A Simple Stock Comparison Model *and* Planck's Law in Quantum Physics: *the birth of an idea*

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Introduction: During the height of the 'tech-bubble' in the late nineties I was the treasurer of a small faculty investment club at my school. One question we always debated was whether we invest in 'value stocks' with low P/E ratio and low growth, or 'growth stocks' with high P/E ratio but also high growth. To help in making investment decisions I formulated a simple mathematical model for stock comparison. All else being equal, I used the (equilibrium) 'break-even time' as a measure for comparison. For simplicity I also assumed a period of exponential growth of earnings for all stocks.

The Model: Let P be the price paid for a stock and E be its earnings (as determined by the P/E ratio). Consider that the earnings grow exponentially at a rate r. Let Δt be the interval of time when the accumulated earnings of the stock equals the price paid for the stock. We will then have that

$$P = \int_s^{s+\Delta t} E_0 e^{ru} du ,$$

integrating we get that

$$P = \frac{1}{r}\left[E_0 e^{r(s+\Delta t)} - E_0 e^{rs} \right]$$

and so

$$P = \frac{E(s)}{r}\left[e^{r\Delta t} - 1 \right] \tag{1}$$

From (1) we have that

$$\Delta t = \ln\left(1 + r \cdot \frac{P}{E} \right)^{1/r} \tag{2}$$

Example: Consider two stocks. Stock A sells for \$105 per share, has a P/E ratio of 15 and a growth rate of 6%. Stock B sells for \$25 per share, has a P/E ratio of 95 and a growth rate of 30%. All else being equal, which is a better value?
Answer: Using (2), we can calculate the 'break-even time' for stock A to be 10.7 years while the 'break-even time' for stock B is 11.3 years. Therefore stock A is a better value!

Although the collapse of the 'tech bubble' invalidated all my assumptions (and put an end to our investment club), this simple model did leave me with something to think about. It led to a surprising connection to Planck's Law in Quantum Physics.

Planck's Law connection in Quantum Physics: We can rewrite eq. (1) above as

$$E(s) = \frac{rP}{e^{r\Delta t} - 1} \tag{3}$$

For exponential functions *E(s)* it is easy to show that $\Delta E = rP$. Always, $\Delta t = \frac{P}{E_{av}}$, where $\Delta t = t - s$, $\Delta E = E(t) - E(s)$ and $E_{av} = \frac{1}{t-s}\int_s^t E(u)du$.

Substituting these in (3) we get,

$$E(s) = \frac{\Delta E}{e^{\Delta E / E_{av}} - 1} \quad (4)$$

Planck's Law for 'blackbody radiation' states that

$$E = \frac{h\nu}{e^{h\nu / kT} - 1} \quad (5)$$

where E is energy, h is Planck's constant, k is Boltzmann's constant, ν is frequency of radiation and T is temperature of the 'blackbody'. It is well know in Quantum Physics that a 'quantum of energy' ΔE is equal to $h\nu$ while the average energy of the 'blackbody' (per degree of freedom) is given by kT. If we where to substitute $\Delta E = h\nu$ and $E_{av} = kT$ in (5), we see that Planck's Law has the exact same mathematical form as (4). We can show that (4) is an *exact mathematical identity (a tautology) that characterizes all exponential functions*. We have the following theorem *(proven elsewhere)*:

$$\textit{Theorem: } E(s) = E_0 e^{rs} \textit{ if and only if } E(s) = \frac{\Delta E}{e^{\Delta E / E_{av}} - 1}$$

Planck's Law derivation without using 'energy quanta': The above results suggest a derivation of Planck's Law for 'blackbody' radiation using continuous processes only and without needing the 'quantization of energy' hypothesis. We will show that Planck's Formula is an exact mathematical identity that describes the 'interaction of measurement' -- i.e. the functional relationship between the energy $E(s)$ at the 'sensor' at time s, the energy ΔE absorbed by the 'sensor' making the measurement, and the average energy E_{av} at the 'sensor' during measurement. We will use the following *notation* in the discussion below, where E is energy and t is time:

$E(t)$ is an integrable function of t, $\Delta E = E(s + \Delta t) - E(s)$, $P = \int_s^{s+\Delta t} E(u)du$ is the 'accumulation of energy' over the interval $[s, s+\Delta t]$, $E_{av} = \frac{1}{\Delta t}\int_s^{s+\Delta t} E(u)du$ is the average value of energy over the interval $[s, s+\Delta t]$, h is Planck's constant, k is Boltzmann's constant, ν is frequency of radiation and T is temperature.

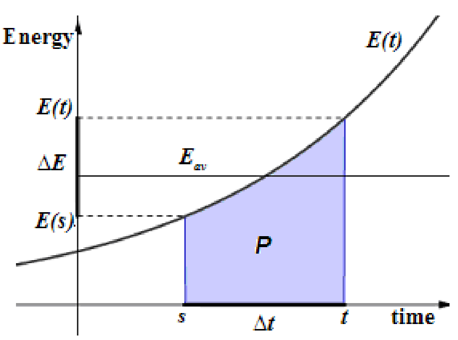

figure 1

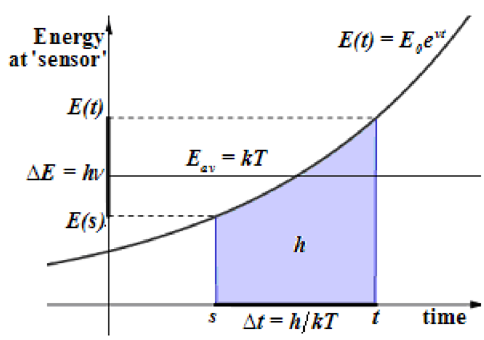

figure 2

It is always true that $\Delta t = \dfrac{P}{E_{av}}$ *(figure 1)*. Consider a 'source' of constant temperature T . Then the average energy will equal *kT*. Let *P=h* be the (minimal) 'accumulation of energy' necessary for a measurement to be made (*figure 2)*. For measurement to occur, the 'source' is in equilibrium with the 'sensor'. And so, the average energy at the 'sensor' will equal to the average energy of the 'source'. We will then have that $E_{av} = kT$. Substituting, we will get that $\Delta t = \dfrac{h}{kT}$. Therefore $h = \int_s^{s+\frac{h}{kT}} E(u)du$, an *exact mathematical identity*. If we assume that $E(u) = E_0 e^{\nu u}$ (as suggested by the Theorem stated above), integrating we get $h = \int_s^{s+\frac{h}{kT}} E_0 e^{\nu u} du = \dfrac{E(s)}{\nu}\left[e^{h\nu/kT} - 1\right]$, and from this we get Planck's Law for 'blackbody' radiation, $E(s) = \dfrac{h\nu}{e^{h\nu/kT} - 1}$.

Using this understanding of Planck's Law and its derivation, we can now easily explain the phenomenon of 'energy quanta'; why energy is proportional to the frequency of radiation; why the energy of a single quantum is $h\nu$; the true meaning of *h* and ν *;* and the energy-time 'uncertainty principle'.

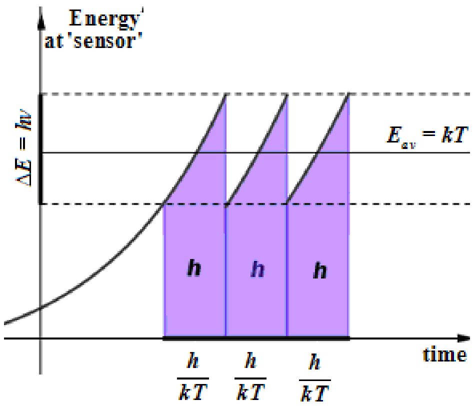

figure 3

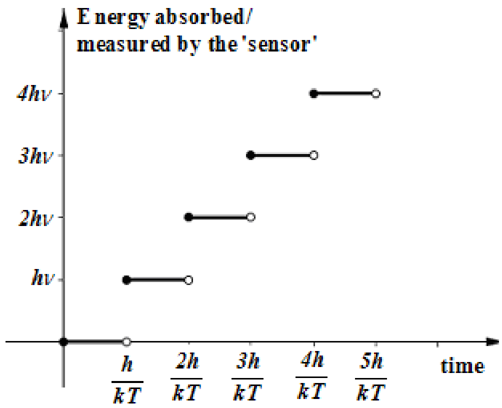

figure 4

When a measurement is made the 'sensor' absorbs a (minimal) amount of energy ΔE and therefore the function *E(t)* 'collapses' *(figure 3).* We can calculate this (minimal) amount of energy ΔE absorbed by the 'sensor' to be

$$\Delta E = E(s+\Delta t) - E(s) = E_0 e^{\nu(s+\Delta t)} - E_0 e^{\nu s} = E_0 e^{\nu s}\left[e^{\nu \Delta t} - 1\right] = E(s)\left[e^{h\nu/kT} - 1\right] = h\nu$$

This explains why the change in energy is proportional to the frequency and is $h\nu$. *Figure 4* shows the energy absorbed by the 'sensor' is in discrete integral units of $h\nu$. This shows that though the propagation of energy can be continuous, the measurement of energy is made in discrete *'equal size sips'*. *Figure 4* also shows that for constant temperature *T* the energy absorbed by the 'sensor' is linear with respect to time, with slope equal to $\nu\, kT$. *(This provides an easy way of experimentally confirming these results).* Note further that if we take $\Delta t \geq \dfrac{1}{\nu}$ (the 'wavelength'), since the minimal energy is $\Delta E = h\nu$ we have that $\Delta E \cdot \Delta t \geq h$ (energy-time 'uncertainty principle').

Furthermore, it can be mathematically shown that no matter what is the value of ΔE absorbed by the 'sensor', Planck's Law always reduces to the same familiar form, $E = \frac{h\nu}{e^{h\nu/kT} - 1}$. That is to say, *the exact mathematical identity* $E = \frac{\Delta E}{e^{\Delta E/E_{av}} - 1}$ *is invariant to* ΔE. This helps explains why Planck's Law is such an exact fit to data.

The assumption $E(t) = E_0 e^{\nu t}$ makes Planck's Formula an *'exact mathematical identity'*. However, if we were to consider that *E(t)* is only an integrable function we will still have that Planck's Formula is the 'best fit' to the experimental data. In theory we cannot do better than Planck's Formula in describing the experimental results, as the following argument shows.

It is easy to mathematically verify that for any integrable function *E(t),* $\lim_{t \to s} \frac{rP}{e^{r(t-s)} - 1} = E(s)$. By setting $P = h$, $\Delta t = \frac{h}{kT}$ and $r = \nu$ as above, we will have $\frac{h\nu}{e^{h\nu/kT} - 1} \approx E(s)$. Since *h* is the (minimal) 'accumulation of energy' and $\Delta t = \frac{h}{kT}$ is the corresponding (minimal) time interval for measurement, this approximation represents the 'best fit' that we can theoretically have to experimental data. The data itself cannot be any different from this.

Summary:

- Planck's Law describes the 'interaction of measurement' – i.e the functional relationship between the energy *E(t)* at the 'sensor' at any time, the energy ΔE 'absorbed' by the 'sensor' making the measurement, and the average energy E_{av} at the 'sensor' during measurement.
- Planck's Law can be derived using continuous processes and without needing the 'quantization of energy' hypothesis.
- Planck's Law is an exact mathematical identity which is invariant to the amount of energy ΔE absorbed by the 'sensor'. No matter what is the value of ΔE, or if Δt is positive or negative, the Formula always reduces to the same familiar form. This proves that Planck's Law is independent of the instrumentation or methodology used in making measurements, and why Planck's Law is such an exact fit to data.
- The experimental phenomenon of 'energy quanta' has a simple and intuitive explanation. Planck's constant *h* is the minimal amount of accumulated energy that can be manifested (absorbed/measured).
- The quantization of energy hypothesis, $\Delta E = h\nu$, can be mathematically proven.
- The time required for an amount *h* of accumulated energy to manifest at temperature T is equal to *h/kT.*
- The energy-time 'uncertainty principle' can be mathematically explained.

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