R_2\end{pmatrix} & \begin{pmatrix}0\\ \bar{\tau}A^T Z_2\end{pmatrix} & \begin{pmatrix}C_z^T\\ 0\end{pmatrix}\\ 0 & 0 & \begin{pmatrix}0\\ A_h^T R_2\end{pmatrix} & \begin{pmatrix}0\\ \bar{\tau}A_h^T Z_2\end{pmatrix} & 0\\ 0 & 0 & A_d^T R_2 & \bar{\tau}A_d^T Z_2 & 0\\ 0 & 0 & 0 & 0 & D_z^T\\ 0 & 0 & F^T R_2 & \bar{\tau}F^T Z_2 & 0 \end{bmatrix}$$

$$P = \begin{bmatrix}P_1 & 0\\ P_2 & P_3\end{bmatrix},\ Z = \begin{bmatrix}Z_1 & 0\\ 0 & Z_2\end{bmatrix}$$

$$\Sigma_{22} = \mathrm{diag}\left(\begin{bmatrix}-Q & 0\\ 0 & -R_1\end{bmatrix},\quad -\bar{\tau}Z_1,\quad -R_2,\quad -\bar{\tau}Z_2,\quad -I\right)$$

Proof. The proof of Lemma 3 is given in the appendix.

Remark 1. It should be noted that a significant advantage of the resulting bounded real lemma representation in this theorem is its permanent efficiency in designing the H_∞ controllers with both A_h and A_d in (3) depending on the controller parameters. To see this advantage, consider the LMI (9). Y is a full-block matrix variable which appears in both matrix elements (1,1) and (1,2) of Σ_{11}. It means that Y affects on the negative definiteness of (1,1) as well as the value of A_d. Hence, $Y=0$ is not the best solution for LMI (9) in the feasibility region. Consequently, despite the BRLs proposed in the literature [31,32], A_d is not forced to be chosen zero in this special case.

Remark 2. In the proof of Lemma 3, the new terms $\int_{-\tau}^{0}\int_{t+\beta}^{t}\dot{y}^T(\alpha)Z_2\dot{y}(\alpha)\,d\alpha\,d\beta$ and $\int_{t-\tau}^{t}\dot{y}^T(\alpha)R_2\dot{y}(\alpha)\,d\alpha$ are seen in the introduced Lyapunov–Krasovskii functional. Following the proof of the above lemma, it can be easily seen that these terms play crucial role in providing advantageous specifications of this BRL mentioned in Remark 1.

3.2. Controller design

Now, we are in the position to derive an H_∞ PD controller to stabilize the closed-loop input delayed system, and furthermore, to achieve the minimization of an H_∞ norm bound of the closed-loop transfer matrix from the disturbance input to the controlled output. This method can be easily used to provide an H_∞ PI controller which is very popular in industrial plants, especially in process control systems. This point will be explained later in more detail. To this aim, we present a delay-dependent sufficient condition to design the parameters of the proposed controller with H_∞ performance for the closed-loop system (3) with uncertain delay. For this purpose, the descriptor representation (7) of the closed loop system is deployed.

Theorem 1. For a time-delay system (1) with $0 \le \tau < \bar{\tau}$ and a prescribed scalar m with $0<m<1$, the system (1) with the control law $u = K_p y_p + K_d \dot{y}_p$ is asymptotically stable and $||T_{zw}||_\infty < \gamma$, if there exist positive definite symmetric matrices $L_1, L_3, Z_1, Z_2, Q, R_1, R_2 \in \mathcal{R}^{n\times n}$ and matrices $L_2 \in \mathcal{R}^{n\times n}$, $M \in \mathcal{R}^{2n\times 2n}$, $K_p, K_d \in \mathcal{R}^{l\times q}$ satisfying matrix inequalities (11)–(13).

$$\begin{bmatrix} \phi & I+\begin{pmatrix} 0 & 0 \\ BK_pC_y & BK_dC_y \end{pmatrix} & 0 & \begin{pmatrix} 0 \\ F \end{pmatrix} & 0 & L^T & L^T\begin{pmatrix} 0 \\ \bar{\tau}I \end{pmatrix} & L^T\begin{pmatrix} 0 \\ A^T \end{pmatrix} & L^T\begin{pmatrix} 0 \\ \bar{\tau}A^T \end{pmatrix} & L^T\begin{pmatrix} C_z^T \\ 0 \end{pmatrix} \\ * & \begin{pmatrix} -Q & 0 \\ 0 & -R_1 \end{pmatrix} & 0 & 0 & 0 & 0 & 0 & \begin{pmatrix} 0 \\ (BK_pC_y)^T \end{pmatrix} & \begin{pmatrix} 0 \\ \bar{\tau}(BK_pC_y)^T \end{pmatrix} & 0 \\ * & * & -R_2 & 0 & 0 & 0 & 0 & (BK_dC_y)^T & \bar{\tau}(BK_dC_y)^T & 0 \\ * & * & * & -m^2\gamma^2 I & 0 & 0 & 0 & 0 & 0 & D_z^T \\ * & * & * & * & -(1-m)^2\gamma^2 I & 0 & 0 & F^T & \bar{\tau}F^T & 0 \\ * & * & * & * & * & \begin{pmatrix} -Q^{-1} & 0 \\ 0 & -R_1^{-1} \end{pmatrix} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & -\bar{\tau}Z_1^{-1} & 0 & 0 & 0 \\ * & * & * & * & * & * & * & -R_2^{-1} & 0 & 0 \\ * & * & * & * & * & * & * & * & -\bar{\tau}Z_2^{-1} & 0 \\ * & * & * & * & * & * & * & * & * & -I \end{bmatrix} < 0 \qquad (11)$$

and

$$\begin{bmatrix} M & -I \\ -I & Z \end{bmatrix} > 0 \qquad (12)$$

and

$$\begin{bmatrix} I & (BK_dC_y)^T \\ BK_dC_y & I \end{bmatrix} > 0 \qquad (13)$$

where

$$\phi = \begin{bmatrix} 0 & I \\ A & -I \end{bmatrix} L + L^T \begin{bmatrix} 0 & A^T \\ I & -I \end{bmatrix} + \bar{\tau}M - L - L^T, \quad L = \begin{bmatrix} L_1 & 0 \\ L_2 & L_3 \end{bmatrix}, \quad Z = \begin{bmatrix} Z_1 & 0 \\ 0 & Z_2 \end{bmatrix}$$

Proof. Substituting $A_h = BK_pC_y$ and $A_d = BK_dC_y$, Assumption 1 transforms to the following condition that requires to be satisfied

$$\left\| BK_dC_y \right\| < 1 \qquad (14)$$

Here, $|.|$ is denoted as the Euclidean norm of the matrix BK_dC_y which is equal to the maximum singular value of BK_dC_y. Hence, (14) is equivalent to the following inequality

$\bar{\sigma}(BK_dC_y) < 1$ or

$$\sqrt{\lambda_{\max}((BK_dC_y)^T BK_dC_y)} < 1 \qquad (15)$$

The above inequality can be rewritten as follows:

$$I - (BK_dC_y)^T BK_dC_y > 0 \qquad (16)$$

Using Schur complement, the LMI (13) is obtained from (16). Furthermore, considering the resulting matrix inequality (9) in Lemma 3, we define $Y = -P^T$, $L = P^{-1}$ and substitute $A_h = BK_pC_y$, $A_d = BK_dC_y$. Pre and post multiplying this matrix inequality by diag $(L^T, I, I, I, I, Q^{-1}, R_1^{-1}, Z_1^{-1}, R_2^{-1}, Z_2^{-1}, I)$ and defining $M = L^TXL$, the matrix inequality (11) is obtained. Similarly, Pre and post multiplying the LMI (10) by diag (L^T, I), the LMI (12) is provided. This completes the proof.

Remark 3. It should be noted that the main merit of the above theorem is to provide a new delay-dependent sufficient conditions for finding H_∞ PD/PI controller in terms of some matrix inequalities in presence of uncertain bounded delay. These conditions enable us to compute H_∞ PD/PI controller for both SISO and MIMO systems.

Remark 4. It is useful to know that the conditions provided in Theorem 1 can also be used for designing a PI controller. To this aim, it suffices to augment the plant model with an integrator. Consequently, design of a PD controller for the augmented plant model is equivalent to designing a PI controller for the plant model. The state space equations of the augmented system is represented as

$$\dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + \bar{B}u(t-\tau) + \bar{F}w(t)$$
$$\bar{z}(t) = \bar{C}_z\bar{x}(t) + \bar{D}_z w(t)$$
$$\bar{y}_p(t) = \bar{C}_y\bar{x}(t)$$

where

$$\bar{A} = \begin{bmatrix} A & B \\ 0_{m\times n} & 0_{m\times m} \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0_{n\times m} \\ 1_{m\times m} \end{bmatrix}, \quad \bar{F} = \begin{bmatrix} F \\ 0_{m\times p} \end{bmatrix}, \quad \bar{C}_y = \begin{bmatrix} C_y & 0_{r\times m} \end{bmatrix}, \quad \bar{C}_z = \begin{bmatrix} C_z & 0_{q\times m} \end{bmatrix}, \quad \bar{D}_z = D_z$$

Replacing the state space matrices A, B, F, C_z, D_z and C_y by the matrices $\bar{A}$, $\bar{B}$, $\bar{F}$, $\bar{C}_z$, $\bar{D}_z$ and $\bar{C}_y$ respectively in the matrix inequalities (11)–(13), a PD controller $C(s) = K_p + K_d s$ can be designed which together with the integrator term acts as a PI controller $C(s) = (K_p/s) + K_d$ for the original process.

Remark 5. It should be noted that the resulting matrix inequality (11) is not LMI condition due to the terms Q^{-1}, Z_i^{-1} and R_i^{-1}. A good remedy to deal with this problem is to apply iterative method presented by Moon et al. [36]. Exploiting this method enables us to replace the existing nonconvex optimization problem with a nonlinear minimization problem with LMI conditions. This iteration algorithm works efficiently for many examples. Moreover, it helps us to find an initial guess in the feasibility region and improve the solution iteratively by applying BMI conditions obtained during the proof of Theorem 1. To elaborate on this method, first, we define new variables F_i, H_i and S for $i = 1, 2$. Then, the original conditions (11)–(13) are represented as:

$$\Upsilon_1 < 0$$

$$\begin{bmatrix} M & -I \\ -I & Z \end{bmatrix} > 0$$

$$\begin{bmatrix} I & \left(BK_dC_y\right)^T \\ BK_dC_y & I \end{bmatrix} > 0,\ R_i^{-1} = H_i,\ Z_i^{-1} = F_i,\ Q^{-1} = S,\ \forall i = 1, 2$$

where

$$\Upsilon_1 = \begin{bmatrix} \phi & I + \begin{pmatrix} 0 & 0 \\ BK_pC_y & BK_dC_y \end{pmatrix} & 0 & \begin{pmatrix} 0 \\ F \end{pmatrix} & 0 & L^T & L^T\begin{pmatrix} 0 \\ \bar{\tau}I \end{pmatrix} & L^T\begin{pmatrix} 0 \\ A^T \end{pmatrix} & L^T\begin{pmatrix} 0 \\ \bar{\tau}A^T \end{pmatrix} & L^T\begin{pmatrix} C_z^T \\ 0 \end{pmatrix} \\ * & \begin{pmatrix} -Q & 0 \\ 0 & -R_1 \end{pmatrix} & 0 & 0 & 0 & 0 & 0 & \begin{pmatrix} 0 \\ (BK_pC_y)^T \end{pmatrix} & \begin{pmatrix} 0 \\ \bar{\tau}(BK_pC_y)^T \end{pmatrix} & 0 \\ * & * & -R_2 & 0 & 0 & 0 & 0 & (BK_dC_y)^T & \bar{\tau}(BK_dC_y)^T & 0 \\ * & * & * & -m^2\gamma^2 I & 0 & 0 & 0 & 0 & 0 & D_z^T \\ * & * & * & * & -(1-m)^2\gamma^2 I & 0 & 0 & F^T & \bar{\tau}F^T & 0 \\ * & * & * & * & * & \begin{pmatrix} -S & 0 \\ 0 & -H_1 \end{pmatrix} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & -\bar{\tau}F_1 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & -H_2 & 0 & 0 \\ * & * & * & * & * & * & * & * & -\bar{\tau}F_2 & 0 \\ * & * & * & * & * & * & * & * & * & -I \end{bmatrix} < 0$$

$$\phi = \begin{bmatrix} 0 & I \\ A & -I \end{bmatrix} L + L^T \begin{bmatrix} 0 & A^T \\ I & -I \end{bmatrix} + \bar{\tau}M - L - L^T, \quad L = \begin{bmatrix} L_1 & 0 \\ L_2 & L_3 \end{bmatrix}, \quad Z = \begin{bmatrix} Z_1 & 0 \\ 0 & Z_2 \end{bmatrix}$$

Now, using a cone complementary problem [37], the following trace minimization problem involving LMI conditions can be solved instead of the original non-convex feasibility problem of Theorem 1.

Minimize $Tr\,(QS + \sum_{i=1}^{2}(Z_iF_i + R_iH_i))$ *subject to*

$$\begin{aligned} &\Upsilon_1 < 0 \\ &\begin{bmatrix} M & -I \\ -I & Z \end{bmatrix} > 0 \\ &\begin{bmatrix} I & (BK_dC_y)^T \\ BK_dC_y & I \end{bmatrix} > 0 \\ &\begin{bmatrix} Q & I \\ I & S \end{bmatrix} \geq 0 \begin{bmatrix} Z_i & I \\ I & F_i \end{bmatrix} \geq 0 \begin{bmatrix} R_i & I \\ I & H_i \end{bmatrix} \geq 0\ \forall i = 1, 2 \end{aligned} \tag{17}$$

If the solution of this minimization problem is $5n$ (n is the system dimension which leads to the dimension of $n \times n$ for the matrices Q, S, Z_i, R_i, F_i, H_i), that is, $Tr\,(QS + \sum_{i=1}^{2}(Z_iF_i + R_iH_i)) = 5n$, from Theorem 1 it can be concluded that the time-delay system (1) with PD control law (2) is asymptotically stable with a noise attenuation level of γ. Although it is still not always possible to find the global optimal solution, the proposed nonlinear minimization problem is easier to solve than the original non-convex feasibility problem. In fact, we can find a suboptimal maximal delay using an iterative algorithm similar to the algorithm presented by Moon, et al. [36]. Since it is numerically quite difficult to obtain the optimal solution such that $Tr\,(QS + \sum_{i=1}^{2}(Z_iF_i + R_iH_i))$ is exactly equal to $5n$, the conditions (11)–(13) are used as a stopping criterion in the algorithm.

Algorithm 1. *Step* 1. Choose a sufficiently large initial γ such that there exists a feasible solution to (17) for a given $\bar{\tau}$. Set $\gamma_{\min} = \gamma$.

Step 2. Find a feasible set $(L_0, M_0, Q_0, S_0, F_{i0}, H_{i0}, Z_{i0}, R_{i0}, K_{p0}, K_{d0},\ i = 1, 2)$ satisfying (17). Set $k = 0$.

Step 3. Solve the following LMI problem for the variables L, M, Q, S, F_i, H_i, Z_i, R_i, K_p, k_d ($i = 1, 2$)

Table 1
Calculation result to obtain suboptimal maximal delay.

$\bar{\tau}$	Feedback gain $[K_p\ K_d]$	Iterations
0.1	[−0.4017 −0.5205]	6
0.2	[−0.3520 −0.4831]	11
0.3	[−0.2440 −0.3782]	25
0.35	[−0.1646 −0.2797]	76
0.393	[−0.1239 −0.0890]	709

Minimize $Tr\,(Q_kS + QS_k + \sum_{i=1}^{2}(Z_{ik}F_i + Z_iF_{ik} + R_{ik}H_i + R_iH_{ik}))$
Subject to (17). Set $Q_{k+1} = Q,\ S_{k+1} = S,\ Z_{ik+1} = Z_i,\ F_{ik+1} = F_i,\ R_{ik+1} = R_i,\ H_{ik+1} = H_i$

Step 4. If the conditions (11)–(13) are satisfied, then set $\gamma_{\min} = \gamma$ and return to Step 2 after decreasing γ to some extent. If conditions (11)–(13) are not satisfied within a specified number of iterations, then exit. Otherwise, set $k = k + 1$, and go to Step 3.

Remark 6. Algorithm 1 obtains a suboptimal minimum disturbance attenuation level γ for a given $\bar{\tau}$. This algorithm can be similarly used to calculate a suboptimal maximal delay upper bound such that there exists an H_∞ PD/PI controller for a prescribed γ.

Remark 7. One advantage of the Algorithm 1 is that no manual initial value for decision variables is required and it can be automatically obtained by solving the LMI (17) in Step 2 using Matlab™ LMI toolbox. Although randomly choosing of the initial matrices in Step 2 cannot guarantee to produce a feasible solution all the time, various simulations have led to a good and satisfying solution starting from a random choice [41].

Remark 8. Theorem 1 can be further modified to cope with stabilization problem, leading to the following corollary.

Corollary 1. For a time-delay $0 \le \tau < \bar{\tau}$, the system (1) with the control law $u = K_py_p + K_d\dot{y}_p$ is asymptotically stable, if there exist positive definite symmetric matrices $L_1, L_3, Z_1, Z_2, Q, R_1, R_2 \in \mathcal{R}^{n\times n}$ and matrices $L_2 \in \mathcal{R}^{n\times n}$, $M \in \mathcal{R}^{2n\times 2n}$, $K_p, K_d \in \mathcal{R}^{l\times q}$ satisfying matrix inequalities (12), (13) and (18).

$$\begin{bmatrix} \phi & I+\begin{pmatrix} 0 & 0 \\ BK_pC_y & BK_dC_y \end{pmatrix} & 0 & L^T & L^T\begin{pmatrix} 0 \\ \bar{\tau}I \end{pmatrix} & L^T\begin{pmatrix} 0 \\ A^T \end{pmatrix} & L^T\begin{pmatrix} 0 \\ \bar{\tau}A^T \end{pmatrix} \\ * & \begin{pmatrix} -Q & 0 \\ 0 & -R_1 \end{pmatrix} & 0 & 0 & 0 & \begin{pmatrix} 0 \\ (BK_pC_y)^T \\ (BK_dC_y)^T \end{pmatrix} & \begin{pmatrix} 0 \\ \bar{\tau}(BK_pC_y)^T \\ \bar{\tau}(BK_dC_y)^T \end{pmatrix} \\ * & * & -R_2 & 0 & 0 & & \\ * & * & * & \begin{pmatrix} -Q^{-1} & 0 \\ 0 & -R_1^{-1} \end{pmatrix} & 0 & 0 & 0 \\ * & * & * & * & -\bar{\tau}Z_1^{-1} & 0 & 0 \\ * & * & * & * & * & -R_2^{-1} & 0 \\ * & * & * & * & * & * & -\bar{\tau}Z_2^{-1} \end{bmatrix} < 0 \quad (18)$$

where

$$\phi = \begin{bmatrix} 0 & I \\ A & -I \end{bmatrix} L + L^T \begin{bmatrix} 0 & A^T \\ I & -I \end{bmatrix} + \bar{\tau}M - L - L^T, \quad L = \begin{bmatrix} L_1 & 0 \\ L_2 & L_3 \end{bmatrix}$$

4. Simulation results

To illustrate the application of the results carried out in this paper, we present following examples. Design of PD controller is investigated in the example 1. In the examples 2 and 3 we concentrate on the design of PI controller due to its popularity in process control. Furthermore, note that plants of examples 1 and 2 are both unstable. The results are compared with well-known methods given in the literature.

Example 1. To illustrate the application of the results carried out in this paper as well as seeing the advantages we mentioned in Remark 1, we investigate the BRL provided by Gao [39], which is an extension of the result proposed by Fridman and Shaked [38]. To this aim, we assume an unstable system (3) with parameters given as follows

$$A = \begin{bmatrix} 0 & 1 \\ 0.1 & -2 \end{bmatrix}, \quad F = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}, \quad C_z = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D_z = 0, \quad (19)$$

and unknown delayed term coefficients (A_h and A_d). Applying Gao's BRL [39] with $\bar{\tau} = 0.2$ and $\gamma = 0.8$, an exact solution equal to zero is obtained for A_d. It means that by setting $A_h = BK_pC_y$ and $A_d = BK_dC_y$ the solution $K_d = 0$ is provided which ends to a P controller. Similarly, the same result is found for the method presented by Fridman and Shaked [38]. To get over this limitation, we apply Corollary 1 with $B = [0\ 1]^T$ and $C_y = [11]$ to find a stabilizing PD controller (non-zero K_p and K_d gains) for the closed-loop system with neutral type equations. For this purpose, we use LMI toolbox in MATLAB™ and utilize iterative Algorithm 1 mentioned in Remark 5. These results are shown in Table 1.

As it is shown in Table 1, the suboptimal maximum value of delay $\bar{\tau}$ has been calculated for which an instantaneous stabilizing PD controller exists. The number of iterations in this table denotes the number of iterations performed before the stopping criterion, i.e. the conditions (11), (12) and (13), becomes effective. From the table we can see that the maximum value of delay $\bar{\tau}$ is near to 0.393.

Table 2
Calculation result to obtain suboptimal minimum γ with $\bar{\tau} = 0.2$.

γ	Feedback gain $[K_p\ K_d]$	Iterations
2	[−1.3001 −0.3899]	30
1.5	[−1.2832 −0.3692]	31
1	[−1.2078 −0.3105]	37
0.8	[−1.0909 −0.2407]	50
0.69	[−0.9211 −0.1182]	1823

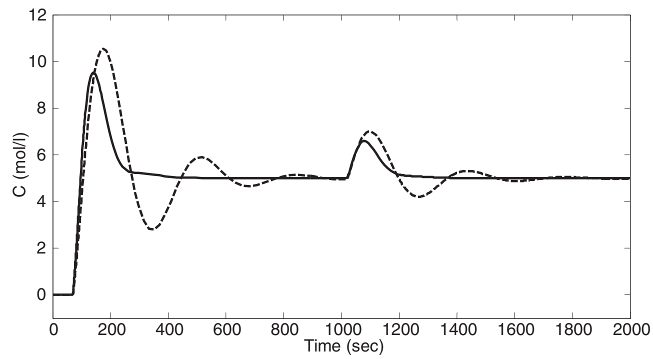

Fig. 1. Step response of the plant output with $\tau = 20$ s for example 2. Proposed controller (solid) and Lee et al.'s controller [4] (dashed).

Now, consider the case that the size of the maximum delay is known and it is equal to 0.2. We apply Theorem 1 to find an H_∞ PD controller for the input-delayed system (1) with the state space matrices given in (19). Using Algorithm 1 and LMI toolbox in MATLAB^{TM}, the minimum value for γ is obtained as 0.69. Table 2 shows the details of this result.

These results confirm the point we mentioned in remark 1 and show that the proposed BRLs in literature [31,32] are not suitable for such design problems, whereas this limitation is completely remedied by our proposed method.

Example 2. Concentration in an unstable reactor

In this example we consider a chemical reactor with unstable dynamical behavior. For control purposes, $C(t)$, the output concentration, is the output variable and $C_i(t)$, the input concentration, is the manipulated variable. Furthermore, the concentration transducer needs a dead time of 20 s to give the output variable. The linearized transfer function from $C_i(t)$ to $C(t)$ at the operating point is obtained as follows [18]:

$$P(s) = \frac{3.433e^{-20s}}{103.1s - 1} \tag{20}$$

with an unstable pole at $s = 1/103.1$. Using Nyquist method presented in [4], a stabilizing PI controller is obtained as $C_1(s) = 0.6554\left(1 + (1/102.04\,s)\right)$. To improve the performance of the closed-loop system, we set $A_h = BK_pC_y$ and $A_d = BK_dC_y$ and use the original BMI conditions come out from Lemma 3. Using $C_1(s)$ as an initial condition and solving the aforementioned BMI conditions iteratively, an H_∞ PI controller is obtained as $C_2(s) = 0.9435\left(1 + (1/157.25\,s)\right)$. Figs. 1 and 2 show the simulation results of the closed-loop system with both controllers $C_1(s)$ and $C_2(s)$. At $t = 50$ s the set-point is changed from 0 to 5 mol/l and a step input disturbance is introduced at $t = 1000$.

As it is seen in Fig. 1, the closed-loop system with controller $C_2(s)$ yields a better step response as well as a good disturbance rejection compared to that of controller $C_1(s)$. Besides, our proposed controller results a lower control effort than the controller $C_1(s)$ as it is shown in Fig. 2.

Defining $e = y - r$ where y is the output signal and r is the set point, the performance indices IAE (Integral of Absolute Error) for servo response, M_s (Maximum sensitivity) and the delay range tolerated by the closed-loop systems with both controllers $C_1(s)$ and $C_2(s)$ are shown in Table 3. τ_t is the delay range tolerated by the closed-loop system. As it is seen in Table 3, using the proposed controller, the

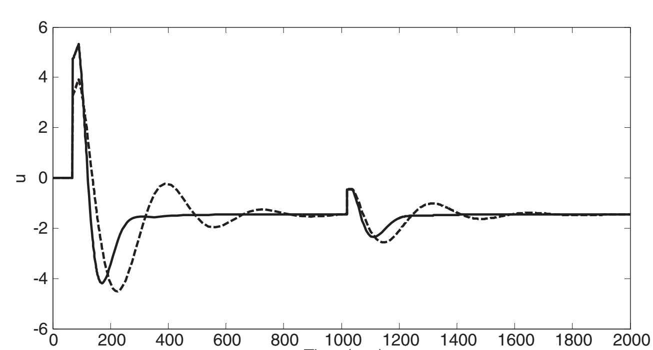

Fig. 2. Control effort with $\tau = 20$ s for example 2. Proposed controller (solid) and Lee et al.'s controller [4] (dashed).

Table 3
Performance indices.

	IAE	M_s	τ_t
Lee et al. [4]	1156	1.2	[0 33.5]
Proposed controller	576.5	1.31	[0 34.6]

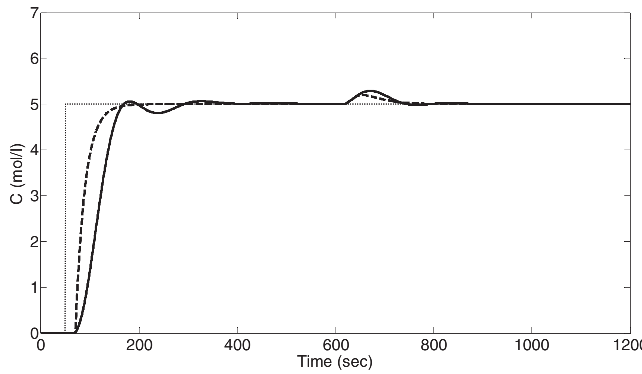

Fig. 3. Step response of the plant output with $\tau = 20$ s for Example 2. Proposed controller (solid), FSP controller [18] (dashed) and Step input (dotted).

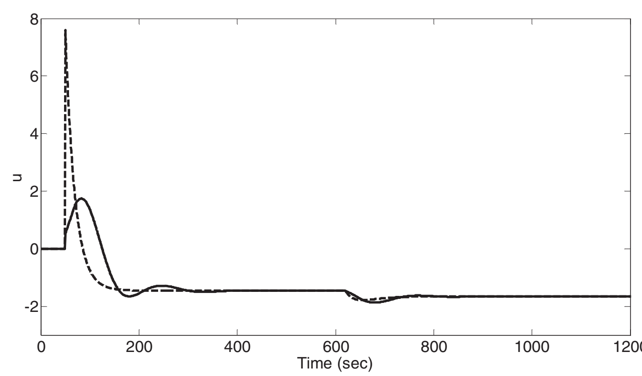

Fig. 4. Control effort with $\tau = 20$ s for Example 2. Proposed controller (solid) and FSP controller [18] (dashed).

performance index IAE has been considerably reduced compared to Lee et al.'s [4] method. Furthermore, although the method presented in Ref. [4] shows a slightly better maximum sensitivity than our proposed method, the delay range tolerated by the closed-loop system with our proposed method is slightly larger than the one tolerated by the method in Ref. [4].

We have repeated the above procedure using matrix inequalities provided in Ref. [26]. The controller $C_1(s)$ does not satisfy these matrix inequalities and furthermore, $C_2(s)$ is not in their feasible region. These results show that the method presented in this paper has a wider feasible region compared to that of [26].

Now, we continue this example to compare our method with the filtered Smith predictor (FSP) method presented by Normey-Richo et al. [18]. In order to obtain comparable results to that of FSP [18], we provide the controller $C_3(s) = 1.1(1 + (1/114.6\,s))$ which results the same maximum tolerated delay as [18] using Corollary 1. Furthermore, an additional filter $F(s) = (10\,s + 1)/(107\,s + 1)$ is introduced which eliminates the effect of the controller zero, as introduced in [18]. Figs. 3 and 4 shows closed-loop responses when the process is simulated with our proposed controller, $C_3(s)$, as well as the FSP controller [18]. As can be seen, with the same maximum tolerated delay $\tau_{max} = 29.4$ s for both $C_3(s)$ and FSP controllers, the FSP controller results a better step response with an overshoot O.S. = 0% as well as a slightly better disturbance rejection compared to our presented controller with an overshoot O.S. = 1.2%. These performance indices for FSP controller are achieved at the expense of considerably larger control effort. Moreover, the closed-loop system with controller $C_3(s)$ remains stable for all time delays $\tau \leq 29.4s$ as expected through design procedure, whereas the closed-loop system with the FSP controller is stable in a more limited range of delay variations $11.61\,\text{s} < \tau \leq 29.4$ s and $\tau \leq 5.46$ s. The aforementioned discussion is summarized in Table 4 together with Integral of Absolute Error (IAE) for both servo response (IAE_s) and load disturbance rejection (IAE_d) as well as Maximum Sensitivity performance index which is represented as M_s. Furthermore, τ_t represents the delay range tolerated by the closed-loop system. As can be seen in this table, the FSP method [18] obtains better IAE performance index, whereas our proposed method results a smaller maximum sensitivity and more robustness in presence of delay variations compared to that of Ref. [18].

Table 4
Comparison of Performance indices between the presented controller and FSP [18] with unsaturated control input.

	IAE_s	IAE_d	M_s	O.S.%	τ_t
Normey-Richo et al. [18]	198.8	14.5	2.1	0	[0,5.46], [11.61,29.4]
Proposed controller	348.2	22.1	1.37	1.2	[0,29.4]

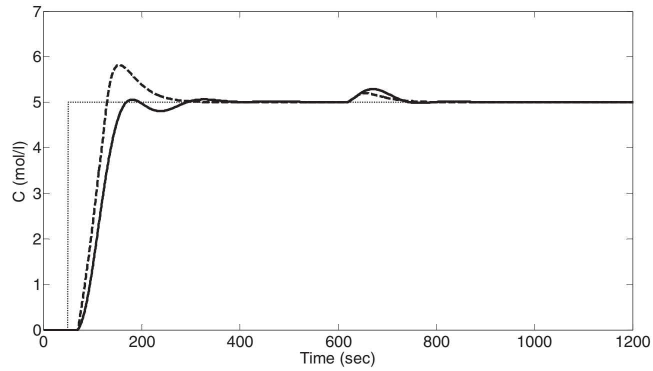

Fig. 5. Step response of the plant output with $\tau = 20$ s and saturation on the control input for Example 2. Proposed controller (solid), FSP controller [18] (dashed) and Step input (dotted).

Table 5
Comparison of Performance indices between the presented controller and FSP [18] with saturated control input.

	IAE_s	IAE_d	O.S. %	τ_t
Normey-Richo et al. [18]	313.8	14.5	16.4	[0,5.45], [11.6,29.5]
Proposed controller	348.2	22.1	1.2	[0,29.5]

Table 6
PI Controller parameters [K_p K_I].

	G_1	G_2	G_3	G_4
Ziegler–Nichols	[0.56 0.11]	[0.63 0.084]	[0.5180 0.0560]	[0.542 0.05]
PIP [16] & FPIP [17]	[11]	[1 0.4]	[11]	[1 0.4]
Theorem 1	[−0.0025 0.0757]	[0.0012 0.0577]	[−0.001 0.0471]	[−0.0006 0.0313]

Since all real actuators in industry are subject to saturation, there is a need for more investigation of the performances of the above closed-loop systems in presence of saturated control input. To this aim, we apply a saturation limit of ± 2 on the control input and then repeat the simulation of the process. As can be seen in Fig. 5 and Table 5, the step response quality of the closed-loop system with FSP controller is deteriorated in presence of the saturation limit, in which O.S. increases to 16.4% and IAE_s increases by around 58% for servo response, whereas the stability region in presence of delay variations and the disturbance rejection of both controllers has not been affected by the limit of the control input.

Example 3. In this example we consider the following set of process models studied in literature [16]:

$$G_1(s) = \frac{e^{-5s}}{s+1}, \quad G_2(s) = \frac{e^{-5s}}{(s+1)^3}, \quad G_3(s) = \frac{e^{-10s}}{s+1}, \quad G_4(s) = \frac{e^{-10s}}{(s+1)^3} \tag{21}$$

Then, we apply Theorem 1 to find H_∞ PI controllers for the closed-loop system of the above plants with neutral type equations. These controllers are designed with $\bar{\tau} = 5$ for G_1, G_2 and $\bar{\tau} = 10$ for G_3, G_4 and a minimum value of γ is obtained as 1.611, 2.06, 2.715, 3.78 for G_1, G_2, G_3 and G_4, respectively. The resulting controller parameters are shown in Table 6. To see the efficiency of our method, we compare the robustness of the above controllers for change in the time-delay as well as the actual value of γ with some well-known methods in the literature. Those are Ziegler–Nichols [40] and two Smith predictor controllers known as predictive PI (PIP) [16] and robust predictive PI (FPIP) [17] controllers which their parameters are shown in Table 6 as well. Although the FPIP controller has the same controller parameters as PIP in smith predictor structure, an additional filter in its closed-loop system acts on the error in order to improve the robustness. Table 7 shows the maximum delay that each closed-loop system can tolerate to remain in the stable range and the actual values of γ obtained directly from the closed-loop systems are shown in Table 8.

As it is shown in Table 7, applying Theorem 1, the maximum delay tolerated by the closed-loop system is significantly increased. Therefore, the PI controller presented by our method improves the robust stability of the closed-loop system compared to the other methods. Furthermore, Theorem 1 guarantees the stability of the closed-loop system for all time-delays smaller than $\bar{\tau}$.

As mentioned earlier, Table 8 shows the actual values of γ for each closed-loop system with respect to G_1 G_2, G_3 and G_4 applying the various mentioned methods. The results show that applying our method provides a better disturbance attenuation level γ compared to the other methods.

Table 7
Maximum delay tolerated by the closed-loop systems applying various methods.

	G_1	G_2	G_3	G_4
Ziegler–Nichols	15.5	18.3	30.7	33.2
PIP [16]	11.14	13.72	11.67	13.4
FPIP [17]	14.2	17.13	27.25	29.9
Theorem 1	19.74	24.4	32.3	47

Table 8
Actual value of γ for the closed-loop systems applying various methods.

	G_1	G_2	G_3	G_4
Ziegler–Nichols	2.05	1.97	2.157	2.11
PIP [16]	1.7	1.72	1.89	1.82
FPIP [17]	1.6	1.43	1.66	1.58
Theorem 1	1.48	1.46	1.60	1.42

5. Conclusions

In this paper, a neutral system approach is given to stabilize and synthesize H_∞ proportional derivative (PD) and proportional integral (PI) controllers for input-delay systems with uncertain delay. To this aim, it is shown that a time-delay plant under a PD controller ends to a neutral closed loop system. Lyapunov–Krasovskii theory and descriptor model transformation is used to analyze the stability of this system. A new and advantageous bounded real lemma is given for neutral systems with efficiency in designing the H_∞ controllers with both state delay and state derivative delay coefficients depending on the controller parameters. Moreover, new delay-dependent sufficient conditions for designing H_∞ PD controller are given in terms of some matrix inequalities. By augmenting the plant model with an integrator and exploiting the proposed method we can easily design a PI controller for the closed-loop system. Using Moon's iterative algorithm to solve bilinear matrix inequalities as a series of LMI's, a suboptimal solution is obtained for the resulting matrix inequalities. A further modification on this result, leads to stabilization sufficient conditions. Some case studies and numerical examples are presented in this paper. Simulation results show superior performance or robustness in all case studies given in this paper compared to some well-known presented methods in the literature.

Appendix A.

Proof of Lemma 3: A Lyapunov–Krasovskii functional for system (8) has the form

$$V = V_1 + V_2 + V_3 \tag{A.1}$$

where, $V_1 = \begin{bmatrix} x^T(t) & y^T(t) \end{bmatrix} EP \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$

$$V_2 = \int_{-\tau}^{0}\int_{t+\beta}^{t} y^T(\alpha)Z_1 y(\alpha)\,d\alpha\,d\beta + \int_{-\tau}^{0}\int_{t+\beta}^{t} \dot{y}^T(\alpha)Z_2\dot{y}(\alpha)\,d\alpha\,d\beta \tag{A.2}$$

$$V_3 = \int_{t-\tau}^{t} x^T(\alpha)Qx(\alpha)\,d\alpha + \int_{t-\tau}^{t} y^T(\alpha)R_1 y(\alpha)\,d\alpha + \int_{t-\tau}^{t} \dot{y}^T(\alpha)R_2\dot{y}(\alpha)\,d\alpha$$

in which,

$$E = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \quad P = \begin{bmatrix} P_1 & 0 \\ P_2 & P_3 \end{bmatrix}, \quad P_1 > 0$$

and Q, R_1, R_2, Z_1, Z_2 are real symmetric positive definite matrices of appropriate dimensions. Note that

$$V_1 = \begin{bmatrix} x^T(t) & y^T(t) \end{bmatrix} EP \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = x^T(t)P_1x(t) \tag{A.3}$$

and, therefore, differentiating V_1 with respect to t results into

$$\dot{V}_1 = 2x^T(t)P_1\dot{x}(t) = 2\begin{bmatrix} x^T(t) & y^T(t) \end{bmatrix} P^T \begin{bmatrix} \dot{x}(t) \\ 0 \end{bmatrix} = 2\begin{bmatrix} x^T(t) & y^T(t) \end{bmatrix} P^T \begin{bmatrix} y \\ -y(t) + Ax(t) + A_h x(t-\tau) + A_d y(t-\tau) \end{bmatrix}$$
$$+2\begin{bmatrix} x^T(t) & y^T(t) \end{bmatrix} P^T \begin{bmatrix} 0 \\ F \end{bmatrix} w(t) \tag{A.4}$$

By definite integral formula known as Leibniz–Newton formula, it is possible to write

$$\begin{aligned} x(t-\tau) &= x(t) - \int_{t-\tau}^{t} \dot{x}(\alpha)\,d\alpha \\ y(t-\tau) &= y(t) - \int_{t-\tau}^{t} \dot{y}(\alpha)\,d\alpha \end{aligned} \tag{A.5}$$

Hence, we have

$$\dot{V}_1 = 2\begin{bmatrix} x^T(t) & y^T(t) \end{bmatrix} P^T \begin{bmatrix} 0 & I \\ A & -I \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + 2\begin{bmatrix} x^T(t) & y^T(t) \end{bmatrix} P^T \begin{bmatrix} 0 & 0 \\ A_h & A_d \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} - 2\begin{bmatrix} x^T(t) & y^T(t) \end{bmatrix} P^T \begin{bmatrix} 0 & 0 \\ A_h & A_d \end{bmatrix} \begin{bmatrix} \int_{t-\tau}^{t} \dot{x}(\alpha)\,d\alpha \\ \int_{t-\tau}^{t} \dot{y}(\alpha)\,d\alpha \end{bmatrix}$$
$$+2\begin{bmatrix} x^T(t) & y^T(t) \end{bmatrix} P^T \begin{bmatrix} 0 \\ Fw(t) \end{bmatrix} \quad \text{(A.6)}$$

On the other hand, using Lemma 1, we have

$$-2\begin{bmatrix} x^T(t) & y^T(t) \end{bmatrix} S \int_{t-\tau}^{t} \begin{bmatrix} y(\alpha) \\ \dot{y}(\alpha) \end{bmatrix} d\alpha \le \int_{t-\tau}^{t} \begin{bmatrix} x(t) \\ y(t) \\ y(\alpha) \\ \dot{y}(\alpha) \end{bmatrix}^T \begin{bmatrix} X & Y-S \\ Y^T - S^T & Z \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \\ y(\alpha) \\ \dot{y}(\alpha) \end{bmatrix} d\alpha \quad \text{(A.7)}$$

$$\begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix} > 0 \quad \text{(A.8)}$$

where $S = P^T \begin{bmatrix} 0 & 0 \\ A_h & A_d \end{bmatrix}$.

The inequality (A.7) can be rewritten as:

$$-2\begin{bmatrix} x^T(t) & y^T(t) \end{bmatrix} S \int_{t-\tau}^{t} \begin{bmatrix} y(\alpha) \\ \dot{y}(\alpha) \end{bmatrix} d\alpha \le \int_{t-\tau}^{t} \left\{ \begin{bmatrix} x^T(t) & y^T(t) \end{bmatrix} X \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + \begin{bmatrix} y^T(\alpha) & \dot{y}^T(\alpha) \end{bmatrix} \left(Y^T - S^T\right) \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \right.$$
$$\left. + \begin{bmatrix} x^T(t) & y^T(t) \end{bmatrix} (Y-S) \begin{bmatrix} y(\alpha) \\ \dot{y}(\alpha) \end{bmatrix} + \begin{bmatrix} y^T(\alpha) & \dot{y}^T(\alpha) \end{bmatrix} Z \begin{bmatrix} y(\alpha) \\ \dot{y}(\alpha) \end{bmatrix} \right\} d\alpha \le \bar{\tau} \begin{bmatrix} x^T(t) & y^T(t) \end{bmatrix} X \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + 2\begin{bmatrix} x^T(t) & y^T(t) \end{bmatrix} (Y-S) \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$
$$-2\begin{bmatrix} x^T(t) & y^T(t) \end{bmatrix} (Y-S) \begin{pmatrix} x(t-\tau) \\ y(t-\tau) \end{pmatrix}$$
$$+\int_{t-\tau}^{t} \begin{bmatrix} y^T(\alpha) & \dot{y}^T(\alpha) \end{bmatrix} Z \begin{bmatrix} y(\alpha) \\ \dot{y}(\alpha) \end{bmatrix} d\alpha$$

Therefore, an upper bound for $\dot{V}_1$ is as follows:

$$\dot{V}_1 \le \begin{bmatrix} x^T(t) & y^T(t) \end{bmatrix} \left(P^T \begin{bmatrix} 0 & I \\ A & -I \end{bmatrix} + \begin{bmatrix} 0 & A^T \\ I & -I \end{bmatrix} P + \bar{\tau}X + Y + Y^T \right) \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} - 2\begin{bmatrix} x^T(t) & y^T(t) \end{bmatrix} (Y-S) \begin{pmatrix} x(t-\tau) \\ y(t-\tau) \end{pmatrix}$$
$$+\int_{t-\tau}^{t} \begin{bmatrix} y^T(\alpha) & \dot{y}^T(\alpha) \end{bmatrix} Z \begin{bmatrix} y(\alpha) \\ \dot{y}(\alpha) \end{bmatrix} d\alpha + 2\begin{bmatrix} x^T(t) & y^T(t) \end{bmatrix} P^T \begin{bmatrix} 0 \\ Fw(t) \end{bmatrix} \quad \text{(A.9)}$$

Also, the time derivative of V_2 and V_3 can be represented as follows:

$$\dot{V}_2 = \tau y^T(t) Z_1 y(t) + \tau \dot{y}^T(t) Z_2 \dot{y}(t) - \int_{t-\tau}^{t} y^T(\alpha) Z_1 y(\alpha)\,d\alpha - \int_{t-\tau}^{t} \dot{y}^T(\alpha) Z_2 \dot{y}(\alpha)\,d\alpha \le \bar{\tau} y^T(t) Z_1 y(t) + \bar{\tau} \dot{y}^T(t) Z_2 \dot{y}(t)$$
$$-\int_{t-\tau}^{t} y^T(\alpha) Z_1 y(\alpha)\,d\alpha - \int_{t-\tau}^{t} \dot{y}^T(\alpha) Z_2 \dot{y}(\alpha)\,d\alpha \quad \text{(A.10)}$$

$$\dot{V}_3 = x^T(t) Q x(t) - x^T(t-\tau) Q x(t-\tau) + y^T(t) R_1 y(t) - y^T(t-\tau) R_1 y(t-\tau) + \dot{y}^T(t) R_2 \dot{y}(t) - \dot{y}^T(t-\tau) R_2 \dot{y}(t-\tau) \quad \text{(A.11)}$$

Since $\dot{V} = \dot{V}_1 + \dot{V}_2 + \dot{V}_3$, the inequalities (A.9)–(A.10) and the equality (A.11) result a new bound of $\dot{V}$ as follows:

$$\dot{V}(t) \le \begin{bmatrix} x^T(t) & y^T(t) \end{bmatrix} \left(P^T \begin{bmatrix} 0 & I \\ A & -I \end{bmatrix} + \begin{bmatrix} 0 & A^T \\ I & -I \end{bmatrix} P + \bar{\tau}X + Y + Y^T + \begin{bmatrix} Q & 0 \\ 0 & R_1 + \bar{\tau}Z_1 \end{bmatrix} \right) \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$
$$-2\begin{bmatrix} x^T(t) & y^T(t) \end{bmatrix} (Y-S) \begin{pmatrix} x(t-\tau) \\ y(t-\tau) \end{pmatrix} - \begin{bmatrix} x^T(t-\tau) & y^T(t-\tau) \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & R_1 \end{bmatrix} \begin{bmatrix} x(t-\tau) \\ y(t-\tau) \end{bmatrix}$$
$$+\dot{y}^T(t)(R_2 + \bar{\tau}Z_2)\dot{y}(t) - \dot{y}^T(t-\tau) R_2 \dot{y}(t-\tau) + \int_{t-\tau}^{t} \begin{bmatrix} y^T(\alpha) & \dot{y}^T(\alpha) \end{bmatrix} Z \begin{bmatrix} y(\alpha) \\ \dot{y}(\alpha) \end{bmatrix} d\alpha$$
$$-\int_{t-\tau}^{t} y^T(\alpha) Z_1 y(\alpha)\,d\alpha - \int_{t-\tau}^{t} \dot{y}^T(\alpha) Z_2 \dot{y}(\alpha)\,d\alpha + 2\begin{bmatrix} x^T(t) & y^T(t) \end{bmatrix} P^T \begin{bmatrix} 0 \\ F \end{bmatrix} w(t) \quad \text{(A.12)}$$

By the assumption of $Z = \begin{bmatrix} Z_1 & 0 \\ 0 & Z_2 \end{bmatrix}$, and substituting $\dot{y}(t)$ from (7), we obtain

$$\dot{V}(t) + z^T z - m^2\gamma^2 w^T w - (1-m)^2\gamma^2 \dot{w}^T \dot{w} \le \xi^T \Phi \xi \tag{A.13}$$

in which, $\xi = col\begin{bmatrix} x(t) & y(t) & x(t-\tau) & y(t-\tau) & \dot{y}(t-\tau) & w(t) & \dot{w}(t) \end{bmatrix}$, $\Phi = \begin{bmatrix} \Phi_{ij} \end{bmatrix}$, $i, j = 1, \ldots, 5$ and

$$\Phi_{11} = P^T \begin{bmatrix} 0 & I \\ A & -I \end{bmatrix} + \begin{bmatrix} 0 & A^T \\ I & -I \end{bmatrix} P + \bar{\tau}X + Y + Y^T + \begin{bmatrix} Q + C_z^T C_z & 0 \\ 0 & R_1 + \bar{\tau}Z_1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & A^T(R_2 + \bar{\tau}Z_2)A \end{bmatrix}$$

$$\Phi_{12} = -Y + P^T \begin{bmatrix} 0 & 0 \\ A_h & A_d \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & A^T(R_2 + \bar{\tau}Z_2)A_h \end{bmatrix}$$

$$\begin{aligned}
&\Phi_{13} = \begin{pmatrix} 0 \\ A^T(R_2 + \bar{\tau}Z_2)A_d \end{pmatrix} && \Phi_{14} = P^T \begin{pmatrix} 0 \\ F \end{pmatrix} + \begin{pmatrix} C_z^T D_z \\ 0 \end{pmatrix} \\
&\Phi_{15} = \begin{pmatrix} 0 \\ A^T(R_2 + \bar{\tau}Z_2)F \end{pmatrix} && \Phi_{22} = -\begin{bmatrix} Q & 0 \\ 0 & R_1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & A_h^T(R_2 + \bar{\tau}Z_2)A_h \end{bmatrix} \\
&\Phi_{23} = \begin{pmatrix} 0 \\ A_h^T(R_2 + \bar{\tau}Z_2)A_d \end{pmatrix} && \Phi_{24} = 0 \\
&\Phi_{25} = \begin{pmatrix} 0 \\ A_h^T(R_2 + \bar{\tau}Z_2)F \end{pmatrix} && \Phi_{33} = -R_2 + A_d^T(R_2 + \bar{\tau}Z_2)A_d \\
&\Phi_{34} = 0 && \Phi_{35} = A_d^T(R_2 + \bar{\tau}Z_2)F \\
&\Phi_{44} = D_z^T D_z - m^2\gamma^2 I && \Phi_{45} = 0 \\
&\Phi_{55} = -(1-m)^2\gamma^2 I + F^T(R_2 + \bar{\tau}Z_2)F
\end{aligned} \tag{A.14}$$

When assuming the zero disturbance input in (22), i.e. $w(t)\equiv 0$, if the condition $\Phi < 0$ holds, then the negative definiteness of $\dot{V}(t)$ is guaranteed and asymptotic stability of system (3) is established. Integrating the inequality (A.13) in t from 0 to ∞ with $w \in \mathcal{W}^{1,2}(0, \infty, \mathcal{R}^p) \cap \mathcal{L}^2(0, \infty, \mathcal{R}^p)$ and $\Phi < 0$, we have

$$V(\infty) - V(0) + \int_0^\infty (z^T z - m^2\gamma^2 w^T w - (1-m)^2\gamma^2 \dot{w}^T \dot{w})\, dt \le 0$$

Since $V(0) = 0$ and $V(\infty) \ge 0$, then considering the above inequality, $J_{zd} < 0$ and therefore, the signal $z(t)$ is bounded by

$$\int_0^\infty \left\|z(t)\right\|^2 dt \le \gamma^2 \int_0^\infty \left(m^2 \left\|w(t)\right\|^2 + (1-m)^2 \left\|\dot{w}(t)\right\|^2 \right) dt$$

This means that $\left\|T_{zd}\right\|_\infty < \gamma$ where $d(t) = [mw^T(t)(1-m)\dot{w}^T(t)]^T$. By Lemma 2, the inequality $\left\|T_{zd}\right\|_\infty < \gamma$ guarantees $\left\|T_{zw}\right\|_\infty < \gamma$ to be satisfied. Therefore, the closed-loop system (3) is stable with disturbance attenuation γ. Using Schur complement on $\Phi < 0$ and considering LMI (A.8), the matrix inequalities (9) and (10) are obtained. This completes the proof.

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