THE NEPALI MATHEMATICAL SCIENCES REPORT

Published By

CENTRAL
DEPARTMENT OF MATHEMATICS
TRIBHUVAN UNIVERSITY
KATHMANDU, NEPAL

VOLUME 24. No. 1. 2005.

THE NEPALI MATHEMATICAL SCIENCES REPORT

Published By

CENTRAL
DEPARTMENT OF MATHEMATICS
TRIBHUVAN UNIVERSITY
KATHMANDU, NEPAL

VOLUME 24. **No. 1.** **2005.**

The Nepali Math. Sci. Report
Vol. 24. No.1. 2005.

CONTENTS

Computer typesetting at UNION COMPUTERS,
Wotu Marg, Ward No. 24. Tadhahbahal, 35, Kathmandu. Tel. 4–252281.

Printed at the TRIBHUVAN UNIVERSITY PRESS.
Kirtipur, Kathmandu, Nepal. Tel. 4–331320, 4–331321

The Nepali Math. Sci. Report
Vol. 24. No.1. 2005.

Hardy's Uncertainty Principle on $\mathbb{R}^+=[0,\infty)$

CHET RAJ BHATTA

Abstract: Hardy's uncertainty principle states that if the function f is "very rapidly decreasing" then the Fourier transform can not also be "very rapidly decreasing" unless f is identically zero. In this paper we discuss some variants of Hardy's theorem on $\mathbb{R}^+=[0,\infty)$.

Keywords: Uncertainty principle, Fourier transform pair, Laplace transform, very rapidly decreasing.

1. Introduction

It is well-known simple fact that if a function f on $\mathbb{R}$ is compactly supported, then its Fourier transform $\hat{f}$ can not also be compactly supported, unless $f = 0$. More generally, we have the following principle in classical Fourier analysis: If the function f is "very rapidly decreasing" then the Fourier transform can not also be "very rapidly decreasing" unless f is identically zero. An important result making this precise is the following theorem. There are several ways of measuring "Concentration". One way of measuring concentration is by considering the decay of the function at infinity and another natural way of measuring 'concentration' is in terms of the supports of the function f and its Fourier transform $\hat{f}$.

Hardy Theorem 1.1 [1] : *Let α, β and C be positive real numbers and suppose that f is measurable function on $\mathbb{R}$ such that*

(i) $|f(x)| \le C \exp(-\alpha\pi x^2)$ for all $x \in \mathbb{R}$

(ii) $|\hat{f}(\xi)| \le C \exp(-\beta\pi\xi^2)$ for all $\xi \in \mathbb{R}$

If $\alpha\beta > 1$ then $f = 0$ almost everywhere. If $\alpha\beta < 1$ then there are infinitely many linearly independent functions satisfying (i) and (ii) and if $\alpha\beta = 1$, then

$$f(x) = C\exp(-\alpha\pi x^2) \text{ for some constant } C.$$

$$\hat{f}(y) = \int_{\mathbb{R}} f(x)\exp(-2\pi i x y)\,dx,\ y \in \mathbb{R}$$

Definition 1.2: A function f is said to be "exponential type" if $|f(\gamma)| \le \text{constant} \times e^{T|\gamma|}$ for some $T < \infty$.

Definition 1.3: For a measurable function f on $\mathbb{R}^+ = [0, \infty)$ the Laplace transform $\mathcal{L}f$ of f is defined by

$$\mathcal{L}f(\gamma) = \int_0^\infty f(t)\exp(-\pi t\gamma)\,dt$$

If $\operatorname{Re}\gamma > 0$ and $f \in L^1(\mathbb{R}^+)$ then $\mathcal{L}f$ is well-defined. If in addition f satisfies $|f(x)| \le C\exp(-\alpha x^2)$ for all $x \in \mathbb{R}^+$ then $\mathcal{L}f(\gamma)$ is defined for all $\gamma \in \mathbb{C}$ and is holomorphic function on $\mathbb{C}$.

The following is a simple deduction from Hardy's theorem on $\mathbb{R}^+$.

Theorem 1.3: *Let f be a measurable function on $\mathbb{R}^+$ satisfying*

(i) $|f(x)| \le C\exp(-\alpha x^2)$ for all $x \in \mathbb{R}^+$

(ii) $|\mathcal{L}f(\gamma)| \le C\exp(-\beta(\operatorname{Im}\gamma)^2)$ for all $\gamma \in i\,\mathbb{R}$

If $\alpha\beta > 1/4$ then $f = 0$ a.e.

Proof: Extend f to $\tilde{f}$ on $\mathbb{R}$ by defining O on $\{x : x < 0\}$ then $\tilde{f}$ is a measurable functions on $\mathbb{R}$ and satisfy $|\tilde{f}(x)| \le C\exp(-\alpha x^2)$ for all $x \in \mathbb{R}$

$$\text{For } \gamma \in \mathbb{R},\ \left|\hat{\tilde{f}}(\gamma)\right| = \left|\int_{-\infty}^{\infty} \tilde{f}(x)\exp(-2\pi i x\gamma)\,dx\right|$$

$$= \left|\int_0^\infty f(x)\exp(-2\pi i x\gamma)\,dx\right|$$

$$= |\mathcal{L}f(2i\gamma)|$$

$$\le C\exp(-4\beta\gamma^2)$$

Since $4\alpha\beta > 1$ so by Hardy's theorem on $\mathbb{R}$, $\tilde{f} = 0$ a.e. Thus $f = 0$ a.e.

Theorem 1.4: *Let f be a measurable function on $\mathbb{R}^+$ satisfying*

(i) $|f(x)| \le C\exp(-\alpha\pi x^2)$ for all $x \in \mathbb{R}^+$

(ii) $|\mathcal{L}f(\gamma)| \le C\exp(-\beta\pi\gamma^2)$ for all $\gamma \in \mathbb{R}$

If $\alpha\beta > 1$ then $f = 0$ a.e.

Proof: Suppose that $\alpha = \beta = 1$ and $\mathcal{L}f(\gamma)$ is a even function i.e. $\mathcal{L}f(\gamma) = \sum C_n\gamma^{2n}$. Since the function $u(\gamma) = \gamma^k$, $k \in \mathbb{R}$ is a holomorphic function in the cut plane $\{\gamma = R\exp(i\theta);\ R > 0, |\theta| < \pi\}$ where $\gamma^k = R^k\exp(ik\theta)$, we can define a function

$$h(\gamma) = \mathcal{L}f(\sqrt{\gamma}) = \sum C_n\gamma^n.$$

$$|h(\gamma)| = \left|\int_0^\infty f(t)\exp\left(-\pi\sqrt{\gamma}\,t\right)dt\right|$$

$$\le C\int_0^\infty \exp\left(-\pi t^2\right)\exp\left(-\pi\sqrt{R}\,\mathrm{Cos}\tfrac{\theta}{2}t\right)dt$$

$$= C\exp\left(\tfrac{\pi}{4} R\,\mathrm{Cos}^2\tfrac{\theta}{2}\right)\int_0^\infty \exp\left(-\pi\left(t+\tfrac{1}{2}\sqrt{R}\,\mathrm{Cos}\tfrac{\theta}{2}\right)^2\right)dt$$

$$\le C'\exp\left(\tfrac{\pi}{4}\mathrm{Cos}^2\tfrac{\theta}{2} R\right)$$

(1) $$\le C'\exp\left(\tfrac{\pi}{4} R\right) \text{ for some } C' > 0$$

Which is independent of θ. Thus h is a exponential type
For $0 < \delta < \pi$

$$\left|\exp\left(\frac{\pi i\gamma e^{-i\delta/2}}{\mathrm{Sin}\,\delta/2}\right)h(\gamma)\right| = \exp\left(\frac{-\pi R\sin(\theta-\frac{\delta}{2})}{\mathrm{Sin}\,\delta/2}\right)|h(\mathrm{Re}^{i\theta})|$$

If $\theta = 0$ then $\mathrm{Re}^{i\theta} = R > 0$ so

$$\left|\exp\left(\frac{\pi i\gamma e^{-i\delta/2}}{\mathrm{Sin}\,\delta/2}\right)h(\gamma)\right| = \exp(\pi R)\,|h(R)|$$

$$\le \exp(\pi R)\,|\mathcal{K}f(\sqrt{R})|$$

$$\le C\exp(\pi R)\exp(-\pi R) \le C$$

If $\theta = \delta$ then, $$\left|\left(\exp\frac{i\pi\gamma e^{-i\delta/2}}{\mathrm{Sin}\,\delta/2}\right)h(\gamma)\right| = \exp(-\pi R)\,|h(\mathrm{Re}^{i\delta})|$$

$$\le C'\exp(-\pi R)\exp(\pi/4\,R) \text{ using (1)}$$

$$\le C'$$

Now we apply Phragmen - Lindelöf's theorem to the sector $0 < \theta < \delta$ to get

$$|h(\gamma)| \le K\exp\left(\frac{\pi R\sin(\theta-\frac{\delta}{2})}{\mathrm{Sin}\,\delta/2}\right), K = \mathrm{Max}(C, C')$$

Now taking $\delta \uparrow \pi$, we have

$$|h(\gamma)| \le K\exp(-\pi R\ \mathrm{Cos}\,\theta) \text{ for } 0 \le \theta \le \pi$$

$$\Rightarrow |\exp(\pi\gamma)\,h(\gamma)| \le K,\ \gamma = \mathrm{Re}^{i\theta},\ 0 \le \theta \le \pi$$

A similar argument will hold for the lower half plane so

$$|\exp(\pi\gamma)\,h(\gamma)| \le K \text{ for } -\pi \le \theta \le 0,\ \gamma = \mathrm{Re}^{i\theta}$$

Therefore $g(\gamma) = \exp(\pi\gamma)\,h(\gamma)$ is bounded and holomorphic in $\mathbb{C}$. Hence by Liouville's theorem there is a constant $M > 0$ such that

$$h(\gamma) = M\exp(-\pi\gamma)$$

Thus $$\mathcal{K}f(\gamma) = M\exp(-\pi\gamma^2).$$

Suppose now that $\mathcal{K}f(\gamma)$ is an odd function i.e.

$$\mathcal{K}f(\gamma) = \sum C_n\,\gamma^{2n+1},\ \mathcal{K}f(0) = 0. \text{ For } \gamma \ne 0,\ \gamma^{-1}\mathcal{K}f(\gamma) = \sum C_n\,\gamma^{2n}.$$

Thus by even case $\gamma^{-1}\mathcal{K}f(\gamma) = M\exp(-\pi\gamma^2)$. But for $\gamma \in \mathbb{R}$, we have

$$|\mathcal{K}f(\gamma)| \le C \exp(-\pi\gamma^2)$$

Therefore, $M|\gamma| \exp(-\pi\gamma^2) \le C \exp(-\pi\gamma^2)$

i.e. $M|\gamma| \le C$ for all $\gamma \in \mathbb{R}$ which is possible only if $M = 0$.

Hence $\mathcal{K}f(\gamma) = 0$.

In general, we break $\mathcal{K}f$ into even and odd part i.e.

$$\mathcal{K}f(\gamma) = \tfrac{1}{2}(\mathcal{K}f(\gamma) + \mathcal{K}f(-\gamma)) + \tfrac{1}{2}(\mathcal{K}f(\gamma) - \mathcal{K}f(-\gamma))$$
$$= g_1(\gamma) + g_2(\gamma) \text{ (say)}$$

$g_1(\gamma)$ is an even function satisfying $|g_1(\gamma)| \le C \exp(-\pi\gamma^2)$ for all $\gamma \in \mathbb{R}$. So the function $h_1(\gamma) = g_1(\sqrt{\gamma})$ is of exponential type as

$$|h_1(\gamma)| \le \tfrac{1}{2}\left(\mathcal{K}f(\sqrt{\gamma}) + \mathcal{K}f(-\sqrt{\gamma})\right)$$

and

$$\left|\mathcal{K}f(-\sqrt{\gamma})\right| \le \int_0^\infty |f(t)|\left|\exp(\pi\sqrt{\gamma}\,t)dt\right|$$

$$\le C\int_0^\infty \exp(-\pi t^2)\exp\left(\pi\sqrt{R}\operatorname{Cos}\tfrac{\theta}{2}t\right)dt$$

$$= C\exp\left(\pi\frac{R}{4}\operatorname{Cos}^2\frac{\theta}{2}\right)\int_0^\infty \exp\left(-\pi\left(t - \frac{\sqrt{R}}{2}\operatorname{Cos}\frac{\theta}{2}\right)^2\right)dt$$

$$\le C'\exp\left(\frac{\pi R}{4}\right)$$

Hence, $|h_1(\gamma)| \le C'\exp\left(\frac{\pi R}{4}\right)$

Thus, $g_1(\gamma) = K\exp(-\pi\gamma^2)$ and $g_2(\gamma) = 0$ and so

$$\mathcal{K}f(\gamma) = K\exp(-\pi\gamma^2)$$

If $\alpha = \beta > 1$. Then

$$|f(x)| \le C\exp(-\alpha x^2) \le C\exp(-x^2)$$

and $|\mathcal{K}f(\gamma))| \le C\exp(-\beta\gamma^2) \le C\exp(-\gamma^2)$

So, $\mathcal{K}f(\gamma)) = K\exp(-\pi\gamma^2)$ by the above case.

For $x \ne 0$.

$$f(x) = M\lim_{a\downarrow 0}\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\left(-\pi(a+ib)^2\right)\exp(ibx)\,db$$

$$= M\lim_{a\downarrow 0}\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\left(-\pi\left(a^2 - b^2 + ib\left(2a - \tfrac{x}{\pi}\right)\right)\right)db$$

$$= M\lim_{a\downarrow 0}\exp(-\pi a^2)\frac{1}{2\pi}\left[\int_{-\infty}^{\infty}\exp(\pi b^2)\cos b\left(2a - \tfrac{x}{\pi}\right)db\right.$$

$$\left. + i\int_{-\infty}^{\infty}\exp(\pi b^2)\right]\sin b\left(2a - \tfrac{x}{\pi}\right)db$$

Therefore, $\lim_{x \to 0+} f(x) = M \cdot \frac{1}{\pi} \int_0^\infty \exp(\pi b^2)\, db \geq \frac{M}{\pi} X$ for all $X \geq 0$

Hence we must have $M = 0$ and therefore $\mathfrak{L} f(\gamma) = 0$ for all γ i.e. f = o a.e

REFERENCES

[1] Conway, J.B. 1973. *Functions of One Complex Variable Springer Ver*lag, New York Inc.

[2] Dym, H. and Mckean, H.P. 1972. *Fourier series and integrals*, Academic Press.

[3] Hardy, G.H. 1933. *A theorem concerning Fourier trans*form. J. London. Math. Soc. 8, 227–237.

[4] Kaniuth, E. and Kumar, A. 2001. *Hardy's theorem for simply connected Liegroups*, Math. Proc. Comb. Phil. Soc. 487–494.

[5] Oberhettinger, F. and Badii, L. *Table of Laplace transforms*, Springer Verlag, Berlin Heidelberg. New York 1995.

[6] Sitaram, M. Sundari and S. Thangavelu. *Uncertainty Principles on Certain Liegroups*, Proc. Indian Acad. Sci. (Math. Sci), Vol 105, No. 2, 135–151.

CHET RAJ BHATTA
Central Department of Mathematics,
Tribhuvan University,
Kirtipur, Kathmandu, Nepal.

The Nepali Math. Sc. Report
Vol. 24, No 1, 2005.

Representation of certain probability groups as orbit spaces of groups

H.N. BHATTARAI

Abstract: The set of double cosets of a group with respect to a subgroup and the set of orbits of a group with respect to a group of automorphisms have structures which can be studied as multigroups, hypergroups or Pasch geometries. When the subgroup or the group of automorphisms are finite, the multivalued products can be provided with some weightages forming so-called Probability Groups. It is shown in this paper that certain abstract probability groups can be realized as orbit spaces of groups.

1.Introduction

The set of double cosets of a group with respect to a subgroup and the set of orbits of a group with respect to a group of automorphisms inherit certain structures from the group which have been studied as multigroups ([6], [9]), hypergroups [10], or Pasch geometries. In particular, the points of the projective space $P(V)$ of a vector space V over a (skew)-field F can be taken as the set of orbits V/F^* and the inherited structure provides the geometry of the projective space. In addition, some of these structures possess probabilistic measures whereby there is a certain probability for a given element to belong to the set of products of two given elements. Such structures have been abstractly studied as Probability groups ([1],[7]). Double cosets and orbits of topological groups with convolution of measures have been studied as convos or hypergroups. A probability group is equivalent to a discrete convo of [8]. The proto-types of probability groups are the double cosets and orbits of groups. So a natural converse question would be: what are the necessary and sufficient conditions for an abstract probability group to be realized as an orbit space or a double coset space of a group? It has been proved in [2] that an elementary abelian Pasch geometry is isomorphic as geometries to an orbit space of an abelian group. Also, a geometric space over a geometric sfield is shown to be isomorphic to the orbit space of a vector space (see [5]). We show here that a finite elementary abelian probability group of proper length is isomorphic as probability group to the orbit space of a group. Similarly a probability space of proper dimension over a geometric sfield is shown to be isomorphic to the probability space of a vector space. In particular, it gives that probability group structure on a projective space of finite order is unique. We wish to point out that the study of projective geometry and related spaces in the framework of Pasch geometry has the convenience of dealing with morphisms and homomorphisms with the availability of homomorphism theorems similar to other algebraic structures (see [3].).

2. Preliminaries

In this section we briefly present the basic concepts and preliminary results on Pasch geometries and probability groups. The details can be found in the references, particularly in [7].

Definition 2.1 By a Pasch geometry is meant a triple (A, e, Δ) where A is a set, $e \in A$, and $\Delta_A = \Delta \subseteq A \times A \times A$ subject to the following axioms:

1. $\forall a \in A, \exists$ a unique $b \in A$ with $(a, b, e) \in \Delta$. Let $b = a^{\#}$.
2. $e^{\#} = e$ and $(a^{\#})^{\#} = a \forall a \in A$.
3. $(a, b, c) \in \Delta \Rightarrow (b, c, a) \in \Delta$.
4. $(a_1, a_2, a_3), (a_1, a_4, a_5) \in \Delta \Rightarrow \exists a_6 \in A$ with $(a_6, a_4^{\#}, a_2), (a_6, a_5, a_3^{\#}) \in \Delta$.

The identity element e and the inverse $a^{\#}$ are unique. Throughout this paper, geometry will mean Pasch geometry.

A geometry is called *abelian* if $(a, b, c) \in \Delta \Rightarrow (b, a, c) \in \Delta$. A geometry is called *sharp* if $(a, b, c), (a, b, d) \in \Delta \Rightarrow c = d$. Also, a geometry is called *projective* if $a^{\#} = a \forall a \in A$ and $(a, a, b) \in \Delta \Rightarrow b = e$ or $b = a$.

Now a structure stronger than the geometry defined above is given in the following.

Definition 2.2. A probability group is a pair (A, p) where A is a set and $p : A \times A \times A \to [0, 1]$ is a map to the unit interval, denoted as $(a, b, c) \to p_c(a, b)$, subject to the following axioms:

1. For $a, b \in A$, $p_x(a,b) = 0$ for all but finitely many $x \in A$ and
$$\sum_{x \in A} p_x(a,b) = 1.$$
2. For $a, b, c, d \in A$,
$$\sum_{x \in A} p_x(a,b) p_d(x,c) = \sum_{y \in A} p_d(a,y) p_y(b,c)$$
3. $\exists e \in A$ such that $p_a(e, a) = 1 = p_a(a,e) \; \forall a \in A$
4. For each $a \in A$, there exists a unique $b \in A$ with $p_e(a,b) \neq 0$. We denote b by $a^{\#}$.
5. $p_c(a,b) = p_{c^{\#}}(b^{\#}, a^{\#}) \; \forall a, b, c \in A$.

It should be noted that $p_c(a,b)$ can be read as the probability for the element c to belong to the multivalued product $a.b$. Also, axiom (1) describes probability distribution, (2) gives associativity, the identity e given by (3) is unique and the unique inverse $a^{\#}$ of a given by (4) satisfies $(a^{\#})^{\#} = a \; \forall a \in A$. When dealing with more than one probability group, we write them as $(A, p^A), (B, p^B)$ etc. or use the same p to let the context distinguish. We may simply write A is a probability group, the associated p being understood. A probability group is called *abelian* if $p_c(a,b) = p_c(b,a) \forall a,b,c \in A$.

The following useful relations are obtained as consequences of the axioms:

Lemma 2.3. For any probability group (A, p), we have:

(i) $p_a(c^\#,a) = p_a(c,a)\ \forall a,c \in A$.

(ii) $p_a(a,a) \neq 1 \forall a \in A^* - \{e\}$

(iii) $p_{c^\#}(a,b)\, p_e(c^\#, c) = p_e(a,a^\#)\, p_{a^\#}(b,c)$. In particular, if $b = c^\#$, we get

$$p_b(a,b)\, p_e(b,b^\#) = p_e(a,a^\#)\, p_{a^\#}(b,b^\#).$$

(iv) $p_c(a,b) \neq 0$ if and only if $p_{a^\#}(b,c^\#) \neq 0$.

For a probability group A, let $\Delta_A = \{(a,b,c) : p_{c^\#}(a,b) \neq 0\}$. Then

Proposition 2.4. (A, e, Δ_A) is a Pasch Geometry.

Thus, when A is a probability group, we speak of the geometry A to mean the induced Pasch geometry structure as described above. Every probability group is a Pasch geometry but the example (4) below shows that the converse is not true.

A probability group is called *sharp (projective)* if it is sharp (projective) as a geometry.

Examples 2.5.

1. Let G be a group. Define p by $p_a(b,c) = 1$ if $a = b.c$ and 0 otherwise. Then (G, p) is a sharp probability group with $a^\# = a^{-1}$. Note that the probability for an element a to be in the product $b.c$ is either 1 or 0. Conversely, every sharp probability group is a group.

2. Let P be the set of points of a finite projective plane of order m. Let $A = P \cup \{e\}$, $e \notin P$. On A, define the map p as follows:

$$p_a(b,c) = \begin{cases} \delta_a(c) & \text{if } b = e, \text{ where } \delta_a(c) = 1 \text{ if } a = c, 0 \text{ otherwise} \\ \delta_a(b) & \text{if } c = e \\ \frac{1}{m-1}\delta_b(c) & \text{if } a = e \\ \frac{m-2}{m-1} & \text{if } a = b = c \neq e \\ \frac{1}{m-1} & \text{if } a, b, c \in P \text{ and } a, b, c \text{ are distinct and collinear} \\ 0 & \text{otherwise} \end{cases}$$

Then (A, p) is a probability group, the induced geometry being that of the projective plane. Note that if $m = 2$, $p_a(a, a) = 0$ and A is sharp.

3. Let G be a finite group and $\hat{G} = \{\chi_1, \chi_2, \ldots \chi_s\}$ be the set of irreducible complex characters of G. For $1 \leq i, j \leq s$, let $\chi_i.\chi_j = \sum_{k=0}^{s} n_k^{i,j} \chi_k$. Let p be defined by

$$p_{\chi_k}(\chi_i,\chi_j)=\frac{n_k^{i,j}\,\chi_k(1)}{\chi_i(1)\,\chi_j(1)}$$

Then $(\hat{G},p)$ is a probability group.

4. The multiplicative group of positive rationals Q^+ acts on the additive group of rationals Q with three orbit elements: $\{[1],[-1],[0]\}$. It forms a *Pasch geometry* of orbits) (cf. 2.3). For elements $a=[1]$, $b=[1]$, we have $c=[-1]$ a unique element such that $([1],[1],[-1])\in\Delta$. So if this geometry were induced by a probability group, then we would have $p_{[1]}([1],[1])=1$, contradicting lemma 2.3 (ii).

Now let (A,p) be a probability group and $S\subseteq A$, a finite subset. Set $n_S=\sum_{x\in S}\frac{1}{p_e(x,x^\#)}$. Note that $p_e(x,x^\#)\neq 0$ and S is finite, so n_s is well defined. In particular, n_A is defined if A is finite. If A is sharp and hence a group, then $n_A=|A|$, the order of the group; if A is projective representing a finite projective plane of order m, then $n_A=m^3$.

2.1. Subgeometry and subprobability group. Let A be a geometry and $B\subseteq A$. Then B is called a subgeometry if $e\in B$ and $(b_1,b_2,x)\in\Delta$, $b_1,b_2\in B\Rightarrow x\in b$. Let $\Delta_B=\Delta_A\cap(B\times B\times B)$. Then (B,e,Δ_B) is a geometry.

Let (A,p) be a probability group and $B\subseteq A$. Then B is called a subprobability group of A if $e\in B$ and (B,p^B) is a probability group on its own where p^B is the restriction of p on $B\times B\times B$. We will simply write p for p^B. It checks that B is a subprobability group of A if and only if B is a subgeometry of A. So B is a subprobability group of A if and only if the following hold: $e\in B$ and $p_a(b_1,b_2)\neq 0$, $b_1,b_2\in B\Rightarrow a^\#\in B$. We call B a normal subprobability group if B is normal as a subgeometry of A.

2.2. Factor Geometry and factor probability group. Let B be a subgeometry of A. For $a,b\in A$, define $a\sim b$ if $\exists b_1,b_2\in B$ and $x\in A$ such that $(a,b_1,x^\#)$, $(x,b^\#,b_2)\in\Delta$. This defines an equivalence relation on A. Let $A//B=\{[a]:a\in A\}$ be the set of all equivalence classes. Let $([a],[b],[c])\in\Delta_{A//B}$ if $\exists x\in[a]$, $y\in[b]$, $z\in[c]$ with $(x,y,z)\in\Delta_A$. Then $A//B$ is a geometry.

In particular, if $A=G$ is a group and $B=H$ is a subgroup, then the set of double cosets $G//H$ is a geometry.

Now suppose B is a finite subprobability group of a probability group A. Then B is a subgeometry of A and so we get the factor geometry $A//B$. For $X,Y,Z\in A//B$ define

$$p_Z(X,Y)=\frac{1}{n_B}\sum_{b\in B}\sum_{z\in Z}\sum_{a\in A}p_a(x,b)\,p_z(a,y)/\,p_e(b,b^\#)$$

where $x\in X$, $y\in Y$ are arbitrary elements. The map p is independent of the choice of x and y and makes $A//B$ into a probability group inducing the factor geometry of $A//B$. In particular, if $A=G$ is a group and $B=H$ is finite subgroup of G, then the geometry of double cosets $G//H$ is a probability group. In this case, the map p simplifies to the following:

$$p_z(X,Y)=\frac{|xHy\cap Z|}{|H|}$$

for some $x\in x$, $y\in Y$.

2.3. Geometry and probability groups of orbits. Let A be a geometry. A group Γ is said to act on A if there is a homomorphism from Γ to the geometry automorphisms of A. Thus for $\alpha\in\Gamma$ and $a\in A$, we get $\alpha a\in A$ satisfying obvious properties. In such cases, we call A a Γ-geometry. For $a\in A$, let $\langle a\rangle=\{\alpha a:\alpha\in\Gamma\}$ denote the orbit of A and $A/\Gamma=\{\langle a\rangle : a\in A\}$ be the set of orbits. Let $(\langle a\rangle,\langle b\rangle,\langle c\rangle)\in\Delta_{A/\Gamma}$ iff $\exists x\in\langle a\rangle$, $y\in\langle b\rangle, z\in\langle c\rangle$ with $(x,y,z)\in\Delta_A$. This makes A/Γ a geometry called the geometry of orbits of A by Γ. In particular, if V is a (left) vector space over a skewfield F, then the geometry of orbits V/F^* is the geometry of the classical projective space $P(V)$.

Now, let A be a probability group. Suppose a finite group Γ acts on the geometry A. Then A is called a Γ- probability group if, in addition, $p_{\alpha a}(\alpha b,\alpha c)=p_a(b,c)\forall a,b,c\in A,\alpha\in\Gamma$. Suppose A is a Γ– probability group. Since Γ acts on the geometry A, we get the geometry of orbits A/Γ as above. Define

$$p_{\langle a\rangle}(\langle b\rangle,\langle c\rangle)=\frac{|\langle a\rangle|}{|\langle b\rangle||\langle c\rangle|}\sum_{y\in\langle b\rangle}\sum_{z\in\langle c\rangle}p_x(y,z)$$

for some $x\in\langle a\rangle$. The map is well defined and makes A/Γ into a probability group inducing the geometry of orbits. Thus, if G is a group and Γ is a finite group of automorphisms, then the geometry of orbits G/Γ is a probability group.

A special important case is given by the following:

Example 2.6. Suppose V is a vector space over a finite filed F containing m elements. The multiplicative group F^* acts on V and the set of orbits V/F^* is a probability group of the corresponding projective space.. If $\langle v\rangle\in V/F^*$, $v\neq 0$, then $\langle v\rangle=F^*v$, so $|\langle v\rangle|=|F^*v|=|F^*|=m-1$. Hence the above formula becomes

$$p_{\langle u\rangle}(\langle v\rangle,\langle w\rangle)=\frac{1}{|F^*|}\sum_{y\in\langle v\rangle}\sum_{z\in\langle w\rangle}p_x(y,z)=\frac{1}{m-1}\sum_{\alpha\in F^*}\sum_{\beta\in F^*}p_x(\alpha v,\beta w)$$

A case by case consideration will give p-values exactly as defined in example 2.5(2).

2.4. Homomorphism. Let A and B be geometries and $f:A\to B$ be a map. Then f is called a morphism if $f(e_A)=e_B$ and $(x,y,z)\in\Delta_A\Rightarrow(f(x),f(y),f(z))\in\Delta_B$. If, in addition, $(f(x),f(y),b)\in\Delta_B\Rightarrow b=f(z)$ for some $z\in A$ with $(x,y,z)\in\Delta$, then the morphism is called a homomorphism.

Let A, B be probability groups and $f:A\to B$ be a map. Then f is called a probability homorphism if $f(e_A)=e_B$ and

$$p_b(f(a_1),f(a_2))=\sum_{x\in f^{-1}(b)}p_x(a_1,a_2)\ \forall a_1,a_2\in A,b\in B.$$

A homomorphism of probability groups is naturally a homomorphism of corresponding geometries. A bijective homomorphism is an isomorphism. So a bijective f is an isomorphism if and only if $p_a(b,c) = p_{f(a)}(f(b), f(c))$ $\forall a, b, c \in A$. Note that the context distinguishes p for A and B.

As in geometry, the natural map $A \to A//B$ is a homomorphism if and only if B is normal in A. There are isomorphism theorems for homomorphisms of probability groups similar to those in geometry. A probability group A is said to be of discrete probability type if $\forall a \in A$, there is a finite set F_a with $p_x(a, b) \in F_a$ $\forall x, b \in A$. For such we have:

Proposition 2.7. Let A be a probability group of discrete probability type and B,C be subprobability groups of A with C normal in A. Let $B.C = \{x : (b, c, x) \in \Delta$, for some $b \in B, c \in C\}$. The $B.C$ is a subprobability group of A and $B.C//C \cong B//B \cap C$ as probability groups.

In particular, the proposition is true if A is finite.

2.5.Geometry and Probability Spaces over Geometric Skewfields. Let $(A, 0_A, \Delta)$ be an abelian geometry. Suppose, in addition, $(A, .)$ is a semigroup with 1 such that $0 . a = a.0 = 0$. It is called a geometric ring if $(a, b, c) \in \Delta, x \in A \Rightarrow (ax, bx, cx)$, $(xa, xb, xc) \in \Delta$. It is called a geometrics sfield if $A^* = A - \{0\}$ is a group. Suppose $(V, 0_V, \Delta)$ is an abelian geometry and the geometric sfield A acts on V compatibly as scalars satisfying: $a(bv) = (ab)v$; $0_A . v = a. 0_V = 0_V$; $1. v = v$; $(u, v, \omega) \in \Delta \Rightarrow$ $(au, a v, a\omega) \in \Delta$; $(a, b, c) \in \Delta \Rightarrow (av, bv, cv) \in \Delta$; $(ab, bv, cv) \in \Delta$, $v \neq 0 \Rightarrow (a, b, c) \in \Delta$; $(av,bv,\omega) \in \Delta \Rightarrow \omega = cv$; where $a, b, c \in A$ and $u, v, w \in V$. Then V is said to be a geometric space over geometric sfield A. For such there is a basis and well defined dimension (see[4], [5]). In case V and A have sharp geometries, the geometric space V is a vector space over the usual skewfield A.

If V is a geometric space over a geometric sfield A, then the geometry of orbits V/A^* is projective and so represents a projective space (including degenerate ones).

Now suppose V is a geometric space over A and in addition, (V, p) is a probability group inducing the given geometry. Then we call V a probability space over A if V is A^* - probability group. Hence, $\forall$ $u,v, \omega \in V$ and $\forall \alpha \in A^*$, we have

$$p_{\alpha u}(\alpha v, \alpha w) = p_u(v,w)$$

If A is finite, then the projective space V/A^* is a probability group.

2.6. Semi-isomorphism. Let V and W be geometric spaces over geometric sfields A and B respectively. A pair of maps $(\sigma, \hat{\sigma}) : (V, A) \to (W, B)$ is called a semi-isomorphism if $\sigma : V \to W$ is an isomorphism of geometries, $\hat{\sigma} : A \to B$ is an isomorphism of geometric sfields and $\sigma(av) = \hat{\sigma}(a)\, \sigma(v) \forall v \in V, \forall a \in A$.

Suppose, in addition, V and W are probability spaces over A and B respectively. Then $(\sigma, \hat{\sigma})$ is called a semi-isomorphism of probability spaces if $p_u(v,w) = p_{\sigma(u)}(\sigma(v),\sigma(w)) \forall u,v,w \in V$.

3. Elementary abelian probability groups

Let (A,p) be an abelian probability group. Recall that an abelian geometry A is elementary if $\forall a \in A$, the subgeometry $\langle a\rangle$ generated by a is simple in the sense that whenever S is a subgeometry of A and $\{e\} \neq S \subseteq \langle a\rangle$, then $S = \langle a\rangle$ (see [2]). In such a geometry, $\langle a\rangle \neq \langle b\rangle \Rightarrow \langle a\rangle \cap \langle b\rangle = \{e\}$. Now, since B is a subprobability group of A if and only if B is a subgeometry of A, we make the

Definition 3.1 An abelian probability group A is called elementary if it is elementary as a geometry.

Also, the length of the probability group A will mean the length of the corresponding geometry [2].

Lemma 3.2. Let A be a finite elementary abelian probability group of length greater than one. Then

(i) $p_e(a, a^{\#}) = p_e(b,b^{\#}) \forall a, b \in A^*$. (ii) $p_a(b, c) = p_{b^{\#}}(c, a^{\#}) \forall a, b, c \in A^*$.

Proof: (i) Let $a,b \in A^*$. Suppose $\langle a\rangle \neq \langle b\rangle$. Let $t \in A$ such that $p_t(a,b) \neq 0$, so $(a, b, t^{\#}) \in \Delta$. Since $\langle a\rangle \neq \langle b\rangle$, we have $t \notin \langle a\rangle$, $t \notin \langle b\rangle$. So $\langle a\rangle \cap \langle t\rangle = \{e\}$ and $\langle a,t\rangle = \langle b, t\rangle$. By proposition (2.7), we get

$$\langle a\rangle \cong \langle a\rangle // \langle a\rangle \cap \langle t\rangle \cong \langle a\rangle.\langle t\rangle // \langle t\rangle = \langle b\rangle.\langle t\rangle // \langle t\rangle \cong \langle b\rangle.$$

If the composite map is σ, then $\sigma(a) = b$, so $\sigma(a^{\#}) = b^{\#}$. Hence, $p_e(a, a^{\#}) = p_{\sigma(e)}(\sigma(a), \sigma(a^{\#}) = p_e(b, b^{\#})$. If $\langle a\rangle = \langle b\rangle$, then the length being greater than one, $\exists c \in A^*$ such that $\langle c\rangle \neq \langle a\rangle$. Then, $p_e(a,a^{\#}) = p_e(c,c^{\#}) = p_e(b,b^{\#})$. (ii) It follows from (i) and lemma 2.3 (iii).

Lemma 3.3 Let A be a finite elementary abelian probability group of length greater than 2. Suppose $p_{a_1}(b_1,c_1) \neq 0$, and $p_{a_2}(b_2, c_2) \neq 0$, where $b_1,b_2,c_1,c_2 \in A^*$, $\langle b_1\rangle \neq \langle c_1\rangle$, $\langle b_2\rangle \neq \langle c_2\rangle$. Then, $p_{a_1}(b_1,c_1) = p_{a_2}(b_2,c_2)$.

Proof: In corresponding geometry, we have $(a_1^{\#}, b_1, c_1), (a_2^{\#}, b_2, c_2) \in \Delta$. Note that $a_1 \neq e$, otherwise it would give $\langle b_1\rangle = \langle c_1\rangle$. Similarly $a_2 \neq e$.

Case (1): $a_2 = a_1$. Suppose first, $b_2 \notin \langle b_1,c_1\rangle$. Then, $(a_1^{\#}, b_1, c_1)$, $(a_1^{\#}, b_2, c_2) \in \Delta$, so $\exists t \in A$ such that $(t, b_2^{\#}, b_1)$, $(t, c_2, c_1^{\#}) \in \Delta$. Note that $t \notin \langle a_1, b_1\rangle$, otherwise $(t, b_2^{\#}, b_1) \in \Delta$ would give $b_2^{\#}$ and hence $b_2 \in \langle a_1,b_1\rangle = \langle b_1, c_1\rangle$. So being elementary, we get $\langle t\rangle \cap \langle a_1,b_1\rangle = \{e\} = \langle t\rangle \cap \langle a_1,b_2\rangle$. So by proposition (2.7), we get

$$\langle a_1,b_1\rangle \cong \langle a_1,b_1\rangle // \langle t\rangle \cap \langle a_1,b_1\rangle \cong \langle a_1,b_1\rangle.\langle t\rangle // \langle t\rangle = \langle a_1,b_2\rangle.\langle t\rangle // \langle t\rangle \cong \langle a_1,b_2\rangle.$$

Suppose σ is the composite map. If $x \in \langle a_1,b_1\rangle$, then $x \in \langle a_1,b_1\rangle.\langle t\rangle = \langle a_1, b_2\rangle.\langle t\rangle$, so $\exists y \in \langle a_1,b_2\rangle$, $t_1 \in \langle t\rangle$ with $(x, y^{\#}, t_1) \in \Delta$. Since $\langle a_1, b_2\rangle \cap \langle t\rangle = \{e\}$, the elements $y^{\#}$ and t_1 are unique. Chasing the above isomorphism, it easily verifies that $\sigma(x) = y$. In particular, we have $\sigma(a_1) = a_1 = a_2$, $\sigma(b_1) = b_2$, $\sigma(c_1) = c_2$. So, $p_{a_1}(b_1, c_1) = p_{\sigma(a_1)}(\sigma(b_1), \sigma(c_1)) = p_{a_2}(b_2, c_2)$.

Now suppose $b_2 \in \langle b_1,c_1\rangle$. Then $\langle b_1,c_1\rangle = \langle b_2,c_2\rangle$. Since the length of $A > 2$, $\exists b_3$ such that $b_3 \notin \langle b_1,c_1\rangle$. Choose $c_3 \in A$ such that $p_{a_1}(b_3,c_3) \neq 0$. Then, $\langle b_3\rangle \neq \langle c_3\rangle$, otherwise $\langle b_3\rangle = \langle a_1\rangle \subseteq \langle b_1,c_1\rangle$. So from the first part, we get

$$p_{a_1}(b_1,c_1) = p_{a_1}(b_3,c_3) = p_{a_1}(b_2,c_2) = p_{a_2}(b_2,c_2).$$

Case (2): Let a_2 be arbitrary. Since $\langle b_1\rangle \neq \langle c_1\rangle$, either $\langle a_2\rangle \neq \langle b_1\rangle$, or $\langle a_2\rangle \neq \langle c_1\rangle$. We may assume $\langle a_2\rangle \neq \langle b_1\rangle$. Now, $\exists d \in A$ with $p_{b_1^\#}(a_2^\#, d) \neq 0$. By lemma 3.2 (ii), we get $p_{a_1}(b_1,c_1) = p_{b_1^\#}(c_1, a_1^\#)$. Now using case (1), we get

$$p_{b_1^\#}(c_1,a_1^\#) = p_{b_1^\#}(a_2^\#,d) = p_{a_2}(d,b_1) = p_{a_2}(b_2,c_2).$$

Thus in every case $p_{a_1}(b_1,c_1) = p_{a_2}(b_2,c_2)$.

Now, suppose (A,p) is a probability group of length greater than two such that the corresponding geometry is projective. Then the geometry A corresponds to a projective space. Suppose the projective space is of order m so that each line contains $m + 1$ points. The following theorem shows that the probability structure on a projective space is unique. This result was proved in [7] by using duality.

Theorem 3.4 *Let (A,p) be a finite probability group such that the induced geometry on A is projective of order m with length $(A) > 2$. Then,*

$$p_a(b,c) = \begin{cases} \delta_a(c) & \text{if } b = e \\ \delta_a(b) & \text{if } c = e \\ \frac{1}{m-1}\delta_b(c) & \text{if } a = e \\ \frac{m-2}{m-1} & \text{if } a = b = c \neq e \\ \frac{1}{m-1} & \text{if } a, b, c \in P \text{ and } a, b, c \text{ are distinct and collinear} \\ 0 & \text{otherwise} \end{cases}$$

Proof: Since A is projective, it is abelian and $(a,a,b) \in \Delta \Rightarrow b = e$ or $b = a$. This implies that $\langle a\rangle = \{e, a\}, \forall a \in A$. So A is elementary abelian.

Now if $b = e$, then $p_a(e,c) = \delta_a(c)$ and if $c = e$, then $p_a(b,e) = \delta_a(c)$ are clear. So suppose $b,c \in A^*$. We consider two cases.

Case (1): $b \neq c$. Then $p_a(b,c) \neq 0$ if and only if $(a,b,c) \in \Delta$. Since A is elementary Abelian, lemma 3.3 gives that $p_a(b,c) = p_x(b,c) \forall x \in A^*$ such that $(x,b,c) \in \Delta$. But $(x,b,c) \in \Delta$ if and only if $x \in L_{bc} - \{b,c\}$, where L_{bc} is the line determined by the points b and c. Since the line L_{bc} has $m + 1$ points, the number of $x \in L_{bc} - \{b,c\}$ will be $\{m + 1\}-2 = m-1$. So

$$1 = \sum_{x\in A} p_x(b,c) = \sum_{x\in L_{bc}-\{b,c\}} p_x(b,c) = (m-1)\, p_a(b,c).$$

Hence, $p_a(b,c) = \frac{1}{m-1}$.

Case (2): $b = c$. Then $p_a(b, c) \neq 0$ only if $a = b = c$ or $a = e$. Suppose first

$a = b = c$ and $p_a(a, a) \neq 0$. Since length $(A) > 2$, $\exists y \in A^{*}$ such that $y \neq a$. Let $z \in A$ such that $(a, y, z) \in \Delta$. Clearly $z \neq e, a, y$. So $p_a(y, z) = \frac{1}{m-1}$. Now consider the equation:

$$\sum_{x \in A} p_x(y,z)p_a(a,x) = \sum_{x \in A} p_a(y,x)p_x(z,a)$$

On the left side, $p_a(a, x) = 0$ except for $x = a$ or e. But if $x = e$, then $p_x(y, z) = p_e(y, z) = 0$, since $y \neq z$. So the only nonzero term in the sum is for $x = a$. So Left side $= p_a(y, z)\, p_a(a, a) = \frac{1}{m-1}\; p_a(a, a)$.

Also on the right a, y, z are distinct and collinear, so $p_a(y, x) = p_x(z, a) = \frac{1}{m-1}$ or 0. The number of elements x such that $p_a(y, x) \neq 0$ is $m - 1$ and includes z, but if $x = z$, then $p_z(z, a) = 0$. So the number of elements for which both factors are not zero is $m - 2$. So Right side $= (m-2)\left(\frac{1}{m-1}\right)\left(\frac{1}{m-1}\right) = (m-2)\left(\frac{1}{m-1}\right)^2$.

Thus, $\frac{1}{m-1}\; p_a(a, a) = (m-2)\left(\frac{1}{m-1}\right)^2$ giving $p_a(a, a) = \frac{m-2}{m-1}$.

Finally, suppose $a = e$. Then, we have $p_e(b,b) + p_b(b, b,) = 1$, so $p_e(b,b) = 1 - \frac{m-2}{m-1} = \frac{1}{m-1}$ and the proof is complete.

4. Probability spaces over geometric sfields as orbits of vector spaces

The following theorem establishes uniqueness of the probability structure which induces a given geometric space over a geometric sfield.

Theorem: 4.1 *Let V be a finite geometric space over a geometric sfield A, $\dim_A(V) > 2$. Suppose (V, p) and (V, q) are probability spaces over A inducing the given geometric space over A. Then, $p = q$.*

Proof: Since V is a geometric space over the geometric sfield A, the orbit space V/A^* is a projective space of length greater than 2. Since (V, p) is an A^* –probability group, it gives a probability structure on the orbits V/A^* as follows (cf. 2.3):

$$p_{\langle v_1 \rangle}(\langle v_2 \rangle, \langle v_3 \rangle) = \frac{1}{|A^*|} \sum_{\alpha \in A^*} \sum_{\beta \in A^*} p_{v_1}(\alpha v_2, \beta v_3)$$

Similarly, the A^*–probability group (V,q) gives

$$q_{\langle v_1 \rangle}(\langle v_2 \rangle, \langle v_3 \rangle) = \frac{1}{|A^*|} \sum_{\gamma \in A^*} \sum_{\delta \in A^*} q_{v_1}(\gamma v_2, \delta v_3)$$

But by (3.4), the two probability structures on the projective space V/A^* must be the same. So

$$\frac{1}{|A^*|} \sum_{\alpha \in A^*} \sum_{\beta \in A} p_{v_1}(\alpha v_2, \beta v_3) = \frac{1}{|A^*|} \sum_{\gamma \in A^*} \sum_{\delta \in A^*} q_{v_1}(\gamma v_2, \delta v_3)$$

i.e

$$\sum_{\alpha \in A^*} \sum_{\beta \in A^*} p_{v_1}(\alpha v_2, \beta v_3) = \sum_{\gamma \in A^*} \sum_{\delta \in A^*} q_{v_1}(\gamma v_2, \delta v_3)$$

Now let $\upsilon_1,\upsilon_2,\upsilon_3 \in V$ be arbitrary. Then $p_{\upsilon_1}(\upsilon_2,\upsilon_3) \neq 0$ if and only if $(\upsilon_2,\upsilon_3,\upsilon_1^{\#})$ $\in \Delta$ if and only if $q_{\upsilon_1}(\upsilon_2,\upsilon_3) \neq 0$. So let $(\upsilon_2,\upsilon_3,\upsilon_1^{\#}) \in \Delta$. We show $p_{\upsilon_1}(\upsilon_2,\upsilon_3) = p_{\upsilon_1}(\upsilon_2,\upsilon_3)$. We consider the following cases.

Case (1). υ_2,υ_3 are independent over A. Then for any $\upsilon \in span(\upsilon_2,\upsilon_3)$, $\exists$ unique $\alpha, \beta \in A$ such that $(\upsilon, \alpha\upsilon_2, \beta\upsilon_3) \in \Delta$. So, since $p_{\upsilon_1}(\upsilon_2, \upsilon_3) \neq 0$, we get $p_{\upsilon_1}(\alpha\upsilon_2, \beta\upsilon_3) = 0$ except for $\alpha = \beta = 1$. So on the left side of the above equation we get only one nonzero term $p_{\upsilon_1}(\upsilon_2,\upsilon_3)$. Similarly, the right side gives $q_{\upsilon_1}(\upsilon_2,\upsilon_3)$. So, $p_{\upsilon_1}(\upsilon_2,\upsilon_3) = q_{\upsilon_1}(\upsilon_2,\upsilon_3)$.

Case (2). υ_2,υ_3 are dependent. If $\upsilon_2 = 0$ or $\upsilon_3 = 0$, it is obvious. So let $\upsilon_2 \neq 0$, $\upsilon_3 \neq 0$.

Suppose first $\upsilon_1 \neq 0$. Then $\exists\, \upsilon \in V$ such that $\upsilon_1 = \upsilon$, $\upsilon_2 = \alpha\upsilon$, $\upsilon_3 = \beta\upsilon$. So we show $p_\upsilon(\alpha\upsilon, \beta\upsilon) = q_\upsilon(\alpha\upsilon,\beta\upsilon)$. Since $\dim(V) > 2$, $\exists u, w \in V$ independent such that $(u, \omega, \alpha\upsilon^{\#}) \in \Delta$. So by case (1), $p_{\alpha\upsilon}(u, w) = q_{\alpha\upsilon}(u,w) \neq 0$. Also, $p_{\delta\upsilon}(u,w) = q_{\delta\upsilon}(u,w) = 0$ for $\delta \neq \alpha$. In (V, p), we have:

$$\sum_{y\in V} p_y(u,w)p_\upsilon(y,\beta\upsilon) = \sum_{x\in V} p_\upsilon(u,x)p_x(w,\beta\upsilon)$$

But $p_\upsilon(y,\beta\upsilon) \neq 0$ implies $y = \gamma\upsilon$ for some $\gamma \in A$ and $p_{\gamma\upsilon}(u, w) \neq 0$ only when $\gamma = \alpha$. Hence the above equality gives

$$p_{\alpha\upsilon}(u,w)\, p_\upsilon(\alpha\upsilon, \beta\upsilon) = \sum_{x\in V} p_\upsilon(u,x)p_x(w,\beta\upsilon)$$

Similarly we get for q:

$$q_{\alpha\upsilon}(u,w)\, q_\upsilon(\alpha\upsilon, \beta\upsilon) = \sum_{x\in V} q_\upsilon(u,x)q_x(w,\beta\upsilon)$$

Since u,υ are independent $p_\upsilon(u,x) \neq 0$ implies u,x are also independent. So by case (1), $p_\upsilon(u,x) = q_\upsilon(u,x)\ \forall x \in V$. Similarly, $p_x(w,\beta\upsilon) = q_x(w,\beta\upsilon)\ \forall x \in V$. So the right sides of the above equalities are equal giving the equality of the left sides:

$$p_{\alpha\upsilon}(u,w)\, p_\upsilon(\alpha\upsilon,\beta\upsilon) = q_{\alpha\upsilon}(u,w)\, q_\upsilon(\alpha\upsilon,\beta\upsilon)$$

But again, $p_{\alpha\upsilon}(u,w) = q_{\alpha\upsilon}(u,w) \neq 0$. So we eventually get $p_\upsilon(\alpha\upsilon,\beta\upsilon) = q_\upsilon(\alpha\upsilon,\beta\upsilon)$. Finally, let $\upsilon_1 = 0$. Then, $1 = \sum_x p_x(\upsilon_2,\upsilon_3) = \sum_x q_x(\upsilon_2,\upsilon_3)$. Since $p_x(\upsilon_2,\upsilon_3) = q_x(\upsilon_2,\upsilon_3)$ $\forall x \neq 0$, we must have $p_0(\upsilon_2, \upsilon_3) = q_0(\upsilon_2,\upsilon_3)$. Thus $p = q$.

Now suppose V is a vector space over a finite field F, $\dim(V) \geq 3$ and Γ is a subgroup of F^*. Then the orbit spaces V/Γ and F/Γ are probability groups in a natural way (cf. 2.3). It can easily be seen that the probability group V/Γ so defined is F^*/Γ - probability group. So the theorem gives:

Corollary 4.2. Let V be a finite dimensional vector space over a finite field F and Γ be a subgroup of F^*. Then the space V/Γ is a probability space over F/Γ in a unique (natural) way.

Definition 4.3. We call a geometric space V over a geometric sfield A to be of *finite order type* if the corresponding projective space $P(V)$ is of finite order.

The following theorem gives the representation of a probability space over a geometric space as orbits of a vector space.

Theorem 4.4. *Suppose (V, p) is a probability space of finite dimension over a geometric sfield A with $\dim_A V \geq 4$. Suppose V is of finite order type. Then there is a finite dimensional vector space W over a finite F, a subgroup Γ of F^* and a semi-isomorphism of probability spaces*:

$$(\psi, \hat{\psi}) : (W/\Gamma, F/\Gamma) \to (V, A)$$

The same is true if $\dim_A V = 3$ and the geometry of V is D-geometry.

Proof : Since V is a geometric space over the geometric sfield A with proper dimension, there is a vector space W over a skewfield F, a normal subgroup Γ of F^* and a semi-isomorphism of geometric spaces $(\psi, \hat{\psi}) : (W/\Gamma, F/\Gamma) \to (V, A)$ (see [5]). We show that it is a semi-isomorphism of probability spaces. Since V is of finite order type, the projective space $P(W)$ is of finite order and so F is a finite field. So W and hence V is finite. We use the isomorphism ψ to make W/Γ into a probability space as follows:

$$p_{\tilde{u}}(\tilde{v}, \tilde{w}) = p_{\psi(\tilde{u})}(\psi(\tilde{v}), \psi(\tilde{w})),\ \forall \tilde{u}, \tilde{v}, \tilde{w} \in W/\Gamma.$$

We have for $\tilde{a} \in F^*/\Gamma$, $p_{\tilde{a}\tilde{u}}(\tilde{a}\tilde{v}, \tilde{a}\tilde{w}) = p_{\hat{\psi}(\tilde{a})\psi(\tilde{u})}(\hat{\psi}(\tilde{a})\psi(\tilde{v}), \hat{\psi}(\tilde{a})\psi(\tilde{w}))$

$$= p_{\psi(\tilde{u})}(\psi(\tilde{v}), \psi(\tilde{w})) = p_{\tilde{u}}(\tilde{v}, \tilde{w}),$$

as V is A^*-probability group. So this makes W/Γ into F^*/Γ - probability group. But by corollary 4.2, such probability structure is uniquely the natural probability structure of W/Γ.

Hence the theorem is proved.

5. Elementary abelian probability groups as orbits of groups

The following theorem gives orbit space representation of probability groups, which are elementary abelian.

Theorem 5.1 *Suppose A is a finite elementary abelian probability group, length $(A) \geq 4$. Then there exists a vector space V over a finite field F and a subgroup Γ of F^* such that $A \cong V/\Gamma$ as probability groups.*

If length $(A) = 3$, then the same is true if A is a D-geometry.

Proof: Since A is elementary abelian geometry of proper length, there is a vector space V over a skewfield F and a subgroup Γ of F^* such that $\sigma : V/\Gamma \to A$ is an isomorphism of geometries which induces isomorphism of projective spaces $P(V)$ and $P(A)$ (see [2]). We show that σ is an isomorphism of probability groups.

Since A is finite, the projective space $P(A)$ and hence $P(V)$ has finite order. So F is finite and hence a held. Then, V/Γ is a geometric space over F/Γ. We make V/Γ a probability group by defining p as follows:

$$p_x(y, z) = p_{\sigma(x)}(\sigma(y), \sigma(z)) \forall x, y, z \in V/\Gamma.$$

The probability group $(V/\Gamma, p)$ so defined is isomorphic to (A,p). To show that it is the natural probability group, it is sufficient to show that V/Γ is F^{*}/Γ-probability group. So, let $\alpha \in F/\Gamma$, $\alpha \neq 0$. We show $p_{\alpha x}(\alpha y, \alpha z) = p_x(y, z) \forall x, y, z \in V/\Gamma$. It is obvious if $x = 0$, or $y = 0$, or $z = 0$. So let $x, y, z \in (V/\Gamma)^{*}$. Suppose first y, z are independent. This means in $P(V)$, $\langle y \rangle \neq \langle z \rangle$, so in $P(A)$, $\langle \sigma(y) \rangle \neq \langle \sigma(z) \rangle$. Similarly, $\langle \sigma(\alpha y) \rangle \neq \langle \sigma(\alpha z) \rangle$. So by lemma 3.3, $p_{\sigma(x)}(\sigma(y), \sigma(z)) = p_{\sigma(\alpha x)}(\sigma(\alpha y), \sigma(\alpha z))$, showing that $p_x(y, z) = p_{\alpha x}(\alpha y, \alpha z)$. Now suppose y, z are dependent. They $y = \beta x$, $z = \gamma x$. Choose $t \in V/\Gamma$ with x, t independent. Then, as in lemma 3.2, $\langle x \rangle \cong \langle t \rangle \cong \langle \alpha x \rangle$, the composite isomorphism being given by $x \to \alpha x$. So for $\beta x, \gamma x \in \langle x \rangle$, we get $p_x(\beta x, \gamma x) = p_{\alpha x}(\alpha \beta x, \alpha \gamma x) = p_{\alpha x}(\alpha y, \alpha z)$.

Thus, (V,p) is the natural probability group and $\sigma: V/\Gamma \to A$ is the required isomorphism.

Now since a vector space over a finite field of characteristic, say p, is a vector space over Z_p, and hence is a finite elementary abelian p-group, we may restate

Theorem 5.2. *A finite elementary abelian probability group of length greater than three is isomorphic to the probability group of orbits of a finite elementary abelian p-group with respect to a finite group of automorphisms.*

Acknowledgement

This paper was partly prepared while visiting the Abdus Salam International Centre for Theoretical Physics, Trieste, Italy as an associate. Financial support from the Swedish International Development Cooperation Agency is appreciated.

REFERENCES

[1.] Bhattarai, H.N. and Fernandez, J.W., Join of Double Coset Spaces, *Pacific J. Math.* 98(2) (1982).

[2.] Bhattarai, H.N., An Orbit Space Representation of A Geometry, *J. Algebra* 84(1) (1983), 142–150.

[3.] _______, Categories of Projective Geometries with Morphisms and Homomorphisms, *Geometriae Dedicata* 78(1999), 111–120.

[4.] _______, Pasch geometric spaces inducing finite projective planes, *J. Geometry* 34(1989), 6–13.

[5.] _______, Pasch Geometric Spaces as Orbits of Vector Spaces, *Nep. Math. Sc. Report*, 20(2003).

[6.] Dresher, M. and Ore, O., Theory of Multigroups, *Amer. J. Math.* 60(1938).

[7.] Harrison, D.K., Double Cosets and Orbit Spaces, *Pacific J. Math.* 80(2), 451–491.

[8.] Jewett, R.I., Spaces with an Abstract Convolution of Measures, *Advances in Math.* 18(2) (1975), 1–101.
[9.] Prenowitz, W., Projective Geometries as Multigroups, *Amer. J. Math.* 65(1943).
[10.] Roth, R.L., Character and Conjugacy Class Hypergroups, *Annali Di Mat. Pura Ed Appl.*, 55(1975).

H.N. BHATTARAI
Cerntral Department of Mathematics,
Tribhuvan University, Kathamandu, Nepal

The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy.

Present Address
University Grants Commision, P.O. Box 10796,
Kathmandu, Nepal,

The Nepali Math. Sci. Report
Vol. 24. No.1. 2005.

The Schrödinger equation associated to 2nd order linear differential equation

J. H. CALTENCO, J. L. LÓPEZ-BONILLA, R. PEÑA-RIVERO

Abstract: We determine the Schrödinger equations associated to Hermite and Laguerre differential equations, hoping that the process here exhibited may be useful in quantum mechanics.

1. Introduction:

It is known [1,2] that the 2nd order linear differential equation :

$$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=0 \tag{1}$$

can be written as an Schrödinger-like equation:

$$\frac{d^2W}{dx^2}+J(x)W=0 \tag{2}$$

via the following change of variable:

$$y=W\exp\left(-\frac{1}{2}\int^x P(t)dt\right) \tag{3}$$

such that:

$$J(x)=Q(x)-\frac{1}{2}\frac{dP}{dx}-\frac{P^2}{4} \tag{4}$$

We here shall apply this procedure to Laguerre and Hermite equations, which has didactic value in the teaching of the elementary quantum mechanics

2. The Schrodinger type equations associated to Hermite and Laguerre equations.

The Hermite equation is given by [1, 2]:

$$y''-2xy'+2ny=0 \quad , \quad n=0,1,2,\ \ldots \tag{5}$$

and its corresponding polynomial solution is denoted by $H_n(x)$. By comparison of (1) with (5) we see that $P = -2x$ and $Q = 2n$, then $J = 2n + 1 - x^2$ and thus the Schrödinger equation (2) adopts the form

(6) $$-\frac{1}{2}W'' + \frac{x^2}{2}W = \left(n + \frac{1}{2}\right)W$$

for the potential $\frac{x^2}{2}$ of the harmonic oscillator in natural units ($\hbar = m = \omega = 1$), resulting thus the energy spectrum $\left(n + \frac{1}{2}\right)$ for the stationary status. The equation (3) implies $W \propto H_n \exp\left(-\frac{x^2}{2}\right)$, then the normalization of the waves functions leads to final result [3–5] :

(7) $$\psi_n(x) = \left(2^n\, n!\sqrt{\pi}\right)^{-\frac{1}{2}} H_n(x)\exp\left(-\frac{x^2}{2}\right).$$

The associated Laguerre equation has the structure [1, 2]:

(8) $$y'' + \frac{k+1-x}{x}\, y' + \frac{N}{x}\, y = 0$$

and the polynomials $L_N^K(x)$ represent their respective solutions. From (1) and (8) it is clear that $P = \frac{k+1-x}{x}$ with $Q = \frac{N}{x}$, then (2), (3) and (4) give us the expressions :

(9) $$W'' + \left(-\frac{1-K^2}{4x^2} + \frac{K+1+2N}{2x} - \frac{1}{4}\right)W = 0$$

with

(10) $$W \propto x^{\frac{K+1}{2}} e^{-\frac{x}{2}}\, L_N^K(x).$$

In (9) is not evident the corresponding potential, thus we make the changes :

(11) $$K = 2\,l + 1\ ,\ N = n - l - 1\ ,\ x = \frac{2r}{bn}\ ,\ b = \frac{4\pi\varepsilon_0}{Ze^2}$$

then (9) takes the known form for the Coulomb potential ($\hbar = m = 1$) :

(12) $$-\frac{1}{2}\left[\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2}\right]W - \frac{Ze^2}{4\pi\varepsilon_0 r}\,W = \frac{Z^2}{32\pi^2\varepsilon_0^2}\,\frac{1}{n^2}W$$

where n and l denote the principal and orbital quantum numbers, respectively.Therefore, (10) and (11) imply the normalized radial wave functions [3,4, 6]:

(13) $$\psi_{nl}(r) = \left(\frac{2r}{n}\right)^{l+1}\left[\frac{(n-l-1)!}{(n+l)!}\right]^{1/2}\frac{e^{-\frac{r}{bn}}}{b^{l+3/2}}\, L_{n-l-1}^{2l+1}\left(\frac{2r}{bn}\right).$$

If in (9) we use changes of variables different to (11), then it is easy to show that (9) reproduces the radial part of the Schrödinger equation for the Morse and two-dimensional harmonic oscillator potentials [7].

REFERENCES

[1] Goswami, A., *Quantum mechanics*, Wm. C. Brown Pub. (U.S.A) (1992)
[2] Hochstadt, H., *The functions of mathematical physics*, Dover N. Y. (1986).
[3] López-Bonilla J., and Ovando, G., *Bull. Irish Math.* Soc. No. **44** (2000) 61.
[4] López-Bonilla, J., Morales, J., and Rosales, M., Int. *J.Quantum Chem*. **53**(1995) 3.
[5] López-Bonilla, J., Navarrete, D., Núñez H., and Sales, A., Int, *J. Quantum Chem.* **62** (1997) 177.
[6] Schwabl, F., *Quantum mechanics*, Springer-Verlag (1990)
[7] Seaborn, J. B., *Hypergeometric functions and their applications*, Springer-Verlag (1991)

J. H. CALTENCO, J. L. LÓPEZ-BONILLA, R. PEÑA-RIVERO

Sección de Estudios de Posgrado e Investigacion
Escuela Superior de Ingenieria Mecanica y Eléctrica
Instituto Politécnica Nacional
Edif. Z, Acc 3-3er Piso, Col. Lindavista, C.P. 07738 Méexico, D.F.
E-mai l : lopezbjl@hotmail.com
: hcalte@maya.esimez.ipn.mx

The Nepali Math. Sci. Report
Vol. 24. No.1. 2005.

Just-in-Time Sequencing Algorithms for Mixed-Model Production Systems

TANKA NATH DHAMALA

Abstract: Obtaining an optimal sequence in a mixed-model production system under the just-in-time philosophy is a challenging problem. The problems in a multi-level facility are strongly NP-hard, however, the single-level problems are pseudo-polynomial solvable. In this paper, we consider more practical just-in-time sequencing problem with given set of sequences as precedence constraints. We propose an efficient algorithm which obtains an optimal solution for the maximum deviation objective in the single-level.

Keywords: nonlinear integer programming, just-in-time scheduling, mixed-model systems, level schedules, balanced words, efficient algorithms, precedence constraints.

1. Introduction

The main goal of mixed-model production systems is to increase profit by reducing costs. The *just-in-time* (JIT) systems, which require producing only the necessary product in the necessary quantities at the necessary time, have been used for controlling systems. These methods satisfy the consumer demands for a variety of products without incurring large shortages or holding large inventories. We consider flexible transfer lines, where negligible switch over costs from one model to another make possible for diversified small-lot production avoiding production of each model in large-lots. One of the most important optimization problems have been considered is to determine the sequence in which different models are scheduled on the line. The sequences refereed as balanced, fair or level always keep the actual production level and the desired production one as close to each other as possible all the times. The objectives may vary.

There has been growing research in JIT sequencing since MONDEN [13]. MILTENBURG [12], formulates the single-level JIT sequencing problem as a nonlinear integer programming. STEINER/YEOMANS [15], give an efficient algorithm for minimizing the maximum deviation. KUBIAK/SETHI [9] reduce the minimization of more general sum deviation to an assignment problem and give efficient algorithm. These algorithms are also applicable for multi-level problems under the pegging assumption [14]. The existence of cyclic schedules have reduced computational time [14], KUBIAK [10].

BRAUNER/CRAMA [1] present an algebraic approach to [15] and formulate the small deviation conjecture. KUBIAK [11] gives a geometric proof and BRAUNER/JOST/ KUBIAK [6] exploit the concept of balanced words to prove the conjecture. Also, KOVALYOV/KUBIAK/YEOMANS[8] consider the computational time issues and COROMINAS /MORENO[3] establish optimality relations between different objectives. The minimization of maximum deviation is co-NP, but the complexity of these problems remains open for the binary encoding [1]. We refer to DHAMALA /KUBIAK [4] for a recent survey of JIT sequencing and the references.

In this paper, we study the JIT sequencing problem with different settings. Given a set of non-overlapping sequences as chain constraints, we give a pseudo-polynomial algorithm which obtains an optimal solution to the whole instance that preserves the customers orders, DHAMALA/KUBIAK[5]. By doing this we have introduced the *first-order first-serve* concept in mixed-model systems.

The paper is organized as follows. A brief review of the existing JIT sequencing algorithms for mixed-model systems is presented in Section 2. In section 3, we give an extension of the existing formulation of the JIT sequencing problem with additional constraints. Section 4 contains our optimization algorithm for the considered sequencing problem. The final section includes conclusions with some possible directions for further research.

2. Just-in-Time Sequencing Problem

For $i = 1, 2, \ldots, n$, given n products (models) i, n positive integers (demands) d_i and n convex-symmetric functions f_i of a single variable, called deviation, all assuming minimum 0 at 0, the following optimization problem have been considered in the literature [9, 12, 15]. Find a sequence $s = s_1 s_2 \ldots s_D$ with total demand $D = \sum_i^n d_i$ of products where product i occurs exactly d_i times that minimizes one of the following objective function (s)

(1) $$F_{MD}(s) = \max_{i,k} f_i(x_{ik} - r_i k)$$

(2) $$F_{SD}(s) = \sum_{i=1}^{n} \sum_{k=1}^{D} f_i(x_{ik} - r_i k)$$

where x_{ik} represents the number of product i occurrences (copies) in the prefix $s_1 s_2 \ldots s_k$, $k = 1, 2, \ldots, D$, and $r_i = \frac{d_i}{D}$, $i = 1, 2, \ldots, n$. Associated to each function F_{MD}, F_{SD}, there have been studied two type of objective functions in the literature:

$$f_i(x_{ik} - r_i k) = \begin{cases} (x_{ik} - r_i k)^2 & \text{the squared - deviation,} \\ |x_{ik} - r_i k| & \text{the absolute - deviation.} \end{cases}$$

A solution of this problem always keeps the *actual production level* x_{ik} and the *desired production level* $r_i k$ as close to each other as possible all the times. The problem is one of the most fundamental problems in flexible JIT mixed-model production systems. Here, the sequencing problems *maximum-deviation* JIT *and sum-deviation* JIT are represented by *min-max* and *min-sum problems*, respectively.

Both min-max and min-sum problems have been formulated as integer programming problem subject to the following cardinality, monotonicity and integrality constraints [9,12,15];

$$\begin{array}{lll} \sum_{i=1}^{n} x_{i,k} = k & k = 1, \ldots, D & \\ x_{i,D} = d_i & i = 1, \ldots, n & \\ x_{i,k} \le x_{i,k+1} & i = 1, \ldots, n, & k = 1, \ldots, D-1 \\ x_{i,k} \in \mathcal{N} & i = 1, \ldots, n, & k = 1, \ldots, D, \end{array}$$

where $\mathcal{N}$ denotes the set of all nonnegative integers. The whole solution region is denoted by $\chi = \{X \mid X = (x_{ik})_{n \times D}\}$. Thus, the JIT sequencing problem is equivalent to the following optimization problem

$$\min \{F(s) \mid X \in \chi\}, \text{ where } F \in \{F_{MD}, F_{SD}\}.$$

A solution $s = s_1 s_2 \ldots s_D$ of the min-max problem of n models is called *B-feasible* if $\max_{i,k} f_i (x_{ik} - r_i k) \le B$ holds for the $n \times D$ matrix variables $X = (x_{ik})$ The set of all B-feasible solutions is denoted by χ_B.

STEINER/YEOMANS [15] study min-max problem reducing to a single machine scheduling decision problem with release times and due dates. They represent the problem as a perfect matching in a V_1-convex bipartite graph $G = (V_1 \cup V_2, E)$ where $V_1 = \{1, 2, \ldots, D\}$ represents positions and $V_2 = \{(i,j) \mid i = 1, \ldots, n\, ; j = 1, \ldots, d_i\}$ represents the copies of the products. There exists an edge $\{k, (i, j)\} \in E$ if and only if k lies in the permissible interval $[E(i, j), L(i, j)] \subseteq V_1$ of release time and due date for the j-th copy of the product i. They prove the following (see also [1]).

Lemma 1 *Let $d_1, d_2, \ldots, d_n$ be any instance of min-max-absolute problem. A sequence $s = s_1 s_2 \ldots s_D$, is B-feasible if and only if for all $i = 1, \ldots, n$ and $j = 1, \ldots d_i$, this sequence assigns the copy (i, j) to the interval $[E(i, j), L(i, j)]$, where*

$$E(i,j) = \lceil \frac{j-B}{r_i} \rceil \quad \text{and} \quad L(i,j) = \lfloor \frac{j-1+B}{r_i} + 1 \rfloor$$

denote the release date and the due date of the copy (i, j) for given upper bound B.

Derivation of similar closed formulas for other measures of deviations remains open. Amongst various versions of the earliest due date algorithms for scheduling unit time jobs with release times and due dates on a single machine, they apply a modified version of GLOVER's [6]$O(|E|)$ Earliest Due Date algorithm for finding a maximum matching in a V_1-convex bipartite graph $G = (V_1 \cup V_2, E)$ such that each ascending $k \in V_1$ is matched to the unmatched copy (i, j) with smallest due date value of $L(i, j)$. They prove the following result.

Theorem 1 *The min-max-absolute sequence s is 1-feasible if and only if the graph G with bound B = 1 has a perfect matching. Moreover, an optimal solution can be determined by a exact pseudo-polynomial algorithm with complexity O (D log D).*

However, $1-r_{\max}$ is the tight lower bound for the min-max-absolute problem.

A binary search finds an optimal solution for the weighted min-max-absolute problem in $O(D \log(D\phi G_{\max}))$ time, where ϕ is a positive integer constant depending upon problem data [14]. They show in this case that the maximum weight $G_{\max} = \max_i G_i(1-r_i)$ gives an upper bound and $LB_W = \min_i G_i(1-r_i)$ gives a lower bound for the optimal objective value.

Note that both release dates and due dates are non-decreasing functions of j for a fixed i. But, they are non-increasing functions of d_i for a fixed j, on the other hand. As $r_{i_0} = r_{i_1}$ for any models i_0 and i_1 with equal product rates $d_{i_0} = d_{i_1}$, the equal quantity products i with densities $\frac{d_i}{D}$ always do competition for their release times and due dates. Moreover, the corresponding positions k for the copy (i, j) are interchangeable in any feasible sequences s.

For any instance $d_i, i = 1, 2, \ldots, n (n \geq 2)$ of the min-max-absolute problem, the optimal value B^* satisfies the inequality $B^* \leq 1 - \max\left\{\frac{1}{D}, \frac{1}{2(n-1)}\right\}$, [1, 16]. For $n \geq 3$, an instance with gcd $(d_1, d_2, \ldots, d_n) = 1$ of this problem has optimal value $B^* = \frac{2^{n-1}-1}{2^n-1} < \frac{1}{2}$ if and only if $d_i = 2^{i-1}$ for $i = 1, 2, \ldots, n$, [1,2,11]. In the case of two products, $B^* < \frac{1}{2}$ if and only if one of the demands is even and the other is odd [1,11].

The existence of cyclic sequences reduces the computational time. The set of optimal sequences for min-sum problem includes *cyclic* sequences [10]. Similar result holds for min-max-absolute problem [14]. We refer to [4] for more discussion on this issue.

For the sake of completeness we mention that the min-sum problem has been efficiently solved by reducing it to the well-known assignment problem [9]. The results in [3,8] either refute or establish relations between different objective functions (see also [4]).

3. JIT Sequencing with Input Sequences

Here we extend the formulation of single-level JIT sequencing problem under a number of chain constraints as follows. We denote the following problem by *JIT-Chain*, (see also DHAMALA / KUBIAK [5]). Let

$$
\begin{aligned}
u(n_1, D_1) &= u(n_1, D_1)_1\, u(n_1, D_1)_2 \quad \ldots \quad u(n_1, D_1)_{D_1} \\
u(n_2, D_2) &= u(n_2, D_2)_1\, u(n_2, D_2)_2 \quad \ldots \quad u(n_2, D_2)_{D_2} \\
&\vdots \\
u(n_t, D_t) &= u(n_t, D_t)_1\, u(n_t, D_t)_2 \quad \ldots \quad u(n_t, D_t)_{D_t} \\
&\vdots \\
u(n_m, D_m) &= u(n_m, D_m)_1\, u(n_m, D_m)_2 \quad \ldots \quad u(n_m, D_m)_{D_m}
\end{aligned}
$$

be $B_1, B_2, \ldots, B_m$-feasible sequences of lengths $D_1, D_2, \ldots, D_m$, where $D_t = \sum_{i=1}^{n_t} d_i^t$, of given any model sets n_t, $t = 1, 2, \ldots, m$, respectively. These sequences represent as $chain_1$, $chain_2$, . . . , $chain_t$, . . . , $chain_m$, respectively. More than a single chain may contain the same type of product models. We call it by *overlapping system*. Here, we consider the problem with *non-overlapping system*.

In this paper we extend the previous results and obtain a B-feasible sequence $s = s_1 s_2 \ldots s_D$, where $D = \sum_{t=1}^{m} D_t$ for min-max-absolute problem such that the restricted mappings satisfy $s_{|u(n_t,D_t)} : s \to u(n_t,D_t)$ for all $t = 1,2, \ldots m$ and has the least maximum deviation, i.e., $F(s) \le F(\bar{s})$ for any sequence $\bar{s} = s_1 s_2 \ldots s_D$ satisfying $\bar{s}_{|s(n_t,D_t)} : \bar{s} \to u(n_t,D_t)$

The restriction $s_{|u(n_t,D_t)}$ of the super sequence s to any given subsequence $u(n_t,D_t)$, $t = 1, 2, \ldots, m$, yields the sequence $u(n_t,D_t)$. Therefore, the super sequence s that contains $u(n_t,D_t)$ as its subsequence is order preserving with respect to the m-chain constraints $u(n_t,D_t)$ as its subsequence is order preserving with respect to the m-chain constraints $u(n_t,D_t)_l \prec u(n_t,D_t)_{l'}$, if $l < l'$ for all $l = 1, 2, \ldots, D_t$ and $t = 1, 2, \ldots m$. We call such a sequence by order-persevering super sequence. By construction each subsequence represents a chain and there exist at most D constraints all together in these chains.

4. An Efficient Scheduling Algorithm

To study the single-level min-max problem with chain constraints, let us consider the collective demand rates of all together $n = \sum_{t=1}^{m} n_t$ models with total demand $D = \sum_{i=1}^{m} D_t$ in the union of all chains. Then for given bound B, we calculate the permissible intervals of time-windows $[E(i, j), L(i, j)]$, where $i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, d_i$, (see Lemma 1) using the known algorithm of STEINER / YEOMANS [15]. But our chain constraints imposed in min-max-absolute sequencing problem are not included for calculating these time windows yet. Therefore, it is straight forward that these time windows must be feasible without chin constraints (see Theorem 1) for this data set.

To ensure that the given bound B which is a target variable value for the objective function is feasible for the super sequence to be delivered, a further test is required. We reduce min-max-absolute-chain sequencing problem to a single machine scheduling decision problem with release times, due dates and chain constraints. Given any bound B for min-max-absolute-chain problem, we ask does there exists a feasible solution of the single processor scheduling problem $1 \mid r_i, chain \mid L_{\max}$ with $L_{\max} \leq 0$?

Here considered the problem $1 \mid r_i, chain \mid L_{\max}$, the time windows are represented by the intervals $[r_i, d_i] = [E(i, j), L(i, j)]$ calculated as a function of the given bound B. The chain constraints are given by the subsequences $\cup_{t=1}^{m} \{u(n_t, D_t)_{l=1}^{D_t}\}$ that may be represented by the following graph. Define a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with the vertex set $\mathcal{V} = \cup_{t=1}^{m} \{u(n_t, D_t)_{l=1}^{D_t}\}$. There exists an arc in $\mathcal{E}$ from $u(n_t, D_t)_k$ to $u(n_t, D_t)_{k'}$ if the precedence relation $u(n_t, D_t)_k \prec u(n_t, D_t)_{k'}$ is satisfied.

HORN[7] formulated $O(n \log n)$ time algorithm to the single machine scheduling problem $1 \mid r_i, chain \mid L_{\max}$. His rule, also called earliest due date (EDD), at any time schedules an available job with the smallest due date. For implementing this rule to $1 \mid r_i, chain \mid L_{\max}$ one needs to modify the due dates. In this modification, if job k is the immediate predecessor of job l in any *chain* and $d'_k = d_l - 1 < d_k$, denoted by $k \to l$, then the due date d_k has to be replaced by the modified due date d'_k. A proof on the validity of optimality on $L_{\max}$ objective makes the use of interchange arguments.

Following algorithm is proposed for the mim-max-absolute-chain sequencing problem, (see also DHAMALA/KUBIAK [5]).

Algorithm 1 *min-max-absolute-chain-algorithm*

Given: d_i^t *for* $i = 1, 2, \ldots, n_t$ *and* $t = 1, 2, \ldots, m$;
an upper bound B for min - max - absolute - chain-problem;
$Chain_1, Chain_2, \ldots, Chain_t, \ldots, Chain_m$;
Update:
number of demands $n = \sum_{t=1}^{m} n_t$;
demand rates d_i *for* $i = 1, 2, \ldots, n$;
total demand $D = \sum_{i=1}^{n} d_i$.

Step 1. *Calculate windows* $[E(i, j), L(i, j)]$ *for* $j = 1, \ldots, d_i$ and $i = 1, \ldots, n$ *by* STEINER / YEOMANS [15].

Step 2. *Modify the due dates* $L(i, j)$;
If $(i, j) \to (i', j')$, *then* $L(i, j) := \min \{L(i, j), L(i', j') - 1\}$.

Step 3. *Schedule the jobs by EDD-Algorithm of* HORN [7].

Output : *B feasible for* (n,D) *if* $L_{\max} \le 0$.

As the first two steps require $O(D)$ time and the Step 3 costs $O(D \log D)$, the overall time complexity of the min-max-absolute-chain-algorithm is $O(D \log D)$. An EDD algorithm in Step 3 applied to modified due dates by Step 2 is called modified EDD algorithm.

Following theorem proves the correctness of Algorithm 1, (see also DHAMALA/ KUBIAK [5]).

Theorem 2 *Let B be a target value for the objective function of min-max-absolute-chain sequencing problem. Then, if the modified EDD algorithm finds an optimal solution with* $L_{\max} \le 0$, *then min-max-absolute-chain-algorithm finds a B-feasible solution to min-max-absolute-chain sequencing problem.*

Proof: Suppose $s = s_1\ s_2\ \ldots\ s_D$ be a sequence obtained by min-max-absolute-chain-algorithm such that $L_{\max} \le 0$. That is, each job k=1, 2, . . . , D is scheduled in the proper window and none of the job is delayed. If s is infeasible to min-max-absolute-chain sequencing problem, then $|x_{ik} - x_i k| > B$ for some product copy (i, j) with k=1,2, . . . , D and i = 1, 2, . . . , n. But this is impossible by the constriction of time windows. □

If the first copy $(i,1)$ of the product i has to be completed at position k, then it holds $|x_{ik} - kr_i| = 1 - r_i$. Therefore, the sharp lower bound $1 - r_{\max}$ on a target value B is still valid. An optimal solution to the min-max-absolute-chain problem has to be determined by applying binary search of the target value B in the interval $[1 - r_{\max}, B]$.

One way to give an upper bound to the obtained super sequence is to put given sequences $\bigcup_{t=1}^{m} \{u(n_t, D_t)_{l=1}^{D_t}\}$ one after another and then calculate

$$B = \max_{i,k} \{\, |x_{ik} - kr_i| : i = 1, 2, \ldots, n \text{ and } k = 1, 2, \ldots, D\}.$$

To calculate an upper bound on the target value B of the super sequence s, we study also the properties of batch sequences. A *batch* w is a factor of sequence s consisting of the same product copies which cannot be extended either to the left or to the right by the same product type copy. The cardinality $|w|$ is the *batch size* of the batch w in s. Clearly, longer batches reduces the number of setups provided sufficiently long buffer size.Given an instance (n,D), we consider a batch sequence s with exactly n-batches, say $s = \sigma_{i_1}\ \sigma_{i_2}\ \ldots\ \sigma_{i_n}$ where σ_{i_t} represents a batch with respect to the product type i_t, $t = 1, 2, \ldots, n$.

Lemma 2. Let $s = \sigma_{i_1}\ \sigma_{i_2}\ \ldots\ \sigma_{i_n}$ *be a sequence with batches* σi_t, *for* $t = 1, 2, \ldots, n$. *Then an upper bound on the target value of s is* $d_{\max}(1 - r_{\max})$.

Proof: For any product model $i = 1, 2, \ldots, n$ with demand d_i and $j = 1, 2, \ldots, d_i$, we have $|j-kr_i| = \begin{cases} j-kr_i & if\ j \geq kr_i \\ kr_i - j & if\ j \leq kr_i \end{cases}$. As $d_i - d_i r_i \geq d_i - (d_i + k') r_i$ for any model i and any positive integer k', we have $|j - kr_i| \leq d_i - d_i r_i$ for all i and j. As $|j - kr_i| \leq d_i(1-r_i)$ for each (i, j), it follows that the deviation with respect to product i is not more than $d_i(1-r_i)$. Then the statement follows by considering the batch of maximum length $d_{\max}$. □

Corollary 1 *An upper bound on the target value of the super sequence s obtained by min-max-absolute-chain-algorithm is* $d_{\max}(1 - r_{\max})$. *Moreover, the tight lower bound is* $1 - r_{\max}$.

The bound $\max_{i,k}\{|x_{i,k} - kr_i| : i = 1, \ldots, n \text{ and } k = 1, \ldots, D\}$ obtained making use of the super sequence improves the bound $d_{\max}(1-r_{\max})$ obtained by batch sequences. However, the latter yields an explicit bound of the super sequence. DHAMALA/ KUBIAK [5] prove the following result.

Theorem 3 *An optimal solution to the min-max-absolute-chain problem can be determined testing at most* $O(D\, d_{\max}(1-r_{\max}))$ *sequences each with time complexity* $O(D \log D)$.

Proof: An optimal solution to the min-max-absolute-chain problem can be determined by applying binary search in the interval $[1-r_{\max}, d_{\max}(1-r_{\max})]$. But a feasibility test requires $O(D \log D)$ time. □

As the HORN's [7] algorithm works for the problem $1|r_i, prec|L_{\max}$, our approach is applicable to the min-max-absolute problem with precedence constraints as well. The time complexity of the algorithm does not increase.

Given two sequences $u(3,11) = bcbbcebcbbc$ and $u(2,9) = aadaaaada$, the super sequence $s =$ **a***bc***a***b***da***bc***a***e***ba***c***b***a***d***a***d***b***c***a** preservers the orders of the subsequences: $s_{|u(3,11)} = bcbbcebcbbc$ and $s_{|u(2,9)} = aadaaaada$. Moreover, the obtained super sequence s is optimal as $B = 1 - r_{\max} = 1 - \frac{7}{20} = 0.65$ is tight. Note that first subsequence of the input subsequences is not optimal: $|3-4 \times \frac{6}{11}| = \frac{9}{11} > \frac{6}{11}$ for the 3rd copy of product b. But the sequence $u(3,11) = bcbcbebcbbc$ is optimal with upper bound $B = \frac{6}{11}$.

Consider an example, on the other hand, to illustrate that not each optimal sequence necessarily preserves the order of the input subsequences. Given two EDD-optimal subsequences $u(2,3) = bab$ and $u(2,3) = dcd$, the super sequence $s = d$**ab***c***b***d* is EDD-optimal but does not preserve the order of the subsequence as $s_{|u(2,3)} = abb \neq bab$. The tight lower bound of the super sequence s is $B = 1 - r_{\max} = 1 - \frac{2}{6} = \frac{2}{3}$. The EDD-optimal super sequence $s = d$**ba***c***b***d* is order preserving.

Following two lemmas illustrate some properties.

Lemma 3. *Any* EDD-*optimal sequence* $s = s_1 s_2 \ldots s_D$ *with total demand* D *satisfies the following properties for a target value in* $[1 - r_{\max}, 1]$

a. *Only the first* (*last*) *copy of a product may start at* $q = 0$ ($q = D - 1$)

b. $E(i, j) < E(i, j+1)$ *and* $L(i, j) < L(i, j+1)$ *for every* j and i.

Lemma 4. *Let* D_t *be total demand of any subsequence u of a super sequence s of total demand D for any floor or ceiling function H on nonnegative numbers, it holds* $H(xD_t) \leq H(yD_t)$ *if and only if* $H(xD) \leq H(yD)$.

Because of which we conjecture that given any number of optimal sequences for min-max-absolute problem with $B = 1$, there exists an optimal super sequence with $B = 1$ which preserves the order of the subsequences.

5. Concluding Remarks

In this paper, we assumed a number of non-overlapping sequences in the mixed-model production systems. With respect to those input sequences as constraints, we presented an efficient algorithm which finds an optimum solution (sequence) to the maximum deviation JIT sequencing problem. Our result is carried out based on the reduction of JIT sequencing problem to a single machine scheduling problem. Earlier results of HORN [7] and STEINER/YEOMANS [15] were applicable for solving our problem. It is open whether the min-sum problem with such constraints and/or min-max problem with overlapping sequences as constraints are efficiently solvable.

REFERENCES

[1] Brauner, N., and Crama, Y., *The maximum deviation just-in-time scheduling problem;* Discrete Applied Mathematics **134** (2004) 25–50.

[2] Brauner, N., Jost, V., and Kubiak,W., *On symmetric Fraenkel's and small deviations conjecture*, Les cahiers du Laboratoire Leibniz-IMAG, No. 54, Grenoble, France (2004).

[3] Corominas, A., and Moreno, N., *On the relations between optimal solutions for different types of min-sum balanced JIT optimization problems*, INFOR **41** (2003) 333–339.

[4] Dhamala, T.N., Kubiak, W., *A brief survey of just-in-time sequencing for mixed-model systems*, International Journal of Operations Research, submitted.

[5] Dhamala, T.N., Kubiak, W., *Optimal just-in-time sequences for mixed-model multi-level production*, Working Paper, Memorial University of Newfoundland, Canada, (2005).

[6] Glover, F., *Maximum matching in a convex bipartite graph*, Naval Research Logistics Quarterely **4**(1967) 313–316.

[7] Horn, W.A., *Some simple scheduling algorithms*, Naval Research Logistics Quarterely **21** (1974) 177–185.

[8] Kovalyov, M.Y., Kubiak, W., and Yeomans, J. S., *A computational study of balanced JIT optimization algorithms*, Information Processing and Operational Research **39** (2001) 299–316.

[9] Kubiak W., and Sethi, S. P., *Optimal just-in-times schedules for flexible tranfer lines*, International Journal of Flexible Manufacturing Systems **6**(1994) 137–154.

[10] Kubiak, W., *Cyclic just-in-time sequences are optimal*, Journal of Global Optimization **27** (2003) 333–347.

[11] Kubiak, W., On small deviations conjecture, Bulletin of the Polish Academy of Sciences **51** (2003) 189–203.

[12] Miltenburg, J., *Level schedules for mixed-model assembly lines in just-in-time production systems*, Management Science **35** (1989) 192–207.

[13] Monden, Y., *Toyota production systems* (Industrial Engineering and Management Press, Norcross, GA, 1983).

[14] Steiner, G., and Yeomans, S., *Optimal level schedules in mixed-model multi-level JIT assembly systems with pegging*, European Journal of Operational Research **95** (1996), 38–52.

[15] Steiner, G., and Yeomans, S., *Level schedules for just in time production processes*, Management Science **39** (1993), 728–735.

[16] Tijdeman, R., *The chairman assignment problem*, Discrete Mathematics **32** (1980) 323–330.

TANKA NATH DHAMALA
Central Department of Mathematics
Institute of Science and Technology
Tribhuvan University,
Kathmandu, Nepal.

The Nepali Math. Sci. Report
Vol. 24. No.1. 2005.

Invariant and non invariant hypersurfaces of almost Lorentzian para contact manifolds

HARSIMRAN GILL AND K. K. DUBE

Goldberg, S. I. and Yano studied and defined Noninvariant Hypersurfaces of almost contact manifolds and has become subject of sufficient interest and Sato (1976) studied about a structure similar to almost contact structure. In present paper our aim is to study Invariant and Noninvariant Hypersurfaces of almost Lorentzian Para contact manifolds.

Introduction. Let V_n be an n-dimensional differentiable manifold endowed with a tensor field ϕ of type (1,1) a vector field U and a 1-form u such that

(1.1) $$\phi^2 = 1 + u \otimes U,\ u(U) = -1,\ \phi\ U = 0$$
$$u \circ \phi = 0, \quad \text{rank}\ \phi = n-1.$$

Then V_n is said to have an almost Lorentzian Para contact structure. If in V_n there exist a Remannian metric g such that

(1.2) $$u(X) = g(X,U),$$
$$g(\phi X, \phi Y) = g(X,Y) + u(X)\,u(Y),$$

Then V_n is said to have an almost Lorentzian Para contact metric structure [4]. We say that the almost Lorentzian Para contact structure is normal if

(1.3) $$[\phi,\phi] - U \otimes du = 0.$$

Where $[\phi,\phi]$ is the Nijenhuis torsion of ϕ.

An almost Lorentzian Para contact metric structure is said to be Lorentzian Para-Sasakian,

if

(1.4) $$(D_X\,\phi)(Y) = u(Y)X + 2\,u(X)u(Y)\,U + g(X,Y)U,$$

where D denotes the Riemannian connexion of g([1],[2]). An almost Lorentzian Para contact metric manifold is said to be a closed almost Lorentzian Para contact metric manifold if u is closed. Further if,

(1.5) $$D_X\,U = \phi X.$$

Then it is called a Lorentzian Para contact metric manifold ([2], [4]). It is observed that a Lorentzian Para-Sasakian manifold is a Lorentzian Para contact metric manifold.

An almost product manifold is a differentiable manifold which has a (1,1) tensor field J satisfying the condition,

(1.6) $$J^2 = I.$$

Moreover if there exist a Riemannian metric g such that,

(1.7) $$g(JX, JY) = g(X, Y),$$

then it is called an almost product metric manifold. Let D be the Riemannian connexion of g. then the manifold is said to be an almost product almost decomposable manifold if,

(1.8) $$(D_X, J)(Y) = 0$$

Consider an almost Lorentzian Para contact manifold V_n and let V_m be an orientable hypersurface of V_n, and B the differential of the immersion I of V_m into V_n. Let X, Y and Z be tangent to V_m and C a unit normal vector.

Then we have

(1.9) $$\phi BX = BFX + \alpha(X)\, C.$$

Where F is a (1,1) tensor field, and α a 1 - form on V_m. If $\alpha \neq 0$, then V_m is called a non-invariant hypersurface of V_n. If α is identically zero, then V_m is said to be an invariant hypersurface, that is, the tangent space of V_m is invariant under ϕ [3].

The metric g of an almost Lorentzian Para contact metric manifold induces a Riemannian metric G on the Hypersurface V_m given by,

(1.10) $$G(X,Y) = g(BX, BY).$$

Further the symmetric affine connexion D on V_n induces a symmetric affine connexion $\overline{D}$ on the hypersurface V_m such that,

(1.11) $$D_{BX}\, C = B(\overline{D}_X\, Y) + h(X,Y)C,$$

where h is a symmetric tensor of type (0,2) called the second fundamental form of the hypersurface V_m. We have,

(1.12) $$D_{BX}\, C = -BHX + W(X)C,$$

where W is a 1 - form on V_m defining the connexion an affine normal bundle and H is a (1,1) tensor field on V_m such that $g(HX, Y) = h(X, Y)$.

2. Noninvariant Hypersurfaces of Almost Lorentzian Para Contact Manifolds.

Let V_n be an almost Lorentzian Para Contact Manifold with the structure tensors (ϕ, U, u), and V_m a non-invariant hypersurface of V_n. In what follows we assume that U is nowhere tangent to V_m and so we can take $C = U$, then (1.9) takes the form,

(2.1) $$\phi BX = BFX + \alpha(X)U.$$

Theorem 1. *If V_m is a noninvariant hypersurface of an almost Lorentzian Para Contact manifold V_n with U nowhere tangent to V_m, then V_m admits an almost product structure.*

Proof : Applying ϕ to both sides of (2.1), we get

$$\phi^2 BX = \phi\ BFX + \phi(\alpha(X)\ Up).$$

From (1.1) & (2.1) we have

$$BX + u(BX)U = BF^2X + \alpha(FX)U.$$

Now equating the co-efficient of above equation we have,

$$F^2X = X, \tag{2.2}$$

$$u(BX) = \alpha\ (FX). \tag{2.3}$$

Thus F acts as an almost product structure on V_m.

Theorem 2. *If V_m is a noninvariant hypersurface of an almost Lorentzian Para contact metric manifold V_n (ϕ, U, u, g), then V_m is an almost product metric manifold.*

Proof : From Theorem (1) it follows that V_m has an almost product structure F. Let G be the induced metric in V_m, that is,

$$g(BX, BY) = G(X,Y)$$

Now we define a metric on V_m by

$$G^*(X,Y) = G(X,Y) - \alpha(X)\alpha(Y).$$

Then we have,

$$G^*(FX,FY) = G(FX,FY) - \alpha(FX)\ \alpha(FY).$$

Applying the condition of equation (1.10), (2.1) & (2.3) in above equation we have,

$$G^*(FX,FY) = g(BFX,\ BFY) - u(BX)\ u(BY),$$

$$= g(\phi\ BX - \alpha(X)U, \phi\ BY - \alpha(Y)U) - u(BX)\ u(BY),$$

$$= g(\phi\ BX, \phi\ BY) - \alpha(X)\alpha(Y) - u(BX)\ u(BY)$$

Applying the condition of equation (1.2) in above equation

$$= g\ (BX,BY) + u\ (BX)\ u\ (BY) - \alpha\ (X)\ \alpha\ (Y) - u\ (BX)\ u\ (BY),$$

$$= g\ (BX,BY) - \alpha\ (X)\alpha\ (Y),$$

$$= G^*\ (X,Y).$$

Hence G^* is the metric which makes V_m an almost product metric manifold.

Theorem 3. *Let V_m be a noninvariant hypersurface of Lorentzian Para contact metric manifold V_n then,*

$$\text{(a) } F = -H, \quad \text{(b) } \alpha = W \tag{2.4}$$

Proof : Since V_n is a Lorentzian Para contact metric manifold, we have,

$$D_{BX}\ U_P = \phi\ BX.$$

Using (1.12) and (2.1) we have

$$-BHX + w(X)U = BFX + \alpha\ (X)U.$$

Which gives

$$F = -H,$$
$$\& \quad \alpha = W.$$

Theorem 4. *If V_m is a noninvariant hypersurface of a Lorentzian Para-Sasakian manifold V_n then,*

$$(\overline{D}_X F)(Y) = \alpha(FY)X - \alpha(Y)FX,$$
$$g(BX,BY) = h(X,FY) + (\overline{D}_X \alpha)(Y) + \alpha(X)\alpha(Y) - 2\alpha(FX)\alpha(FY)$$

Proof : We know that

$$(D_{BX}\phi)(BY) = D_{BX}\phi BY - \phi(D_{BX} BY).$$

Using equation (1.4), (1.9) & (1.11) in the above equation we get,

$$U(BY)BX + 2u(BX)u(BY)U + g(BX,BY)U = D_{BX}(BFY + \alpha(Y)U) - \phi(B\overline{D}_X Y + h(X,Y)U)$$
$$= D_{BX} BFY + D_{BX}\alpha(Y)U - \phi BD_X Y,$$
$$= B(\overline{D}_X FY) + h(X,FY)U + (D_{BX}\alpha(Y)) + \alpha(Y)D_{BX}U - BF\overline{D}_X Y - (\overline{D}_X Y)U,$$
$$= B(\overline{D}_X F)(Y) + \{h(X,FY) + D_{BX}\alpha(Y) - \alpha(\overline{D}_X Y)\}U + \alpha(Y)D_{BX}U.$$

In consequence of equation (1.5) we have,

$$= B(\overline{D}_X F)(Y) + \{h(X,FY) + D_{BX}\alpha(Y) - \alpha(\overline{D}_X Y)\}U + \alpha(Y)\phi BX,$$
$$= B(\overline{D}_X F)(Y) + \{h(X,FY) + D_{BX}\alpha(Y) - \alpha(\overline{D}_X Y) + \alpha(Y)\alpha(X)\}U + \alpha(Y)BFX,$$

equating the components we get

$$(\overline{D}_X F)(Y) = -\alpha(Y)FX + u(BY)X,$$

and

$$h(X,FY) + (\overline{D}_X \alpha)(Y) + \alpha(X)\alpha(Y) = 2u(BX)u(BY) + g(BX,BY).$$

From equation (2.3) we have

$$(\overline{D}_X F)(Y) = \alpha(FY)X - \alpha(Y)FX,$$
$$h(X,FY) + (\overline{D}_X \alpha)(Y) + \alpha(X)\alpha(Y) = 2\alpha(FX)\alpha(FY) + g(BX,BY).$$

As an immediate consequence we have the following:

COROLLARY : Let V_m be a noninvariant hypersurface of Lorentzian Para-Sasakian manifold V_n with the induced almost product structure F. Then V_m is an almost product almost decomposable manifold if and only if

$$\alpha(Y)FX = \alpha(FY)X.$$

3. Invariant hypersurface of almost Lorentzian paracontact manifolds.

Let $V_n(\phi, U, u)$ be an almost Lorentzian Para contact manifold and let V_m be an invariant hypersurface of V_n. Then, equation (1.9) becomes

$$\phi BX = BFX.$$

In what follows we study the invariant hypersurface with the following conditions:

(a) When U is nowhere tangent to V_m.
(b) When U is everywhere tangent to V_m.

When U is nowhere tangent to V_m.

Theorem 5. *Let V_m be an invariant hypersurface of an almost Lorentzian Para contact manifold V_n. Then V_m is an almost product manifold with $u(BX) = 0$.*

Proof : The proof follows from theorem (1), for invariant hypersurface $\alpha = 0$, then we will get

(3.1) $$u(BX) = 0.$$

Theorem 6. *Let V_m be an invariant hypersurface of an almost Lorentzian Para contact manifold V_n. If V_n is normal then the almost product structure induced on V_m is integrable.*

Proof : We know that,

$$[\phi,\phi](BX,BY) = \phi^2[BX,BY] + [\phi BX,\phi BY] - \phi[\phi BX,BY] - \phi[BX,\phi BY].$$

Using equation (3.1) and $B[X,Y] = [BX, BY]$ in above equation we have

$$[\phi,\phi](BX,BY) = BF^2[X,Y] + B[FX,FY] - BF[FX,Y] - BF[X,FY],$$
$$= B[F,F](X,Y),$$

further we have

$$du(BX,BY) = BX.\,u(BY) - BYu(BX) - u(B[X,Y])$$
$$= 0.$$

Thus we can write,

$$[\phi,\phi](BX,BY) - du(BX,BY)U = B[F,F](X,Y).$$

Hence the theorem is proved.

Theorem 7. *An invariant hypersurface V_m of a Lorentzian Para-Sasakian manifold V_n is an almost product almost decomposable manifold.*

Proof : From theorem (5) it follows that V_m is an almost product manifold. Further, theorem -3 gives that it is metric also. Now from (1.11) we have,

$$D_{BX}BFY = B\overline{D}_X FY + h(X,FY)U,$$
$$= B[(\overline{D}_X F)(Y) + F\overline{D}_X Y] + h(X,FY)U,$$
$$D_{BX}BFY = B(\overline{D}_X F)(Y) + BF\overline{D}_X Y + h(X,FY)U,$$

(3.2) $$D_{BX}BFY - BF\,\overline{D}_X Y = B(\overline{D}_X F)(Y) + h(X,FY)\,U,$$

From equation (1.4) we have

$$(D_X\phi)(Y) = u(Y)X + 2u(X)\,u(Y) + g(X,Y)\,U,$$

$$(D_{BX}\phi)(BY) = u(BY)BX + 2u(BX)\,u(BY) + g(BX,BY)U,$$
$$(D_{BX}\phi)(BY) = g(BX, BY)\,U,$$
$$D_{BX}\,\phi\,BY - \phi D_{BX}\,\phi\,BY = g(BX,BY)U,$$
$$D_{BX}\phi\,BFY - \phi[B\overline{D}_X Y + h(X,Y)U] = g(BX,BY)U,$$
$$D_{BX}\,\phi BFY - \phi\,B\overline{D}_X Y = g(BX,BY)\,U,$$
$$D_{BX}\,\phi\,BFY - BF\,\overline{D}_X Y = g(BX,BY)U.$$

From equation (3.2) we have,

$$g(BX,BY)U = B((\overline{D}_X F)(Y)) + h(X, FY)U.$$

Hence

$$(\overline{D}_X F)(Y) = 0.$$

$$g(BX, BY) = h(X,FY).$$

Which completes the proof.

When U is everywhere tangent to V_m.

Theorem 8. *Let V_m be an invariant hypersurface of an almost Lorentzian Para contact manifold V_n. Then V_m is almost Lorentzian Para contact manifold. Further, if V_n is normal then V_m is normal.*

Proof : Since U is everywhere tangent to V_m then in a unique vector field U^* such that

$$BU^* = U. \text{ Set}$$

$$u^*(BX) = u(BX)$$

Then u^* is a 1-form on V_m. Further, we have,

$$BFX = \phi\ BX,$$

Which implies

$$BF^2 X = \phi^2 BX = BX + u^*(X)\,BU^*,$$

That is $$F^2 X = X + u^*(X)U^*.$$

Also $$u^*(FX) = u(B\,FX) = u(\phi\,BX) = 0,$$

$$u^*(U^*) = u(BU^*) = u(U) = -1,$$

and

$$BF(U^*) = \phi\,BU^* = \phi\,U = 0.$$

Which gives that,

$$F(U^*) = 0.$$

Thus V_m be is an almost Lorentzian Para contact manifold with the structure tensors (F, U^*, u^*).

Finally we have,

$$N(BX, BY) = [\phi,\phi](BX, BY) - du(BX, BY)U,$$

$$= \phi^2 B[X,Y]+[\phi BX,\phi BY]-\phi[BX,\phi BY]-\phi[\phi BX,BY]$$
$$-\{BX.u(BY)+BY.u(BX)+u(B[X,Y])U\},$$
$$= BF^2[X,Y]+[BFX,BFY]-BF[X,FY]-BF[FX,Y]$$
$$-\{BX.u^*(Y)+B.Y.u^*(X)+U^*([X,Y]\}BU^*,$$
$$= B\{[F,F](X,Y)-du^*(X,Y)U^*\}.$$

Hence, if V_n normal, then V_m. is also normal.

Theorem 9. *If V_m is an invariant hypersurface of a Lorentzian Para contact metric manifold V_n. Then V_m is also Lorentzian Para contact metric manifold.*

Proof : From theorem 8 it follows that V_m is an almost Lorentzian Para contact manifold with structure tensors (F,U^*,u^*). Let g^* be the induced metric on V_m. Then we have,

$$g^*(FX,FY)=g(BFX,BFY)=g(\phi BX,\phi BY)$$

Further we easily show that u^* is closed. Finally, since V_n is a Lorentzian Para contact metric manifold, we have-

$$D_{BX}U=\phi BX,$$

that is

$$D_{BX}BU^*=BFX,$$

Which is in consequence of (1.11) becomes,

$$B\overline{D}_X U^*+h(X,U^*)C=BFX,$$

Where C is normal to V_n.
Equating the components of above equation we have,

$$\overline{D}_X U^*=FX,$$
$$\&\ h(X,U^*)=0$$

Which completes the proof.

Theorem 10. *Let V_m is an invariant hypersurface of a Lorentzian Para Sasakian manifold V_n. Then V_m is a Lorentzian Para Sasakian manifold.*

Proof : we have proved that V_m is a Lorentzian Para contact manifold, Now we have,

$$B(\overline{D}_X F)(Y)=B\overline{D}_X FY-BF(\overline{D}_X Y).$$

Which is in consequence of equation (1.11) and (3.1) becomes,

$$B(\overline{D}_X F)(Y)=D_{BX}BFY-h(X,FY)C-\phi(B\overline{D}_X Y),$$
$$=D_{BX}\phi BY-\phi(D_{BX}BY-h(X,Y)C)-h(X,FY)C,$$
$$=(D_{BX}\phi)BY+\phi(h(X,Y)C)-h(X,FY).C.$$

Using equation (1.4) in above equation we have,

$$B(\overline{D}_X F)(Y)=u(BY)BX+2u(BX)u(BY)U$$
$$+g(BX,BY)U+\phi(h(X,Y)C)-h(X,FY)C,$$

$$= B[u^*(Y)X + 2u^*(X)\, u^*(Y)U^* + g^*(X,Y)U^*] + h(X,Y)\phi C - h(X, FY)\, C.$$

Since C and U are linearly independent unit vectors, C can be thought of as eigen vector of ϕ corresponding to eigen + or – and $\phi C = \pm$ C.
Equating the tangential and normal components we have

$$(\overline{D}_X F)(Y) = u^*(Y)\, X + 2u^*(X)\, \dot{u}^*(Y)U^* + g^*(X,Y)U^*.$$

$$h(X,Y)\phi C = h(X,FY)\, C.$$

Hence V_m is Lorentzian Para Sasakian manifold, Now we have.

REFERENCES

[1] Goldberg, S.I. and Yano, K. (1970). *Non-invariant Hypersurfaces of almost contact manifolds*, J. Math. Soc. Japan 22, 25–34.

[2] Indu Kala Mishra and Ram Hit. Ojha (2000). *On Lorentzian Para Contact manifolds*, Bull. Cal. Math. Soc. 92 (5) 357–360.

[3] Latika Bhatt, N. K. Joshi and K. K. Dube (1999). *Invariant Hypersurfaces of Kenmotsu manifolds*, Acta Cinca Indica Vol xxv M, No 2. 183–186.

[4] Lakshmi Kant Pandey and Ram Hit Ojha (2001). *Semi-symmetric metric and non metric connections in Lorentzian Para contact manifolds*, Bull. Cal. Math. Soc. 93 (6) 497–504.

[5] Sato, I. (1976). *On a structure similar to the almost contact structure*, Tensor, N. S, 30, 203–219

[6] Sato, I. (1977). *On a structure similar to the almost contact structure*, Tensor, N. S, 31, 199–203.

[7] Sharfuddin, A and Husain S. I. (1977). *On almost Para Contact manifolds* Alig. Bull. Maths, (103–107).

[8] V. K. Jaiswal, R. H. Ojha and B. Prasad (2002). *A Semi symmetric nonmetric ϕ connection in a Lorentzian Para Sasakian manifolds*. Bull. Cal. Math. Soc. 94 (4) 287–290.

HARSIMRAN GILL AND K. K. DUBE
Department Of Mathematics
Statistics And Computer Science
Cbsh, G. B. Pant Univ. Of Ag. And Tech.
Pantnagar - 263145, Uttaranchal, India
E-Mail:- Simran_Gill@Rediffmail.Com.

The Nepali Math. Sc. Report
Vol. 24, No. 1, 2005.

Submanifold of Codimension p of a HSU-Structure Manifold

RAM NIVAS AND DHARMENDRA SINGH

Abstract : Hsu-structure manifolds have been defined and studied by Prof. Mishra [2]. Islam and others. The purpose of the present paper is to study the submanifolds of such a manifold. It has been shown that a submanifold of condimension p of such a manifold admits a para p-contact Hsu-metric structure. Certain other interesting results have also been proved.

1. Preliminaries

Let V_n be an n-dimensional differentiable manifold of class C^∞. Suppose there exists on V_n a tensor field $F(\neq 0)$ of type $(1,1)$ satisfying

$$F^2 = a^r I \tag{1.1}$$

where 'a' is any non-zero complex number and r a positive integer. Suppose further that V_n admits a hermite metric G satisfying

$$G(FX, FY) + a^r G(X,Y) = 0 \tag{1.2}$$

for arbitrary vector fields X and Y on V_n. Thus, in view of the equations (1.1) and (1.2) V_n will be said to possess a Hsu-metric structure.

Let $'F(X,Y)$ is the tensor field of type (1.2) given by

$$'F(X,Y) = G(FX,Y). \tag{1.3}$$

The following results can be proved easily

(1.4)

(i) $'F(FX,Y) = -'F(X,FY) = a^r G(X,Y)$

(ii) $'F(FX,FY) + a^r\, 'F(X,Y) = 0$ and

$'F(X,Y) + 'F(Y,X) = 0$

Let $\overline{D}$ be the Riemannian connection on V_n ; then

(1.5) $$\overline{D}_X Y - \overline{D}_Y Y = [X,Y] \qquad \overline{D}_X G = 0$$

Let $\overline{N}$ be the Nijenhuis tensor formed with F ; then

(1.6) $$\overline{N}(X,Y) = [FX,FY] - F[FX,Y] - F[X,FY] + F^2[X,Y]$$

A Hsu-structure manifold V_n will be called a HK-manifold if the structure tensor F is parallel i.e.

(1.7) $$(\overline{D}_X F)(Y) = 0.$$

A sub manifold V_{n-p} of condimesion p of the Hsu-structure manifold V_n will be said to possess a para p-contact Hsu-structure if there exists a tensor field f of type (1,1) p(c*) contravariant vector fields U p(c*) 1-forms $\overset{X}{u}$ (p some finite integer) satisfying

(1.8) $$f^2 = a^r I - \sum_{X=1}^{p} \overset{X}{u} \otimes \underset{X}{U}$$

Also,

$$\overset{Y}{u} \circ f + \sum_{X=1}^{p} \theta^{Y}_{X} \overset{X}{u} = 0 \qquad f\underset{X}{U} + \sum_{Y=1}^{p} \theta^{Y}_{X} \underset{X}{U} = 0$$

(1.9) $$\overset{Y}{u}(\underset{Y}{U}) + \sum_{Z=1}^{p} \theta^{X}_{Z}\theta^{Z}_{Y} = a^r \delta^{X}_{Y}$$

where X, Y = 1, 2, . . . p, δ^{X}_{Y} denotes the Kronecker delta and δ^{X}_{Y} are scalar fields.

If in addition, the submanifold V_{n-p} admits a Riemannian metric g satisfying

(1.10) $$g(fX, fY) + a^r g(X,Y) + \sum_{X=1}^{p} \overset{X}{u}(X)\,\overset{X}{u}(Y) = 0$$

we say that V_{n-p} admits a para p-contact Hsu-metric structure.

2. Submanifolds of Codimension p

Let V_{n-p} be the submanifold of codimension p of a Hsu-structure menifold V_n If B denotes the differential of the immersion $\tau : V_{n-p} \to V_n$ a vector field X in the tangent space of V_{n-p} determines a vector field BX in that of V_n . Let $\underset{X}{N}$, X = 1,2, . . . , p be p mutually orthogonal fields of unit normal vectors defined on V_{n-p}. Thus, we have

(2.1) $G(BX,BY)=g(X,Y) \quad G(BX,N)=0 \quad G(\underset{X}{N},\underset{Y}{N})=\delta_Y^X$

The vector fields FBX and $F\underset{X}{N}$ can be expressed by

(2.2)

(i) $FBX = B\,fX - \sum_{X=1}^{p} \overset{X}{u}(X)\underset{X}{N}$

(ii) $F\underset{X}{N} = -B\underset{X}{U} + \sum_{Y=1}^{p} \theta_X^Y \underset{X}{N}$

where f is a (1,1) tensor field, $\overset{X}{u}$ 1-forms and $\underset{X}{U}$ vector fields on the submanifold V_{n-p}. Operating by F on both the sides of (2.2)(i) and making use of equation (1,1) and (2.2), we obtain

$$a^r BX = Bf^2X - \sum_{Y=1}^{p} \overset{Y}{u}(fX)\underset{Y}{N} - \sum_{X=1}^{p} \overset{X}{u}(X)\left\{-B\underset{x}{U} + \sum_{Y=1}^{p}\theta_x^y \underset{y}{N}\right\}.$$

Comparison of tangential and normal vectors gives

(2.3) $f^2 = a^r I - \sum_{X=1}^{p} \overset{Z}{u} \otimes \underset{X}{U} \qquad \overset{Y}{u} \circ f + \sum_{X=1}^{p} \theta_X^Y \overset{X}{u} = 0$

Multiplying both the sides of equation (2.2)(ii) by F and using again equation (1.1) and (2.2), we get

$$a^r \underset{X}{N} = \left\{-Bf\underset{X}{U} - \sum_{Z=1}^{p} \overset{X}{u}(\underset{X}{U})\underset{Y}{N}\right\} + \sum_{Y=1}^{p} \theta_X^Y \left\{-B\underset{Y}{U} + \sum_{Z=1}^{p} \theta_Y^Z \underset{Z}{N}\right\}$$

Comparison of tangential and normal vectors gives

(2.4) $f\underset{Z}{U} + \sum_{Y=1}^{p} \theta_X^Y \underset{Y}{U} = 0 \qquad \overset{X}{u}(\underset{X}{U}) + \sum_{Y=1}^{p} \theta_Y^Z \theta_X^Y = a^r \delta_X^Z$

Further in view of the equations (1.1), (2.1) and (2.2), if g is the induced metric on V_{n-p} then we have

(2.5) $g(fX, fY) + a^r g(X,Y) + \sum_{X=1}^{p} \overset{X}{u}(X)\,\overset{X}{u}(Y) = 0$

In view of the equation (2.3), (2.4) and(2.5), we have

Theorem 2.1. *The submanifold* V_{n-p} *of codimension p of a Hsu-structure manifold* V_n *admits a para p-contact Hsu-metric structure.*

Suppose further that $\overline{D}$ is the Riemannian connection on V_n and D the induced connection on the submanifold V_{n-p}. Then the equations of Gauss and Weingarten can be expressed as

$$\overline{D}_{BX} BY = BD_X Y + \sum_{X=1}^{p} \overset{X}{h}(X,Y) \underset{X}{N} \tag{2.6}$$

$$\overline{D}_{BX} \underset{X}{N} = -B\overset{X}{H}(X) + \sum_{Y=1}^{p} \theta_X^Y \underset{Y}{N} \tag{2.7}$$

where $\overset{X}{h}(X,Y)$ are second fundamental forms, and

$$\overset{X}{h}(X,Y) = g(\overset{X}{H}(X),Y). \tag{2.8}$$

Suppose that the enveloping manifold V_n is a HK-manifold. Hence we have $(\overline{D}_{BX} F)(BY) = 0$ or equivalently

$$\overline{D}_{BX} FBY = F\overline{D}_{BX} BY.$$

In view of the equations (2.2), (2.6) and (2.7), the last equation takes the form

$$D_{BX} = \left\{ BfY - \sum_{X=1}^{p} \overset{X}{u}(Y) \underset{X}{N} \right\} = F\left\{ BD_X Y + \sum_{X=1}^{p} \overset{X}{h}(X,Y) \underset{Y}{N} \right\}$$

or equivalently

$$BD_X f Y + \sum_{X=1}^{p} \overset{X}{h}(X,fY) \underset{X}{N} - \sum_{X=1}^{p} \overset{X}{u}(Y) \left\{ -B\overset{X}{H}(X) + \sum_{Z=1}^{p} \theta_X^Y \underset{Y}{N} \right\}$$

$$= BD_x Y - \sum_{X=1}^{p} \overset{X}{u}(D_x Y) \underset{X}{N} + \sum_{X=1}^{p} \overset{X}{h}(X,Y) \left\{ -B\underset{X}{U} + \sum_{Y=1}^{p} \theta_X^Y \underset{Y}{N} \right\}$$

The comparison of the tangential vectors gives

$$D_X Y + \sum_{X=1}^{p} \overset{X}{u}(Y) \overset{X}{H}(X) = fD_X Y - \sum_{X=1}^{p} \overset{X}{h}(X,Y) \underset{X}{U}$$

or equivalently

$$(D_x f)(Y) + \sum_{X=1}^{p} + \left\{ \overset{X}{u}(Y) \overset{X}{H}(X) + \overset{X}{u}(X,Y) \underset{X}{U} \right\} = 0 \tag{2.9}$$

If N(X,Y) is the Nijenhuis tensor for the submanifold V_{n-p} we can write

$$N(X,Y) = (D_{fX} f)(Y)\ (D_{fY} f)(X) + f(D_Y f)(X) - f(D_X f)(Y)$$

A necessary and sufficient condition that the submanifold V_{n-p} be totally geodesic is that $\overset{x}{h}(X,Y) = 0$ $(X = 1,2,\ldots p)$. Thus, in view of the equations (2.8) and (2.9) it follows that $D_X f = 0$. Hence from (2.1) we have $N(X,Y) = 0$.

But V_{n-p} is said to be integrable if and only if $N(X,Y) = 0$. Thus, we have

Theorem 2.2. *A totally geodesic submanifold V_{n-p} with a para p-contact Hsu-structure of a Hsu-structure manifold is integrable*

3. Curvature Tensor

Suppose that W, X, Y, Z are arbitrary vector fields on an open set A in the neighbourhood of a point of the sub manifold V_{n-p}. If $\overline{L}$ and L are the Riemann Chrostoffel curvature tensors of V_n and V_{n-p} respectively, we have

$$(3.1)\qquad \overline{L}(BW, BX, BY, BZ) = L(W,X,Y,Z) + \sum_{X=1}^{p}\{\overset{X}{h}(X,Z)\overset{X}{h}(W,Y) - \overset{X}{h}(X,Y)\overset{X}{h}(W,Z)\}.$$

If the manifold V_n admits constant holomorphic sectional curvature C, we have

$$(3.2)\qquad \overline{L}(BW, BX, BY, BZ) = \frac{C}{4}[G(BW, BZ)G(BX,BY) - G(BX, BZ)G(BW,BY) + {}'F(BX,BZ)\,{}'F(BX,BY) - {}'F(BX,BY)\,{}'F(BW,BZ) + 2\,{}'F(BW,BX)\,{}'F(BY,BZ)].$$

From equation (1.3) and (2.2), it can be proved that

$${}'F(BX,BY) = f(X,Y) \underset{=}{def}\ g(fX,Y)$$

Hence in view of the equations (2.1), (3.1) and (3.3) the equations (3.2) takes the

$$(3.4)\qquad L(W,X,Y,Z) = \frac{C}{4}[g(W,Z)g(X,Y) - g(X,Z)g(W,Y) + {}'f(X,Z)\,{}'f(W,Y) - {}'f(X,Y)\,{}'f(W,Z) + 2\,{}'f(W,X)\,{}'f(Y,Z) + \sum_{X=1}^{p}\{\overset{X}{h}(X,Y)\overset{X}{h}(W,Z) - \overset{X}{h}(X,Z)\overset{X}{h}(W,Y)\}]$$

Thus, we have

Theorem 3.1. Let V_n *be an Hsu-structure manifold of constant holomorphic sectional curvature* C. *Then the curvature tensor of the submanifold* V_{n-p} *satisfies the equation* (3.4).

REFERENCES

[1] K.L. Duggal, *On differentiable structures defined by algebraic equation* (1). Nijenhuis Tensor, **22**(1971), 238–242.
[2] R. S. Mishra, *Structures in differentiable manifold and their applications.* Published by Chandrama Prakashan, Allahbad, India, 1984.
[3] H.B. Pandey, *On GF-Hypersurfaces*, India J. Pure Appl. Math. (5) **10**(1979), 549–557.
[4] R.P. *Rai, Submanifolds of Codimension 2 of an almost hyperbolic hermite manifold*, J. Indian Acad. Math. (1) (1987), 1–10.
[5] Vanzura, Jiri, *Almost r-contact structures*, Ann.Seuola Norm. Sup. Pisa **26**(1972), 97–115.

RAM NIVAS AND DHARMENDRA SINGH
Department of Mathematics & Astronomy,
Lucknow University,
Lucknow – 226007,
India.

E-mail: `rnivas@sify.com`

The Nepali Math. Sci. Report
Vol. 24. No.1. 2005.

On the compact support of solutions to a nonlinear long internal waves model

MAHENDRA PANTHEE*

Abstract: We use complex analysis techniques to prove that, if a sufficiently regular solution to a model that governs the unidirectional propagation of long internal waves in a rotating homogeneous incompressible fluid is supported compactly in a non trivial time interval then it vanishes identically.

Key words: Dispersive equations; unique continuation property; smooth solution; compact support.

2000 Mathematics Subject Classification: 35Q35, 35Q53.

1. Introduction:

In this work we are interested in studying the following initial value problem (IVP):

$$(1.1)\qquad \begin{cases} (u_t - \beta u_{xxx} + (u^2)_x)_x - \gamma u = 0, & x, t \in \mathbb{R} \\ u(x,0) = u_0(x), \end{cases}$$

where $u = u(x,t)$ is a real valued function and γ, β are constants. This model was introduced by Ostrovsky in [12] which describes the propagation of weakly nonlinear long surface and internal waves of small amplitude in a rotating homogeneous incompressible fluid. In literature, this model is also called as Ostrovsky equation. The parameters $\gamma > 0$ and β describe the effect of rotation and type of dispersion respectively. The value $\beta = -1$ describes negative dispersion for surface and internal waves in the Ocean and Surface waves in a shallow channel with uneven bottom. The value $\beta = 1$ describes positive dispersion for capillary waves on the liquid surface or for magneto-acoustic oblique waves in plasma [1], [5], [6].

* This work was partially supported by CAMGSD through FCT / POCTI / FEDER, IST, Portugal.

Several authors have studied this model in recent literature, see for example [1],[5],[6],[11],[17] and references there in. In particular, Cauchy problem associated with (1.1) has been studied in [17].

In this work we are concerned about the unique continuation property (UCP) for the model (1.1). There are various forms of UCP in the literature, see for example [2],[8],[9],[10],[15]and references there in. The following is the definition of UCP given in [15], where the first result of UCP for a dispersive model is proved.

Definition [15]. Let L be an evolution operator acting on functions defined on some connected open set Ω of $\mathbb{R}^n \times \mathbb{R}_t$. The operator L is said to have unique continuation property if every solution u of $Lu = 0$ that vanishes on some nonempty open set $\mathcal{O} \subset \Omega$ vanishes in the horizontal component of $\mathcal{O}$ in Ω.

Much effort has been used in studying UCP for various models in recent literature, for example [2],[3],[4],[7],[8],[9],[10],[13],[14],[15],[16]and [18] are just few to mention. In most cases Carleman type estimates are used to prove UCP. Recently Bourgain in [2] introduced a new method based on complex analysis to prove UCP for dispersive models. Although, by using Paley-Wiener theorem, the UCP for linear dispersive models, with this method, is almost immediate, the same is not so simple when one considers full nonlinear model. Some extra and technical efforts are necessary to address the case of nonlinear model. In this work we use method in [2] to prove that, if a sufficiently smooth solution to the IVP (1.1) it supported compactly in a non trivial time interval then it vanishes identically. In some sense it is a weak version of the UCP given in the above definition. Due to technical reason (see proof of Theorem 1.1, below) we consider the negative dispersion case i.e. $\beta = -1$, in (1.1). The main result of this work reads as follows:

Theorem 1.1: Let $u \in C(\mathbb{R}, H^s(\mathbb{R}))$ *be a solution to the* IVP (1.1) *with* $s > 0$ *large enough. If there exists a non trivial time interval* $I = [-T, T]$ *such that for some* $B > 0$,

$$\operatorname{supp} u(t) \subseteq [-B, B], \ \forall \ t \in I,$$

then $u \equiv 0$.

To prove this theorem we write the IVP (1.1) as

$$\begin{cases} u_t - \beta u_{xxx} + (u^2)_x - \gamma D_x^{-1} u = 0, \\ u(x,0) = u_0(x). \end{cases} \tag{1.2}$$

Now, we use Duhamel's formula to write the IVP (1.2) in the equivalent integral form

$$u(t) = U(t)u_0 - \int_0^t U(t-t')(u^2)_x(t')dt'. \tag{1.3}$$

where $U(t)$ is the unitary group describing the solution to the linear problem

$$\begin{cases} u_t - \beta u_{xxx} - \gamma D_x^{-1} u = 0 \\ u(x,0) = u_0(x), \end{cases} \tag{1.4}$$

and is given by

(1.5) $$U(t)u_0(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i(x\xi - (\beta\xi^3 + \frac{\gamma}{\xi})t)} \hat{u}_0(\xi) d\xi.$$

Note that following are the conserved quantities satisfied by the flow of (1.1):

(1.6) $$\int_{\mathbb{R}} |u(x,t)|^2 dx. \qquad \text{(momentum)}$$

(1.7) $$\int_{\mathbb{R}} \beta\, u_x^2 + \tfrac{\gamma}{2}\left(D_x^{-1}u\right)^2 + \tfrac{1}{3}\, u^3 dx. \qquad \text{(energy)}$$

We organise this article as follows. We establish some preliminary estimates in section 2 and in section 3 we supply the proof of the main result of this work, Theorem 1.1.

Now we introduce some notations that will be used throughout this article. The Fourier transform of a function f denoted by $\hat{f}$ is defined as

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} f(x) dx.$$

We use H^s to denote L^2-based Sobolev space with index s. The various constants whose exact values are immaterial will be denoted by c. We use supp f to denote support of a function f and $f * g$ to denote the usual convolution product of f & g. Also, we use the notation $A \lesssim B$, if there exists a constant $c > 0$ such that $A \le cB$.

2. Preliminary estimates

In this section we record some preliminary estimates that are essential in the proof of our main result. The details of the proof of these estimates can be found in [2] and the author's previous works [13] & [14]. For the shake of clearness we just sketch the idea of the proofs.

Let us start by recording the following result.

Lemma 2.1: *Let $u \in C([-T,T]; H^s(\mathbb{R}))$ be a sufficiently smooth solution to the* IVP (1.1). *If for some $B > 0$, supp $u(t) \subseteq [-B, B]$, then for all $\xi, \theta \in \mathbb{R}$, we have*

(2.1) $$|\hat{u}(t)(\xi + i\theta)| \lesssim e^{C|\theta|B}.$$

Proof: The proof follows by using the Cauchy-Schwarz inequality and the conservation law (1.6). The argument is similar to the 2-dimensional case presented in [13] & [14].

Now we define

(2.2) $$u^*(\xi) = \sup_{t \in I} |u(t)(\xi)|.$$

and

(2.3) $$m(\xi) = \sup_{\xi' \ge \xi} |u^*(\xi')|.$$

Considering the initial data $u(0)$ sufficiently smooth and taking into account the well-posedness theory for the IVP(1.1) (see for e.g., [17] we have the following result.

Lemma 2.2: *Let $u \in C([-T,T]; H^s(\mathbb{R}))$ be a sufficiently smooth solution to the* IVP (1.1) *with supp $u(t) \subseteq [-B, B]$, $\forall\, t \in I$, then for some constant B_1, we have*

$$(2.4) \qquad m(\xi) \leq \frac{B_1}{1+|\xi|^4}$$

Proof: The proof follows by using Cauchy-Schwarz inequality, conservation law (1.6) and well-posedness theory with the similar argument in the author's previous works [13] & [14].

Proposition 2.3: Let $u(t)$ be compactly supported and suppose that there exists $t \in I$ with $u(t) \neq 0$. Then there exists a number $C > 0$ such that for any large number $Q > 0$ there are arbitrary large ξ-values such that

$$(2.5) \qquad m(\xi) > C\,(m * m)(\xi)$$

and

$$(2.6) \qquad m(\xi) > e^{-\frac{|\xi|}{Q}}$$

Proof: The main ingredient in the proof of this Lemma is the estimate (2.4) in Lemma 2.2. The detail argument is similar to the one given in the proof of lemma in page 440 in [2], so we omit it.

Now, using the definition of $m(\xi)$ and Proposition 2.3 we can choose ξ large enough and $t_1 \in I$ such that

$$(2.7) \qquad |\hat{u}(t_1)(\xi)| = u^*(\xi) = m(\xi) > C\,(m * m)(\xi) + e^{\frac{-|\xi|}{Q}}.$$

In what follows we prove some derivative estimates for entire function. We start with the following result whose proof is given in [2].

Lemma 2.4: Let $\phi : \mathbb{C} \to \mathbb{C}$ *be an entire function which it bounded and integrable on the real axis and satisfies*

$$|\phi(\xi + i\theta)| \leq e^{|\theta| B}, \qquad \xi, \theta \in \mathbb{R}.$$

Then, for $\xi_1 \in \mathbb{R}^+$ we have

$$(2.8) \qquad |\phi'(\xi)| \leq B\Big(\sup_{\xi' \geq \xi_1} |\phi(\xi')|\Big)\Big[1 + \big|\log\Big(\sup_{\xi' \geq \xi_1} |\phi(\xi')|\Big)\Big].$$

Corollary 2.5. Let $\theta \in \mathbb{R}$ be such that

$$(2.9) \qquad |\theta| \leq B^{-1}\Big[1 + \Big|\log\Big(\sup_{\xi' \geq \xi_1 > 0} |\phi(\xi')|\Big)\Big|\Big]^{-1}$$

Then,

$$(2.10) \qquad \sup_{\xi' \geq \xi_1} |\phi(\xi' + i\theta)| \leq 2 \sup_{\xi' \geq \xi_1} |\phi(\xi')|,$$

and

(2.11) $$\sup_{\xi'\geq\xi_1} |\phi(\xi'+i\theta)| \leq B\left(\sup_{\xi'\geq\xi_1}|\phi(\xi')|\right)\left[1+|\log\left(\sup_{\xi'\geq\xi_1}|\phi(\xi')|\right)|\right].$$

Proof: Detailed proof of this corollary can be found in Corollary 2.9 in [2]. So we omit it.

Now we state the last result of this section whose proof can be found in the author's previous works [13] & [14].

Corollary 2.6. Let $t \in I$, $\phi(z) = \widehat{u(t)}(z)$, θ be as in Corollary 2.5 and $m(\xi)$ be as in definition (2.3). Then for $|\theta'| \leq |\theta|$ fixed, we have

(2.12) $$|\phi'(\xi-\xi'+i\theta')| \leq B\left[m(\xi)+m(\xi-\xi')\right]\left[1+|\log m(\xi)|\right]$$

3. Proof of the main result

Now we are in position to supply proof of the main result of this work. The main idea in the proof is similar to the one employed in [2], [13] and [14], but the structure of the Fourier symbol associated with the linear part of the IVP (1.1) demands special attention and some basic modifications.

Proof of Theorem 1.1: We prove this theorem by contradiction.
If possible, suppose that there is some $t \in I$ such that $u(t) \neq 0$. Now our goal is to use the estimates derived in the previous section to arrive at a contradiction.

Let $t_1, t_2 \in I$ with t_1 as in (2.7). Using Duhamel's formula, we have

(3.1) $$u(t_2) = U(t_2-t_1)u(t_1) - c\int_{t_1}^{t_2} U(t_2-t')(u^2)_x(t')dt'$$

Taking Fourier transform in the space variable in (3.1) we get

(3.2) $$\begin{aligned}\widehat{u(t_2)}(\xi) &= e^{-i(t_2-t_1)(\beta\xi^3+\frac{\gamma}{\xi})}\hat{u}(t_1)(\xi)\,dt - \\ &\quad - ci\xi\int_{t_1}^{t_2} e^{-i(t_2-t')(\beta\xi^3+\frac{\gamma}{\xi})}\widehat{u^2(t')}(\xi)\,dt'.\end{aligned}$$

Let $\Delta t = t_2 - t_1$ and make a change of variable $s = t' - t_1$ to obtain,

(3.3) $$\begin{aligned}\widehat{u(t_2)}(\xi) &= e^{-i\Delta t(\beta\xi^3+\frac{\gamma}{\xi})}\widehat{u(t_1)}(\xi) - \\ &\quad - ci\xi\int_0^{\Delta t} e^{-(i\Delta t-s)(\beta\xi^3+\frac{\gamma}{\xi})}\widehat{u^2(t_1+s)}(\xi)\,ds \\ &= e^{-i\Delta t(\beta\xi^3+\frac{\gamma}{\xi})}\left[\widehat{u(t_1)}(\xi) - ci\xi\int_0^{\Delta t} e^{is(\beta\xi^3+\frac{\gamma}{\xi})}\widehat{u^2(t_1+s)}(\xi)\right].\end{aligned}$$

Since $u(t)$, $t \in I$ is compactly supported, by Paley-Wiener theorem, $u(t_2)(\xi)$ has analytic continuation in $\mathbb{C}$ and we have

$$\widehat{u(t_2)}(\xi+i\theta)=e^{-i\Delta t\left\{\beta(\xi+i\theta)^3+\frac{\gamma}{\xi+i\theta}\right\}}\Big[\widehat{u(t_1)}(\xi+i\theta)-$$

$$-ci(\xi+i\theta)\int_0^{\Delta t} e^{is\left\{\beta(\xi+i\theta)^3+\frac{\gamma}{\xi+i\theta}\right\}}\,\widehat{u^2(s+t_1)}(\xi+i\theta)\,ds\Big]. \tag{3.4}$$

Since,

$$\beta(\xi+i\theta)^3+\frac{\gamma}{\xi+i\theta}=\beta(\xi^3-3\xi\theta^2)+\frac{\gamma\xi}{\xi^2+\theta^2}+i\Big(3\beta\xi^2\theta-\theta^3-\frac{\gamma\theta}{\xi^2+\theta^2}\Big),$$

using Lemma 2.1, we obtain from (3.4)

$$ce^{-\Delta t\left(3\beta(\xi^2\theta-\theta^3)-\frac{\gamma\theta}{\xi^2+\theta^2}\right)}\geq|\widehat{u(t_1)}(\xi+i\theta)|-$$

$$-ci\,|\xi+i\theta|\int_0^{\Delta t} e^{-s\left(3\beta\xi^2\theta-\theta^3-\frac{\gamma\theta}{\xi^2+\theta^2}\right)}\,|\widehat{u^2(s+t_1)}(\xi+i\theta)|\,ds. \tag{3.5}$$

Now, let us select ξ very large and $\theta=\theta(\xi)$ such that $|\theta|\approx 0$. i.e.

$$\frac{1}{|\xi|}\ll|\theta|. \tag{3.6}$$

Also, let us choose sign of θ in such a way that

$$\theta\Delta t<0. \tag{3.7}$$

Now using these choices we get from (3.5)

$$ce^{-\Delta t\left(3\beta\xi^2\theta-\frac{\gamma\theta}{\xi^2+\theta^2}\right)}\geq|\widehat{u(t_1)}(\xi+i\theta)|-$$

$$-|\xi|\int_0^{\Delta t} e^{-s\left(3\beta\xi^2\theta-\frac{\gamma\theta}{\xi^2+\theta^2}\right)}\,|\widehat{u^2(s+t_1)}(\xi+i\theta)|\,ds. \tag{3.8}$$

Now considering the negative dispersion case, i.e., $\beta=-1$ and taking into account of (3.6) and (3.7), we obtain from (3.8).

$$ce^{-|\Delta t|\left(3\xi^2|\theta|+\frac{\gamma|\theta|}{\xi^2+\theta^2}\right)}\geq|\widehat{u(t_1)}(\xi+i\theta)|-$$

$$-|\xi|\int_0^{|\Delta t|} e^{-s\left(3\xi^2|\theta|+\frac{\gamma|\theta|}{\xi^2+\theta^2}\right)}\,|\widehat{u^2(t_1\pm s)}(\xi+i\theta)|\,ds. \tag{3.9}$$

where '+' sign corresponds to $\Delta t>0$ and '$-$' sign to $\Delta t<0$. From here onwards we consider the $\Delta t>0$ case only, the other case follows similarly. Since $e^{-x}<0$ for $x>0$ we can write the estimate (3.9) as,

$$ce^{\left(3\xi^2+\frac{\gamma}{\xi^2+\theta^2}\right)|\theta\Delta t|} \geq |\widehat{u(t_1)}(\xi+i\theta)| -$$

$$-|\xi| \int_0^{\Delta t} e^{-s\left(3\xi^2+\frac{\gamma}{\xi^2+\theta^2}\right)|\theta|} |\widehat{u^2}(t_1+s)(\xi+i\theta)|\,ds. \tag{3.10}$$

Finally we write the estimate (3.10) in the following way

$$\begin{aligned} ce^{-\left(3\xi^2|\theta|+\frac{\gamma|\theta|}{\xi^2+\theta^2}\right)|\theta\Delta t|} &\geq |u(t_1)(\xi)| - |\xi| \int_0^{\Delta t} e^{-s\left(3\xi^2+\frac{\gamma}{\xi^2+\theta^2}\right)|\theta|} |\widehat{u^2}(t_1+s)(\xi)|\,ds - \\ &\quad -|\widehat{u(t_1)}(\xi+i\theta) - \widehat{u(t_1)}(\xi)| - \\ &\quad -|\xi| \int_0^{\Delta t} e^{-s\left(3\xi^2+\frac{\gamma}{\xi^2+\theta^2}\right)|\theta|} |\widehat{u^2}(t_1+s)(\xi+i\theta) - \\ &\quad -\widehat{u^2}(t_1+s)(\xi)|\,ds. \\ &:= I_1, - I_2 - I_3 \end{aligned} \tag{3.11}$$

In sequel we use the preliminary estimates from the previous section to get appropriate estimates for I_1, I_2 and I_3 to arrive at a contradiction in (3.11).

Now we use definition of $u^*(\xi)$ and the estimate (2.5) to obtain

$$\begin{aligned} |\xi| \int_0^{\Delta t} e^{-s\left(3\xi^2+\frac{\gamma}{\xi^2+\theta^2}\right)|\theta|} |\widehat{u}(t_1+s)| * |\widehat{u}(t_1+s)|(\xi)\,ds &\leq \\ &\leq |\xi|(u^* * u^*)(\xi) \int_0^{\Delta t} e^{-s\left(3\xi^2+\frac{\gamma}{\xi^2+\theta^2}\right)|\theta|} ds \\ &\leq |\xi|(m*m)(\xi) \frac{1-e^{-\Delta t\left(3\xi^2+\frac{\gamma}{\xi^2+\theta^2}\right)|\theta|}}{\left(3\xi^2+\frac{\gamma}{\xi^2+\theta^2}\right)|\theta|} \\ &\leq \frac{|\xi|(m*m)(\xi)}{3\xi^2|\theta|} \\ &\leq \frac{m(\xi)}{3|\xi\theta|}. \end{aligned}$$

Therefore we get,

$$I_1 \geq m(\xi) - \frac{m(\xi)}{3|\xi\theta|} \geq \frac{m(\xi)}{3}. \tag{3.12}$$

To obtain estimate for I_2 we define $|\phi(z)| = \widehat{u(t_1)}(z)$, for $z \in \mathbb{C}$. Using (2.7) we get,

$$(3.13) \qquad |\theta(z)=|\widehat{u(t_1)}(\xi)|=\sup_{|\xi'|\geq|\xi|}|\phi(\xi')|=m(\xi).$$

Now, choose θ such that

$$(3.14) \qquad |\theta|\leq B^{-1}[1+|\log m(\xi)|]^{-1}.$$

Using Corollary 2.5 we obtain

$$\begin{aligned} I_2 &\leq |\theta| \sup_{|\xi'|\geq|\xi|}|\partial \widehat{u(t_1)}(\xi'+i\theta)|.\\ &\leq |\theta|\, B m(\xi)[1+|\log m(\xi)|]^{-1}\\ &\leq m(\xi)\leq \tfrac{1}{15}m(\xi). \end{aligned}$$

Finally to get estimate for I_3 we use Proposition 2.3, Corollary 2.6 and θ as in (3.14) to obtain

$$\begin{aligned} &|\widehat{u^2(t_1+s)}(\xi+i\theta)-\widehat{u^2(t_1+s)}(\xi)|\leq\\ &\leq \int_{\mathbb{R}}|\widehat{u(t_1+s)}(\xi-\xi'+i\theta)-\widehat{u(t_1+s)}(\xi-\xi')||\widehat{u(t_1+s)}(\xi')|\,d\xi'\\ &\leq |\theta|\int_{\mathbb{R}}\sup_{|\xi'|\leq|\xi|}|\partial\widehat{u(t_1+s)}(\xi-\xi'+i\theta')|\,m(\xi')\,d\xi'\\ &\leq \int_{\mathbb{R}}[m(\xi)-m(\xi-\xi')]\,m(\xi')\,d\xi'\\ &\leq m(\xi)\,c_2+(m*m)(\xi)\\ &\leq m(\xi)(c_2+c^{-1})\leq m(\xi). \end{aligned}$$

Therefore,

$$\begin{aligned} I_3 &\leq |\xi|\,m(\xi)\int_0^{\Delta t} e^{-s\left(3\xi^2+\frac{\gamma}{\xi^2+\theta^2}\right)|\theta|}\,ds\\ &= |\xi|\,m(\xi)\,\frac{1-e^{-\Delta t\left(3\xi^2+\frac{\gamma}{\xi^2+\theta^2}\right)|\theta|}}{\left(3\xi^2+\frac{\gamma}{\xi^2+\theta^2}\right)|\theta|}\\ &\leq \frac{|\xi|\,m(\xi)}{3\xi^2|\theta|}\\ &\leq \frac{m(\xi)}{3|\xi\theta|}\leq \tfrac{1}{15}m(\xi). \end{aligned}$$

Now, using (3.12), (3.15) and (3.16) in (3.11) and using the estimate (2.6) one gets.

$$(3.17) \qquad e^{-\left(3\xi^2+\frac{\gamma}{\xi^2+\theta^2}\right)|\theta\Delta t|}\geq \frac{m(\xi)}{3}-\frac{m(\xi)}{15}-\frac{m(\xi)}{15}=\frac{1}{3}m(\xi)\geq e^{\frac{-|\xi|}{Q}}.$$

On the other hand, with the choice of ξ and θ we have

$$e^{-\left(3\xi^2+\frac{\gamma}{\xi^2+\theta^2}\right)|\theta||\Delta t|} \le e^{-|\xi||\Delta t|}. \tag{3.18}$$

Now from (3.17) and (3.18) we obtain

$$e^{-|\xi||\Delta t|} \ge e^{\frac{-|\xi|}{Q}}, \tag{3.19}$$

which is false for $|\xi|$ large if we choose Q large enough such that $\frac{1}{Q} < |\Delta t|$. This contradiction completes the proof of the theorem.

REFERENCES

[1] Benilov E. S. ; *On the surface waves in a shallow channel with uneven bottom*, Stud. Appl. Math. **87** (1992) 1–14.

[2] Bourgain J. ; *On the Compactness of the support of solutions of dispersive equations*, IMRN, 9(1997) 437–447.

[3] Carvajal X., Panthee M.; *Unique continuation property for a higher order nonlinear Schrödinger equation*, J. Math. Anal. Appl., **303** (2005) 188–207.

[4] Carvajal X., Panthee M., *On uniqueness of solution for a nonlinear Schrödinger Airy equation*, to appear in Nonlinear Analysis, TMA.

[5] Galkin V.N., Stepanyants Y.A., ; *On the existence of stationary solitary waves in a rotating* fluid, J. Appl. Math. Mech., **55**(6) (1991) 939–943.

[6] Gilman O.A., Grimshaw R., Stepanyants Y.A., ; *Approximate and numerical solutions of the stationary Ostrovsky equation*, Stud. Apl. Math., **95**(1995) 115–126.

[7] Iório Jr. R.J.; *Unique continuation principles for some equations of Benzamin-Ono type, preprint*, IMPA (2002).

[8] Kenig C.E., Ponce G, Vega L.; *On the support of solutions to the generalized KdV equation*, Ann I.H. Poincare, **19** (2) (2002) 191–208.

[9] Kenig C.E., Ponce G, Vega L.; *On unique continuation for nonlinear Schrödinger equation*, Comm. Pure Appl. Math., **56**(2003) 1247–1262.

[10] Kenig C.E., Ponce G, Vega L.; *On unique continuation of solutions to the generalized KdV equation*, Math. Res. Lett., **10** (2003) 833–846.

[11] Linares F., Milaès A. ; *A note on solutions to a model for long internal waves in a rotating fluid*, Preprint (2005).

[12] Ostrovsky L. A.; *Nonlinear internal waves in a rotating Ocean*, Okeanlogia, **18**(2) (1978) 181–191.

[13] Panthee M., *A note on the unique continuation property for Zakharov-Kuznetsov equation*, Nonlinear Analysis; TMA, **59** (2004) 425–438.

[14] Panthee M.; *Unique continuation property for the Kadomtsev-Petviashvili* (KP-II) *equation*, Electronic J. Diff. Equations, **2005** (2005) 1–12.

[15] Saut J-C., Scheurer B.; *Unique continuation for some evolution equations*, J. Diff Equations, **66**(1987) 118-139.

[16] Tataru D., ; *Carleman type estimates and unique continuation for the Schrödinger equation*, Diff. and Int. Equations, **8**(4) (1995) 901–905.

[17] Valamov V., Liu Y.; *Cauchy problem for the Ostrovsky equation*, Discrete & Cont. Dyn. Syst. **10**(3) (2004) 731–753.

[18] Zhang B.Y.; *Unique continuation for the Korteweg-de Vries equation*, SIAM J. Math. Anal., **23**(1992) 55–71.

MAHENDRA PANTHEE
Central Department of Mathematics
Tribhuvan University
Kirtipur, Kathmandu, Nepal.

Currently :
Post–Doctoral fellow
Department of Mathematics
CAMGSD, IST.
1049–001, Lisbon, Portugal.

The Nepali Math. Sci. Report
Vol. 24. No.1. 2005.

Wandering Domains of Meromorphic Functions

A.P. SINGH AND A. SINGH

Abstract: We show in this paper that there exists a function $f \in \mathbf{M}$ which has a k-connected bounded and a k-connected unbounded connected wandering domains and other properties.

Keywords: Transcendental Meromorphic function, bounded and unbounded k-connected wandering domains.

1.Introduction

Let **M** denote the set of all transcendental meromorphic functions with at lest two polls or exactly one poll which is not omitted value. for $n \in N, f \in M$. Let f^n denote the nth iteration of f. Thus $f^1 = f, f^n = f(f^{n-1}),\ n = 2, 3, 4 \ldots$.

The set $\{z : f^n)$ is normal in some neighborhood of $z\}$ denoted by $F(f)$ is the Fatou set and its complement denoted by $J(f)$ is the Julin set. Clearly $F(f)$ is open. Also it is known that $F(f)$ is completely invariant: $z \in F(f)$ is and only if $f(z) \in F(f)$. Consequently if U is a component of $F(f)$ then $f(U)$ is in a component V of $F(f)$. In fact $V \setminus f(U)$ contains at most one point [6]. If $U_n \cap Um = \phi$ for $n \neq m$ where, U_n denotes the component of $F(f)$ which contains $f(U)$ then U is called a *wandering domain.* If $U_n = U$ for some n, then U is called a periodic domain (of period n if $U_n = U$ and $U \neq U$ for $I = 1, 2, \ldots, n-1$).

The structure of periodic domain is well understood [6], and in contrast to it, the results corresponding to the wandering domain are still far from in 1976 by Baker [1]. Since then several examples of wandering domains with different properties have been given by various authors such as [2, 3, 5, 8, 9, 10, 11].

For an entire function f, it is known that a wandering component may be simply connected or multiply connected. Also it is known that a multiply connected wandering component is always bounded. Further that example of infinite connectivity can occur [3]. Also it is open whether the connectivity may be finite but different from one. However this need not be true for functions in class **M**. In fact, Baker, Kotus, Yinian [4] have proved

Theorem 1: *Let $k \in N$. Then there are meromorphic functions f_i, $1 \le i \le 4$, in the class* **M** *and are such that*

(i) $N(f_1)$ has a k-connected bounded wandering component.

(ii) $N(f_2)$ has a k-connected unbounded wandering component

(iii) $N(f_3)$ has a bounded wandering component of infinite connectivity

(iv) $N(f_4)$ has an unbounded wandering component of infinite connectivity.

In the construction of the proof, the authors have exhibited meromorphic functions f_1 and f_2 with one k-connected wandering domain U_o satisfying the condition (i) and (ii) of the above Theorem. Later Singh [11] constructed a meromorphic, function which has infinitely many wandering component whose paths do not intersect. He proved

Theorem 2: *Let $k \in N$. There exists meromorphic function $f \in$* **M** *such that $F(f)$ has infinitely many k-connected bounded wandering domains each having distinct paths .*

Theorem 3: *Let $k \in N$. These exist meromorphic function $f \in$* **M** *such that $F(f)$ has infinitely many k-connected unbounded wandering domains, each having distinct paths.*

Here the two functions of Theorem 2 and Theorem 3 may be different. In our theorem we shall show the existence of one meromorphic function f which contains both k-connected bounded as well as k-connected unbounded wandering components having distinct paths.

2 Lemmas

In this paper we shall require the following lemmas:

Lemma 2.1: (p.131) *Suppose that K is a compact in $\mathbb{C}$ and f is holomorphic on K, let also $\varepsilon > 0$. Let E be a set such that E meets every component of $\mathbb{C}_\infty - K$. Then there exists a rational function r with poles in E such that*

$$|f(z) - r(z)| < \varepsilon, \; z \in K.$$

Suppose that E is a closed set in $\mathbb{C}$ and f is a function defined on E. Then f can be uniformly approximated on E by meromorphic functions without poles in E if and only if f can be uniformly approximated by rational functions on each compact subset of E.

Lemma 2.2: ([9],p.137): *Suppose that E is a closed set in $\mathbb{C}$ and that z_1 z_2 lie in the same component of $\mathbb{C} - E$. Then for each function m meromorphic in $\mathbb{C}$ with a pole at z_1 and for each $\varepsilon > 0$ there exists a function m^* meromorphic in $\mathbb{C}$ which is analytic at z_1 has a pole at z_2, has no other poles except those of m and for which*

$$|m(z) - m^*(z)| < \varepsilon, \; z \in E$$

Lemma 2.3. ([9], p. 140): *Suppose that E is a closed set in $\mathbb{C}$ such that*

i) $\mathbb{C}_\infty \setminus E$ is locally connected at ∞. If the meromorphic function m has no poles on E, then for each $\varepsilon > 0$ there exist a rational function r with poles outside E and an entire function g such that

$$|m(z) - (r+g)(z)| < \varepsilon, \; z \in E.$$

Main Results

Theorem 3.1: *For any $k \in$ N, there is a meromorphic function f, which lies in class* **M** *and is such that Fatou set has k-connected bounded and k-connected unbounded wandering domains.*

Proof: Let us choose $n, k \in N$. Where k is a fixed integer.

Define $\varepsilon_n = 10^{-n}$, $\varepsilon'_n = 10^{-(n+2)}$

$$\eta_n = \begin{cases} 0 & n=1 \\ \sum_{m=1}^{n-1} \varepsilon_m & n>1 \end{cases}.$$

Let Q_n denote the rectangle

$$Q_n = \{z : -10n - 4k - 5 < \operatorname{Re} z < 10n + 5,\ |I_m z| < 10n + 5\}$$

and set

$$D_n = \{z : -(10+4k) + \eta_n < \operatorname{Re} z < -(10+\eta_n),\ |I_m z - 10n| < 4 - \eta_n\}$$
$$\cup \{z : \operatorname{Re} z > 10 + \eta_n,\ |I_m z - 10n| < 4 - \eta_n\}.$$

Let G_n, G'_n, G''_n contain two k- connected bounded and k- connected unbounded domains defined as follows:

$$G_n = D_n - \left[\bigcup_{l=1}^{k-1} \{ z \in ¢ : | z - (10ni + 4l) | \le 1 + \eta_n \} \cup \left\{ \bigcup_{l=1}^{k-1} \{ z \in ¢ : | z + (10\,ni + 4l)| \le 1 + \eta_n \} \right\} \right].$$

$$G'_n = \{ z \in G_n : d(z, \partial G_n) > \varepsilon'_n \}.$$
$$G''_n = \{ z \in G_n : d(z, \partial G_n) > \varepsilon_n \}.$$

Then clearly

$$G''_n \subset G'_n \subset G,\ G_m \cap G_n = \phi \ \text{for}\ m \neq n.$$

Further $\quad G^{*} = \{ z \in G_1 : d(z, \partial G_1) > \tfrac{1}{3} \}$

Set $\psi(z) = z + 10i$ so that ψ maps G''_n onto G_{n+1} and $\psi^n(G^*) \subset G''_{n+1}$, $n \in$ N.

Let $\phi(z) = \frac{1}{2}(z+5)^2 - 5$ so that $B_0 = B(-5,1)$ and $B'_0(-5, ½)$ and let χ be the constant map defined by $\chi(z) = -5$.

Let the function f be defined on $F_1 = \overline{B}_0 \cup \overline{G'_1} \cup \partial G_1$ by

$$f = \phi \text{ on } \overline{B}_0$$
$$f = \psi \text{ on } \overline{G'_1}$$
$$f = \chi \text{ on } \partial G_1.$$

Hence the assumptions of lemma 2.1 are satisfied by the function. So there is a meromorphic function m_1 with no poles on F_1 such that

(1) $$|m_1(z) - \phi(z)| < \tfrac{1}{2}\varepsilon_1^3,\ z \in \overline{B}_0$$

(2) $$|m_1(z) - \psi(z)| < \tfrac{1}{2}\varepsilon_1^3,\ z \in \overline{G_1'}$$

(3) $$|m_1(z) - \chi(z)| < \tfrac{1}{2}\varepsilon_1^3,\ z \in G_1$$

Since F_1 satisfies the assumption of Lemma 2.3 there exists rational function r_1 with poles outside F_1 and entire function g_1 such that

(4) $$|m_1(z) - (r_1 + g_1)(z)| < \tfrac{1}{2}\varepsilon_1^3,\ \text{for } z \in F_1$$

Applying lemma 2.2, we can choose r_1 so that it has exactly one pole in $\mathbb{C}\,(\overline{B}_0 \cup \overline{D}_n)$ at say $a_1=10i \in Q_1 \setminus (\overline{B}_0 \cup \overline{D}_1)$. We can clearly suppose that r_1 really has a pole a_1 since the addition of $\lambda/(z-a_1)$, where λ is a sufficiently small constant will bring this about without spoiling the approximation properties listed above.

By similar arguments to those which lead to (1) to (4) we obtain a sequence of functions $m_n = r_n + g_n$, where r_n is rational and g_n entire and such that

(5) $$|r_n(z) + g_n(z)| < \varepsilon_n^3,\ z \in \overline{Q}_{n-1} \cup \{\bigcup_{m=1}^{n-1} \overline{D}_m\}$$

(6) $$\left|\sum_{m=1}^{n}(r_m(z)+g_m(z)) - \psi(z)\right| < \varepsilon_n^3,\ z \in \overline{G}_n'$$

(7) $$\left|\sum_{m=1}^{n}(r_m(z)+g_m(z)) - \chi(z)\right| < \varepsilon_n^3,\ z \in \partial G_n$$

Moreover we may assume that $r_n(z)$ has precisely one pole in the component of

$\not{c} \setminus (\overline{Q}_{n-1} \cup \bigcup_{m=1}^{n} \overline{D}_m)$, which contains

$$a_n = 10\,ni \in Q_n \setminus (\overline{Q}_{n-1} \cup (\bigcup_{m=1}^{n} \overline{D}_m)).$$

and that this pole is indeed at a_n.

It follows from (5) that $f(z) = \sum_{m=1}^{\infty} |r_m(z)+g_m(z)|$ is meromorphic in $\not{c}$, with infinitely many poles. The disk B_o is f-invariant and $B_o \subset N(f)$ and hence $G^* \subset N(f)$.

From the above construction we know that G^* contains two components, say G_1^* and G_2^*, k-connected bounded and k-connected unbounded components respectively. If H_1, H_2 are the components of $N(f)$ containing G_1^* and G_2^* respectively, we have

$$f_n \to \infty \ \text{for } z \in H_1$$

and

$$f_n \to \infty \quad \text{for } z \in H_2.$$

Since $f(\partial G_{n+1}) \subset B_o$ and $f^m(\partial G_{n+1}) \subset B_0$ for all m, it follows that $f^n(H_1) \subset G_{n+1}, f^n(H_2) \subset G_{n+1}$ and H_1 and H_2 are wandering components of $N(f)$.

As in [4] we can show that f is univalent on the set L_m and H_1 and H_2 are k connected are bounded and k connected unbounded components.

From (4) and (5.5), $f(L_n) \supset G_{n+1} \supset f^n(H_1 \cup H_2)$.

$$G^* \subset (H_1 \cup H_2) \subset D_n = f^{-n}(G_{n+1}) \subset f^{-n}(f(L_n)).$$

where f^{-n} is univalent in $f(L_n)$ and H_1 and H_2 are bounded by k-Jordan curves.

Let

$$a_j = -(10ni + 4j), \text{ where } 1 \le j \le k-1.$$

$$b_j = (10ni + 4j), \text{ where } 1 \le j \le k-1.$$

Denote by $A_{i,n}$, $B_{i,n}$, C_n (where $i = 1,2,\ldots, k-1$) the components of the complements of D_n which contains a_i, b_i and ∞ respectively. Then the complement of $H_1 \cup H_2$ is the union of

$$\widetilde{A}_i = (\bigcup_{n=1}^{\infty} A_{i,n}),\ \widetilde{B}_i = (\bigcup_{n=1}^{\infty} B_{i,n}),\ C = (\bigcup_{n=1}^{\infty} C_n),\quad i = 1,2,\ldots, k\text{-}1.$$

Then $\widetilde{A}_i \neq \widetilde{A}_j, i \neq j,\ \widetilde{B}_i \neq \widetilde{B}_j, j \neq i.$

Since $\widetilde{A}_i$, $\widetilde{B}_i$ are bounded by H_1 and H_2 respectively, H_1 and H_2 are k-connected components.

Theorem 3.2: *For any $k \in N$. Then there is a mermorphic function f in the class* **M** *such that Fatou set of f i.e. $F(f)$ has infinitely many k-connected bounded and infinitely many k-connected unbounded wandering components*

Proof : Let us choose $k,n \in$ N. Define

$$\varepsilon_n = 10^{-n},\ \varepsilon'_n = 10^{-(n+2)},$$

$$\eta_n = \begin{cases} 0 & n = 1 \\ \sum_{m=1}^{n} \varepsilon_m & n > 1 \end{cases}.$$

Let Q_n denote the rectangle

$$Q_n = \{ z : -10n - 4k - 5 < \text{Re } z < 10n + 5, |I_m z| < 10n + 5\}$$

and set

$$D_n = \{z : -(10+4k)+\eta_n < \text{Re} z < -(10+\eta_n)$$

$$|(I_m z) - 10n| < 4 - \eta_n\} \cup \{z : \text{Re} z > 10 + \eta_n, |(I_m z) - 10n| < 4 - \eta_n\}.$$

Let

$$G_n = D_n \setminus [\{ \bigcup_{l=1}^{k-1} \{ z \in \not{c} : | z-(10ni+4l) | \le 1+\eta_n \}\}$$

$$\cup \{ \bigcup_{l=1}^{k-1} \{ z \in \not{c} : | z-(10ni+4l) | \le 1+\eta_n \}\}].$$

Let

$$G'_n = \{ z \in G_n : d(z,\partial G_n) > \varepsilon'_n \}$$

$$G''_n = \{ z \in G_n : d(z,\partial G_n) > \varepsilon_n \}.$$

Then clearly $G''_n \subset G'_n \subset G_n$ and $G_n \cap G_m = \phi$ for $n \neq m$. and G_n, G'_n, G''_n contain k-connected bounded and k-connected domains.

Let $B_0 = B(-5, 1)$, and $B'_o = (-5, \frac{1}{2})$.

The set of all natural numbers can be arranged as follows:

1	2	4	7	...
3	5	8	12	...
6	9	13	18	...
.	.	.	.	...
.	.	.	.	...
.	.	.	.	...

that is of the form $(\frac{q(q-1)}{2}) + 1 + pq + (\frac{p(p+1)}{2}); p = 0,1,2, \ldots, q = 1,2,3, \ldots\}$

In fact a natural number lying in p[th] row and q[th] column ($p = 0,1,2,\ldots, q = 1, 2,\ldots$) would be $\frac{q(q+1)}{2} + 1 + pq + \frac{p(p+1)}{2}$.

Next, if $n \in N$, let $r = (r_n)$ be the least positive integer such that $\frac{r(r+1)}{2} \ge n$ and let $s = \frac{r(r+1)}{2} - n$. Then n lies in row $n_r = r-s-1$ and column $n_c = s+1$. Thus without loss of generality we may denote the set G_n by its place position G_{n_r,n_c} or simply $G_{i,j}$.

Let

$$G^*_{0,q} = \{ z \in G_{o,q} : d(z,\partial G_{o,q}) > \tfrac{1}{3}\}$$

For any $z \in G_{p,q}$, define

$$\psi_{p,q}(z) = z + \frac{1}{10^r} + \frac{1}{10^{r+1}} + \ldots + \frac{1}{10^{r+p+q}} + (p+q+1)10i$$

where $r = \frac{p(p+1)}{2} + 1 + pq + \frac{q(q-1)}{2}$.

Then $\psi_{p,q}$ maps $G''_{p,q}$ onto $G_{p+1,q}$, and

$$(\psi_{p,q} \circ \psi_{p-1,q} \circ \ldots \circ \psi_{0,q})(G''_{0,q}) \subset G_{p+1,q}.$$

Let $\phi(z) = \frac{1}{2}(z+5)^2-5$, then ϕ is a mapping from B_0 to B'_0

Let $\chi(z) = -5$,

$$F_{0,1} = \overline{B}_0 \cup \overline{G}'_{0,1} \cup \partial G_{0,1} \text{ and}$$

$$f_{0,1} = \begin{cases} \phi(z), & z \in \overline{B}'_0 \\ \psi_{0,1}(z), & z \in \overline{G}'_{0,1} \\ \chi(z), & z \in \partial G_{0,1} \end{cases}$$

Then by Lemma 2.1, there exists a mermorphic function $m_{0,1}$, with no poles on $F_{0,1}$ such that

$$(8) \qquad |m_{0,1}(z) - \phi(z)| < \frac{\varepsilon_{0,1}^3}{2}, \; z \in \overline{B}_0$$

$$(9) \qquad |m_{0,1}(z) - \psi_{0,1}(z)| < \frac{\varepsilon_{0,1}^3}{2}, \; z \in \overline{B}_0$$

$$(10) \qquad |m_{0,1}(z) - \chi(z)| < \frac{\varepsilon_{0,1}^3}{2}, \; z \in \partial G_{0,1}$$

Also $F_{0,1}$ satisfies the condition of Lemma 2.3, so that there exists a rational function $r_{0,1}$ with poles outside $F_{0,1}$ and entire function $g_{0,1}$ such that

$$(11) \qquad |m_{0,1}(z) - (r_{0,1}(z) + g_{0,1}(z))| < \frac{\varepsilon_{0,1}^3}{2}, \; z \in F_{0,1}$$

Applying Lemma 2.2 we can choose $r_{0,1}$ so that it has exactly one pole in $\mathbb{C} \setminus (\overline{B}'_0 \cup \overline{D}_{0,1})$ say at $a_{0,1} = 10i \in Q_{0,1} - (\overline{B}'_0 \cup \overline{D}_{0,1})$. Also we can suppose that $r_{0,1}$ really has a pole at $a_{0,1}$ for it is possible to choose λ sufficiently small so that (11) holds with $r_{0,1}(z)$ replaced by the rational function $r_{0,1}(z) + \dfrac{\lambda}{z - a_{0,1}}$

By considering $F_n = \overline{Q}_{n-1} \cup (\bigcup_{m=1}^{n-1} \overline{D}_m) \cup \overline{G}'_n \cup \partial G_n$ and

$$f_n = \begin{cases} 0, & z \in \overline{Q}_{n-1} \cup (\bigcup_{m=1}^{n-1} \overline{D}_m) \\ \psi_n(z) - \sum_{k=1}^{n-1} (r_k(z) + g_k(z)), & z \in \overline{G}'_n \\ \chi(z) - \sum_{k=1}^{n-1} (r_k(z) + g_k(z)), & z \in \partial G_n \end{cases}$$

and by the above argument, it is possible to find an entire function $g_n(z)$ and rational function $r_n(z)$ which has exactly one pole in the component of $\mathbb{C} \setminus \overline{Q}_{n-1} \cup (\bigcup_{m=1}^{n} D_m)$ which contains $a_n = 10i$, and this pole is at a_m and further

$$| r_n(z) + g_n(z) | < \varepsilon_n^3,\ z \in \overline{Q}_{n-1} \cup (\bigcup_{m=1}^{n-1} \overline{D}_m)$$

$$| \sum_{m=1}^{n} | r_m(z) + g_m(z) - \psi_n(z)| < \varepsilon_n^3,\ z \in \overline{G}_n'$$

$$| \sum_{m=1}^{n} | (r_m(z) + g_m(z)) - \chi(z)| < \varepsilon_n^3,\ z \in \partial G_n.$$

Let $f(z) = \sum_{m=1}^{\infty} (r_m(z) + g_m(z))$. Then f is meromorphic function having infinitely many poles and so belongs to class M. Also $f(B_0') \subset B_0'$ and $B_0' \subset F(f)$. Also $f^p(G_{0,q}^*) \subset G_{p,q}^*$ and hence $G_{0,p}^* \subset F(f)$.

If $H_{0,q}$, $H'_{0,q}$ are components of $F(f)$ which contains components of $G_{0,q}^*$ we have $f^p(H_{0,q}) \to \infty$ and $f^p(H'_{0,q}) \to \infty$ as $p \to \infty$. Since $f(\partial G_{p,q}) \subset B_0'$ and $f^m(\partial G_{p,q}) \subset B_o$ for all m. It follows that $f^n(H_{o,q}) \subset G_{p,q}$ and hence $H_{o,q}$ and $H'_{o,q}$ are wandering components of $F(f)$. As in Theorem 5.2.1 f is univalent in the set

$$L_{p,q} = \{ z \in G_{p,q} : d(z, \partial G_{p,q}) > \frac{1}{10^a} \}$$

where $a = \frac{p(p+1)}{2} + 1 + \frac{p(p+1)}{2} + p(q+1)$ which contains $f^p(H_{o,q})$ and $f^p(H'_{0,q})$ and $H_{o,q}$, $H'_{0,q}$ *are of exactly k-connectivity.*

Theorem 3.3: *For any $k \in$ N. There exist two meromorphic function, say f and g such that f has a k-connected bounded and a k-connected unbounded wandering domains, say $B_{1,1}$ and $B_{2,1}$ such that $B_{1,m} \cap B_{1,n} = \phi$, $m \neq n$, $B_{2,m} \cap B_{2,n} = \phi$, $m \neq n$, where $B_{i,m} = f^{m-1}(B_{i,1})$ and $f^n(B_{i,1}) \to \infty$ as $n \to \infty$, $i = 1,2$ and there exists a doubly connected wandering domain A of g such that*

$$A \subset B_{1,1},\ g(A) \subset B_{2,1},$$
$$g^2(A) \subset f(B_{1,1}),\ g^3(A) \subset f(B_{2,1}),\ g^4(A) \subset f^2(B_{1,1})$$
$$g^5(A) \subset f^2(B_{2,1}) \ldots g^{2n}(A) \subset f^n(B_{1,1}),\ g^{2n+1}(A) \subset f^n(B_{2,1}) \ldots$$

Proof: We first do the construction of 'g'. Define $\varepsilon_n = 10^{-n}$, $\varepsilon_n' = 10^{-(n+2)}$, $n \in$ **N**.

$$\eta_n = \begin{cases} 0, & n = 1 \\ \sum_{m=1}^{n-1} \varepsilon_m, & n > 1 \end{cases}$$

Let B_n denote the disc $B(0,10n)$ and set

$$G_n = \{ z : |z-10n+5| < 3-\eta_n, |z-10n+5| > \tfrac{1}{2}+\eta_n \}$$

$$G'_{n} = \{z \in G_n : d(z,\partial G_n) > \varepsilon'_n\}.$$

$$G''_n = \{z \in G_n : d(z,\partial G_n) > \varepsilon_n\}.$$

$$L_n = \{z \in G_n : d(z,\partial G_n) > 10^{-(n+1)}\}, n \in \mathbf{N}.$$

Thus G_n, G'_n, G''_n are doubly connected domain such that

$$G''_n \subset G'_{n} \subset G_n,\ G_m \cap G_n = \phi \text{ for } m \neq n.$$

write $a_n = 10n-5$ and further $C_{1,n} = 10n-9 + \eta_n + \dfrac{\varepsilon'_n}{2}$

$$C_{2,n} = a_n + \tfrac{1}{2} + \eta_n + \frac{\varepsilon'_n}{2}$$

so that $C_{1,m}, C_{2,n}$ lie in one of the components of $C_n \setminus G'_n$.

We set $B_o = B(-5,1)$ and $B'_0 = B(-5, \frac{1}{2})$.

Denote $G^* = \{z \in G_1 : d(z, \partial G_1) > \frac{1}{3}\}$.

Note that $\psi(z) = z+10$ maps G''_n univalently onto G_{n+1} and $\psi^n(G^*) \subset G''_{n+1}, n \in \mathbf{N}$.

Let $\phi(z) = \frac{1}{2}(z+5)^2-5$ so that $B_o = B(-5, 1)$ and $B'_0 = B(-5, ½)$. Let $\chi(z) = -5$, so that in particular $\chi(\bigcup_1^\infty \partial G_n) \subset B'_0$. By Lemma 2.1 applied to $K = (\overline{B}'_0 \cup \partial G_1 \cup \overline{G'_1})$, there is a rational function R_1 such that

(12) $$|R_1(z) - \phi(z)| < \varepsilon_1^3, z \in \overline{B}_0$$

(13) $$|R_1(z) - \psi(z)| < \varepsilon_1^3, z \in \overline{G'_1}$$

(12) $$|R_1(z) - \chi(z)| < \varepsilon_1^3, z \in \partial G_1$$

and R_1 has poles in $\{a_1, C_{1,1}\ C_{2,1}, \infty\}$. For $n > 1$, there is a rational function R_n such that

(15) $$|R_n(z)| < \varepsilon_n^3, z \in \overline{B}_{n-1}$$

(16) $$\left|\sum_{m=1}^{n} R_m(z) - \psi(z)\right| < \varepsilon_n^3, z \in \overline{G}'_n$$

(17) $$\left|\sum_{m=1}^{n} R_m(z) - \chi(z)\right| < \varepsilon_n^3, z \in \partial G_n$$

and R_n has poles in $\{a_n, a_{1,n}, C_{2,n}, \infty\}$.

It follows from (15) that $g(z) = \sum_{n=1}^{\infty} R_n(z)$ defines a function meromorphic in $\not{c}$.

Since by (15) and (17), $|g(z) + 5| < \sum_{n=1}^{\infty} \varepsilon_n^3$ (so that $g(z) \in B_0'$) on every ∂G_n while g takes values larger than 5 on G_n', it follows that g has poles in each G_n and so is transcendental and is in class **M**.

We note that in B_o, $|g(z)+5| < \frac{1}{2} + \sum \varepsilon_n^3 < 1$ so that B_o is g-invariant and $B_o \subset F(g)$. Now $g^n(G^*) \subset G''_{n+1}$. Hence $G^* \subset F(g)$, $g^n(z) \to \infty$ as $n \to \infty$ for $z \in G^*$ or more generally, in the whole component A of $F(g)$ such that $A \supset G^*$. Since $g(\partial G_{n+1}) \subset B_o$, $g^m(\partial G_{n+1}) \subset B_o$, for all m, it follows that $g^n(A) \subset G_{n+1}$ and A is a wandering component of $F(g)$. Also as in Theorem 3.1, we can show A is doubly connected. We now do the construction of f.

As in Theorem 3.1 we choose $k \in \mathbf{N}$. Define $\varepsilon_n = 10^{-n}$, $\varepsilon_n' = 10^{-(n+2)}$,

$$\eta_n = \begin{cases} 0, & n = 1 \\ \sum_{m=1}^{n-1} \varepsilon_m, & n > 1 \end{cases}.$$

Let Q_n denote the rectangle

$$Q_n = \{ z : |\operatorname{Re} z| < 20n, -10n-5 < I_m z < 5 + 4k + 10n \}.$$

And set

$$D_{1,n} = \{z : |\operatorname{Re} z - (20n-15)| < 4 - \eta_n, -5 < I_m z < 5 + 4k\}.$$

$$D_{2,n} = \{z : |\operatorname{Re} z - (20n-5)| < 4 - \eta_n, -5 < I_m z \}$$

Let $B_o = B(-5,1)$ and $B_0' = (-5, \frac{1}{2})$. Let $H_{1,n}$, $H_{1,n}'$, $H_{1,n}''$ and $H_{2,n}$, $H_{2,n}'$, $H_{2,n}''$ be the k-connected bounded and k-connected unbounded domains defined as follows:

$$H_{1,n} = D_{1,n} \setminus \bigcup_{p=1}^{k-1} \{z \in \not{c} : |z-((20n-15) + i(5+4p))| \leq 1 + \eta_n\}$$

$$H_{1,n}' = \{z \in H_{1,n} : d(z, \partial H_1, n) > \varepsilon_n'\}.$$

$$L_{1,n} = \{z \in H_{1,n} : d(\partial H_{1,n}) > 10^{-(n+1)}\}.$$

And

$$H_{2,n} = D_{1,n} \setminus \bigcup_{p=1}^{k-1} \{ z \in \not{c} : |z - ((20n-5) + I(5+4p))| \leq 1 + \eta_n\}$$

$$H_{2,n}' = \{ z \in H_{2,n} : d(z, \partial H_{2,n}) > \varepsilon_n' \}.$$

$$H_{2,n}'' = \{ z \in H_{2,n} : d(z, \partial H_{2,n}) > \varepsilon_n \}.$$

$$L_{2,n}'' = \{ z \in H_{2,n} : d(z, H_{2,n}) > 10^{-(n+1)} \}.$$

Then clearly $H''_{1,n} \subset H'_{1,n} \subset H_{1,n}$, $H_{1,m} \cap H_{1,n} = \phi$, for $m \neq n$, and
$H''_{2,n} \subset H'_{2,n} \subset H_{2,n}$, $H_{2,m} \cap H_{2,n} = \phi$, for $m \neq n$ and

$$G_1^* = \{ z \in H_{1,1} : d(z, \partial H_{1,1}) > \tfrac{1}{3} \}.$$

$$G_2^* = \{ z \in H_{2,1} : d(z, \partial H_{2,1}) > \tfrac{1}{3} \}.$$

Set $\psi(z) = z + 20$, so that ψ maps $H''_{1,n}, H''_{2,n}$ on to $H_{1,n+1}, H_{2,n+1}$ respectively

$$\psi^n(G_1^*) \subset H''_{1,n+1},\ \psi^n(G_2^*) \subset H''_{2,n+1},\ n \in \mathbf{N}.$$

Let $\phi(z) = \frac{1}{2}(z+5)^2 - 5$, so that

$$\phi : B_0 \to B'_0 \text{ and } \chi(z) = -5$$

Let $F_1 = \overline{B}_0 \cup \overline{H}'_{1,1} \cup \partial H_{1,1} \cup \overline{H}'_{2,1} \cup \partial H_{2,1}$.

Define a function f on F_1 by

$$F(z) = \begin{cases} \phi(z) & \text{on} \quad \overline{B}_0 \\ \psi(z) & \text{on} \quad \overline{H}'_{1,1} \cup \overline{H}'_{2,1} \\ \chi(z) & \text{on} \quad \partial H_{1,1} \cup \partial H_{2,1}. \end{cases}$$

Exactly similar arguments as theorem [3.1] we can construct a transcendental meromorphic functions f which has k connected bounded and unbounded wandering domains say, $B_{1,1}$ and $B_{2,1}$ such that $B_{1,m} \cap B_{1,n} = \phi$, $m \neq n$, $B_{2,m} \cap B_{2,n} = \phi$, $m \neq n$, where $B_{i,m} = f^{m-1}(B_{i,1})$ and $f^n(B_{i,1}) \to \infty$ as $n \to \infty$, $i = 1,2$.

After construction of two meromorphic functions g and f and its wandering domains A and $B_{1,1}$ and $B_{2,1}$ respectively, it is clear from the way of construction

$$A \subset B_{1,1},\ g(A) \subset B_{2,1},$$

$$g^2(A) \subset f(B_{1,1}),\ g^3(A) \subset f(B_{2,1}),\ g^4(A) \subset f^2(B_{1,1})$$

$$g^5(A) \subset f^2(B_{2,1}) \ldots\ g^{2n}(A) \subset f^n(B_{1,1}),\ g^{2n+1}(A) \subset f^n(B_{2,1}) \ldots$$

Cor 1: For any $k \in N$. there exists two meromorphic functions, say f and g such that f has a k-connected bounded and a k-connected unbounded wandering domains say $B_{1,1}$ and $B_{2,1}$ such that $B_{1,m} \cap B_{1,n} = \phi$, $B_{2,m} \cap B_{2,n} = \phi$, $m \neq n$, where $B_{i,m} = f^{m-1}(B_{i,1})$ and $f^m(B_{i,1}) \to \infty$ as $n \to \infty$ as $n \to \infty$, $i = 1,2,\ldots$ *and there is a k-connected subset (wande-ring domain of g) say A such that*

$$A \subset B_{1,1},\ g(A) \subset B_{2,1},$$

$$g^2(A) \subset f(B_{1,1}),\ g^3(A) \subset f(B_{2,1}),\ g^4(A) \subset f^2(B_{1,1})$$

$$g^5(A) \subset f^2(B_{2,1}) \ldots\ g^{2n}(A) \subset f^n(B_{1,1}),\ g^{2n+1}(A) \subset f^n(B_{2,1}) \ldots$$

Exactly on similar arguments we can prove

Theorem 3.4: *For any $k \in N$. There exist two meromorphic functions, say f and g such that f has infinitely many k-connected bounded and infinitely many k-connected unbounded wandering domains and g has a doubly connected wandering domain say A such that under the iteration of g, the domain A passes through all k-connected bounded and k-connected unbounded wandering domains of f.*

REFERENCES

[1] Baker, I. N., *An entire function which has wandering domains*, J. Austral. Math. Soc. **22**(1976) 173–176.

[2] Baker, I. N., *Wandering domains in the iteration of entire functions*, Proc. London Math. Soc. (3) **49**(1984) 563–576.

[3] Baker, I. N., *Some entire functions with multiply connected wandering domains*, Ergod. Th. and Dynam. System **5**(1985) 163–167.

[4] Baker, I. N., J. Kotus, Y.Lu., *Iteration of the meromorphic functions II*, J.London. Math. Soc. **42**(1990) 267–278.

[5] Baker, I. N. and A.P. Singh, *Wandering domain in the iteration of composition of entire functions*, Ann. Acad. Sci Fenn. Ser A, I Math **20**(1995) 149–153.

[6] Bergweiler, W., *Iteration of meromorphic functions*, Bull Amer. Math. Soc. **19**(1993) 151–188.

[7] Bergweiler, W., Wang, Y., *On the dynamics of composite entire functions Ark.* Mat. **36**(1998) 31–39.

[8] Eremenko, A. and Lyubich, M. Yu, *Examples of entire functions with pathological dynamics*, J. London Math. Soc. (2) 36, 1987, 458–468.

[9] Gaier, D., *Lectures on Complex approximation*, Birkhauser, Boston 1987.

[10] Singh, A. P., *Unbounded components of Fatou sets*, Complex Variables **41**(2000) 133–144.

[11] Singh, A.P., *Dynamics of transcendental meromorphic functions*, *Finite or infinite dimensional complex analysis*, Ed. J. Kajiwara, Z. Li and K.H. Shan, Marcel Dekker (2000)479–494.

A. P. SINGH AND A. SINGH
University of Jammu,
Jammu.,
J & K., India.

A. SINGH
Central Department of Mathematics
Mathematics Tribhuvan,
Kirtipur , Kathmandu, Nepal.

The Nepali Math. Sci. Report
Vol. 24. No.1. 2005.

On Hypersurfaces of H Hsu - Manifold

GEETA VERMA

Abstract: In these papers [2], [3] and [4] we have studied some properties of hypersurfaces of *H* Hsu - manifold. In this paper we have defined hyperbolic almost kahler manifold and studied its hypersurfaces. It has been found that the hypersurface of hyperbolic almost Kahler manifold is locally quasi-Sasakian manifold. Some results regarding the hypersurfaces of a flat *H* Hsu-manifold have also been obtained.

Keywords: Hyperbolic Almost Kahler manifold, Curvature tensor, Reimannian connection.

1. Introduction

We consider a differentiable manifold M^n of class C^∞. Let there be a vector valued linear function F of C^∞, satisfying the algebraic equation

$$F^2 = -a^r I_n \tag{1.1}$$

where 'a' is a complex number.

Then F is said to give to M^n a hyperbolic differentiable structure, briefly H Hsu-structure, defined by algebraic equation (1.1) and the manifold M^n is called HHsu-manifold [5]. The equation (1.1) gives different algebraic structures for different values of a. If $a \neq 0$, it is a hyperbolic π-structure, $a = \pm 1$, it is an almost complex or an almost hy6perbolic product structure. $a = \pm 1$, it is an almost product or an almost hyperbolic complex structure and $a = 0$, it is an almost tangent or a hyperbolic almost tangent structure. In the second case n has to be even and in the second and third cases $a^{2r} = 1$.

If the H Hsu-structure is endowed with Hermite metric G. such that

$$G(F\lambda, F\mu) = a^r G(\lambda, \mu) \tag{1.2}$$

Then $\{F,G\}$ is said to give to M^n hyperbolic Hermitee structure, briefly known as H Hsu-structure subordinate to H Hsu-structure.

In a hyperbolic H-structure, if

(1.3) $$(E_\lambda, F)(\mu)=0 \text{ or } (E_\lambda, F)(F\mu)=0$$

is satisfied, then M^n is said to be a hyperbolic Kahler manifold. E is the Reimannian connexion.

If a hyperbolic H-structure, if

(1.4) $$(E_\lambda, F)(\mu)+(E_\mu F)(\lambda)=0$$

is satisfied, then M^n is said to be a hyperbolic nearly Kahler manifold.

Let us consider M^n and M^n as the H Hsu-manifold and its hypersurface respectively. Let $b: M^m \to M^n$ be the embedding map, such that
$p \in M^m \Rightarrow bp \in M^n$,

Let B be the corresponding Jacobian map such that a vector field X in M^n at p, BX in M^n at bp. Let g be the induced Reimannian metric in M^m. Thus we have

(1.5) $$G(BX, BY)ob = g(X, Y)$$

for arbitrary vector fields X, Y in M^n.

(1.6a) $$G(N, N)ob = 1$$

(1.6b) $$G(N, BX)ob = 0$$

for a unit normal to M^m.

If we put

(1.7a) $$FBX = B(fX) + u(X)N$$

(1.7b) $$FN = -BU$$

Then it can be easily seen that

(1.8a) $$\overline{\overline{X}} = a^r X + u(X)U$$

(1.8b) $$u(fX) = 0$$

(1.8c) $$u(U) = a^r$$

(1.8d) $$fU = 0 \text{ and}$$

(1.9) $$g(\overline{X}, \overline{Y}) = a^r g(X, Y) - u(X)u(Y)$$

where $X \underset{=}{def} fX$ and $u(X) = g(X, U)$

i.e. the induced structure in a general contact metric structure.

Let E and D be the Reimannian connexions in M^n and M^m respectively, Gauss and Weingarten equations are

(1.10a) $$E_{BX}BY = BD_X Y + 'H(X, Y)N$$

(1.10b) $$E_{BX}N = -BHX, \text{ respectively [1].}$$

where

$$H(X, Y) \underset{=}{def} g(HX, Y)$$

Let R and K denote the curvature tensors with respect to the connexions E and D respectively. The generalized Gauss and Mainardi-Codazzi equations are given by [5].

(1.11a) $$'R(BX,BY,BZ,BW)\,ob = 'K(X,Y,Z,W) + \\ + a^r\,'H(X,Z)'H(Y,W) - a^r\,'H(Y,Z)'H(X,W)$$

(1.11b) $$'R(BX,BY,BZ,N)\,ob = a^r\{(D_X H)(Y,Z) - (D_Y H)(X,Z)\}$$

where

$$'R(BX,BY,BZ,BW) \underset{=}{def} G(R(BX,BY,BY),BW)$$

On the hypersurface of a hyperbolic Kahler manifold subordinate of H Hsu-manifold the following results hold [2].

(1.12a) $$(D_X f)Y = u(Y)HX - 'H(X,Y)U$$

(1.12b) $$(D_X u)(Y) = -\,'H(X,\bar{Y})$$

Agreement (1.1): In the above and sequal $\lambda, \mu, \nu \ldots$ will be taken as arbitrary vector fields in the enveloping manifold and $X, Y, Z, \ldots$ as arbitrary vector fields in the hypersurface.

2.Hyperbolic Almost Kahler Manifold

Definition (2.1): Hyperbolic Hermite manifold satisfying

(2.1a) $$(E_\lambda F)(\mu,\nu) + (E_\mu F)(\nu,\lambda) + (E_\nu F)(\lambda,\mu) = 0$$

where $$'F(\lambda,\mu) \underset{=}{def} G(F\lambda,\mu)$$

Will be called hyperbolic almost Kahler manifold, sub ordinate to H Hsu-manifold.

From the equation (1.7a), we have

(2.1)b) $$G(FBX, BY) = G(BfX,BY) + u(X)\,G(N, BY)$$

Differentiating equation (2.1b), covariantly with respect to BZ, then using the equations (1.5), (1.6), (1.7a) and (1.10)a), we have

(2.2) $$(E_{BZ}'F)(BX,BY)ob = (D_Z,'f)(X,Y) + 'H(X,Z)u(Y) - 'H(Y,Z)u(X)$$

Writing two other equations by cyclic permutations of X, Y, Z, we have

(2.3) $$(E_{BY}'F)(BZ,BX)ob = (D_Y,'f)(Z,X) + 'H(Z,Y)u(X) - 'H(X,Y)u(Z)$$

and

(2.4) $$(E_{BX}'F)(BY,BZ)ob = (D_X,'f)(Y,Z) + 'H(Y,X)u(Z) - 'H(X,Z)u(Y)$$

Thus we have the following theorem:

Theorem (2.1): *If the enveloping manifold is a hyperbolic almost Kahler manifold, its hypersurface is given by*

(2.5) $(D_X,'f)(Y,Z)+(D_Y,'f)(Z,X)+(DZ,'f)(X,Y)=0$

Proof: Adding the equations (2.2), (2.3) and (2.4), we get

(2.6) $$\{(E_{BZ}'F)(BX,BY)+(E_{BY}'F)(BZ,BX)+(E_{BX}'F)(BY,BZ)\}ob$$
$$=(D_Z,'f)(Y,Z)+(D_Y,'f)(Z,X)+(D_Z,'f)(X,Y)$$

Using the equation (2.1)a) in the equation (2.6), we get the equation (2.5).

Corollary (2.1): Hypersurface of Hyperbolic almost Kahler manifold is locally Quassi Sasakian manifold.

Proof: Equation (2.5) proves the statement.

Theorem (2.2): *For the hypersurface of hyperbolic almost Kahler manifold, we have*

(2.7) $$(D_X,'f)(\overline{Y},\overline{Z})+(D_Y,'f)(\overline{Z},\overline{X})+(D_Z,'f)(\overline{X},\overline{Y})+$$
$$+'f((D_Z,f)X-(D_X,f)Z,\overline{Y})+'f((D_Y,f)Z-(D_Z,f)Y,\overline{X})+$$
$$+'f((D_X,f)Y-(D_Y,f)(X,\overline{Z})=0$$

Proof: We have

(2.8a) $$'f(X,Y)=g(\overline{X},Y)=-'f(X,Y)$$

and

(2.8b) $$'f(\overline{X},\overline{Y})=a^r\,'f(X,Y)$$

Differentiating (2.8)b) covariantly with respect to Z and Using the equation (2.8) again,
We get

(2.9a) $$(D_Z'f(\overline{X},\overline{Y})+'f((D_Z f)(X,\overline{Y})+'f(\overline{X},(D_Z f)Y)=a^r(D_Z'f)(X,Y)$$

Similarly, writing two other equations, we have

(2.9b) $$(D_Y\,'f(\overline{Z},\overline{X})+'f((D_Y f)Z,\overline{X})+'f(\overline{Z},(D_Y f)X)=a^r(D_Y\,'f)(Z,X)$$

(2.9c) $$(D_X\,'f)(\overline{Y},\overline{Z})+'f((D_X f)Y,\overline{Z})+'f(\overline{Y},(D_X f)Z)=a^r(D_Z'f)(X,Y)$$

Adding the equations (2.9a), (b) and (c) then using the equations (2.8a) and (2.5), we get the required result.

3. Hypersurfaces of Flat *H* Hsu-manifold

Theorem (3.1): *The umbilical hypersurface of a hyperbolic General Differentiable (H Hsu) manifold is of constant Reimannian curvature, iff the enveloping manifold is* flat.

Proof: Let the hypersurface be umbilic, i.e.

$$'H(X,Y)=g(X,Y) \qquad [1],$$

then (1.11)a), gives

(3.1) $$'R(BX, BY, BZ, BW)\,ob = {}'K(X,Y,Z,W) + d^r g(X,Y)\, g(Y,W) - d^r g(Y,Z)\, g(X,W)$$

If the enveloping manifold is flat, then (3.1) reduces to

(3.2) $$'K(X,Y,Z,W) = d^r\{g(Y,Z)\, g(X,W) - g(X,Z)\, g(Y,W)\}$$

This shows that the hypersurface is constant Reimannian Curvature.

Conversely, if (3.2) holds, then using (3.2) in (3.1), we have $'R(BX, BY, BZ, BW) = 0$, that is the manifold is flat.

Theorem (3.2): *The scalar curvature of the umbilical hypersurface M^n of a flat H Hsu-manifold M^n is given by*

(3.3) $$r = m\,(m-1)\, d^r$$

Proof: The unbilical hypersurface is of constant *R* reimannian curvature (by theorem (3.1)). We have

$$K(X,Y,Z,) = d^r\{g(Y,Z)\,X - g(X,Z)Y\}$$

From this we at once get the equation (3.3).

Theorem (3.3): *The quasi-umbilical hypersurface of a flat H Hsu-manifold can never be of constant Reimannian cuevature.*

Proof: Let the hypersurface of a flat *H* Hsu-manifold be quasi-umbilical, then we can always write.

(3.4) $$'H(X,Y) = g(X,Y) + u(X)u(Y)$$

Using (3.4) in (1.11a), we have

(3.5) $$'R(BX, BY, BZ, BW)\,ob = {}'K(X,Y,Z,W) + d^r g(X,Z)\, g(Y,W) - d^r g(Y,Z)\, g(X,W)$$
$$d^r g(Y,W)u(X)\, u(Z) + d^r g(X,Z)\, u(Y)\, u(W) - d^r g(Y,Z)\, u(X)\, u(W)$$

Now,

$$'K(X,Y,Z,W) = d^r\{g(Y,Z)\, g(X,W) - g(X,Z)\, g(Y,W)\}$$

If $$d^r\{g(Y,W)\, u(X)\, u(Y)\, u(Z) + g(X,Z)\, u(Y)\, u(W) - g(Y,Z)u(X)\, u(W) - g(X,W)u(Y)\, u(Z)\} = 0$$

Let $d^r \neq 0$ then

$$\{g(Y,W)u(X)\, u(Z) + g(X,Z)u(Y)u(W) - g(Y,Z)u(X)\, u(W) - g(X,W)u(Y)\, u(Z)\} = 0$$

or $$g(Y,W)\, u(X)\, U + u(Y)\, u(W)\, X - u(X)\, u(W)Y - g(X,W)\, u(Y)\, U = 0$$

or $$d^r g(Y,W) + mu(Y)\, u(W) - u(Y)\, u(W) - u(Y)\, u(W) = 0$$

or $$d^r g(Y,W) + (m-2)\, u(Y)\, u(W) = 0$$

or $$d^r\, Y + (m-2)\, u(Y)\, U = 0$$

or $$md' + (m-2)a' = 0$$

or $$2a'(m-1) = 0$$

or $$a' = 0$$

But $a' \neq 0$, thus the quasi-umbilical hypersurface can not be of constant Reimannian curvature.

Theorem (3.4): *If the hypersurface of a flat H Hsu-manifold of minimal variety, then* $div\, H = 0$.

But the converse is not true in general.

Proof: Let the hypersurface be of minimal variety, then

$$\text{tr. } H = 0 \text{ , [1]}$$

Since the enveloping manifold is flat, equation (1.11b) implies that

$$(D_X H)Y - (D_Y H)X = 0$$

Contracting this equation, we get

$$(\text{div } H)Y = Y \text{ tr. } H = 0 \quad (\text{since Tr. } H = 0)$$

Conversely, if div. $H = 0$, then from the last equation, we get

$$Y \text{ tr. } H = 0 \qquad \text{i.e.} \qquad \text{tr. } H = \text{constant}$$

REFERENCES

[1] Mishra, R. S. (1984): *Structures on Differentiable manifold and their Applications*. Chandrama Prakashan, Allahabad (U.P.)

[2] Pandey, S.B. and Singh, Puran (1993): *On general contact hypersurface of H Hsu-manifold*. Jour. Sci. Vol. 5 no.2.

[3] Pandey, S.B. and Singh, Puran (1994): *On H Hsu-manifold of contact Reimannian curvature*. JTSI, Vol. 8, 9, 10. pp.70–75.

[4] Singh, Puran (1995): *On H Hsu-manifold*. JTSI, Vol. 13. pp.12–14.

[5] Singh, Puran (1993): *On differentiable manifolds and their Applications*. Ph. D. Thesis, K.U., Nainital (U.P.) India.

[6] Nivas, Ram and Khan, Nazrul Islam (2003): *On Submanifold Immersed in Hsu-Quaternion Manifold*, The Nepali Mathematical Science report, Vol. 21, No. 1 & 2, pp 73–79.

GEETA VERMA
Dept. of Applied Sciences & Humanities,
Azad Institute of Engineering & Technology,
Natkur, post: Chandrawal, Bangla Bazar Road,
Lucknow - 226002 (U.P.)
INDIA
E-mail: geeta_verma1@yahoo.com

The Nepali Math. Sci. Report
Vol. 24. No.1. 2005.

CONTENTS

Computer typesetting at UNION COMPUTERS,
Wotu Marg, Ward No. 24. Tadhahbahal, 35, Kathmandu. Tel. 4–252281.

Printed at the TRIBHUVAN UNIVERSITY PRESS.
Kirtipur, Kathmandu, Nepal. Tel. 4–331320, 4–331321