1   The Graphical Method: An Example

Consider the following linear program:

$$
\begin{array}{lrrcll}
\text{Max} & 4x_1 & +3x_2 & & & \\
\text{Subject to:} & & & & & \\
& 2x_1 & +3x_2 & \leq & 6 & (1) \\
& -3x_1 & +2x_2 & \leq & 3 & (2) \\
& & 2x_2 & \leq & 5 & (3) \\
& 2x_1 & +x_2 & \leq & 4 & (4) \\
& x_1, & x_2 & \geq & 0 \, , &
\end{array}
$$

where, for ease of reference, the four functional constraints have been labelled as (1), (2), (3), and (4). Our goal is to produce a pair of values for x_1 and x_2 that (i) satisfies all constraints and (ii) has the greatest objective-function value.

A pair of specific values for (x_1, x_2) is said to be a **feasible solution** if it satisfies all the constraints. For example, $(x_1, x_2) = (0, 0)$ and $(x_1, x_2) = (1, 1)$ are feasible solutions, while $(x_1, x_2) = (1, -1)$ and $(x_1, x_2) = (1, 2)$ are not. Note that the objective-function value associated with solution $(0, 0)$ is equal to 0 $(= 4 \cdot 0 + 3 \cdot 0)$ and that value associated with solution $(1, 1)$ is equal to 7 $(= 4 \cdot 1 + 3 \cdot 1)$. Hence, $(1, 1)$ is the better of these two feasible solutions. Since it is impossible to generate and compare *all* feasible solutions one by one, we must develop a systematic method to identify the best, or optimal, solution.

The basic idea behind the graphical method is that each pair of values (x_1, x_2) can be represented as a point in the two-dimensional coordinate system. With such a representation, we will be able to visualize the set of all feasible solutions as a graphical region, called the **feasible region** or the **feasible region**, and then to identify the optimal solution.

To construct the feasible region, we examine the constraints one at a time, starting with constraint (1). Suppose first that the inequality in constraint (1) is replaced by an equality. Then, it is easily seen that the set of points satisfying $2x_1 + 3x_2 = 6$ corresponds to a straight line in the two-dimensional plane. To plot this line, we need to identify the coordinates of two distinct points on this line. The standard approach for doing this is to set, alternatingly, one of the two variables to the value 0 and then solve the resulting equation with one unknown to obtain the corresponding value of the other variable. Thus, if we let $x_1 = 0$, then x_2 must equal to 2, since $x_2 = (6 - 2x_1)/3$. This yields the point $(0, 2)$. Similarly, by setting $x_2 = 0$, we obtain the point $(3, 0)$ (since $x_1 = (6 - 3x_2)/2$). The resulting line that passes through these two points is shown in Figure 1. We shall refer to this line as "line (1)".

Observe that line (1) cuts the plane into two half-planes. All points on one side of this line (including the line itself) satisfy $2x_1 + 3x_2 \leq 6$; whereas on the other side, $2x_1 + 3x_2 \geq 6$. Note that $(x_1, x_2) = (0, 0)$ satisfies $2x_1 + 3x_2 \leq 6$, and that $(0, 0)$ lies in the lower-left direction of line (1). Therefore, we are only interested in the lower-left portion of the plane. (What if $(x_1, x_2) = (0, 0)$ happens to satisfy a given constraint as an equality?) In Figure 1, next to line (1), we also placed two arrows that point toward the lower-left direction. This is intended to indicate what will be called the "feasible direction". In summary, after discarding points that violate constraint (1), we are left with the half plane that sits at the lower-left side of the line defined
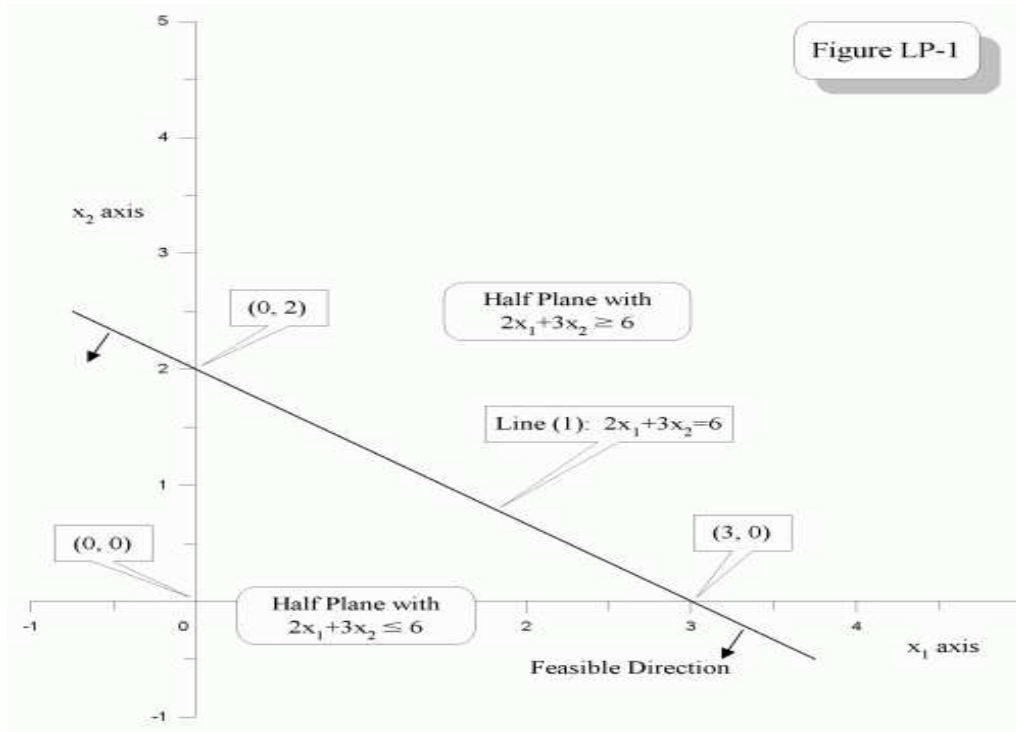
Figure 1: Constraint (1) cutting the plane into two parts.

by $2x_1 + 3x_2 = 6$.

Each constraint makes a "cut" on the two-dimensional plane, and repeating the above process for the remaining five constraints (the lines $x_1 = 0$ and $x_2 = 0$ correspond to the x_2 -axis and the x_1-axis, respectively) yields the feasible region. This is shown in Figure 2, where we have shaded the feasible region. Note that constraint (3) is *redundant*, in that its removal will not have an impact on the shape of the feasible region. Can you tell which constraint makes (3) redundant?

The next question is: Which point within the feasible region is optimal? To answer this, we now look at the objective function. Observe that for any given constant c, the set of points satisfying $4x_1 + 3x_2 = c$ is again a straight line; and more importantly, that by varying c, we can generate a family of lines (one for each c) with the same slope (i.e., members of this family of lines are parallel). Hence, we can choose an arbitrary c, draw its corresponding line $4x_1 + 3x_2 = c$, and then visualize the entire family by "sliding" this line without turning it. In Figure 3, we show two such lines: $4x_1 + 3x_2 = c_1$ and $4x_1 + 3x_2 = c_2$, where $c_1 = 12$ and $c_2 = 16$. Since these lines do not touch or cross the feasible region, points on these two lines are not feasible. Note however that the line with the smaller c, namely c_1, is closer to the feasible region. Therefore, if we decrease c further, from c_1, then at some point the resulting parallel line will eventually touch the feasible region at exactly one point. More precisely, this takes place when the family of objective-function lines hits the intersection of lines (1) and (4), a **corner point** (or an **extreme point**) of the feasible region. It is now clear that this corner point, marked as D in Figure 3, must be optimal, since we need to lower c still further to "reach" the other points within the feasible region. In other words, all other points within the feasible region correspond to lower values of c. Finally, we note that point D has the coordinates $(3/2, 1)$ (Why?) and therefore, it has an objective-function value of $9 \, (= 4 \cdot (3/2) + 3 \cdot 1)$. This completes the solution of the problem.

More generally, observe that *regardless* of the slope of the objective function, the above process will
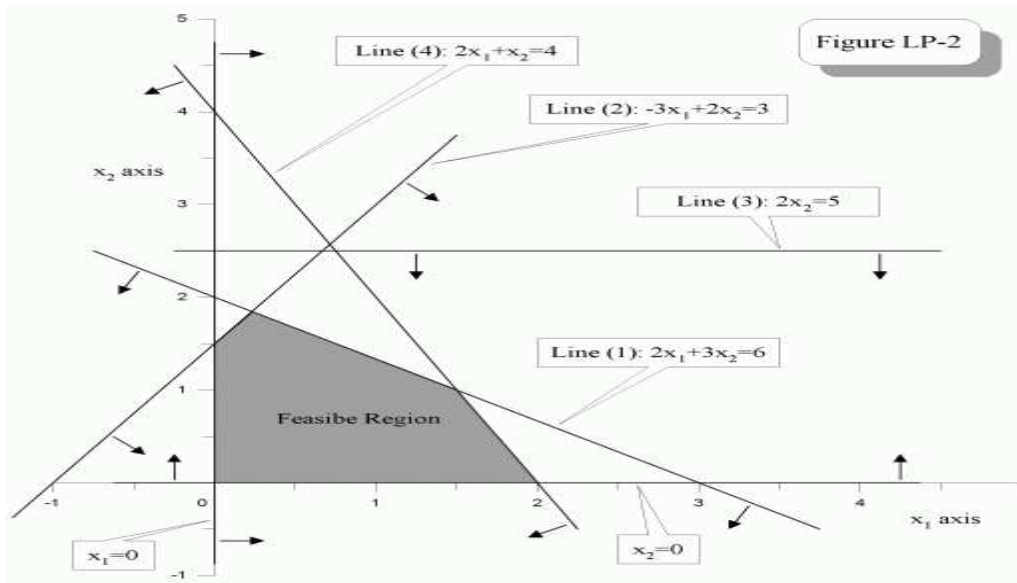
Figure 2: Feasible region obtained by putting all functional and nonnegativity constraints together.
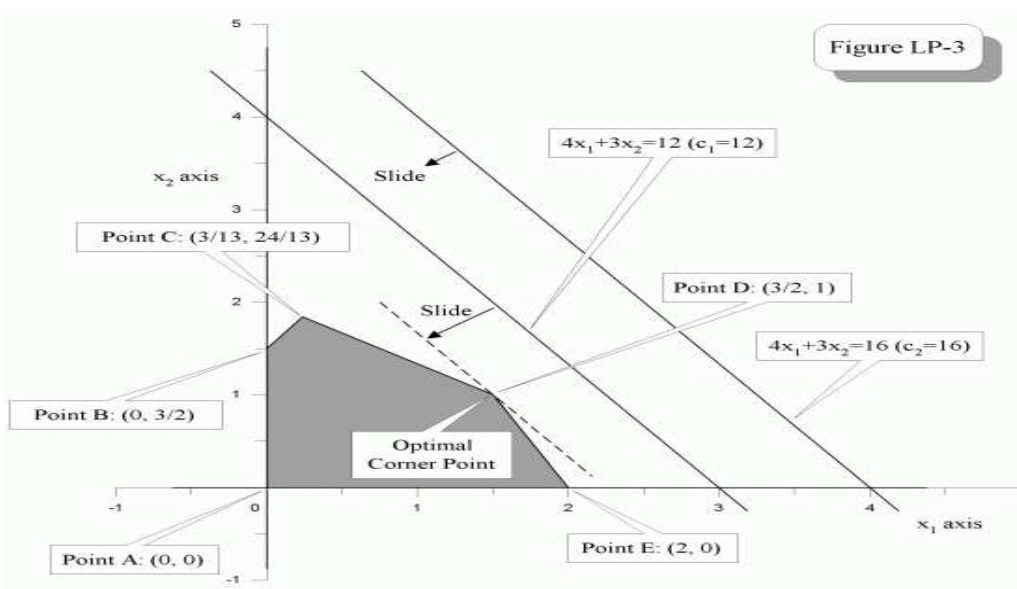

Figure 3: Optimal solution obtained by sliding the objective function towards the feasible region.

always lead to an optimal corner-point solution. (This is true even when the objective-function line first touches the feasible region at an entire edge; in such a case, we actually have two optimal corner-point solutions at the end points of that edge.) This implies that we can be assured that an optimal solution (whenever it exists) can always be found at one of the corner points of the feasible region. This fact is remarkable, as it says that we have managed to reduce the task of finding an optimal solution within the entire feasible set, which is a set containing an *uncountable* number of points, to that of finding an optimal solution within the set of corner points, which is a set containing a *finite* (why?) number of points. In this particular example, there are only five corner-point solutions, namely points A, B, C, D, and E (see Figure 3). Therefore, the reduction in the search effort is truly significant.

1.1 Procedure Solve LP

The above observation (sufficiency of looking at *only* corner points naturally suggests the following simple three-step procedure for solving a linear program:

> 1. Identify the coordinates of all corner points of the feasible region.
> 2. Evaluate the objective function at all of these corner points.
> 3. Pick the best corner point.

In applying Step 1 to the particular problem above, the location of the corner points can be identified via the graph. For problems with three decision variables, one can still attempt to draw three-dimensional graphs. With four or more variables, it becomes impossible to implement the procedure. Therefore, in order for this search procedure to work, we must develop a general method for determining the coordinates of the corner points *without relying* on a graphical representation of the feasible set; and this will be our next task.

In Figure 3, the coordinates of all five corner points have been explicitly specified. For example, the coordinates of the optimal corner point, point D, is given as $(3/2, 1)$. How does one determine these coordinates? Of course, we can attempt to determine the coordinates of a given point on a graph by "visual inspection", but in general, this cannot be expected to yield accurate answers.

More constructively, recall that point D is sitting at the intersection of lines (1) and (4) so D has to satisfy equalities obtained by replacing "≤" with "=" in both (1) and (4). Thus, we can algebraically solve the system of two **defining equations**:

$$\begin{array}{rcll} 2x_1 & +3x_2 & = & 6 \qquad (1) \\ 2x_1 & +x_2 & = & 4 \qquad (4) \end{array}$$

To obtain the coordinates of this intersection, subtract equation (4) from equation (1) to get $2x_2 = 2$, implying that $x_2 = 1$. Next, substitution of $x_2 = 1$ into equation (1) (or alternatively, into equation (4)) shows that $x_1 = 3/2$. This verifies that $(x_1, x_2) = (3/2, 1)$ is indeed the precise location of point D. You should now verify the coordinates of points A, B, C, and E yourself.

It is important to realize that this analysis still depends on the availability of the graph. Specifically, we need to identify the defining equations for each corner point (i.e., we need to know that equations (1) and (4) together define point D, equations (1) and (2) together define point C, equation (2) and $x_1 = 0$ together define point B, and so on). And we need to know that **only** these five (Out of how many? See below.) particular combinations of two defining equations give rise to corner points of the feasible region.

1.2 Procedure Generate Corner Points

A fundamental question is: Can we free ourselves from the reliance on the graph? A little bit of reflection now leads us to the following procedure:

1. From the given set of six equations (including $x_1 = 0$ and $x_2 = 0$), choose an arbitrary combination of two equations. Solve this chosen set of two equations to obtain the coordinates of their intersection. (A unique solution may not exist in general; see the Remarks below.)
2. If the solution in Step 1 satisfies all other constraints, then accept it as a corner-point solution; otherwise, discard the solution and the corresponding combination of constraints.
3. Repeatedly loop through Steps 1 and 2 until all possible combinations are exhausted.

Despite its brutal flavor, it turns out that this procedure, indeed, will allow us to generate the coordinates of all corner-point solutions.

To convince ourselves that this procedure rejects infeasible points, consider, for example, the choice of equations (1) and (3):

$$
\begin{array}{rrll}
2x_1 & +3x_2 & = & 6 & \quad (1) \\
& 2x_2 & = & 5 \,. & \quad (3)
\end{array}
$$

From equation (3), we have $x_2 = 5/2$, which implies that $x_1 = (6 - 3x_2)/2 = -3/4$. Since the resulting solution $(x_1, x_2) = (-3/4, 5/2)$ has a negative value for x_2, it is not feasible. Therefore, the combination of equations (1) and (3) does not lead to a corner-point solution.

With six equations, it is easily seen that the total number of subsets of two equations is, 2 choose out of 6, equal to

$$
\binom{6}{2} = \frac{6!}{2!4!} = 15
$$

After cycling through all 15 of these combinations and discarding combinations that do not yield a feasible solution, it can be shown that only five combinations remain; moreover, these five combinations correspond precisely to the pairs of defining equations for points A, B, C, D, and E. This empirically establishes the validity of the above procedure.

Procedure Solve LP and Procedure Generate Corner Points form the backbone of the Simplex method, which is an algebraic procedure for solving linear programs with any (finite) number of variables and constraints. We will return to a full development of this method later.

1.3   Remarks

1. Infeasibility: In general, the feasible region of a linear program may be empty. Procedure Solve LP is meaningful only if the feasible region is not empty. Discovery of an empty feasible region will be discussed later. A linear program of this type is said to be **infeasible**.

2. Unboundedness: Consider the linear program: Maximize $x_1 + x_2$, subject to $x_1, x_2 \geq 0$. Since the values of x_1 and x_2 are allowed to be arbitrarily large, this problem does not have an optimal solution. Note that the feasible region has exactly one corner point, at $(0,0)$; and that this corner point is not optimal. This clearly is a disturbing factor for Procedure Solve LP. A linear program of this type is said to be **unbounded**.

   In general, a given pair of straight lines on the plane may not have a unique intersection point. This could occur if the two lines are parallel, in which case there is no intersection, or if the two lines coincide, in which case every point on the line is an intersection point. Algebraically, this means that a given pair of equations may not have a unique solution. For example, the equation pair

$$
\begin{array}{rrll}
x_1 & +2x_2 & = & 3 \\
x_1 & +2x_2 & = & 6
\end{array}
$$

has no solution; and the equation pair

$$
\begin{aligned}
x_1 & +2x_2 & = & \ 3 \\
2x_1 & +4x_2 & = & \ 6
\end{aligned}
$$

has an infinite number of solutions. This issue is relevant in Step 1 of Procedure Generate Corner Point.

2 A Characterization of the Corner-Point Solutions

Consider the example of the previous section again. Our first step is to "rewrite" that problem as:

$$
\begin{array}{llll}
\text{Max} & 4x_1 & +3x_2 \\
\text{Subject to:} \\
& 2x_1 & +3x_2 & +s_1 & & & & = & 6 & \quad (1) \\
& -3x_1 & +2x_2 & & +s_2 & & & = & 3 & \quad (2) \\
& & 2x_2 & & & +s_3 & & = & 5 & \quad (3) \\
& 2x_1 & +x_2 & & & & +s_4 & = & 4 & \quad (4)
\end{array}
$$

$$
x_1,\ x_2 \geq 0 \ \text{ and } \ s_1,\ s_2,\ s_3,\ s_4 \geq 0\,,
$$

where we have (i) introduced four new variables, s_1, s_2, s_3, and s_4, (ii) converted all functional (inequality) constraints into equalities, and (iii) added nonnegativity requirements for the new variables.

We will use examples to argue that this new problem is indeed equivalent to the previous one. Consider the feasible solution $(x_1, x_2) = (1, 1)$ to the original problem. This solution is feasible because

$$
\begin{array}{rcrcl}
2 \cdot 1 & +3 \cdot 1 & = & 5 & \leq & 6 \\
-3 \cdot 1 & +2 \cdot 1 & = & -1 & \leq & 3 \\
& 2 \cdot 1 & = & 2 & \leq & 5 \\
2 \cdot 1 & +1 \cdot 1 & = & 3 & \leq & 4\,,
\end{array}
$$

and the two values in $(1, 1)$ are both nonnegative. Note that for each functional constraint, there is a nonnegative "slack" — the difference between the constant on the right-hand side and the evaluation on the left-hand side. Explicitly, these slacks are given by 1 ($= 6 - 5$), 4 ($= 3 - (-1)$), 3 ($= 5 - 2$), and 1 ($= 4 - 3$), respectively. Now, if we let $s_1 = 1$, $s_2 = 4$, $s_3 = 3$, and $s_4 = 1$, then the **augmented solution** $(x_1, x_2, s_1, s_2, s_3, s_4) = (1, 1, 1, 4, 3, 1)$ is, by construction, a feasible solution to the new problem. In general, by appropriately augmenting a given feasible solution to the original problem, we can always construct a corresponding feasible solution for the new problem.

Next, consider the feasible solution $(x_1, x_2, s_1, s_2, s_3, s_4) = (0, 0, 6, 3, 5, 4)$ to the new problem. This solution is feasible because

$$
\begin{array}{rclll}
2 \cdot 0 & +3 \cdot 0 & +6 & & = & 6 \\
-3 \cdot 0 & +2 \cdot 0 & +3 & & = & 3 \\
& 2 \cdot 0 & & +5 & = & 5 \\
2 \cdot 0 & +1 \cdot 0 & & +4 & = & 4\,,
\end{array}
$$

and all six assigned values are nonnegative. Now, if we drop the last four values in $(0, 0, 6, 3, 5, 4)$ and consider the solution $(x_1, x_2) = (0, 0)$, then, since these four values are nonnegative, we obviously have

$$
\begin{array}{rcrcl}
2 \cdot 0 & +3 \cdot 0 & \leq & 6 \\
-3 \cdot 0 & +2 \cdot 0 & \leq & 3 \\
& 2 \cdot 0 & \leq & 5 \\
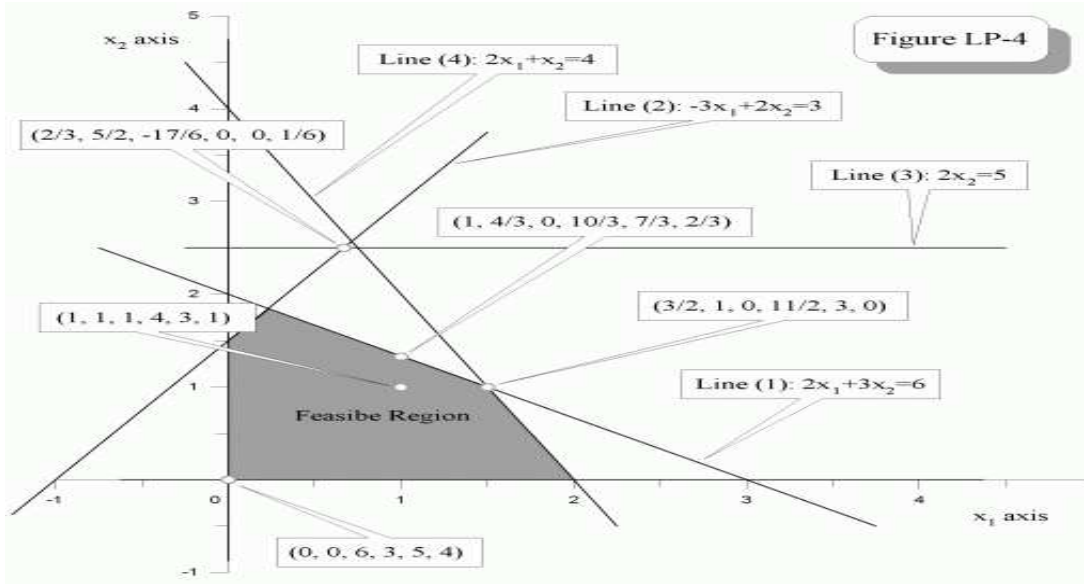2 \cdot 0 & +1 \cdot 0 & \leq & 4\,,
\end{array}
$$

Figure 4: Basic solutions, basic feasible solutions and an interior solution.

which means that the solution $(0, 0)$ is feasible to the original problem. Therefore, by removing the last four components of a given feasible solution to the new problem, we can always obtain a corresponding feasible solution to the original problem. In summary, we have empirically argued that the feasible regions of the original and the new linear programs are identical. Finally, since these two problems also have the same objective function, they are equivalent.

Above discussion shows that a point in the feasible region can be described either in the original form of two coordinates (x_1, x_2) or in the augmented form with six coordinates $(x_1, x_2, s_1, s_2, s_3, s_4)$. We shall henceforth refer to the new variables s_1, s_2, s_3, and s_4 as the **slack variables**, and the new problem as the **augmented problem**.

An important reason for constructing the augmented problem is that the value of the slack variable associated with a new equality constraint provides **explicit** information on the "tightness" of the corresponding original (inequality) constraint. We will illustrate this via a series of examples. Refer to Figure 4 while going over these examples.

Consider the solution $(x_1, x_2) = (3/2, 1)$, which is feasible for the original problem. This solution has the corresponding augmented solution $(x_1, x_2, s_1, s_2, s_3, s_4) = (3/2, 1, 0, 11/2, 3, 0)$, where the values for the slack variables are determined by substituting $x_1 = 3/2$ and $x_2 = 1$ into equations (1), (2), (3), and (4). Notice that in the augmented solution, we have $s_1 = 0$ and $s_4 = 0$, which corresponds to the fact that this solution satisfies constraints (1) and (4) as equalities in the original problem. In other words, this solution is such that the original inequality constraints (1) and (4) are tight, or **binding**. Notice further that, in contrast, both s_2 and s_3 are strictly positive; hence, the inequality constraints (2) and (3) in the original problem are not binding. From Figure 4, we see that, in this example, we have a solution that corresponds to a corner point of the feasible region.

As a second example, consider the feasible solution $(x_1, x_2) = (1, 4/3)$ and its corresponding augmented solution $(x_1, x_2, s_1, s_2, s_3, s_4) = (1, 4/3, 0, 10/3, 7/3, 2/3)$. In this case, we have $s_1 = 0$ and the values of the other slack variables are all positive; therefore, constraint (1) in the original problem is binding, whereas

the other functional constraints are not. Note that point $(1, 4/3)$ lies on an edge of the feasible region.

As a third example, consider the (candidate) solution $(x_1, x_2) = (2/3, 5/2)$ and its corresponding augmented solution $(x_1, x_2, s_1, s_2, s_3, s_4) = (2/3, 5/2, -17/6, 0, 0, 1/6)$. Since $s_2 = 0$ and $s_3 = 0$, the corresponding original constraints (2) and (3) are binding. Note however that $s_1 = -17/6$, a negative value. This means that constraint (1) in the original problem is violated. Hence, while the point $(2/3, 5/2)$ lies at the intersection of two of the original constraint lines, it is not a feasible solution to the original problem.

In the above three examples, we focussed our attention on whether or not any given original functional constraint is binding. In the next two examples, we will also consider the relevance of the original nonnegative constraints, $x_1 \geq 0$ and $x_2 \geq 0$.

Consider, as a fourth example, the solution $(x_1, x_2) = (0, 0)$, with corresponding augmented solution $(x_1, x_2, s_1, s_2, s_3, s_4) = (0, 0, 6, 3, 5, 4)$. Since x_1 and x_2 both equal to 0 and since all slack variables are positive, we see that $(0, 0)$ is feasible, and that the nonnegativity constraints $x_1 \geq 0$ and $x_2 \geq 0$ are the only binding constraints. This solution, like $(x_1, x_2) = (3/2, 1)$, corresponds to a corner point.

As our final example, consider the solution $(x_1, x_2) = (1, 1)$ and its corresponding augmented solution $(x_1, x_2, s_1, s_2, s_3, s_4) = (1, 1, 1, 4, 3, 1)$. This solution is clearly feasible. The fact that all variables (including x_1 and x_2) assume a positive value tells us that every point that is located in the vicinity of $(1, 1)$ is also feasible. More formally, let δ_1 and δ_2 be two arbitrary (not necessarily positive) values; then, as long as the absolute values of δ_1 and δ_2 are sufficiently small, every solution of the form $(x_1, x_2) = (1 + \delta_1, 1 + \delta_2)$ will also be feasible. In other words, at point $(1, 1)$, none of the six inequality constraints in the original problem is tight. A point such as $(1, 1)$ will be called an **interior point**.

We are now ready to revisit Procedure Corner Points. Recall that in the first step of that procedure, we are to choose two of the six equations as defining equations and solve the selected pair of equations to obtain the coordinates of their intersection. In the setting of the augmented problem, this can be rephrased as follows. From our discussion above, we see that declaring an equation in the original problem as a defining equation is the same as assigning a value of zero to its slack variable in the augmented problem. For example, this means that: (i) if we are interested in solutions in the original problem that make the inequality $2x_1 + 3x_2 \leq 6$ binding, we could simply assign the value zero to the variable s_1 and consider augmented solutions of the form $(x_1, x_2, 0, s_2, s_3, s_4)$; (ii) if we are interested in solutions that make the inequality $x_1 \geq 0$ binding, we could let $x_1 = 0$ and consider augmented solutions of the form $(0, x_2, s_1, s_2, s_3, s_4)$; and (iii) if we are interested in solutions that make both $x_1 \geq 0$ and $2x_1 + 3x_2 \leq 6$ binding, we could let $x_1 = s_1 = 0$ and consider augmented solutions of the form $(0, x_2, 0, s_2, s_3, s_4)$. Thus, what we are looking for in the first step of Procedure Corner Points corresponds to the set of augmented solutions that have the value zero assigned to two of their coordinates. Formally, this results in the following restatement.

Revised Step 1: Choose an arbitrary pair of variables in the augmented problem, and assign the value zero to these variables. This reduces the functional constraints in the augmented problem to a set of four equations in four unknowns. Solve (if a unique solution exists) this equation system to obtain an explicit augmented solution.

A solution produced by this step will be called a **basic solution**.

By construction, an augmented solution produced by the above step necessarily satisfies all functional constraints in the augmented problem. This leaves us with the verification of the nonnegativity constraints for decision variables and slacks, which will be done in the next step.

Revised Step 2: If all of the values in the augmented solution are feasible, that is

if all nonnegativity constraints are satisfied,
accept it as a corner-point solution; otherwise, discard the solution.

If there are no nonnegativity constraints on decision variables, confirm nonnegativity only for slacks to establish feasibility. A solution produced by this step will be called a **basic feasible solution**. Finally, as in the original Procedure Generate Corner Points, repeating these two steps will produce, one by one, all of the corner-point solutions.

What we have established is that the set of corner-point solutions is characterized, or is identical, to the set of basic feasible solutions in the augmented problem. Because this characterization is algebraic (and hence is not dependent on a graphical representation of the feasible region), it is extremely important. In the next section, we will show that it forms the foundation of the Simplex method.

Due to their importance, we now conclude this section with a formal definition of the concepts of basic solutions and basic feasible solutions. Consider a system of m linear equations in n unknowns, denoted by x_1, x_2, ..., x_n, and assume that the system has the following format:

$$
\begin{aligned}
a_{11}x_1 &+ a_{12}x_2 + && \cdots && + a_{1n}x_n &&= b_1 \\
a_{21}x_1 &+ a_{22}x_2 + && \cdots && + a_{2n}x_n &&= b_2 \\
& && \cdots && && \\
a_{m1}x_1 &+ a_{m2}x_2 + && \cdots && + a_{mn}x_n &&= b_m\,,
\end{aligned}
$$

where the a_{ij}'s and the b_j's are given constants. We shall assume that $m \leq n$, which means the number of variables is at least as many as the number of equations. Under this assumption, the equation system will typically have an infinite number of solutions. If we arbitrarily select $n - m$ variables and set their values to zero, then the system will be reduced to a set of m equations in m unknowns. The selected set of $n - m$ variables will be called **nonbasic variables**, since they do not "participate" in the reduced equation system; and the remaining m variables will be called **basic variables**. Now, the reduced equation system may, in general, have: (i) a unique solution, (ii) no solution, or (iii) an infinite number of solutions. Suppose, for a given selected set of nonbasic variables, the reduced set of m equations in m (basic) variables has a unique solution. Then, the resulting "full" solution, with an explicit value for each of the basic and nonbasic variables, is called a *basic solution*. Note that we have not made any assumption on the sign of any of the (basic) variables. A basic solution that happens to have all nonnegative values will be called a **basic feasible solution**.

In the augmented problem above, we have four functional equality constraints and six decision variables. Hence, $m = 4$ and $n = 6$. Suppose we declare, for example, s_1 and s_4 as nonbasic variables. Then, it is easy to check that the reduced equation system,

$$
\begin{aligned}
2x_1 &+ 3x_2 && &&= 6 \\
-3x_1 &+ 2x_2 &+ s_2 && &&= 3 \\
& 2x_2 && &+ s_3 &&= 5 \\
2x_1 &+ x_2 && && &&= 4\,,
\end{aligned}
$$

has the (unique) solution $(x_1, x_2, s_2, s_3) = (3/2, 1, 11/2, 3)$. After supplementing this with two zeros for the nonbasic variables, we obtain the basic solution $(x_1, x_2, s_1, s_2, s_3, s_4) = (3/2, 1, 0, 11/2, 3, 0)$. Moreover, since all of these values are nonnegative, what we have is a basic feasible solution as well.

In general, with m equations and n unknowns, the total number of basic solutions cannot exceed

$$
\binom{n}{n - m} = \frac{n!}{(n - m)!m!}\,.
$$

For $m = 4$ and $n = 6$, this evaluates to 15. Thus, potentially, we could have up to 15 basic solutions.

2.1   Remarks

1. *Anology between basic (feasible) solutions and sport teams.* Consider a team sport, say basketball. Say 12 players dress up for a game but the coach uses 5 of them at any particular time. Selecting those 5 players out of 12 is very similar to picking a basic solution in Linear Programming. The selected 5 players make some contributions to the game and correspond to basic variables. The remaining seven players are outside the game and make zero contribution so they correspond to nonbasic variables. If the selected five are all making nonnegative contributions, we can say that the selected team (the set of basic variables) is a (basic) feasible (solution) team. Otherwise, we say that it is an infeasible team (solution). The coach's objective in basketball and our objective in solving Linear Programs are the same: Selecting the best 5 players (variables) out of 12 to include in the team (basic solution).

3   Solved Exercises

Find all the augmented solutions (corresponding to basic solutions) for the following feasible region, also identify which of these are feasible.

$$
\begin{array}{rrrcr}
x_1 & +x_2 & +x_3 & \leq & 5 \\
-x_1 & +x_2 & +2x_3 & \leq & 6 \\
x_1 & & & \geq & 0 \\
& x_2 & & \geq & 0 \\
& & x_3 & \geq & 0
\end{array}
$$

Solution: We add slacks s_1 and s_2 to inequalities to make them equality. Then we have $n = 5$ variables, $m = 2$ equations so we set $n - m$ variables to zero to find a basic solution. Say we set $x_1 = x_2 = x_3 = 0$, the solution of equations is $s_1 = 5$ and $s_2 = 6$. this is the first row in the table below. The remaining rows are obtained similarly.

x_1	x_2	x_3	s_1	s_2	Feasible ?
0	0	0	5	6	Yes
0	0	5	0	-4	No
0	0	3	2	0	Yes
0	5	0	0	1	Yes
0	6	0	-1	0	No
0	4	1	0	0	Yes
5	0	0	0	11	Yes
-6	0	0	11	0	No
4/3	0	11/3	0	0	Yes
-1/2	11/2	0	0	0	No

Since variables must be nonnegative, a negative variable in an augmented solution indicates infeasibility.

4   Exercises

1. Consider

$$
\begin{array}{lrr}
\text{Max} & 6x_1 & +x_2
\end{array}
$$

Subject to:

$$
\begin{array}{rrrcrl}
3x_1 & +2x_2 & \leq & 6 & & (1) \\
4x_1 & +x_2 & \leq & 4 & & (2) \\
& x_1,\ x_2 & \geq & 0 &, &
\end{array}
$$

a) Find the optimal solution and the optimal objective function value graphically.
b) Write down all the basic solutions and basic feasible solutions, point these out on your graph.
c) Introduce slack variables s_1 and s_2 for constraints (1) and (2) respectively. Find basic solutions of the form $(x_1, x_2, s_1 = 0, s_2 = 0)$ and $(x_1 = 0, x_2, s_1, s_2 = 0)$, i.e. what are the values of x_1, x_2, s_1, s_2? Are these solutions feasible?
d) What constraints are binding for each basic solution of part (c)?

2. Refer to the previous problem. Parts (a) and (b) are independent.
   a) Add the constraint $x_2 \leq 0.5$ and resolve the problem.
   b) Add the constraint $x_1 \leq 0.5$ and resolve the problem.
   c) What happens to the feasible region and the optimal objective function value when a constraint is added? Any general conclusions, i.e. can we say that the optimal objective function value is increasing, decreasing, nonincreasing or nondecreasing?
   d) What happens to the feasible region and the optimal objective function value when we delete the constraint $3x_1 + 2x_2 \leq 6$. Any general conclusions?

3. Textbook H-L: pp. 94-95 3.4-6.

4. Consider

$$\text{Max} \qquad x_1 \quad -x_2$$
$$\text{Subject to:}$$
$$\begin{array}{rrcll} 2x_1 & -x_2 & \leq & 3 & (1) \\ -0.5x_1 & +x_2 & \leq & 1.5 & (2) \\ -x_1 & & \leq & 1 & (3) \\ & -x_2 & \leq & 1 & (4) \end{array}$$

   a) Draw the feasible region.
   b) Find the optimal solution graphically.
   c) Write down all basic solutions (i.e. intersection points) and basic feasible solutions (i.e. corner points), show these out on your graph. Note that this problem does not have nonnegativity constraints so x_1, x_2 can be negative and feasible.
   d) Introduce slack variables s_1, s_2, s_3 and s_4 for constraints (1), (2), (3) and (4) respectively. Find basic solutions of the form $(x_1, x_2, s_1 = 0, s_2 = 0, s_3, s_4)$ and $(x_1, x_2, s_1, s_2 = 0, s_3, s_4 = 0)$, i.e. what are the values of $x_1, x_2, s_1, s_2, s_3, s_4$? Are these solutions feasible?
   e) What constraints are binding for each basic solution of part (d)?

5. Refer to the previous problem. Parts (a) and (c) are independent.
   a) Add the constraint $x_2 \geq 2$ and resolve the problem graphically.
   b) What happens to the feasible region and the optimal objective function value when a constraint is added? Any general conclusions, i.e. can we say that the feasible region is smaller/larger or the optimal objective function value is increasing, decreasing, nonincreasing or nondecreasing?
   c) Remove the constraint $2x_1 - x_2 \leq 3$ and resolve the problem graphically.
   d) What happens to the feasible region and the optimal objective function value when we delete a constraint. Any general conclusions?

6. Remember the Thomas & Brown Accounting from Plano High School problems and the formulation:
   Max $400A + 125T$
   s.t. $6A + 3T \leq 120$
   $2A + \frac{1}{4}T \leq 20$
   $A \geq 0$
   $T \geq 0$

a) Draw the feasible region.

b) Find the optimal solution graphically.

c) Write down all basic solutions and basic feasible solutions, point these out on your graph.

d) Introduce slack variables s_1 and s_2 for constraints (1)and (2) respectively. Find basic solutions of the form $(x_1, x_2, s_1 = 0, s_2 = 0)$ and $(x_1 = 0, x_2, s_1, s_2 = 0)$, i.e. what are the values of x_1, x_2, s_1, s_2? Are these solutions feasible?

e) What constraints are binding for each of the two basic solution of part (d)?

7. Refer to the previous problem. Parts (a) and (c) are independent.

a) Add the constraint $A \geq 8$ and resolve the problem graphically.

b) What happens to the feasible region and the optimal objective function value when a constraint is added? Any general conclusions, i.e. can we say that the feasible region is smaller/larger or the optimal objective function value is increasing, decreasing, nonincreasing or nondecreasing?

c) Remove the constraint $2A + T/4 \leq 20$ and resolve the problem graphically.

d) What happens to the feasible region and the optimal objective function value when we delete a constraint. Any general conclusions?

8. Consider

$$
\begin{array}{rrrcr}
x_1 & +x_2 & +x_3 & \leq & 10 \\
x_1 & +x_2 & -x_3 & \leq & 6 \\
x_1 & & & \geq & 0 \\
& x_2 & & \geq & 0 \\
& & x_3 & \geq & 0
\end{array}
$$

How many basic solutions can there be? Introduce slack variables and find all the augmented solutions corresponding to basic solutions, also identify which of these solutions are feasible.

9. Find all the basic solutions for the following system of linear equations :

$$
\begin{array}{rrrrcl}
-x_1 & +2x_2 & +x_3 & +x_4 & = & 4 \\
x_1 & -2x_2 & & +2x_4 & = & 3
\end{array}
$$

5   The Simplex Method: An Example

Our first step is to introduce one more new variable, which we denote by z. The variable z is **defined** to be equal to $4x_1 + 3x_2$. Doing this will allow us to have a unified statement of the objective function, namely: Maximize z. Of course, we must at the same time introduce $z = 4x_1 + 3x_2$ into the constraint set, so that the new variable z is properly linked to the other variables. Following our convention of putting all variables on the left-hand side of the constraints, we will write this new constraint as: $z - 4x_1 - 3x_2 = 0$. With these revisions, we now have the following equivalent problem:

$$
\begin{array}{llrrrrrrcll}
\text{Maximize} & z & & & & & & & & & \\
\text{Subject to:} & & & & & & & & & & \\
& z & -4x_1 & -3x_2 & & & & & = & 0 & (0) \\
& & 2x_1 & +3x_2 & +s_1 & & & & = & 6 & (1) \\
& & -3x_1 & +2x_2 & & +s_2 & & & = & 3 & (2) \\
& & & 2x_2 & & & +s_3 & & = & 5 & (3) \\
& & 2x_1 & +x_2 & & & & +s_4 & = & 4 & (4)
\end{array}
$$

$$x_1,\, x_2,\, s_1,\, s_2,\, s_3,\, s_4 \geq 0 ,$$

where we have labelled the new constraint as equation (0). Note that the new variable z is unrestricted in sign.

More generally, it can be shown that any given linear program can be converted into the above format, which will henceforth be referred to as the **standard form**. The standard form provides a unified starting configuration for the solution of a linear program by the Simplex method. We will discuss how to convert problems into the standard form later.

Our next step is to construct an initial basic feasible solution based on the configuration of equations (1)–(4). Clearly, with equation (0) excluded (why?), we have a set of 4 functional equality constraints in 6 unknowns. Therefore (from our discussion in the previous section), each basic feasible solution will have two nonbasic variables, whose values are set to zero, and four basic variables, whose (nonnegative) values are determined (uniquely) by solving four equations in the four remaining unknowns. In principle, any particular choice of a pair of "candidate" nonbasic variables can *potentially* lead to a basic feasible solution, and this will indeed be the case if the values in the solution of the resulting four equations turn out to be all nonnegative. Since, in general, this outcome cannot be guaranteed a priori, the question is: Which pair should we choose? To answer this question, we will take a close look at equations (1)–(4). Notice that each of these four equations contains a variable that appears in that equation only. These variables are s_1, s_2, s_3, and s_4. Now, if we designate x_1 and x_2 as nonbasic and set their values to zero, then the resulting four equations assume the following most simple form:

$$
\begin{aligned}
s_1 &&&&= 6 \\
&s_2 &&&= 3 \\
&&s_3 &&= 5 \\
&&&s_4 &= 4
\end{aligned}
$$

and hence, no effort is required in solving these equations. Moreover, the fact that all four right-hand side constants are nonnegative implies that the resulting augmented solution, $(x_1, x_2, s_1, s_2, s_3, s_4) = (0, 0, 6, 3, 5, 4)$, is necessarily feasible. Thus, the particular configuration of equations (1)–(4) allows us to "conveniently" generate (or read out) a basic feasible solution, namely $(0, 0, 6, 3, 5, 4)$. We shall say that $(0, 0, 6, 3, 5, 4)$ is the basic feasible solution "associated" with equations (1)–(4), and we will also let it be our starting basic feasible solution. Note that this solution corresponds to point A in the graph.

To determine the value of z associated with this starting basic feasible solution, we turn to equation (0). Note that the variable z appears only in equation (0), and that none of the current basic variables, s_1, s_2, s_3, s_4, appears in that equation. Therefore, with both x_1 and x_2 assigned the value zero, z must equal to the constant sitting at the right-hand side of equation (0). This shows that the objective-function value associated with $(0, 0, 6, 3, 5, 4)$ is equal to 0.

We shall refer to the set of basic variables in a basic solution as the **basis**. Thus, the variables s_1, s_2, s_3, and s_4 constitute the basis in the solution $(x_1, x_2, s_1, s_2, s_3, s_4) = (0, 0, 6, 3, 5, 4)$. Note that every constraint equation has exactly one associated basic variable.

The next question is whether or not the current basic feasible solution, or point A, is optimal. To answer this, it is helpful to imagine yourself standing at point A and attempt to travel toward the direction of either point B or point E, along the x_2 axis or the x_1 axis, respectively. In the discussion below, we will switch freely between the graphical viewpoint and a corresponding "algebraic development." The graphical viewpoint helps us understand the algebraic development, but in the end, only the latter constitute the final solution procedure.

Consider point B first. Algebraically, "travelling along the x_2 axis" means that we will attempt to increase x_2 from its current value 0, while holding the value of x_1 at 0. We will "emulate" this action parametrically, as follows. Suppose x_2 is increased by a very small amount, say δ (≥ 0). Notice that, in order to maintain feasibility while making such a move, we must also readjust the values of the current basic variables, s_1, s_2, s_3, and s_4, appropriately. Specifically, maintaining feasibility means that we must dynamically (or continuously) enforce the equalities in

$$
\begin{array}{rllllll}
2\cdot 0 & +3\cdot(0+\delta) & +s_1 & & & = & 6 \\
-3\cdot 0 & +2\cdot(0+\delta) & & +s_2 & & = & 3 \\
& 2\cdot(0+\delta) & & & +s_3 & = & 5 \\
2\cdot 0 & +(0+\delta) & & & & +s_4 & = & 4
\end{array}
$$

as we increase δ. In the first equation here, s_1 was equal to 6; but as a result of the new contribution 3δ from the nominal increase in x_2 (from 0 to $0+\delta$), the value of s_1 must be adjusted downward by the same amount to offset this new contribution. That is, s_1 must assume the new value $6-3\delta$. Similar reasoning shows that the other current basic variables s_2, s_3, and s_4 must assume the new values $3-2\delta$, $5-2\delta$, and $4-\delta$, respectively. This results in the new augmented solution $(x_1,\,x_2,\,s_1,\,s_2,\,s_3,\,s_4) = (0,\,\delta,\,6-3\delta,\,3-2\delta,\,5-2\delta,\,4-\delta)$. Now, since all variables must remain nonnegative, we also need to require that $6-3\delta \geq 0$, $3-2\delta \geq 0$, $5-2\delta \geq 0$, and $4-\delta \geq 0$. This implies that the value of δ should not exceed

$$
\min\left[\frac{6}{3},\,\frac{3}{2},\,\frac{5}{2},\,\frac{4}{1}\right] = \frac{3}{2}.
$$

Note that, with $\delta = 3/2$, the new augmented solution becomes $(x_1, x_2, s_1, s_2, s_3, s_4) = (0, 3/2, 3/2, 0, 2, 5/2)$; and this is precisely point B. (What happens if δ is taken smaller than $3/2$, say $\delta = 1$, identify the corresponding point on the graph?)

What we have shown is that by increasing the value of δ from 0 to $3/2$, the family of augmented solutions $(0,\,\delta,\,6-3\delta,\,3-2\delta,\,5-2\delta,\,4-\delta)$ "trace out" all points on the AB edge of the feasible region. Note that with $\delta = 3/2$, the value of s_2 is reduced to 0, which means that the second original inequality constraint becomes binding. Thus, if we allow δ to go beyond $3/2$, then the resulting augmented solutions will be infeasible.

To determine whether or not point B is better than point A, observe that from equation (0), we have

$$
z \quad -4\cdot 0 \quad -3\cdot(0+\delta) \quad = \quad 0.
$$

Hence, with $\delta = 3/2$, the value of z will be revised from 0 to $3\delta = 9/2$, a better value. Note in addition that, in response to an increase in δ, the value of z goes up at a rate of 3.

The analysis for point E is similar. Consider a nominal increase of size δ for x_1. Since the value of x_2 will be held at 0, we see from constraints (1)–(4) that the corresponding values for s_1 s_2, s_3, and s_4 should be revised to: $6-2\delta$, $3+3\delta$, $5-0\cdot\delta$, and $4-2\delta$, respectively. Thus, the new augmented solution is $(x_1,\,x_2,\,s_1,\,s_2,\,s_3,\,s_4) = (\delta,\,0,\,6-2\delta,\,3+3\delta,\,5,\,4-2\delta)$. Again, the nonnegativity requirements imply that δ should not exceed

$$
\min\left[\frac{6}{2},\,\frac{4}{2}\right] = 2
$$

(s_2 and s_3 cannot be made negative by increasing δ). With $\delta = 2$, the new augmented solution becomes $(x_1,\,x_2,\,s_1,\,s_2,\,s_3,\,s_4) = (2, 0, 2, 9, 5, 0)$; and this corresponds to point E. Finally, from equation (0), we have

$$
z \quad -4\cdot(0+\delta) \quad -3\cdot 0 \quad = \quad 0.
$$

Hence, with $\delta = 2$, the value of z will be increased from 0 to $4\cdot\delta = 8$, which again is a better value. In addition, note that the value of z increased at a rate of 4.

The above analysis shows that point A is not optimal, and we can travel to either point B or point E to achieve a better objective-function value. We will select point E, since it offers a greater improvement ($9/2$ for B versus 8 for E). This choice, by coincidence, also offers a greater *rate* of improvement in the value of z (3 for B versus 4 for E).

We will now focus our attention on point E, and the immediate question again is: Is E optimal? The above analysis for point A can be viewed as an **optimality test**, which we plan to duplicate for point E. Before proceeding, it is however important to realize that the optimality test for point A was greatly facilitated by the particular algebraic configuration of the constraints. Specifically, recall that the choice of point A as the initial basic feasible solution was motivated by the following three convenient attributes of the standard form:

1. Each of constraints (1)–(4) contains a variable that has a coefficient of 1 and appears in that equation only. These variables, s_1, s_2, s_3, and s_4, serve as candidate basic variables.

2. The constants on the right-hand side of equations (1)–(4) are all nonnegative.

3. The candidate basic variables do not appear in constraint (0).

Therefore, it seems natural to attempt to recast the original constraint set into an equivalent standard form that is better suited for an optimality test for point E. This will be our next task.

Observe that the augmented solution that corresponds to point E, namely $(2, 0, 2, 9, 5, 0)$, assigns a zero to both x_2 and s_4. This means that at point E, the nonbasic variables are x_2 and s_4, and the basic variables are x_1, s_1, s_2, and s_3. Thus, switching from point A to point E can be reinterpreted as changing (i) the status of x_1 from being nonbasic to being basic and (ii) the status of s_4 from being basic to nonbasic. In other words, as we switch from A to E, the variable x_1 is to enter the basis and the variable s_4 is to leave the basis. Now, with x_1, s_1, s_2, and s_3 being the new basis, what we would like to do, in light of the three attributes of the standard form discussed above, is to convert the constraint set into the following "target" configuration:

$$
\begin{array}{rllllll}
z & +? & & & +? & = & ? & \quad (0) \\
& +? & +s_1 & & +? & = & ? & \quad (1) \\
& +? & & +s_2 & +? & = & ? & \quad (2) \\
& +? & & & +s_3 & +? & = & ? & \quad (3) \\
x_1 & +? & & & & +? & = & ?, & \quad (4)
\end{array}
$$

where the ?'s represent blanks whose entries are to be determined. Note that in this new configuration, s_1, s_2, and s_3 continue to serve, respectively, as the basic variables associated with equations (1), (2), and (3); and that the variable x_1 now replaces s_4 as the basic variable associated with equation (4).

To create this target equation system, we will employ two types of **row operations**: (i) multiplying an equation by a nonzero number, and (ii) adding one equation to another. While such row operations do create a change in the appearance of an equation system, they clearly do not have any effect on its solution.

Since the variables s_1, s_2, and s_3 are to retain their original configuration, we will focus on manipulating terms that involve x_1; and while doing so, we will aim to leave terms that involve s_1, s_2, and s_3 undisturbed. This will be done via a series of row operations, described below.

Consider equation (4) first. In the original equation system, x_1 has a coefficient of 2; whereas in the target equation system, a coefficient of 1. Hence, dividing the original equation (4) by 2, which is a row operation, will yield a new equation with the desired coefficient for x_1. Doing this leads to

$$ x_1 \ + (1/2)x_2 \qquad +(1/2)s_4 \ = \ 2. $$

We now look at the other equations one at a time. Equation (0) has a coefficient of -4 for x_1 and we wish to eliminate the term $-4x_1$ to arrive at a new equation (0). To accomplish this, we multiply the original equation (4) by 2 and add the outcome of this operation to the original equation (0). The combination of these two row operations leads to

$$z \quad -x_2 \quad +2s_4 \;=\; 8\,.$$

Next, x_1 has a coefficient of 2 in equation (1). To eliminate the term $2x_1$, we multiply the original equation (4) by -1 and add the outcome to equation (1). This yields

$$+2x_2 \;+s_1 \quad -s_4 \;=\; 2\,.$$

Similarly, multiplying the original equation (4) by $3/2$ and adding the outcome to equation (2) yields

$$+(7/2)x_2 \quad +s_2 \quad +(3/2)s_4 \;=\; 9\,.$$

Finally, since x_1 does not appear in equation (3), there is no need to revise that equation; and in summary, we have arrived at the following explicit (target) equation system:

$$
\begin{array}{rlllll}
z & -x_2 & & & +2s_4 & = & 8 & (0) \\
& +2x_2 & +s_1 & & -s_4 & = & 2 & (1) \\
& +(7/2)x_2 & & +s_2 & +(3/2)s_4 & = & 9 & (2) \\
& +2x_2 & & +s_3 & & = & 5 & (3) \\
x_1 & +(1/2)x_2 & & & +(1/2)s_4 & = & 2\,. & (4)
\end{array}
$$

Notice that, indeed, the configuration of terms involving s_1, s_2, and s_3 (which are to remain in the basis) is preserved throughout these row operations. This is a consequence of the fact that these three variables do not participate in equation (4) in the original configuration.

What we have just completed is a procedure that is often referred to as a **Gaussian elimination**. In this particular example, we eliminated the presence of the variable x_1 in every equation except equation (4), and we converted the coefficient of x_1 in equation (4) into a 1. This paves the way for declaring x_1 a basic variable in a new solution.

With the missing entries in the target equation system explicitly constructed, we can now associate a basic feasible solution to this configuration in the same way as we did for the original configuration. That is, we will declare x_2 and s_4 as nonbasic and x_1, s_1, s_2, and s_3 as basic. Doing this immediately yields the basic feasible solution $(x_1,\,x_2,\,s_1,\,s_2,\,s_3,\,s_4) = (2,\,0,\,2,\,9,\,5,\,0)$. This solution corresponds to point E in the graph.

As before, the objective-function value associated with the new solution $(2,\,0,\,2,\,9,\,5,\,0)$ can be read directly from equation (0). That is, it is equal to the constant 8, located at the right-hand side of that equation.

We are now ready to carry out an optimality test for the current basic feasible solution, $(2,\,0,\,2,\,9,\,5,\,0)$. From the analysis for point A, we observed that boosting the value of each nonbasic variable leads to an adjacent corner-point, or basic feasible, solution. This observation is valid in general. In the current solution, the nonbasic variables are x_2 and s_4. We will therefore consider the effect of increasing the values of these variables from their current level 0. The analysis will be essentially the same as what we did for point A, but we will now do things in a more compact manner.

A careful examination of our analysis at point A shows that it is, in fact, not necessary to mirror the entire analysis at A to determine whether or not point E is optimal. We begin with a simple inspection of

the new equation (0), and observe that the current nonbasic variables x_2 and s_4 have coefficients -1 and $+2$, respectively. This implies that an *attempt* to increase (an actual increase may not be possible, in general) the value of x_2 (while holding s_4 at 0) will result in a corresponding increase in the value of z at a rate of 1 ; and that the corresponding potential rate of increase in z relative to an attempt to increase s_4 (while holding x_2 at 0) is -2. Hence, the objective-function value can, potentially, be improved by increasing the value of x_2, and we cannot conclude that the current basic feasible solution is optimal. Checking the *current* equation (0) for negative nonbasic variable coefficients is called the **optimality test**. The current solution is declared optimal if all nonbasic variable coefficients are nonnegative. This condition will be referred to as the **optimality criterion**. When the optimality criterion fails, the nonbasic variable with the negative coefficient (any one of them, if there are multiple) has to be introduced into the basis with an attempt to boost its value. For our example, this means that x_2 is about to enter the basis, we shall refer to x_2 as the **entering variable**.

In general, if two or more nonbasic variables have a negative coefficient in equation (0), then we need to specify a rule for choosing the entering variable. Observe that, in such a situation, a nonbasic variable with a more negative coefficient, if chosen, will immediately produce a greater rate of improvement in the objective-function value. For this reason, the standard Simplex method calls for selecting the nonbasic variable with the smallest (or the largest in absolute value) coefficient as the entering variable (although this can be construed as being myopic).

At this point, we have also completed what is called an **iteration** in the Simplex method. An iteration begins with a basic feasible solution that is not yet optimal, goes through a round of Gaussian elimination to generate a new basic feasible solution, and ends with an optimality test for the new solution. The need for further iterations will be determined by the outcome of the optimality test. In our example here, the starting basic feasible solution is (0, 0, 6, 3, 5, 4), and this solution does not pass the optimality test. After going through one round of Gaussian elimination, we arrive at the new basic feasible solution (2, 0, 2, 9, 5, 0). This solution, again, fails the optimality test, and this marks the end of the first iteration. Since the new solution is not optimal, further iterations are necessary.

Now, consider an increase in x_2 of size δ. As we increase δ, from 0, we must continuously adjust the values of the current basic variables to maintain feasibility. This means that each of equations (1)–(4) imposes an upper bound on the value of δ. More explicitly, equation (1) implies that δ cannot exceed the ratio 2/2, where the numerator comes from the right-hand side constant and the denominator comes from the coefficient of x_2 in that equation; and similarly, equations (2), (3), and (4) imply, respectively, that δ cannot exceed the ratios $9/(7/2)$, $5/2$, and $2/(1/2)$. Since the smallest of these four ratios is 1, which comes from equation (1), the best we can do is to let $\delta = 1$. Notice that with $\delta = 1$, the value of the current basic variable for equation (1), namely s_1, is reduced from 2 to 0; and this makes s_1 nonbasic in the new solution. Since s_1 is about to leave the basis, we shall refer to s_1 as the **leaving variable**. In addition, we shall refer to the process of calculating a set of ratios to determine an upper bound for δ as the **ratio test**. In general, the basic variable in the most limiting constraint (the one yielding the smallest upper bound for δ during the ratio test) leaves the current basis.

Since the upper bound at $\delta = 1$ is positive, we see that an increase in x_2 by 1 will result in a new feasible solution whose z value is greater than that of the current solution (2, 0, 2, 9, 5, 0) by a magnitude of 1 (this comes from $1 \cdot 1$, where the first 1 is the rate of increase in z and the second 1 is the amount of increase in x_2; see the new equation (0)). We can now conclude that the current basic feasible solution is indeed not optimal. This completes the optimality test.

With x_2 entering and s_1 leaving, the new basis will be x_1, x_2, s_2, and s_3. As before, we will carry out a round of Gaussian elimination to explicitly derive this new basic feasible solution. With x_1, x_2, s_2, and s_3

as the new basis, the target configuration of the equations is:

$$
\begin{array}{llllll}
z & +? & & +? & = & ? & (0)\\
& +x_2 & +? & +? & = & ? & (1)\\
& +? & +s_2 & +? & = & ? & (2)\\
& +? & +s_3 & +? & = & ? & (3)\\
x_1 & +? & & +? & = & ?\,; & (4)
\end{array}
$$

and our task is to convert the previous configuration into this new form.

We will begin with equation (1). In the previous configuration, x_2 has a coefficient of 2 in equation (1). Hence, we will multiply this equation by $1/2$ to create a coefficient of 1. This yields

$$+x_2 \quad +(1/2)s_1 \qquad -(1/2)s_4 \;=\; 1$$

as the new equation (1). In general, the procedure is to always begin with the equation in the previous configuration that is targeted to contain the entering variable in the new configuration after Gaussian elimination. This equation is the one that has the smallest ratio in the ratio test.

What we will do next is to multiply equation (1) in the previous configuration by a set of numbers, called *multipliers*, and add the individual outcomes to the remaining equations. (Thus, this equation plays the role of a "template.") Each of the remaining equations will have an associated multiplier, and the value of this multiplier will be chosen so that the variable x_2, if present, is eliminated from that equation as a result of this operation.

In the previous configuration, x_2 has a coefficient of -1 in equation (0). Therefore, the multiplier for this equation should be $1/2$. After multiplying equation (1) by $1/2$ and adding the outcome to equation (0), we obtain

$$z \qquad +(1/2)s_1 \qquad +(3/2)s_4 \;=\; 9\,;$$

and this will be the new equation (0).

In the previous configuration, x_2 has a coefficient of $7/2$ in equation (2); therefore, the multiplier for that equation should be $-7/4$. After multiplying equation (1) by $-7/4$ and adding the outcome to equation (2), we obtain

$$-(7/4)s_1 \quad +s_2 \qquad +(13/4)s_4 \;=\; 11/2\,;$$

and this will be the new equation (2).

Finally, performing similar operations for equations (3) and (4), with multipliers -1 and $-1/4$, yields

$$
\begin{array}{lllll}
& -s_1 & +s_3 & +s_4 & = & 3\\
x_1 & -(1/4)s_1 & & +(3/4)s_4 & = & 3/2
\end{array}
$$

respectively; and these will be the new equations (3) and (4). Summarizing, we have arrived at the following new configuration of the equations:

$$
\begin{array}{lllllll}
z & & +(1/2)s_1 & & +(3/2)s_4 & = & 9 & (0)\\
& +x_2 & +(1/2)s_1 & & -(1/2)s_4 & = & 1 & (1)\\
& & -(7/4)s_1 & +s_2 & +(13/4)s_4 & = & 11/2 & (2)\\
& & -s_1 & +s_3 & +s_4 & = & 3 & (3)\\
& +x_1 & -(1/4)s_1 & & +(3/4)s_4 & = & 3/2\,. & (4)
\end{array}
$$

Here, the nonbasic variables are s_1 and s_4 and the basic variables are x_1, x_2, s_2, and s_3. Therefore, the basic feasible solution associated with this configuration is readily seen as $(3/2, 1, 0, 11/2, 3, 0)$, which corresponds (as expected) to point D in the graph. Moreover, the objective function value associated with this solution equals 9, taken from the right-hand side of equation (0).

An inspection of equation (0) shows that the coefficients of both nonbasic variables are nonnegative, which, according to the optimality criterion, means that further improvements in z is not possible. Hence, the current basic feasible solution is optimal, and the solution of this example by the Simplex method is complete.

At this point, you should reflect carefully on the entire solution process to convince yourself that it does not explicitly rely on the availability of a graphical representation of the feasible set. That is, the solution procedure is completely algebraic. Furthermore, the applicability of this procedure does not depend on the number of decision variables.

5.1 Remarks

1. It is often not necessary for the Simplex method to visit all of the basic feasible solutions before determining which one is optimal. However, both extremes are possible: Simplex visiting all the basic feasible solutions (Give an example.) and Simplex stopping at the initial solution (Give an example). In our example, there are five basic feasible solutions, but only three out of these five are (explicitly) visited. Thus, the Simplex method, indeed, offers a significant reduction in the search effort, when compared with Procedure Search discussed in the previous section. For arbitrary linear programs, the degree of reduction in effort will be problem specific. However, empirical experience with a wide range of problems has shown that the reduction in effort is often substantial.

2. Basic (feasible) solution and sport team analogy: The starting players in basketball corresponds to an initial basic solution. While selecting the starting players, a coach's objective may be to assemble a defensive team, a fast-break team, etc. The coach makes player substitutions according to his objective just the way simplex introduces entering variables into the basis and takes out the leaving variables according to the objective function. Both during the Simplex iterations and a basketball game the number of basic variables and players in the game is constant. There is a slight difference though: a coach often goes back to the same set of players several times during the game whereas Simplex does not generally return to the same basic feasible solution. Can you explain this difference with Simplex terminology ? (Hint: Think about the coach's objective function.)

6   The Simplex Method in Tabular Form

In its original **algebraic form**, our problem is:

$$
\begin{array}{llllllll}
\text{Maximize} & z \\
\text{Subject to:} \\
z & -4x_1 & -3x_2 & & & & = 0 & (0) \\
& 2x_1 & +3x_2 & +s_1 & & & = 6 & (1) \\
& -3x_1 & +2x_2 & & +s_2 & & = 3 & (2) \\
& & 2x_2 & & & +s_3 & = 5 & (3) \\
& 2x_1 & +x_2 & & & & +s_4 = 4 & (4)
\end{array}
$$

$$x_1,\ x_2,\ s_1,\ s_2,\ s_3,\ s_4 \geq 0\,.$$

Since the nonnegativity constraints do not explicitly participate in the mechanics of the solution procedure, we only need to present the coefficients in the objective function and the constraint equations. In tabular form, this problem will be represented as follows.

	z	x_1	x_2	s_1	s_2	s_3	s_4	RHS
$R_0:$	1	-4	-3	0	0	0	0	0
$R_1:$	0	2	3	1	0	0	0	6
$R_2:$	0	-3	2	0	1	0	0	3
$R_3:$	0	0	2	0	0	1	0	5
$R_4:$	0	2	1	0	0	0	1	4

Thus, the objective function and each constraint equation is translated into a row of coefficients; and the coefficients of a variable in different equations are listed in the same column, with the name of that variable specified at the top of that column as heading. In addition, to facilitate reading, we have delimited the table into several areas, e.g., (i) the z-column at the left, (ii) the coefficients in equation (0) at the top row, and (iii) the right-hand side constants of the equations at the right-most column.

The above table will be referred to as the initial Simplex tableau. To simplify statements, we will refer to the successive rows in the tableau as R_0, R_1, and so on; this numbering, of course, corresponds to that of the original equations. In addition, we will refer to the last column as the RHS column (since it comes from the right-hand-side constants in the equations).

Associated with this initial tableau, the nonbasic variables are x_1 and x_2 and the basic variables are s_1, s_2, s_3, and s_4. Therefore, the initial (or current) basic feasible solution is: $(x_1, x_2, s_1, s_2, s_3, s_4) = (0, 0, 6, 3, 5, 4)$. This solution has an objective-function value 0, which is the right-most number in R_0.

Consider R_0. Since the coefficients of x_1 and x_2 (the nonbasic variables) in that row are both negative, the current solution is not optimal. Furthermore, since the coefficient of x_1, namely -4, is more negative than that of x_2, we will select x_1 as the entering variable.

We will refer to the x_1-column as the **pivot column**. This terminology is suggested by the fact that a round of Gaussian elimination is also called a **pivot**.

To determine the maximum possible increase in x_1, we conduct a ratio test. The ratio test will involve the coefficients in the pivot column and in the RHS column. This is worked out on the right margin of the tableau, as shown below.

Basic	z	$\mathbf{x_1}$	x_2	s_1	s_2	s_3	s_4	RHS	Ratio Test	
Variable	1	**-4**	-3	0	0	0	0	0		
s_1	0	**2**	3	1	0	0	0	6	$6/2 = 3$	
s_2	0	**-3**	2	0	1	0	0	3	$-$	
s_3	0	**0**	2	0	0	1	0	5	$-$	
s_4	0	**2**	1	0	0	0	1	4	$4/2 = 2$	$\leftarrow$ Minimum

Note that we did not compute a ratio for R_2 and R_3, since both of these two rows have a nonpositive coefficient in the pivot column (indicating that the corresponding equations (2) and (3) do not impose any bound on x_1). Since the minimum ratio appears in R_4, the basic variable currently associated with that row, s_4 (indicated at the left margin), will be the leaving variable. We will refer to R_4 as the *pivot row*.

With s_4 leaving and x_1 entering, the new basis will be x_1, s_1, s_2, and s_3. Therefore, we are now interested

in constructing a new tableau in the configuration specified below.

Basic	z	x_1	x_2	s_1	s_2	s_3	s_4	RHS
Variable	1	0	?	0	0	0	?	?
s_1	0	0	?	1	0	0	?	?
s_2	0	0	?	0	1	0	?	?
s_3	0	0	?	0	0	1	?	?
x_1	0	1	?	0	0	0	?	?

As before, the ?'s in the tableau represent blanks whose entries are to be determined.

To create this target tableau, we will employ row operations. As examples, the new row 4 will be generated by multiplying R_4 by $1/2$; and the new row 0 will be generated by multiplying R_4 by 2 and adding the outcome into R_0. Repeating similar operations for the other rows yields the new tableau below.

	z	x_1	x_2	s_1	s_2	s_3	s_4	RHS
$2 \cdot R_4 + R_0$:	1	0	-1	0	0	0	2	8
$(-1) \cdot R_4 + R_1$:	0	0	2	1	0	0	-1	2
$(3/2) \cdot R_4 + R_2$:	0	0	$7/2$	0	1	0	$3/2$	9
$0 \cdot R_4 + R_3$:	0	0	2	0	0	1	0	5
$(1/2) \cdot R_4$:	0	1	$1/2$	0	0	0	$1/2$	2

Note that on the left margin of this tableau, we have explicitly indicated how individual new rows are derived from those in the initial tableau. For example, the operations leading to the new row 0 is listed as $2 \cdot R_4 + R_0$, which corresponds to the earlier description.

In this round of Gaussian elimination, or pivot, the entry 2 located at the intersection of the pivot column and the pivot row in the initial tableau plays a "pivotal role", in that it is repeatedly used to generate all five multipliers of R_4. We shall refer to this entry as the **pivot element**.

The basis associated with the new tableau is: x_1, s_1, s_2, and s_3. Therefore, the new basic feasible solution is: $(x_1, x_2, s_1, s_2, s_3, s_4) = (2, 0, 2, 9, 5, 0)$. This solution has an objective-function value 8. Since there is a negative coefficient in the new R_0, namely the -1 in the x_2-column, the current solution is not optimal. This completes the first iteration of the Simplex method.

Next, since x_2 is now the entering variable, the x_2-column is the new pivot column. To determine the pivot row, we again conduct a ratio test, which is shown below.

Basic	z	x_1	$\mathbf{x_2}$	s_1	s_2	s_3	s_4	RHS	Ratio Test	
Variable	1	0	**-1**	0	0	0	2	8		
s_1	0	0	**2**	1	0	0	-1	2	$2/2 = 1$	$\leftarrow$ Minimum
s_2	0	0	**7/2**	0	1	0	$3/2$	9	$9/(7/2) = 18/7$	
s_3	0	0	**2**	0	0	1	0	5	$5/2$	
x_1	0	1	**1/2**	0	0	0	$1/2$	2	$2/(1/2) = 4$	

This shows that the new pivot row will be R_1, and the basic variable associated with that row, s_1, will be the leaving variable.

With the entry 2 (Which 2?) as the pivot element, we now go through another pivot to obtain the new

tableau below.

	z	x_1	x_2	s_1	s_2	s_3	s_4	RHS
$(1/2) \cdot R_1 + R_0$:	1	0	0	$1/2$	0	0	$3/2$	9
$(1/2) \cdot R_1$:	0	0	1	$1/2$	0	0	$-1/2$	1
$(-7/4) \cdot R_1 + R_2$:	0	0	0	$-7/4$	1	0	$13/4$	$11/2$
$(-1) \cdot R_1 + R_3$:	0	0	0	-1	0	1	1	3
$(-1/4) \cdot R_1 + R_4$:	0	1	0	$-1/4$	0	0	$3/4$	$3/2$

The basic feasible solution associated with this new tableau is (3/2, 1, 0, 11/2, 3, 0), with a corresponding objective-function value of 9. Moreover, since the coefficients of s_1 and s_4 in the new R_0 are positive, this solution is optimal. This completes the second Iteration of Simplex yielding the optimal solution of this problem.

6.1 Remarks

1. Convince yourself that the tableau iterations of this section correspond to algebraic iterations of the previous section. Then each tableau iteration transforms the current basic feasible solution (a corner point) to another basic feasible solution (another corner point).

2. In both of the Simplex iterations of this section, we could find the leaving basic variable with the ratio test because the entering variable column (printed in **boldface**) had at least one positive coefficient. Is this true in general? If not, what happens when all the coefficients in the entering column are nonpositive?

7 Exercises

1. Consider the following problem:

$$\begin{array}{llll}
\text{Maximize} & 60x_1 & +30x_2 & +20x_3 \\
\text{Subject to:} & & &
\end{array}$$

$$\begin{array}{lllll}
8x_1 & +6x_2 & +x_3 & \leq & 48 & (1) \\
4x_1 & +2x_2 & +1.5x_3 & \leq & 20 & (2) \\
2x_1 & +1.5x_2 & +0.5x_3 & \leq & 8 & (3)
\end{array}$$

$$x_1, x_2, x_3 \geq 0.$$

a) Put the problem into the standard form.
b) Use Simplex tableaux to solve it to optimality. Stop after the third iterartion even if the third tableau is not optimal. In every iteration enter the variable with the most negative Row 0 coefficient.

2. Set up the Simplex tableau for the following LP and do a Simplex iteration:

Max $3x_1 + x_3$
ST:
$x_1 + x_3 + x_4 = 6$
$x_1 - x_3 \leq 10$
$2x_1 + x_2 + x_3 = 8$
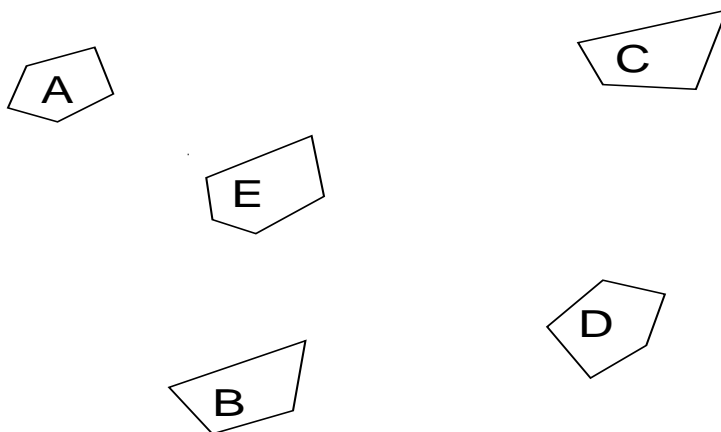$x_1, x_2, x_3, x_4 \geq 0$

3. Solve H-L p.176 4.4-9.

Figure 5: Lucky Billy's Diamond Lakes map.

4. A diamond miner called Lucky Billy wants to check out Diamond Lakes (Arkansas, about 4 hrs northeast of Dallas) for diamonds. There are five lakes he is interested in A, B, C, D and E. Before setting up his mining equipment, he makes an initial evaluation (examining the rock formations) of each lake one by one and assigns a number between 0 and 1 to each lake to indicate the likelihood of finding diamonds (0 means no chance of finding diamonds, 1 means finding diamonds for sure). He wants to maximize his chances of finding diamonds. Refer to the 3-step *Solve LP Procedure* of the lecture notes.

a) Write down Lucky Billy's lake search procedure as a 3-step procedure. Point out the analogous steps between the Solve LP Procedure and the lake search procedure.

b) Can Lucky Billy terminate his lake search procedure without evaluating all the spots? Can Simplex terminate without checking all the basic feasible solutions?

c) Suppose that for the likelihood of finding diamonds Lucky Billy assigns 0, 0, 0, 0 and 1 to lakes A, B, C, D and E respectively. Given this assignment, can the likelihood function be linear?

d) If A, B, C, D, and E are points in a feasible region for a given LP problem, what assumption of LP guarantees that E cannot be the only optimal point for **any** objective function?

5. Consider the Simplex tableau below, borrowed from the last section.

	z	x_1	x_2	s_1	s_2	s_3	s_4	RHS
R_0	1	0	-1	0	0	0	2	8
R_1	0	0	2	1	0	0	-1	2
R_2	0	0	7/2	0	1	0	3/2	9
R_3	0	0	2	0	0	1	0	5
R_4	0	1	1/2	0	0	0	1/2	2

a) Enter s_4 into the basis to obtain the next tableau. What happened to the objective function value?

b) Consider the above tableau again. Enter x_2 into the basis. But suppose that you computed ratios incorrectly and found R_3 as the pivot row. Use R_3 as your pivot row and find the next tableau. Has anything unexpected happened to RHS values? Read the numerical values of your augmented solution from the next tableau and indicate what point the solution correspond to on Figure 4. Is your solution basic and/or feasible?