CAUCHY TETRAHEDRON ARGUMENT AND THE PROOFS FOR THE EXISTENCE OF STRESS TENSOR, A COMPREHENSIVE REVIEW, CHALLENGES, AND IMPROVEMENTS

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Abstract. Cauchy in 1822 presented the idea of traction vector that contains both the normal and tangential components of the internal surface forces per unit area and gave the tetrahedron argument to prove the existence of stress tensor. These great achievements form the main part of the foundation of continuum mechanics. During nearly two centuries, some versions of tetrahedron argument and a few other proofs for the existence of stress tensor are presented in every text in continuum mechanics, fluid mechanics, and the related subjects. In this article, we show the birth, importance, and location of these Cauchy's achievements, then by presenting the formal tetrahedron argument in detail, for the first time we extract some fundamental challenges. These conceptual challenges are related to the result of applying the conservation of linear momentum to any mass element and the order of its surface and volume terms, the definition of traction vectors on the surfaces that pass through the same point, the limiting and approximating processes in the derivation of stress tensor, and some others. In a comprehensive review, we present the different tetrahedron arguments and the proofs for the existence of stress tensor, consider the challenges in each one, and classify them in two general approaches. In the first approach that is followed in most texts, the traction vectors do not define exactly on the surfaces that pass through the same point so, most of the challenges hold. But in the second approach, the traction vectors are defined on the surfaces that pass exactly through the same point, so some of the related challenges are removed. We also represent the improved works of Hamel and Backus, and show that the original work of Backus removes most of the challenges. This article shows that the foundation of continuum mechanics is not a finished subject and there are still some fundamental challenges.

1. Introduction

For the first time, Cauchy in 1822 in his lecture announced the forces on the surface of an internal mass element in continuum media in addition to the normal component on the surface can have the tangential components. An abstract of his lecture was published in 1823, [17]. In translation of Cauchy's lecture from the French by Maugin (2014, [63]), on page 50:

> *However, the new "pressure" will not always be perpendicular to the faces on which it act, and is not the same in all directions at a given point. ...Furthermore, the pressure or tension exerted on any plane can easily*

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> *be deduced, in both amplitude and direction, from the pressures or tensions exerted on three given orthogonal planes. I had reached this point when M. Fresnel, who came to me to talk about his works devoted to the study of light ...*

Here the *new pressure* is the traction vector on the internal surface that it acts and contains both the normal and tangential components. Cauchy's works in continuum mechanics during 1822 to 1828 lead to derivation of *Cauchy lemma* for traction vectors, the existence of *stress tensor*, *Cauchy equation of motion*, *symmetry of stress tensor*, and some other achievements in the foundation of continuum mechanics [99]. Cauchy's proof for the existence of stress tensor is called *Cauchy tetrahedron argument*. From Truesdell (1971, [97]), on page 8:

> *CAUCHY's theorem of the existence of the stress tensor, published in* 1823*. CAUCHY, who knew full well the difference between a balance principle and a constitutive relation, stated the result clearly and proudly; he gave a splendid proof of it, which has been reproduced in every book on continuum mechanics from that day to this; and he recognized the theorem as being the foundation stone it still is.*

On the importance of Cauchy idea for traction vector and tetrahedron argument for the existence of stress tensor, Truesdell (1968, [96]), on page 188, says:

> *Clearly this work of Cauchy's marks one of the great turning points of mechanics and mathematical physics, even though few writers on the history of that subject seem to know it, a turning point that could well stand comparison with Huygens's theory of the pendulum, Newton's theory of the solar system, Euler's theory of the perfect fluid, and Maxwell's theories of the monatomic gas and the electromagnetic field.*

This article gives a comprehensive review on the tetrahedron arguments and the proofs for the existence of stress tensor that represented during nearly two centuries, from 1822 until now, in many books and articles in continuum mechanics, fluid mechanics, solid mechanics, elasticity, strength of materials, etc. There are different methods and processes to prove the existence of stress tensor and presentation of the Cauchy tetrahedron argument in the literature. We extract some fundamental challenges on these proofs and consider these challenges in each one. To enter the subject, we first show the location of the Cauchy tetrahedron argument for the existence of stress tensor in the general steps of the foundation of continuum mechanics. Then, a formal proof of the Cauchy tetrahedron argument based on the accepted reference books will be given. We extract some fundamental challenges on this proof and their importance in the foundation of continuum mechanics. Then we review and consider different proofs in the literature and their challenges. During this review, we also show the general approaches, important works and their enhancements.

2. Location of Cauchy tetrahedron argument in the foundation of continuum mechanics

Although the birth of modern continuum mechanics is considered as the idea of Cauchy in 1822 [63], but some remarkable achievements were obtained earlier by famous mathematical physicians like Daniel Bernoulli, Euler, D'Alembert, Navier, Poisson and the others. In general, these achievements can be addressed as the splitting of forces to the body forces and surface forces, definition of pressure as the normal surface force per unit area, the concept of considering an internal mass element in continuum media, the Euler equation of motion, etc. But this was the genius of Cauchy to use the idea of his friend Fresnel -that worked on optics- in continuum mechanics and develop the idea of traction vector, the existence of stress tensor, its properties and the general equation of motion [17], [18], and [63]. Cauchy's achievements rapidly stand as the foundation of continuum mechanics and related subjects such as fluid mechanics, solid mechanics, elasticity, mechanics of deformable bodies, strength of materials, etc., [95]. A good representation to describe the Cauchy's papers and the situation of continuum mechanics at that time was given by Maugin (2014, [63]), recently.

The general steps that lead to the general concept of stress in continuum mechanics can be described as the followings steps. Some of these steps were developed before Cauchy and some of them were developed or revised by Cauchy based on the new idea of traction vector that contains both the normal and tangential components on the surface.

- The forces that apply to a fluid or solid element in continuum media can split to the *surface forces* ($\boldsymbol{F}_s$) and the *body forces* ($\boldsymbol{F}_b$), (before Cauchy).

$$\boldsymbol{F} = \boldsymbol{F}_s + \boldsymbol{F}_b \tag{2.1}$$

- The surface force can be formulated as surface force per unit area that is called *pressure* and is normal to the surface that it acts, (before Cauchy).

- The surface force per unit area in addition to the normal component (t_n) can have tangential components (t_t). This general surface force per unit area is called *traction vector*, (by Cauchy in 1822).

$$\boldsymbol{t} = t_n\boldsymbol{e}_n + t_t\boldsymbol{e}_t \tag{2.2}$$

- The traction vector depends only on the position vector ($\boldsymbol{r}$), time (t) and the outward unit normal vector of the surface ($\boldsymbol{n}$) that acts on it in continuum media, (by Cauchy).

$$\boldsymbol{t} = \boldsymbol{t}(\boldsymbol{r}, t, \boldsymbol{n}) \tag{2.3}$$

- The traction vectors acting on opposite sides of the same surface at a given point and time are equal in magnitude but opposite in direction. This is called *Cauchy lemma*, (by Cauchy).

$$\boldsymbol{t}(\boldsymbol{r}, t, \boldsymbol{n}) = -\boldsymbol{t}(\boldsymbol{r}, t, -\boldsymbol{n}) \tag{2.4}$$

- *Cauchy tetrahedron argument* tells us that the relation between traction vector and the unit normal vector is linear and this leads to the existence of a second order tensor that is called *stress tensor*. The stress tensor $\boldsymbol{T}$ depends only on the position vector and time, (by Cauchy).

$$\boldsymbol{t} = \boldsymbol{T}^T.\boldsymbol{n} \tag{2.5}$$

where

$$\boldsymbol{T} = \boldsymbol{T}(\boldsymbol{r}, t) = \begin{bmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{bmatrix} \tag{2.6}$$

- Applying the conservation of linear momentum to a mass element in continuum media leads to the general differential equation of motion that is called *Cauchy equation of motion*, (by Cauchy).

$$\rho\boldsymbol{a} = \nabla.\boldsymbol{T} + \rho\boldsymbol{b} \tag{2.7}$$

or

$$\rho(\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}.\nabla)\boldsymbol{v}) = \nabla.\boldsymbol{T} + \rho\boldsymbol{b} \tag{2.8}$$

where ρ, $\boldsymbol{b}$, $\boldsymbol{a}$, and $\boldsymbol{v}$ are the density, body force per unit mass, acceleration, and velocity, respectively.

- The conservation of angular momentum shows the stress tensor is symmetric, (by Cauchy).

$$T_{xy} = T_{yx} \qquad T_{xz} = T_{zx} \qquad T_{yz} = T_{zy} \tag{2.9}$$

or

$$\boldsymbol{T} = \boldsymbol{T}^T \tag{2.10}$$

These steps show the location of Cauchy tetrahedron argument for the existence of stress tensor in the foundation of continuum mechanics.

3. Cauchy tetrahedron argument and the challenges

The following representation of Cauchy tetrahedron argument is based on the two important reference books in continuum mechanics i.e. "Truesdell and Toupin, The Classical Field Theories, pp. 542-543" (1960, [99]) and "Malvern, Introduction to the Mechanics of a Continuous Medium, pp. 73-77" (1969, [59]). Here we give more details to show the process clearly.

3.1. **Cauchy tetrahedron argument.**

Imagine a tetrahedron element in continuum media that its vortex is at point $\boldsymbol{o}$ and its three orthogonal faces are parallel to the three orthogonal planes of the Cartesian coordinate system. The fourth surface of the tetrahedron, i.e. its base, has the outward unit normal vector $\boldsymbol{n}_4$. The geometry parameters and the average values of the traction vectors on the faces of tetrahedron are shown in Figure 1. The integral equation of conservation of linear momentum on a mass element $\mathcal{M}$ in continuum media is:

$$\int_{\partial\mathcal{M}} \boldsymbol{t}\, dS + \int_{\mathcal{M}} \rho\boldsymbol{b}\, dV = \int_{\mathcal{M}} \rho\boldsymbol{a}\, dV \tag{3.1}$$

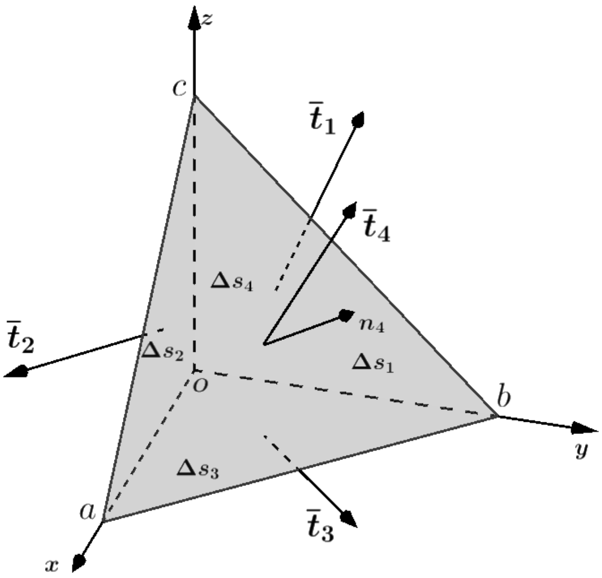

FIGURE 1. Tetrahedron geometry and the average traction vectors on the faces.

Now this law applies to the tetrahedron mass element. By averaging variables on the volume and faces of tetrahedron element, it becomes:

$$\bar{\boldsymbol{t}}_4\Delta s_4 + \bar{\boldsymbol{t}}_1\Delta s_1 + \bar{\boldsymbol{t}}_2\Delta s_2 + \bar{\boldsymbol{t}}_3\Delta s_3 + \overline{\rho\boldsymbol{b}}\Delta V = \overline{\rho\boldsymbol{a}}\Delta V \tag{3.2}$$

where the overline indicates the average values of these terms. The following geometrical relations for the areas and volume of tetrahedron hold:

$$\begin{aligned} &\Delta s_1 = n_x\Delta s_4, \qquad \Delta s_2 = n_y\Delta s_4, \qquad \Delta s_3 = n_z\Delta s_4 \\ &\Delta V = \frac{1}{3}h\Delta s_4 \end{aligned} \tag{3.3}$$

where n_x, n_y, and n_z are components of the outward unit normal vector on Δs_4, i.e. $\boldsymbol{n}_4 = n_x\boldsymbol{e}_x + n_y\boldsymbol{e}_y + n_z\boldsymbol{e}_z$. Here h is the altitude of the tetrahedron. By substituting these geometrical relations into the equation (3.2):

$$\bar{\boldsymbol{t}}_4\Delta s_4 + \bar{\boldsymbol{t}}_1(n_x\Delta s_4) + \bar{\boldsymbol{t}}_2(n_y\Delta s_4) + \bar{\boldsymbol{t}}_3(n_z\Delta s_4) + \overline{\rho\boldsymbol{b}}(\frac{1}{3}h\Delta s_4) = \overline{\rho\boldsymbol{a}}(\frac{1}{3}h\Delta s_4)$$

dividing through by Δs_4

$$\bar{\boldsymbol{t}}_4 + n_x\bar{\boldsymbol{t}}_1 + n_y\bar{\boldsymbol{t}}_2 + n_z\bar{\boldsymbol{t}}_3 + \overline{\rho\boldsymbol{b}}(\frac{1}{3}h) = \overline{\rho\boldsymbol{a}}(\frac{1}{3}h) \tag{3.4}$$

Now decrease the volume of tetrahedron element $\Delta V \to 0$ in the way that $\boldsymbol{n}_4$ and the position of vertex point of tetrahedron do not change. As a result, $h \to 0$ and the tetrahedron shrinks to a point. So, in this limit, the body force and inertia term in the equation (3.4) go to zero and the average traction vectors go to the exact values. The result is:

$$\boldsymbol{t}_4 + n_x\boldsymbol{t}_1 + n_y\boldsymbol{t}_2 + n_z\boldsymbol{t}_3 = \boldsymbol{0} \tag{3.5}$$

The traction vector $\boldsymbol{t}_1$ is applied to the surface Δs_1 by the unit normal vector of $\boldsymbol{n}_1 = -1\boldsymbol{e}_x$. Using the Cauchy lemma, i.e. $\boldsymbol{t}(\boldsymbol{r}, t, \boldsymbol{n}) = -\boldsymbol{t}(\boldsymbol{r}, t, -\boldsymbol{n})$:

$$\boldsymbol{t}(\boldsymbol{n}_1) = -\boldsymbol{t}(-\boldsymbol{n}_1) \tag{3.6}$$

but $-\boldsymbol{n}_1 = +1\boldsymbol{e}_x$ is the unit normal vector on the positive side of coordinate plane yz. If $\boldsymbol{t}_x$ is the traction vector on the positive side of coordinate plane yz, then by using the relation (3.6):

$$\boldsymbol{t}_1 = -\boldsymbol{t}_x \tag{3.7}$$

This strategy for $\boldsymbol{t}_2$ and $\boldsymbol{t}_3$ leads to:

$$\boldsymbol{t}_2 = -\boldsymbol{t}_y, \qquad \boldsymbol{t}_3 = -\boldsymbol{t}_z \tag{3.8}$$

By substituting these relations into the equation (3.5):

$$\boldsymbol{t}_4 + n_x(-\boldsymbol{t}_x) + n_y(-\boldsymbol{t}_y) + n_z(-\boldsymbol{t}_z) = \boldsymbol{0}$$

so

$$\boldsymbol{t}_4 = n_x\boldsymbol{t}_x + n_y\boldsymbol{t}_y + n_z\boldsymbol{t}_z \tag{3.9}$$

The traction vectors $\boldsymbol{t}_x$, $\boldsymbol{t}_y$, and $\boldsymbol{t}_z$ can be shown by their components as:

$$\begin{aligned} \boldsymbol{t}_x &= T_{xx}\boldsymbol{e}_x + T_{xy}\boldsymbol{e}_y + T_{xz}\boldsymbol{e}_z \\ \boldsymbol{t}_y &= T_{yx}\boldsymbol{e}_x + T_{yy}\boldsymbol{e}_y + T_{yz}\boldsymbol{e}_z \\ \boldsymbol{t}_z &= T_{zx}\boldsymbol{e}_x + T_{zy}\boldsymbol{e}_y + T_{zz}\boldsymbol{e}_z \end{aligned} \tag{3.10}$$

By substituting these definitions into the equation (3.9):

$$\begin{aligned} \boldsymbol{t}_4 = {} & n_x(T_{xx}\boldsymbol{e}_x + T_{xy}\boldsymbol{e}_y + T_{xz}\boldsymbol{e}_z) + n_y(T_{yx}\boldsymbol{e}_x + T_{yy}\boldsymbol{e}_y + T_{yz}\boldsymbol{e}_z) \\ & + n_z(T_{zx}\boldsymbol{e}_x + T_{zy}\boldsymbol{e}_y + T_{zz}\boldsymbol{e}_z) \end{aligned} \tag{3.11}$$

or

$$\begin{aligned} \boldsymbol{t}_4 = {} & (n_xT_{xx} + n_yT_{yx} + n_zT_{zx})\boldsymbol{e}_x + (n_xT_{xy} + n_yT_{yy} + n_zT_{zy})\boldsymbol{e}_y \\ & + (n_xT_{xz} + n_yT_{yz} + n_zT_{zz})\boldsymbol{e}_z \end{aligned} \tag{3.12}$$

This relation can be shown by a second order tensor and vector relation as:

$$\boldsymbol{t}_4 = \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}_4 = \begin{bmatrix} n_xT_{xx} + n_yT_{yx} + n_zT_{zx} \\ n_xT_{xy} + n_yT_{yy} + n_zT_{zy} \\ n_xT_{xz} + n_yT_{yz} + n_zT_{zz} \end{bmatrix} = \begin{bmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{bmatrix}^T \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix}_4 \tag{3.13}$$

therefore

$$\boldsymbol{t}_4 = \boldsymbol{T}^T.\boldsymbol{n}_4 \tag{3.14}$$

By forming the tetrahedron element, no one of the components of $\boldsymbol{n}_4$ is zero. For the unit normal vectors that one or two of their components are equal to zero, the tetrahedron element does not form but due to the continuous property of the traction vectors on $\boldsymbol{n}$ and the arbitrary choosing for any orthogonal basis for the coordinate system, the relation (3.14) is valid for these cases, as well. So, the subscript 4 can be removed from this relation:

$$\boldsymbol{t} = \boldsymbol{T}^T.\boldsymbol{n} \tag{3.15}$$

This relation shows that there is a second order tensor that is called stress tensor for describing the state of stress. This tensor, $\boldsymbol{T} = \boldsymbol{T}(\boldsymbol{r}, t)$ depends only on the position vector and time. Also, the relation between the traction vector on any surface and the unit normal vector of the surface that it acts on it, is linear.

Here the tetrahedron argument is finished. This argument and its result have a great importance and role in the foundation of continuum mechanics.

The following statements are not the elements of the tetrahedron argument and we state them to show that the two other important achievements of Cauchy in the foundation of continuum mechanics. Cauchy then applied the conservation of linear momentum to a "cubic element" and using his previous achievements, derived the general equation of motion that is called *Cauchy equation of motion* [63].

$$\rho \boldsymbol{a} = \nabla.\boldsymbol{T} + \rho \boldsymbol{b} \tag{3.16}$$

or

$$\rho(\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}.\nabla)\boldsymbol{v}) = \nabla.\boldsymbol{T} + \rho \boldsymbol{b} \tag{3.17}$$

Also, by applying the conservation of angular momentum to a "cubic element", he showed that the stress tensor is symmetric [63].

$$\boldsymbol{T} = \boldsymbol{T}^T \tag{3.18}$$

3.2. **The challenges.**

During consideration of the presented tetrahedron argument we find some conceptual challenges on it. In the following we present and discuss them.

- Challenge 1: Note that applying the conservation of linear momentum to any mass element with any shape must lead to the general equation of motion that contains all of the effective terms including inertia, body forces, and surface forces (Cauchy equation of motion). But in this argument applying the conservation of linear momentum to the tetrahedron element leads to the equation (3.5), i.e. $\boldsymbol{t}_4 + n_x\boldsymbol{t}_1 + n_y\boldsymbol{t}_2 + n_z\boldsymbol{t}_3 = \boldsymbol{0}$, that differs from the equation of motion (3.16). Because the inertia and body forces do not exist in it. We saw that after presenting the tetrahedron argument, Cauchy and most of the authors derived the equation of motion by applying the conservation of linear momentum to a cubic element. What is the problem, applying the conservation of linear momentum to a tetrahedron element leads to the equation $\boldsymbol{t}_4 + n_x\boldsymbol{t}_1 + n_y\boldsymbol{t}_2 + n_z\boldsymbol{t}_3 = \boldsymbol{0}$ and to a cubic element leads to the Cauchy equation of motion?

- Challenge 2: The tetrahedron argument is based on the limit $\Delta V \to 0$, that is stated by all of the authors who presented this argument by the expressions like "$\Delta V \to 0$", "$h \to 0$", *"when the tetrahedron shrinks to a point"* or *"when the tetrahedron shrinks to zero volume"*, while it must be proved that the existence of stress tensor at a point does not depend on the size of the considered mass element. In other words, the stress tensor exists for any size of mass element in continuum media where the volume of element increases, decreases or does not change. By these proofs the result is only for the infinitesimal volumes and they did not show that this result can be applied to the mass elements with any volume in continuum media.

- Challenge 3: This tetrahedron argument is based on the average values of the effective terms in the integral equation of conservation of linear momentum and even for the limit $\Delta V \to 0$, this trend remains. While the stress tensor and the traction vectors relations are point-based and must be derived from the exact point values not from average or approximate values.

- Challenge 4: During the tetrahedron argument we have the equation (3.4):

$$\bar{\boldsymbol{t}}_4 + n_x\bar{\boldsymbol{t}}_1 + n_y\bar{\boldsymbol{t}}_2 + n_z\bar{\boldsymbol{t}}_3 + \overline{\rho\boldsymbol{b}}(\frac{1}{3}h) = \overline{\rho\boldsymbol{a}}(\frac{1}{3}h)$$

If we rewrite this equation as following, in the limit:

$$\lim_{h\to 0}\left(\frac{\bar{\boldsymbol{t}}_4 + n_x\bar{\boldsymbol{t}}_1 + n_y\bar{\boldsymbol{t}}_2 + n_z\bar{\boldsymbol{t}}_3}{\frac{1}{3}h}\right) = \lim_{h\to 0}(\overline{\rho\boldsymbol{a}} - \overline{\rho\boldsymbol{b}}) \tag{3.19}$$

clearly the right hand side limit exists, because $\overline{\rho\boldsymbol{a}} - \overline{\rho\boldsymbol{b}}$ is bounded and generally is not equal to zero in continuum media. So, the left hand side limit must be existed and is not equal to zero, in general. This means that the numerator has at least one term in the same order of the denominator, i.e.:

$$O(\bar{\boldsymbol{t}}_4 + n_x\bar{\boldsymbol{t}}_1 + n_y\bar{\boldsymbol{t}}_2 + n_z\bar{\boldsymbol{t}}_3) = O(h) \tag{3.20}$$

so, $\bar{\boldsymbol{t}}_4 + n_x\bar{\boldsymbol{t}}_1 + n_y\bar{\boldsymbol{t}}_2 + n_z\bar{\boldsymbol{t}}_3$ and $\overline{\rho\boldsymbol{a}}(\frac{1}{3}h) - \overline{\rho\boldsymbol{b}}(\frac{1}{3}h)$ have the same order of h. This means that by $h \to 0$ these two parts decrease by the same rate to zero and we can not tell that the inertia and body terms go to zero by a rate faster than the surface terms. Since $O(\Delta s_4) = O(h^2)$ and $O(\Delta V) = O(h^3)$, we have:

$$\begin{aligned} &O(\Delta s_4(\boldsymbol{t}_4 + n_x\boldsymbol{t}_1 + n_y\boldsymbol{t}_2 + n_z\boldsymbol{t}_3)) \\ &= O(\boldsymbol{t}_4\Delta s_4 + \boldsymbol{t}_1\Delta s_1 + \boldsymbol{t}_2\Delta s_2 + \boldsymbol{t}_3\Delta s_3) = O(h^3) \end{aligned} \tag{3.21}$$

and

$$O(\overline{\rho\boldsymbol{a}}\Delta V - \overline{\rho\boldsymbol{b}}\Delta V) = O(h^3) \tag{3.22}$$

so, we can not tell that if $\Delta V \to 0$ or $h \to 0$ then the surface terms go to zero by $O(h^2)$ and the inertia and body terms go to zero by $O(h^3)$, because these two parts have the same order i.e. h^3, as shown above in (3.21) and (3.22).

- Challenge 5: The purpose of Cauchy tetrahedron argument is to show that the traction vector at a point on a surface is a linear combination of the traction vectors on the three orthogonal surfaces that pass through that point. So, the four surfaces must pass through the same point to prove this relation between their traction vectors. But in the tetrahedron argument $\boldsymbol{t}_4$ is defined on the surface Δs_4 that does not pass through the vertex point of tetrahedron where the three surfaces Δs_1, Δs_2, and Δs_3 pass through it, see Figure 2.

- Challenge 6: The stress tensor is a point function. This means at any point in continuum media the stress tensor exists. So, in the equation $\boldsymbol{t}_4 + n_x\boldsymbol{t}_1 + n_y\boldsymbol{t}_2 + n_z\boldsymbol{t}_3 = \boldsymbol{0}$ the four traction vectors must belong to a unit point to conclude from the tetrahedron argument that $\boldsymbol{t}_4$ is related to a tensor that forms by the components of $\boldsymbol{t}_1$, $\boldsymbol{t}_2$, and $\boldsymbol{t}_3$. While in this proof, the surface that $\boldsymbol{t}_4$ is defined on it, i.e. Δs_4, does not pass through point $\boldsymbol{o}$, even for an infinitesimal tetrahedron element, see Figure 2.

- Challenge 7: The result of this argument is the relation (3.5), i.e. $\boldsymbol{t}_4 + n_x\boldsymbol{t}_1 + n_y\boldsymbol{t}_2 + n_z\boldsymbol{t}_3 = \boldsymbol{0}$, for an infinitesimal tetrahedron. Here the traction vectors are the average values on the faces of this infinitesimal tetrahedron. If we multiply

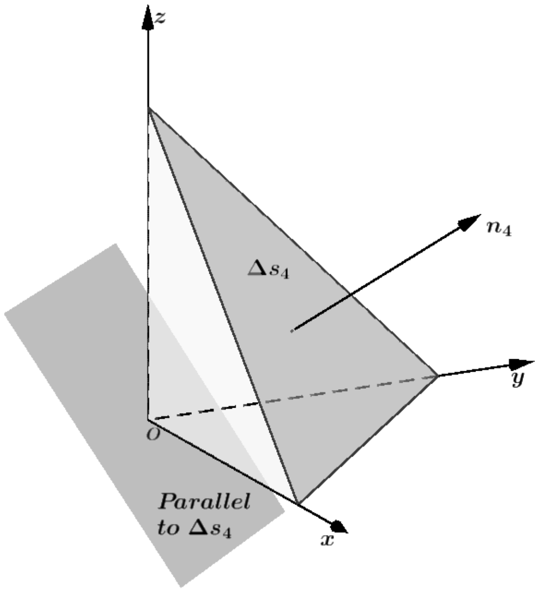

FIGURE 2. Inclined plane that is parallel to Δs_4 and passes through point $\boldsymbol{o}$.

this relation by Δs_4 that is the infinitesimal base area of this tetrahedron, as:

$$\Delta s_4(\boldsymbol{t}_4 + n_x\boldsymbol{t}_1 + n_y\boldsymbol{t}_2 + n_z\boldsymbol{t}_3) = \boldsymbol{t}_4\Delta s_4 + \boldsymbol{t}_1\Delta s_1 + \boldsymbol{t}_2\Delta s_2 + \boldsymbol{t}_3\Delta s_3 = \boldsymbol{0} \tag{3.23}$$

but this is equal to the integral of $\boldsymbol{t}$ over the surface of $\mathcal{M}$, so:

$$\boldsymbol{t}_4\Delta s_4 + \boldsymbol{t}_1\Delta s_1 + \boldsymbol{t}_2\Delta s_2 + \boldsymbol{t}_3\Delta s_3 = \int_{\partial\mathcal{M}} \boldsymbol{t}\, dS = \boldsymbol{0} \tag{3.24}$$

where $\mathcal{M}$ is the infinitesimal tetrahedron element. This equation tells us that for the infinitesimal tetrahedron element the sum of the traction vectors on the surfaces of this element is zero. This means the surface forces have not any effect on the motion and acceleration of the element because their sum on the faces of element is zero. But this is not correct, since for any volume of mass element, even infinitesimal volume, the equation of conservation of linear momentum (3.1), the following equation, is valid and tells us that this sum is not zero:

$$\int_{\partial\mathcal{M}} \boldsymbol{t}\, dS + \int_{\mathcal{M}} \rho\boldsymbol{b}\, dV = \int_{\mathcal{M}} \rho\boldsymbol{a}\, dV$$

- Challenge 8: In the previous challenge the relation (3.23), $\boldsymbol{t}_4\Delta s_4+\boldsymbol{t}_1\Delta s_1+\boldsymbol{t}_2\Delta s_2+\boldsymbol{t}_3\Delta s_3 = \boldsymbol{0}$, tells us that the sum of surface forces on faces of the infinitesimal tetrahedron element is zero. So, it tells nothing about the relation between the traction vectors at a point on four different surfaces that pass through that point, since clearly $\boldsymbol{t}_4$ is defined on Δs_4 and this surface does not pass through point $\boldsymbol{o}$, even for an infinitesimal tetrahedron element, see Figure 2.

Here we do not need to state more challenges.

4. A COMPREHENSIVE REVIEW

The tetrahedron argument for the existence of stress tensor followed by many significant scientists and authors during about two centuries from 1822 to now by some different versions. These proofs lead to the linear relation between the traction vector and the unit outward normal vector of the surface. This argument shows that the stress tensor exists and is independent of the surface characters. In the following we show the different processes to prove this argument that exist in many textbooks of continuum mechanics and the related subjects such as fluid dynamics, solid mechanics, elasticity, plasticity, strength of materials, mathematical physics, etc.

4.1. **The first approach.**
Stokes in the famous article (1845, [90]), uses the Cauchy tetrahedron argument. On page 295:

> *...Suppose now the dimensions of the tetrahedron infinitely diminished, then the resolved parts of the external and of the effective moving forces will vary ultimately as the cubes, and those of the pressures and tangential forces as the squares of homologous lines. ...*
> *The method of determining the pressure on any plane from the pressures on three planes at right angles to each other, which has just been given, has already been employed by MM. Cauchy and Poisson.*

So, from the part of *"now the dimensions of the tetrahedron infinitely diminished"* we can tell that Stokes's proof is based on infinitesimal volume. In this expression the inertia and body terms *"vary ultimately as the cubes"* and surface terms vary *"as the squares of homologous lines"*. While we showed in the challenge 4 that the surface terms and the inertia and body term must vary by the same order.

Let see what is presented in the important book by Love, 1908. On pages 76-78 of the fourth edition of this book (1944, [57]), during the tetrahedron argument:

$$\iiint \rho f_x \, dxdydz = \iiint \rho X \, dxdydz + \iint X_v \, dS \qquad (1)$$

> *[where f_x, X, and X_v are acceleration, body force, and surface traction, respectively, all in direction x.]*[1]
> *46. Law of equilibrium of surface tractions on small volumes.*
> *From the forms alone of equations (1) ... we can deduce a result of great importance. Let the volume of integration be very small in all its dimensions, and let l^3 denote this volume. If we divide both members of equation (1) by l^2, and then pass to a limit by diminishing l indefinitely, we find the equation*
>
> $$\lim_{l\to 0} l^{-2} \iint X_v \, dS = 0$$
>
> *... The equations of which these are types can be interpreted in the statement:*
> *"The tractions on the elements of area of the surface of any portion of a body, which is very small in all its dimensions, are ultimately, to a first*

[1]The comments in the brackets [ ] are given by the author of this article.

> *approximation, a system of forces in equilibrium."*
> *...For a first approximation, when all the edges of the tetrahedron are small, we may take the resultant traction of the face [Δs_4]...*

So, here on these pages of Love's book, we see clearly the important challenges that are stated in the previous section. For example, *"For a first approximation"*, *"when all the edges of the tetrahedron are small"*, *"Law of equilibrium of surface tractions on small volumes"*, *"Let the volume of integration be very small"* and clearly in the important statement between "" that means for a first approximation, the summation of traction vectors on the surfaces of any portion of a body is zero when the portion is very small. We find that the Love's book is very important because it clearly and correctly represents the classical continuum mechanics in detail. For example, on these pages he correctly stated that the results of Cauchy tetrahedron argument and the relation of traction vectors are approximately, for very small portion of body and the relation between traction vectors is for the surfaces of mass element that do not pass through the same point. If instead of *"divide both members of equation (1) by l^2"* we divide them by l^3, then the limit $l \to 0$ gives:

$$\lim_{l\to 0} l^{-3} \iint X_v \, dS = \lim_{l\to 0} l^{-3} \iiint \rho(f_x - X)\, dxdydz = \overline{\rho(f_x - X)}$$

Similar to the challenge 4, here $\overline{\rho(f_x - X)}$ is a bounded value and generally is not equal to zero in continuum media. So, for the existence of the limit in the left hand side the order of surface integral in the limit must be equal to l^3, i.e.:

$$O\Big(\iint X_v \, dS\Big) = l^3$$

that is equal to the order of the volume integrals, i.e., l^3. So, the surface tractions are not in equilibrium even on small volumes, but are equal to the volume terms including inertia and body forces. By dividing *"both members of equation* (1) *by l^2,"* then these two parts have the same order l, thus in the *"limit by diminishing l indefinitely"*, these two parts go to zero by the same rate. This is the trivial solution of the equation and can not be a rigorous base for the existence of stress tensor. The proofs in some books are similar to the Love's proof, for example Planck (1932, [71]), Serrin (1959, [79]), Aris (1989, [3]), Marsden and Hughes (1994, [60]), Ogden (1997, [70]), Leal (2007, [54]), Gonzalez and Stuart (2008, [33]). As a sample, in the book "Vectors, Tensors, and the Basic Equations of Fluid Mechanics" (1989, [3]) by Aris, the proof on pages 100-101 is:

> *The principle of the conservation of linear momentum ...*
>
> $$\frac{d}{dt}\iiint \rho \boldsymbol{v}\, dV = \iiint \rho \boldsymbol{f}\, dV + \iint \boldsymbol{t}_{(\boldsymbol{n})}\, dS \qquad (5.11.3)$$
>
> *...Suppose V is a volume of given shape with characteristic dimension d. Then the volume of V will be proportional to d^3 and the area of S to d^2, with the proportionality constants depending only on the shape. Now let V shrink on a point but preserve its shape, then the first two integrals in Eq.* (5.11.3) *will decrease as d^3 but the last will be as d^2. It follows*

that

$$\lim_{d\to 0} \frac{1}{d^2} \iint \boldsymbol{t}_{(\boldsymbol{n})}\, dS = \boldsymbol{0} \qquad (5.11.5)$$

or, the stresses are locally in equilibrium.
To elucidate the nature of the stress system at a point P we consider a small tetrahedron with three of its faces parallel to the coordinate planes through P and the fourth with normal $\boldsymbol{n}$. ... Then applying the principle of local equilibrium [Eq. (5.11.5)] to the stress forces when the tetrahedron is very small we have

$$\boldsymbol{t}_{(\boldsymbol{n})}\, dA - \boldsymbol{t}_{(1)}\, dA_1 - \boldsymbol{t}_{(2)}\, dA_2 - \boldsymbol{t}_{(3)}\, dA_3$$
$$= (\boldsymbol{t}_{(\boldsymbol{n})} - \boldsymbol{t}_{(1)}\boldsymbol{n}_1 - \boldsymbol{t}_{(2)}\boldsymbol{n}_2 - \boldsymbol{t}_{(3)}\boldsymbol{n}_3) dA = \boldsymbol{0}.$$

Now let T_{ji} denote the i^{th} component of $\boldsymbol{t}_j$ and $t_{(n)i}$ the i^{th} component of $\boldsymbol{t}_{(\boldsymbol{n})}$ so that this equation can be written

$$t_{(n)i} = T_{ji} n_j.$$

Let us see what is presented for the existence of stress tensor in the Timoshenko's books. In the book "Timoshenko and Goodier, Theory of Elasticity, 1934", on page 213 based on the 1951 publication, [94]:

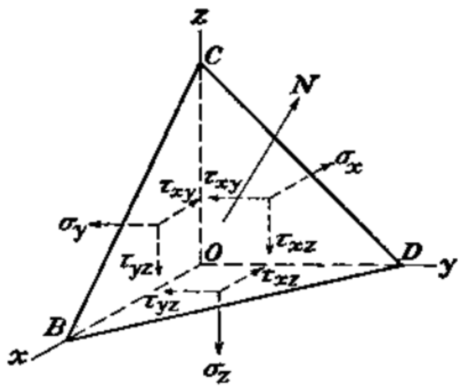

FIG. 132.

... If these components of stress at any point are known, the stress acting on any inclined plane through this point can be calculated from the equations of statics [They considered only the case where acceleration is zero and the body forces can be neglected so, there is no volume integral]. Let O be a point of the stressed body and suppose the stresses are known for the coordinate planes ... (Fig. 132). To get the stress for any inclined plane through O, we take a plane BCD parallel to it at a small distance from O, so that this latter plane together with the coordinate planes cuts out from the body a very small tetrahedron $BCDO$. Since the stresses vary continuously over the volume of the body, the stress acting on the plane BCD will approach the stress on the parallel plane through O as the element is made infinitesimal.
In considering the conditions of equilibrium of the elemental tetrahedron [acceleration is zero] the body forces can be neglected. Also as the element

> *is very small we can neglect the variation of the stresses over the sides and assume that the stresses are uniformly distributed ...*

Timoshenko repeated nearly the same process and comments in the other book "History of Strength of Materials", (1953, [93]). So, in these books we have the average values of the traction vectors on tetrahedron's faces and the traction vector on the fourth surface of the tetrahedron (plane BCD [Δs_4]) is regarded as the traction vector on the inclined plane that passes through point O in an infinitesimal tetrahedron. Therefore, most of the challenges hold. Also, this proof is limited to the cases that the mass element is in equilibrium (acceleration is zero) and the body forces are neglected. Nearly similar process and assumptions are presented for the tetrahedron argument by Prandtl and his coauthors (English translation 2004, [69]). The proofs for the existence of stress tensor in some books are based on nearly similar process and assumptions to the above process, for example Sommerfeld (1950, [88]), Biot (1965, [10]), Feynman, Leighton, and Sands (1965, [27]) (using a wedge rather than a tetrahedron), Borg (1966, [12]), Calcote (1968, [15]), Flügge (1972, [28]), Arfken (1985, [2]), Brekhovskikh and Goncharov (1994, [13]), Salencon (2001, [75]), Kundu, Cohen, and Dowling (2012, [51]), and Chaves (2013, [20]).

Let us see what Truesdell and Toupin presented in the very important reference book "The Classical Field Theories, pp. 542-543" (1960, [99]):

> *...Let the altitude of the tetrahedron be h; the area of the inclined face [Δs_4],... We may then estimate the volume integrals in* (200.1) *[the integral equation of conservation of linear momentum] and apply the theorem of mean value to the surface integral:*
>
> $$\Delta s_4(n_1\boldsymbol{t}_1^* + n_2\boldsymbol{t}_2^* + n_3\boldsymbol{t}_3^* + \boldsymbol{t}_{(\boldsymbol{n})}^*) + h\Delta s_4 K = \boldsymbol{0}, \qquad (203.1)$$
>
> *where K is a bound and where $\boldsymbol{t}_{(\boldsymbol{n})}^*$ [traction vector on Δs_4] and $\boldsymbol{t}_a^*$ [$\boldsymbol{t}_1^*$, $\boldsymbol{t}_2^*$ and $\boldsymbol{t}_3^*$] are the stress vectors at certain points upon the outsides of the respective faces. We cancel Δs_4 and let h tend to zero, so obtaining*
>
> $$\boldsymbol{t}_{(\boldsymbol{n})} = -(\boldsymbol{t}_1 n_1 + \boldsymbol{t}_2 n_2 + \boldsymbol{t}_3 n_3), \qquad (203.2)$$
>
> *where all stress vectors are evaluated at the vertex of the tetrahedron.*

So, the expressions *"then estimate the volume integrals"*, *"apply the theorem of mean value to the surface integral"* and *"let h tend to zero"* show the challenges that we presented in the before section. Here in the last line *"where all stress vectors are evaluated at the vertex of the tetrahedron"* is not obtained exactly and is only an approximating result by this process. Because $\boldsymbol{t}_{(\boldsymbol{n})}$ is defined on the fourth surface of the tetrahedron [Δs_4] and this surface does not pass exactly through the vertex of the tetrahedron when h tend to zero.

In the book "Introduction to the Mechanics of a Continuous Medium" (1969, [59]) by Malvern, on pages 73-76:

> *...Imagine ... a tetrahedron or triangular pyramid bound by parts of the three coordinate planes through O and a fourth plane ABC not passing*

through O, ...
... The asterisks indicate average values; thus $\boldsymbol{b}^*$ is the average value of the body force per unit mass in the tetrahedron. $\boldsymbol{t}^{(\boldsymbol{n})*}$ is the average value of the surface traction per unit area on the oblique face; ...
... then the altitude h will be allowed to approach zero so that the volume and the four surface areas simultaneously approach zero, while the orientation of ON and the position of O do not change. We postulate the continuity of all the components of the stress vectors and the body force and the density as functions of position; it follows that the average values will approach the local values at the point O, and the result will be an expression for the traction vector $\boldsymbol{t}^{(\boldsymbol{n})}$ at the point O in the terms of the three special surface stress vectors $\boldsymbol{t}^{(k)}$ at O ...

$$\boldsymbol{t}^{(\boldsymbol{n})*}\Delta S + \rho^*\boldsymbol{b}^*\Delta V - \boldsymbol{t}^{(1)*}\Delta S_1 - \boldsymbol{t}^{(2)*}\Delta S_2 - \boldsymbol{t}^{(3)*}\Delta S_3 = \rho^*\Delta V\frac{d\boldsymbol{v}^*}{dt}.$$

... dividing through by ΔS, and rearranging terms we obtain

$$\boldsymbol{t}^{(\boldsymbol{n})*} + \frac{1}{3}h\rho^*\boldsymbol{b}^* = \boldsymbol{t}^{(1)*}n_1 + \boldsymbol{t}^{(2)*}n_2 + \boldsymbol{t}^{(3)*}n_3 + \frac{1}{3}h\rho^*\frac{d\boldsymbol{v}^*}{dt}.$$

We now let h approach zero. The last term in each member then approaches zero, while the vectors in the other terms approach the vectors at the point O as is indicated by dropping the asterisks. The result is in the limit

$$\boldsymbol{t}^{(\boldsymbol{n})} = \boldsymbol{t}^{(1)}n_1 + \boldsymbol{t}^{(2)}n_2 + \boldsymbol{t}^{(3)}n_3 = \boldsymbol{t}^{(k)}n_k. \qquad (3.2.7)$$

This important equation permits us to determine the traction $\boldsymbol{t}^{(\boldsymbol{n})}$ at a point, acting on an arbitrary plane through the point, when we know the tractions on only three mutually perpendicular planes through the point. Note that this result was obtained without any assumption of equilibrium. It applies just as well in fluid dynamics as in solid mechanics.

This proof is similar to the one that we presented in the previous section for introducing the Cauchy tetrahedron argument. So, all of the stated challenges hold in this proof. For example, *"plane ABC not passing through O"*, *"asterisks indicate average values"*, *"the average values will approach the local values at the point O"* and *"let h approach zero"*. Note that the postulate in the last paragraph is not exactly but as Love has been told [57], is by a first approximation.

The tetrahedron arguments in many books are nearly similar to the presented proofs by Truesdell and Toupin (1960, [99]) and Malvern (1969, [59]), for example Ilyushin and Lensky (1967, [46]), Jaunzemis (1967, [48]), Rivlin (1969, [73]), Wang (1979, [100]), Eringen (1980, [26]), Narasimhan (1993, [66]), Chandrasekharaiah and Debnath (1994, [19]), Shames and Cozzarelli (1997, [80]), Mase (1999, [62]), Kiselev, Vorozhtsov, and Fomin (1999, [50]), Batchelor (2000, [7]), Basar and Weichert (2000, [6]), Guyon, Hulin, Petit, and Mitescu (2001, [40]), Haupt (2002, [43]), Talpaert (2002, [91]), Jog (2002, [49]), Spencer (2004, [89]), Hutter and Jöhnk (2004, [45]), Han-Chin (2005, [42]), Antman (2005, [1]), Batra (2006, [8]), Dill (2007, [23]), Graebel (2007, [34]), Irgens (2008, [47]), Bonet and Wood (2008, [11]), Nair (2009, [65]), Wegner and Haddow (2009, [101]), Epstein (2010, [25]), Slawinski (2010, [85]), Lai, Rubin, and Krempl (2010, [52]), Reddy (2010, [72]), Lautrup (2011, [53]), Dimitrienko (2011, [24]), Capaldi (2012, [16]), Byskov (2013, [14]), Rudnicki (2015, [74]), and others.

In the book "Introduction to the Mechanics of a Continuous Medium" (1965, [77]) by Sedov, on pages 130-131:

> *... Consider the volume V as an infinitesimal tetrahedron ... with faces MCB, MAB, and MAC perpendicular to the coordinate axes and with face ABC arbitrarily determined by an externally directed unit normal vector ... The stresses on the areas with the normals $\ni_1$, $\ni_2$, $\ni_3$, and $\boldsymbol{n}$ are denoted by $\boldsymbol{p}^1$, $\boldsymbol{p}^2$, $\boldsymbol{p}^3$, and $\boldsymbol{p_n}$, respectively.*
> *... In fact, applying (4.7) [the integral equation of conservation of linear momentum] to the masses of the volume that are inside the infinitesimal tetrahedron $MABC$ at the instant in question, we obtain*
>
> $$(\rho\boldsymbol{a} - \rho\boldsymbol{F}).\frac{1}{3}Sh$$
> $$= (-\boldsymbol{p}^1 S\cos(\widehat{\boldsymbol{n}\ \ni_1}) - \boldsymbol{p}^2 S\cos(\widehat{\boldsymbol{n}\ \ni_2}) - \boldsymbol{p}^3 S\cos(\widehat{\boldsymbol{n}\ \ni_3}) + \boldsymbol{p_n}.S) + S.O(h),$$
>
> *where S is the area of the bounding surface ABC [Δs_4], and h is the infinitesimal height of the tetrahedron; $O(h)$, is a quantity which tends to zero for $h \to 0$. Approaching the limit, as $h \to 0$, we obtain*
>
> $$\boldsymbol{p_n} = \boldsymbol{p}^1\cos(\widehat{\boldsymbol{n}\ \ni_1}) + \boldsymbol{p}^2\cos(\widehat{\boldsymbol{n}\ \ni_2}) + \boldsymbol{p}^3\cos(\widehat{\boldsymbol{n}\ \ni_3}) \qquad (4.10)$$

In this book, we see $O(h)$ that represent the first order approximation in the tetrahedron argument. In addition, the *"infinitesimal height of the tetrahedron"* and *"tends to zero for $h \to 0$"* show that this proof similar to earlier books is only for an infinitesimal tetrahedron. Nearly the same process is given in the other Sedov's book (1971, [78]).

In the book of "Theoretical Elasticity" (1968, [35]) by Green and Zerna, on page 70:

$$\int_\tau \rho(\boldsymbol{F}_i - \dot{\boldsymbol{\omega}}_i)\, d\tau + \int_A \boldsymbol{t}_i\, dA = \boldsymbol{0}, \qquad (2.7.7)$$

> *... We consider a tetrahedron element bounded by the coordinate planes at the point y_i and a plane whose unit normal is $\boldsymbol{n}_k$ measured from inside to outside of the tetrahedron. If we apply (2.7.7) to this tetrahedron and take the limit as the tetrahedron tends to zero with $\boldsymbol{n}_k$ being unaltered we have*
>
> $$\boldsymbol{t}_i = \boldsymbol{n}_k\boldsymbol{\sigma}_{ki}, \qquad (2.7.9)$$
>
> *Provided the contributions from the volume integrals may be neglected compared with the surface integrals, in the limit.*

So, the challenges related to the *"tetrahedron tends to zero"* and *"volume integrals may be neglected compared with the surface integrals, in the limit"*, definitions of traction vectors on the surfaces that do not pass through the same point remain.

A more general proof was given by Gurtin and his coauthors [39], [61], [37], [38]. Here it is represented from the book "The Mechanics and Thermodynamics of Continua" (2010, [38]) by Gurtin, Fried, and Anand. On pages 137-138:

> *A deep result central to all of continuum mechanics is ... Cauchy's theorem ...*
>
> $$\boldsymbol{t}(\boldsymbol{a},\boldsymbol{x}) = -\sum_{i=1}^{3}(\boldsymbol{a}.\boldsymbol{e}_i)\boldsymbol{t}(-\boldsymbol{e}_i,\boldsymbol{x}) \qquad (19.24)$$
>
> *PROOF. Let $\boldsymbol{x}$ belong to the interior of $\mathcal{B}_t$. Choose $\delta > 0$ and consider the (spatial) tetrahedron Γ_δ with the following properties: The faces of Γ_δ are S_δ, $S_{1\delta}$, $S_{2\delta}$, and $S_{3\delta}$, where $\boldsymbol{a}$ and $-\boldsymbol{e}_i$ are the outward unit normals on S_δ and $S_{i\delta}$, respectively; the vertex opposite to S_δ is $\boldsymbol{x}$; the distance from $\boldsymbol{x}$ to S_δ is δ. Then, Γ_δ is contained in the interior of $\mathcal{B}_t$ for all sufficiently small δ, say $\delta \leqslant \delta_0$.*
> *Next, if we assume that $\boldsymbol{b}$ [generalized body term including the inertia and body force] is continuous, then $\boldsymbol{b}$ is bounded on Γ_δ. If we apply the force balance (19.16) to the material region P occupying the region Γ_δ in the deformed region at time t, we are then led to the estimate*
>
> $$\Big|\int_{\partial\Gamma_\delta}\boldsymbol{t}(\boldsymbol{n})\,da\Big| = \Big|\int_{\Gamma_\delta}\boldsymbol{b}\,dv\Big| \leqslant k\,vol(\Gamma_\delta) \qquad (19.25)$$
>
> *for all $\delta \leqslant \delta_0$, where k is independent of δ.*
> *Let $A(\delta)$ denote the area of S_δ. Since $A(\delta))$ is proportional to δ^2, while $vol(\Gamma_\delta)$ is proportional to δ^3, we may conclude from (19.25) that*
>
> $$\frac{1}{A(\delta)}\int_{\partial\Gamma_\delta}\boldsymbol{t}(\boldsymbol{n})\,da \to \boldsymbol{0}$$
>
> *as $\delta \to 0$. But*
>
> $$\int_{\partial\Gamma_\delta}\boldsymbol{t}(\boldsymbol{n})\,da = \int_{S_\delta}\boldsymbol{t}(\boldsymbol{a})\,da + \sum_{i=1}^{3}\int_{S_{i\delta}}\boldsymbol{t}(-\boldsymbol{e})\,da$$
>
> *and, assuming that $\boldsymbol{t}(\boldsymbol{n},\boldsymbol{x})$ is continuous in $\boldsymbol{x}$ for each $\boldsymbol{n}$, since the area of $S_{i\delta}$ is $A(\delta)(\boldsymbol{a}.\boldsymbol{e}_i)$,*
>
> $$\frac{1}{A(\delta)}\int_{\partial S_\delta}\boldsymbol{t}(\boldsymbol{a})\,da \to \boldsymbol{t}(\boldsymbol{a},\boldsymbol{x})$$
>
> *and*
>
> $$\frac{1}{A(\delta)}\int_{\partial S_{i\delta}}\boldsymbol{t}(-\boldsymbol{e}_i)\,da \to (\boldsymbol{a}.\boldsymbol{e}_i)\boldsymbol{t}(-\boldsymbol{e}_i,\boldsymbol{x}).$$
>
> *Combining the relations above we conclude that (19.24) is satisfied.*

This proof is based on the infinitesimal volume and in the limit $\delta \to 0$ the traction vector on the base surface of tetrahedron is regarded as the traction vector on the oriented surface that passes through the vertex point of tetrahedron. The process that leads to

$$\frac{1}{A(\delta)} \int_{\partial\Gamma_\delta} \boldsymbol{t}(\boldsymbol{n})\, da \to \boldsymbol{0}$$

is similar to the Love's proof that is discussed before in detail. So, some of the challenges remain. The proofs in some books are nearly similar to this process, for example Ciarlet (1988, [21]), Smith (1993, [86]), Huilgol and Phan-Thien (1997, [44]), Atkin and Fox (2005, [4]), Oden (2011, [68]), Bechtel and Lowe (2015, [9]).

There is a new proof in the literature that was introduced by this statement: *"This proof was furnished by W. NOLL (private communication) in 1967."*, in the chapter "The Linear Theory of Elasticity" by Gurtin in the book [36]. Then this proof was presented in the two other books by Truesdell (1997, [98]) and Liu (2002, [56]). This proof is based on the properties of a linear transformation on vector space [67]. In the book by Leigh (1968, [55]), it is stated that if a transformation such as $\boldsymbol{T}$ on a vector space has the following properties then it is usually called a *"linear transformation or tensor"*. On page 28:

> *... linear transformation* $\boldsymbol{T}$ *... is defined by*
>
> $$(a) \quad \boldsymbol{T}(\boldsymbol{u}+\boldsymbol{v}) = \boldsymbol{T}(\boldsymbol{u}) + \boldsymbol{T}(\boldsymbol{v})$$
> $$(b) \quad \boldsymbol{T}(\alpha\boldsymbol{v}) = \alpha\boldsymbol{T}(\boldsymbol{v}) \qquad (2.8.1)$$

So, in the Noll's proof, it is tried to prove these properties for the traction vectors. These properties must be derived using the integral equation of conservation of linear momentum. This proof is nearly the same in the three books that are presented it [36], [98], and [56]. Here we represent it from the first book [36]. On pages 48-49:

> *... for any* $\boldsymbol{x} \in B$ *we can extend the function* $\boldsymbol{s}(\boldsymbol{x}, \boldsymbol{.})$ *to all of* V *as follows:*
>
> $$\boldsymbol{s}(\boldsymbol{x}, \boldsymbol{v}) = |\boldsymbol{v}|\boldsymbol{s}(\boldsymbol{x}, \frac{\boldsymbol{v}}{|\boldsymbol{v}|}) \quad \boldsymbol{x} \neq \boldsymbol{0},$$
> $$\boldsymbol{s}(\boldsymbol{x}, \boldsymbol{0}) = \boldsymbol{0}. \qquad (a)$$
>
> *Let* α *be a scalar. If* $\alpha > 0$*, then*
>
> $$\boldsymbol{s}(\alpha\boldsymbol{v}) = |\alpha\boldsymbol{v}|\boldsymbol{s}\left(\frac{\alpha\boldsymbol{v}}{|\alpha\boldsymbol{v}|}\right) = \alpha|\boldsymbol{v}|\boldsymbol{s}\left(\frac{\boldsymbol{v}}{|\boldsymbol{v}|}\right) = \alpha\boldsymbol{s}(\boldsymbol{v}), \qquad (b)$$
>
> *where we have omitted the argument* $\boldsymbol{x}$*. If* $\alpha < 0$*, then* (b) *and Cauchy's reciprocal theorem* (2) *[*$\boldsymbol{s}(\boldsymbol{n}) = -\boldsymbol{s}(-\boldsymbol{n})$*] yield*
>
> $$\boldsymbol{s}(\alpha\boldsymbol{v}) = \boldsymbol{s}(|\alpha|(-\boldsymbol{v})) = |\alpha|\boldsymbol{s}(-\boldsymbol{v}) = \alpha\boldsymbol{s}(\boldsymbol{v}).$$
>
> *Thus* $\boldsymbol{s}(\boldsymbol{x}, \boldsymbol{.})$ *is homogeneous.*
> *To show that* $\boldsymbol{s}(\boldsymbol{x}, \boldsymbol{.})$ *is additive we first note that*
>
> $$\boldsymbol{s}(\boldsymbol{x}, \boldsymbol{w}_1 + \boldsymbol{w}_2) = \boldsymbol{s}(\boldsymbol{x}, \boldsymbol{w}_1) + \boldsymbol{s}(\boldsymbol{x}, \boldsymbol{w}_2)$$
>
> *whenever* $\boldsymbol{w}_1$ *and* $\boldsymbol{w}_2$ *are linearly dependent. Suppose then that* $\boldsymbol{w}_1$ *and* $\boldsymbol{w}_2$ *are linearly independent. Fix* $\epsilon > 0$ *and consider* π_1*, the plane through*

$\boldsymbol{x}_0$ with normal $\boldsymbol{w}_1$; π_2, the plane through $\boldsymbol{x}_0$ with normal $\boldsymbol{w}_2$; and π_3, the plane through $\boldsymbol{x}_0 + \epsilon \boldsymbol{w}_3$ with normal $\boldsymbol{w}_3$, where

$$\boldsymbol{w}_3 = -(\boldsymbol{w}_1 + \boldsymbol{w}_2). \qquad (c)$$

Consider the solid $\mathcal{A} = \mathcal{A}(\epsilon)$ bounded by these three planes and two

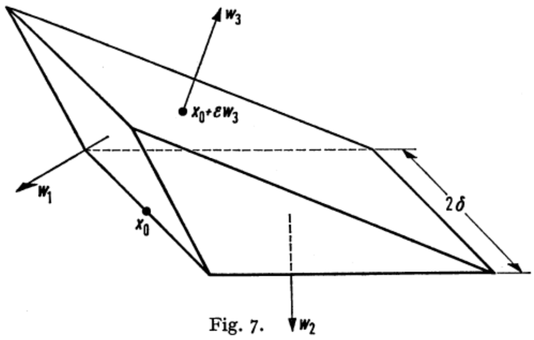

Fig. 7.

planes parallel to both $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ and a distance δ from $\boldsymbol{x}_0$ (see Fig. 7). Let ϵ and δ be sufficiently small that $\mathcal{A}$ is a part of B. Then

$$\partial \mathcal{A} = \bigcup_{i=1}^{5} \mathcal{W}_i,$$

where $\mathcal{W}_i$, is contained in π_i ($i = 1, 2, 3$), and $\mathcal{W}_4$ and $\mathcal{W}_5$ are parallel faces. Moreover,

$$a_i = \frac{|\boldsymbol{w}_i|}{|\boldsymbol{w}_3|} a_3 \qquad (i = 1, 2),$$
$$a_3 = O(\epsilon) \qquad as \qquad \epsilon \to 0,$$
$$\upsilon(\mathcal{A}) = \frac{\epsilon}{2} |\boldsymbol{w}_3| a_3 = 2\delta a_4 = 2\delta a_5,$$

where a_i, is the area of $\mathcal{W}_i$. Thus, by the continuity of $\boldsymbol{s}_{\boldsymbol{n}}$,

$$\boldsymbol{c} \equiv \frac{|\boldsymbol{w}_3|}{a_3} \int_{\partial \mathcal{A}} \boldsymbol{s}_{\boldsymbol{n}} \, da = \sum_{i=1}^{3} \frac{|\boldsymbol{w}_i|}{a_i} \int_{\mathcal{W}_i} \boldsymbol{s}(\boldsymbol{x}, \frac{\boldsymbol{w}_i}{|\boldsymbol{w}_i|}) \, da_{\boldsymbol{x}} + O(\epsilon) \quad as \quad \epsilon \to 0,$$

and (a) implies

$$\boldsymbol{c} = \sum_{i=1}^{3} \boldsymbol{s}(\boldsymbol{x}_0, \boldsymbol{w}_i) + o(\boldsymbol{1}) \quad as \quad \epsilon \to 0.$$

On the other hand, we conclude from estimate (a) [

$$|\int_{\partial P} \boldsymbol{s}_{\boldsymbol{n}} \, da| \leqslant k\upsilon(P)$$

where $\upsilon(P)$ is the volume of P] in the proof of (2) [$\boldsymbol{s}(\boldsymbol{n}) = -\boldsymbol{s}(-\boldsymbol{n})$] that

$$\boldsymbol{c} = O(\epsilon) \quad as \quad \epsilon \to 0.$$

The last two results yield

$$\sum_{i=1}^{3} \boldsymbol{s}(\boldsymbol{x}_0, \boldsymbol{w}_i) = \boldsymbol{0};$$

since $\boldsymbol{s}(\boldsymbol{x}, \boldsymbol{.})$ *is homogeneous, this relation and* (*c*) *imply that* $\boldsymbol{s}(\boldsymbol{x}_0, \boldsymbol{.})$ *is additive. Thus* $\boldsymbol{s}(\boldsymbol{x}_0, \boldsymbol{.})$ *is linear, and, since* $\boldsymbol{x}_0 \in B$ *was arbitrarily chosen, NoLL's proof is complete.*

This is a creative proof by Noll that shows a new insight to the mathematical aspects of traction vectors. But this proof is based on the limiting volume and is for infinitesimal mass element. The expression "$\epsilon \to 0$" showed this. Also, the average values of the traction vectors are used and the traction vector on the surface π_3 is regarded as the traction vector on the parallel surface that passes through point $\boldsymbol{x}_0$ in the limit. So, some of the challenges remain.

Leigh in the book "Nonlinear Continuum Mechanics" (1968, [55]) uses the properties of linear transformation to prove the existence of stress tensor by a different construction that is used in the Noll's proof. On pages 129-130:

$$\boldsymbol{t} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{n}) \qquad (7.5.5)$$

$$\int_{\partial\chi} \boldsymbol{t}\, da + \int_{\chi} \boldsymbol{b}\rho\, dv = \int_{\chi} \ddot{\boldsymbol{x}}\rho\, dv \qquad (7.5.6)$$

Next we prove Cauchy's fundamental theorem for the stress

$$\boldsymbol{t} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{n}) = \boldsymbol{T}(\boldsymbol{x})\boldsymbol{n} \qquad (7.5.7)$$

that is, the stress vector $\boldsymbol{t}$ *at* $\boldsymbol{x}$ *acting on the surface with direction* $\boldsymbol{n}$

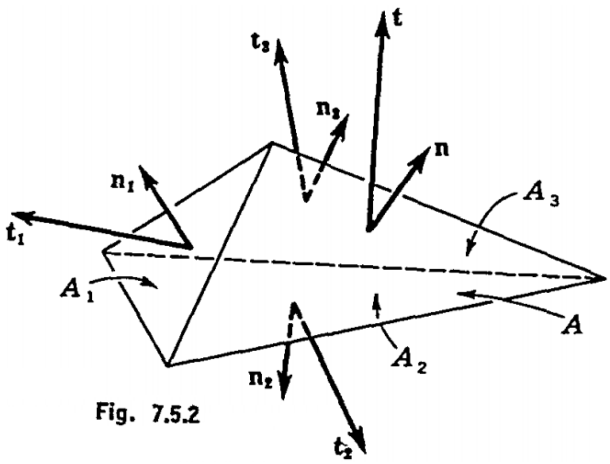

Fig. 7.5.2

is a linear transformation of $\boldsymbol{n}$*. The linear transformation or tensor* $\boldsymbol{T}$ *is called the stress tensor. Consider the elemental tetrahedron of Fig.* 7.5.2 … *The values of the stress vectors on the faces are given by* (7.5.5)*, where we use the same* $\boldsymbol{x}$*, since we are going to allow the tetrahedron to shrink to the point* $\boldsymbol{x}$ *in the limit. Thus we have*

$$\boldsymbol{t} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{n}) \qquad \boldsymbol{t}_i = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{n}_i) \qquad (7.5.8)$$

Thus applying (7.5.6) *to the elemental tetrahedron in the limit as* A*,* $A_i \to 0$*, we note that volume integrals are negligible compared with the surface* *integrals. The surface integral yields*

$$\boldsymbol{t} = -\frac{1}{A}(A_1\boldsymbol{t}_1 + A_2\boldsymbol{t}_2 + A_3\boldsymbol{t}_3) \qquad (7.5.9)$$

Now a closed surface S satisfies the condition

$$\int_S \boldsymbol{n}\, da = \boldsymbol{0} \qquad (7.5.10)$$

Applying (7.5.10) *to our elemental tetrahedron, we get*

$$\boldsymbol{n} = -\frac{1}{A}(A_1\boldsymbol{n}_1 + A_2\boldsymbol{n}_2 + A_3\boldsymbol{n}_3) \qquad (7.5.11)$$

Combining (7.5.8), (7.5.9), *and* (7.5.11), *we have, suppressing $\boldsymbol{x}$,*

$$\boldsymbol{f}\Big(-\frac{1}{A}A_i\boldsymbol{n}_i\Big) = -\frac{1}{A}A_i\boldsymbol{f}(\boldsymbol{n}_i) \qquad (7.5.12)$$

and we see that $\boldsymbol{f}(\boldsymbol{n})$ satisfies the definition (2.8.1) *of a linear transformation [two properties for linear transformation that we presented them before the Noll's proof], which proves* (7.5.7).

In this proof, Leigh uses three linearly independent traction vectors rather than two linearly traction vectors as used in the Noll's proof. As compared with the previous proofs that use a tetrahedron element with three orthogonal faces, in the Leigh's proof it is not needed the faces be orthogonal. But as previous proofs, this proof is based on the infinitesimal volume and is the sequence of the limit $A \to 0$. Here the *"volume integrals are negligible compared with the surface integrals"* shows the challenge 4. So, some of the challenges remain. The proof in the book by Lurie (2005, [58]) is similar to this proof.

4.2. **The second approach.**

During the comprehensive review on a large number of books in continuum mechanics and the related subjects we find that there are two general approaches in the tetrahedron arguments and the proofs for the existence of stress tensor. In the first approach the traction vectors and body terms are not defined on the same point. So, the traction vector on the base surface of the tetrahedron (Δs_4) considered as the traction vector on the parallel oriented surface that passes through the vortex point ($\boldsymbol{o}$) of the infinitesimal tetrahedron element. So, the challenges on the relation $\boldsymbol{t}_4 + n_x\boldsymbol{t}_1 + n_y\boldsymbol{t}_2 + n_z\boldsymbol{t}_3 = \boldsymbol{0}$ and most of the other stated challenges hold. Nearly all the proofs in the previous subsection can be regarded in the first approach. Most of the tetrahedron arguments and the proofs for the existence of stress tensor are based on the first approach.

But in the second approach, the traction vectors and body terms are defined explicitly at the same point, e.g. in the tetrahedron arguments the vortex point ($\boldsymbol{o}$). Then by an approximating process for infinitesimal tetrahedron the relation $\boldsymbol{t}_{4_o} + n_x\boldsymbol{t}_{1_o} + n_y\boldsymbol{t}_{2_o} + n_z\boldsymbol{t}_{3_o} = \boldsymbol{0}$ is obtained. So, in this approach all the traction vectors in this relation are exactly defined at the same point ($\boldsymbol{o}$) on different planes that pass exactly through this point. Here some of the challenges, for example challenges 6, 7, and 8, that are related to the definition of traction vectors at different points in the relation

$\boldsymbol{t}_4+n_x\boldsymbol{t}_1+n_y\boldsymbol{t}_2+n_z\boldsymbol{t}_3=\mathbf{0}$ are removed. But these proofs are based on the approximating process and are for infinitesimal tetrahedron, so, the other related challenges remain. A few of scientists and authors in continuum mechanics followed this approach. They are Muskhelishvili, 1933 (English translation 1977, [64]), Sokolnikoff (1946, [87]), Fung (1965, [30] and 1969, [31]), Godunov and Romenskii (1998, [32]), and Temam and Miranvilli (2000, [92]). The proofs that are presented in all these books are nearly similar. Here, we present Muskhelishvili's proof and Fung's proof as two samples of these books. In the book "Some Basic Problems of the Mathematical Theory of Elasticity, 1933" by Muskhelishvili on pages 8-10 from the English translation, (1977, [64]):

> *Through the point M draw three planes, parallel to the coordinate planes, and in addition, another plane having the normal $\boldsymbol{n}$ and lying a distance h from M. These four planes form a tetrahedron, three faces of which are parallel to the coordinate planes, while the fourth ABC [Δs_4] is the face to be considered. ...the transition to the limit $h \to 0$ the size of the tetrahedron will be assumed infinitely small.*
> *[here (X_x, Y_x, Z_x), (X_y, Y_y, Z_y), (X_z, Y_z, Z_z), and (X_n, Y_n, Z_n) are the components of traction vectors at point M on the four surfaces with unit normal vectors $\boldsymbol{e}_x$, $\boldsymbol{e}_y$, $\boldsymbol{e}_z$, and $\boldsymbol{n}$, respectively. X, Y, Z are the components of body terms at point M.]*
> *...The projection of the body force equals $(X+\epsilon)dV$, where dV is the volume of the tetrahedron. The value X refers to the point M and ϵ is an infinitely small quantity ...Further, the projection of the tractions, acting on the face ABC is $(X_n+\epsilon')\sigma$ where σ denotes the area of the triangle ABC [Δs_4] and ϵ' is again infinitely small; X_n, Y_n, Z_n, as will be remembered, are the components of the stress vector acting on the plane through M with normal $\boldsymbol{n}$.*
> *Finally the projection of the external forces acting on MBC, normal to Ox, is $(-X_x+\epsilon_1)\sigma_1$ where σ_1 is the area of MBC. ...For the sides MCA and MAB one obtains similarly $(-X_y+\epsilon_2)\sigma_2$ and $(-X_z+\epsilon_3)\sigma_3$ respectively. Here ϵ_1, ϵ_2 and ϵ_3 denote again infinitesimal quantities. [So, the conservation of linear momentum in x direction:]*
>
> $$\begin{aligned}&(X+\epsilon)\frac{1}{3}h\sigma+(X_n+\epsilon')\sigma+(-X_x+\epsilon_1)\sigma\cos(\boldsymbol{n},x)\\&+(-X_y+\epsilon_2)\sigma\cos(\boldsymbol{n},y)+(-X_z+\epsilon_3)\sigma\cos(\boldsymbol{n},z)=0.\end{aligned}$$
>
> *Dividing by σ and taking the limit $h \to 0$ one obtains the following formulae ...[similarly for y and z directions]:*
>
> $$\begin{aligned}X_n&=X_x\cos(\boldsymbol{n},x)+X_y\cos(\boldsymbol{n},y)+X_z\cos(\boldsymbol{n},z)\\Y_n&=Y_x\cos(\boldsymbol{n},x)+Y_y\cos(\boldsymbol{n},y)+Y_z\cos(\boldsymbol{n},z)\\Z_n&=Z_x\cos(\boldsymbol{n},x)+Z_y\cos(\boldsymbol{n},y)+Z_z\cos(\boldsymbol{n},z)\end{aligned}$$

So, the traction vector on the oriented plane that passes exactly through point M is obtained by an approximating process and in *"taking the limit $h \to 0$"*.

In the book “A First Course in Continuum Mechanics, 1969” by Fung, on pages 69-71 of the third edition, (1994, [31]):

> *Let us consider an infinitesimal tetrahedron formed by three surfaces parallel to the coordinate planes and one normal to the unit vector* $\boldsymbol{v}$*. Let the area of the surface normal to* $\boldsymbol{v}$ *be* dS*. ...*
> *The forces in the positive direction of* x_1*, acting on the three coordinate surfaces, can be written as*
>
> $$(-\tau_{11}+\epsilon_1)dS_1, \qquad (-\tau_{21}+\epsilon_2)dS_2, \qquad (-\tau_{31}+\epsilon_3)dS_3,$$
>
> *where* τ_{11}*,* τ_{21}*,* τ_{31} *are the stresses at the vertex* P *opposite to* dS*. The negative sign is obtained because the outer normals to the three surfaces are opposite in sense with respect to the coordinate axes, and the* ϵ*'s are inserted because the tractions act at points slightly different from* P*. If we assume that the stress field is continuous, then* ϵ_1*,* ϵ_2*,* ϵ_3 *are infinitesimal quantities. On the other hand, the force acting on the triangle normal to* $\boldsymbol{v}$ *has a component* $(T_1+\epsilon)dS$ *in the positive* x_1*-axis direction, the body force has an* x_1*-component equal to* $(X_1+\epsilon')dv$*, and the rate of change of linear momentum has a component* $\rho\dot{V}_1 dv$*, where* $\dot{V}_1$*, is the component of acceleration in the direction of* x_1*. Here,* T_1 *and* X_1 *refer to the point* P*, and* ϵ *and* ϵ' *are again infinitesimal. The first equation of motion is thus*
>
> $$(-\tau_{11}+\epsilon_1)v_1 dS + (-\tau_{21}+\epsilon_2)v_2 dS + (-\tau_{31}+\epsilon_3)v_3 dS$$
> $$+ (T_1+\epsilon)dS + (X_1+\epsilon')\frac{1}{3}hdS = \rho\dot{V}_1\frac{1}{3}hdS. \qquad (3.3-3)$$
>
> *Dividing through by* dS*, taking the limit as* $h \to 0$*, and noting that* ϵ_1*,* ϵ_2*,* ϵ_3*,* ϵ*,* ϵ' *vanish with* h *and* dS*, one obtains*
>
> $$T_1 = \tau_{11}v_1 + \tau_{21}v_2 + \tau_{31}v_3, \qquad (3.3-4)$$

Fung has also discussed the error of this approximating process. On page 71, [31]:

> ***Checking Acceptable Errors***
> *... We claimed that the sum of the terms*
>
> $$\epsilon_1 v_1 + \epsilon_2 v_2 + \epsilon_3 v_3 + \epsilon + \frac{1}{3}(\epsilon' - \rho\dot{V}_1) \qquad (3.3-5)$$
>
> *is small, compared with the terms that are retained; i.e.,*
>
> $$T_1,\ \tau_{11}v_1,\ \tau_{21}v_2,\ \tau_{31}v_3, \qquad (3.3-6)$$
>
> *when we take Eq.* $(3.3-3)$ *to the limit as* $h \to 0$ *and* $\Delta S \to 0$*. Now, if we are not allowed to take the limit as* $h \to 0$ *and* $\Delta S \to 0$*, but instead we are restricted to accept* h *no smaller than a constant* h^* *and* ΔS *no smaller than a constant multiplied by* $(h^*)^2$*, then the quantity listed in line* $(3.3-5)$ *must be evaluated for* $h = h^*$ *and* $\Delta S = const.(h^*)^2$ *and must be compared with the quantities listed in line* $(3.3-6)$*. A standard of how small is negligible must be defined, and the comparison be made under that definition. If we find the quantity in line* $(3.3-5)$ *negligible*

compared with those listed in line $(3.3-6)$*, then we can say that Eq.* $(3.3-3)$ *or Eq.* $(3.3-2)$ *[*$T_i = v_j\tau_{ji}$*] is valid. This tedious step should be done, in principle, to apply the continuum theory to objects of the real world.*

4.3. **Advanced mathematical works.**

In recent decades, some proofs for the existence of Cauchy stress tensor or general Cauchy fluxes are presented in the geometrical language mathematics and advanced analysis. For example, using variational method [29], considering general Cauchy fluxes under weaker conditions [81], [83], [82], or represented by measures [84], considering contact interactions as maps on pairs of subbodies and the possibility of handling singularities due to shocks and fracture [76], considering contact actions in N-th gradient generalized continua [22], etc. Each of these articles shows some aspects of the contact interactions in continuum physics. Here considering these attempts is outside the scope of this article that is based on the review of the proofs for the existence of stress tensor and their challenges in continuum mechanics and the related subjects.

5. The work of Hamel, its improvements and challenges

Let us see what is presented by Hamel in the famous book "Theoretische Mechanik, pp. 513-514" (1949, [41]). We present this proof completely:

Dann soll (I) nach Division mit dV *die genauere Form*

$$\varrho\overline{\boldsymbol{\omega}} = \sum \overline{\boldsymbol{\chi}} + \lim_{\Delta V \to 0} \frac{1}{\Delta V} \oint \overline{\boldsymbol{\sigma}}_{\boldsymbol{n}}\, dF \qquad (I_A)$$

bekommen und dieser Grenzwert existieren. Das Integral eistreckt sich über die Oberfläche des kleinen Volumens um den betrachteten Punkt, gegen den ΔV *konvergiert.*

Aus der Existenz des Grenzwertes folgen die Sätze:

$$\mathbf{1)} \qquad \overline{\boldsymbol{\sigma}}_n = \overline{\boldsymbol{\sigma}}_x \cos(\boldsymbol{n}, x) + \overline{\boldsymbol{\sigma}}_y \cos(\boldsymbol{n}, y) + \overline{\boldsymbol{\sigma}}_z \cos(\boldsymbol{n}, z)$$

$\overline{\boldsymbol{\sigma}}_x$ *usw. bedeuten die Spannungen an Flächenelementen, deren äußere Normalen Parallelen zur* x*,* y*,* z*-Achse sind. Setzt man*

$$\overline{\boldsymbol{\sigma}}_x = X_x\overline{\boldsymbol{i}} + Y_x\overline{\boldsymbol{j}} + Z_x\overline{\boldsymbol{k}},$$
$$\overline{\boldsymbol{\sigma}}_y = X_y\overline{\boldsymbol{i}} + Y_y\overline{\boldsymbol{j}} + Z_y\overline{\boldsymbol{k}},$$
$$\overline{\boldsymbol{\sigma}}_z = X_z\overline{\boldsymbol{i}} + Y_z\overline{\boldsymbol{j}} + Z_z\overline{\boldsymbol{k}}$$

mit $\overline{\boldsymbol{i}}$*,* $\overline{\boldsymbol{j}}$*,* $\overline{\boldsymbol{k}}$ *als Einheitsvektoren in den drei Achsenrichtungen, so erscheint hier der Spannimgstensor*

$$\begin{Bmatrix} X_x & Y_x & Z_x \\ X_y & Y_y & Z_y \\ X_z & Y_z & Z_z \end{Bmatrix},$$

und man kann **1***) auch schreiben:*

$$\overline{\boldsymbol{\sigma}}_{\boldsymbol{n}} = \overline{\overline{\sigma}}\,\overline{\boldsymbol{n}}$$

wenn

$$\overline{\boldsymbol{n}} = \overline{\boldsymbol{i}}\cos(\boldsymbol{n}, x) + \overline{\boldsymbol{j}}\cos(\boldsymbol{n}, y) + \overline{\boldsymbol{k}}\cos(\boldsymbol{n}, z)$$

den Einheitsvektor der äußeren Normalen angibt. (An der gedachten Fläche wird also die Existenz einer solchen im allgemeinen vorausgesetzt.)

1*a) In* **1***) ist insbesondere enthalten*

$$\overline{\boldsymbol{\sigma}}_{-\boldsymbol{n}} = -\overline{\boldsymbol{\sigma}}_{\boldsymbol{n}},$$

d. h. das Gegenwirkungsprinzip für die inneren Spannungen, das also hier beweisbar ist.

2*) Die Ausführung des Grenzüberganges in* (I_A) *ergibt*

$$\varrho\overline{\boldsymbol{\omega}} = \sum \overline{\boldsymbol{\chi}} + \frac{\partial \overline{\boldsymbol{\sigma}}_x}{\partial x} + \frac{\partial \overline{\boldsymbol{\sigma}}_y}{\partial y} + \frac{\partial \overline{\boldsymbol{\sigma}}_z}{\partial z}$$

Hamel's proof is based on the existence of the limit in the conservation of linear momentum equation (I_A). This is the best part of this proof and is the main improvement of his work. The original difference of this stage from the other previous similar works is that they divided the equation by ΔS that leads to the trivial solution. Because, as we showed before, the two parts (surface and volume integrals) of the equation have the same order (l^3) and by dividing by ΔS they still have the same order (l), that go to zero in the limit when the element goes to infinitesimal volume, and this is a trivial result. But Hamel divided the equation by ΔV and this leads to the logical result of the existence the limit in the equation (I_A). So, some of the important challenges are removed by Hamel's proof.

But this proof is only for $\Delta V \to 0$ and does not exist any statement for a mass element with any volume size in continuum media. Because we must prove that the existence of stress tensor does not depend on the volume size of the considered mess element. So, the challenge 2 remains. In addition, there is no process to show how the relation $\overline{\boldsymbol{\sigma}}_n = \overline{\boldsymbol{\sigma}}_x \cos(\boldsymbol{n}, x) + \overline{\boldsymbol{\sigma}}_y \cos(\boldsymbol{n}, y) + \overline{\boldsymbol{\sigma}}_z \cos(\boldsymbol{n}, z)$ is obtained from the existence of the limit in equation (I_A). This will be an important step for the existence of stress tensor.

6. The work of Backus, its improvements and challenges

Now let us see the Backus's proof from the book "Continuum Mechanics" (1997, [5]). Unfortunately, the Backus's work seems to have attracted no attention of the scientists and authors in continuum mechanics, so far. But this proof removes most of the challenges. First, we represent some notations based on this book. On page 163:

> *...(P, A_P) is oriented real physical space. ... The open set in P occupied by the particles at time t will be written $K(t)$, and the open subset of $K(t)$ consisting of the particles ... will be written $K'(t)$.*

On pages 171-172:

> *...* $\vec{S}(\vec{r}, t, \hat{n}_P) = \vec{S}_{force}(\vec{r}, t, \hat{n}_P) + \vec{S}_{mfp}(\vec{r}, t, \hat{n}_P)$ *is called the stress on the surface* $(S, \hat{n}_P)$. *The total force exerted by the material just in front of* $dA_P(\vec{r})$ *on the material just behind* $dA_P(\vec{r})$ *is*
>
> $$d\vec{\mathcal{F}}_S(\vec{r}) = dA_P(\vec{r})\vec{S}(\vec{r}, t, \hat{n}_P). \qquad (13.2.7)$$
>
> *This is called the surface force on* $dA_P(\vec{r})$*....*
> *... Combining the physical law* (13.2.1) *with the mathematical expressions* (13.2.3) *and* (13.2.9) *gives*
>
> $$\int_{K'} dV_P(\vec{r})(\rho\vec{a} - \vec{f})^E(\vec{r}, t) = \int_{\partial K'} dA_P(\vec{r})\vec{S}(\vec{r}, t, \hat{n}_P(\vec{r})). \qquad (13.2.10)$$
>
> *where* $K' = K'(t)$ *and* $\hat{n}_P(\vec{r})$ *is the unit outward normal to* $\partial K'$ *at* $\vec{r} \in \partial K'$.

In the following paragraphs, Backus has discussed some challenges. These are some aspects of the improvements of this work. On pages 172-173:

> *To convert* (13.2.10) *to a local equation, valid for all* $\vec{r} \in K(t)$ *at all times* t*, (i.e., to "remove the integral signs") we would like to invoke the vanishing integral theorem, ... The surface integral in* (13.2.10) *prevents this. Even worse,* (13.2.10) *makes our model look mathematically self-contradictory, or internally inconsistent. Suppose that* K' *shrinks to a point while preserving its shape. Let* λ *be a typical linear dimension of* K'. *Then the left side of* (13.2.10) *seems to go to zero like* λ^3*, while the right side goes to zero like* λ^2. *How can they be equal for all* $\lambda > 0$?
> *Cauchy resolved the apparent contradiction in* 1827. *He argued that the right side of* (13.2.10) *can be expanded in a power series in* λ*, and the validity of* (13.2.10) *for all* λ *shows that the first term in this power series, the* λ^2 *term, must vanish. In modern language, Cauchy showed that this can happened iff at every instant* t*, at every* $\vec{r} \in K(t)$*, there is a unique tensor* $\overleftrightarrow{S}^E(\vec{r}, t)$ *... such that for each unit vector* $\hat{n}$ *...,*
>
> $$\vec{S}(\vec{r}, t, \hat{n}) = \hat{n}.\overleftrightarrow{S}^E(\vec{r}, t). \qquad (13.2.11)$$
>
> *... The physical quantity* $\overleftrightarrow{S}$ *is also called the Cauchy stress tensor.*

Then on page 173 the Cauchy's theorem for the existence of stress tensor is stated:

> *The argument which led Cauchy from* (13.2.10) *to* (13.2.11) *is fundamental to continuum mechanics, so we examine it in detail....*
>
> ***Theorem 13.2.28 (Cauchy's Theorem)*** *... Suppose that for any open subset* K' *of* K *whose boundary* $\partial K'$ *is piecewise smooth, we have*
>
> $$\int_{K'} dV_U(\vec{r})\vec{f}(\vec{r}) = \int_{\partial K'} dA_U(\vec{r})\vec{S}(\vec{r}, \hat{n}_U(\vec{r})), \qquad (13.2.13)$$

... Then for each $\vec{r} \in K$ *there is a unique* $\overleftrightarrow{S}(\vec{r})$ *...such that for all* $\hat{n}$ *...,*

$$\vec{S}(\vec{r}, \hat{n}) = \hat{n}.\overleftrightarrow{S}(\vec{r}). \qquad (13.2.14)$$

Backus uses two lemmas to prove the "Cauchy's Theorem". The first lemma on pages 174-176:

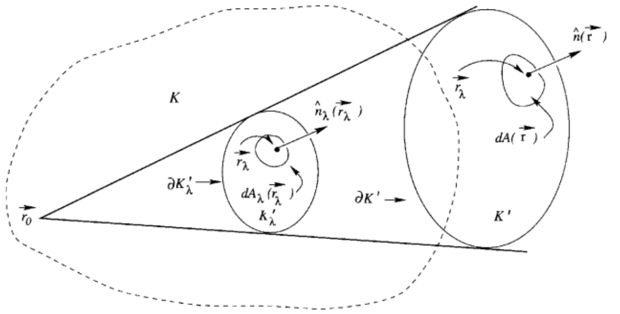

Figure 13.3:

Two lemmas are required. [The first lemma:]
***Lemma* 13.2.29** *Suppose* $\vec{f}$ *and* $\vec{S}$ *satisfy the hypotheses of theorem* 13.2.28*. Let* $\vec{r}_0$ *be any point in* K *and let* K' *be any open bounded (i.e., there is a real* M *such that* $\vec{r} \in K' \Rightarrow \|\vec{r}\| \leqslant M$*) subset of* U*, with piecewise smooth boundary* $\partial K'$*. We don't need* $K' \subseteq K$*. Then*

$$\int_{\partial K'} dA_U(\vec{r})\vec{S}(\vec{r}_0, \hat{n}_U(\vec{r})) = \vec{0}_V, \qquad (13.2.15)$$

if $\hat{n}_U(\vec{r})$ *is the unit outward normal to* $\partial K'$ *at* $\vec{r} \in \partial K'$ *and* $dA_U(\vec{r})$ *is the element of area on* $\partial K'$*.*

Proof of Lemma* 13.2.29*: *For any real* λ *in* $0 < \lambda < 1$*, define* $\boldsymbol{\vec{r}}_\lambda : U \to U$ *by* $\boldsymbol{\vec{r}}_\lambda(\vec{r}) = \vec{r}_0 + \lambda(\vec{r} - \vec{r}_0)$ *for all* $\vec{r} \in U$*. Since* $\vec{r}_\lambda(\vec{r}) - \vec{r}_0 = \lambda(\vec{r} - \vec{r}_0)$*,* $\boldsymbol{\vec{r}}_\lambda$ *shrinks* U *uniformly toward* $\vec{r}_0$ *by the factor* λ*. The diagram above is for* $\lambda \approx 1/2$*. Define* $K'_\lambda = \boldsymbol{\vec{r}}_\lambda(K')$ *so* $\partial K'_\lambda = \boldsymbol{\vec{r}}_\lambda(\partial K')$*. Choose* $\vec{r} \in \partial K'$ *and let* $\vec{r}_\lambda = \boldsymbol{\vec{r}}_\lambda(\vec{r})$*. Let* $dA(\vec{r})$ *denote a small nearly plane patch of surface in* $\partial K'$*, with* $\vec{r} \in dA(\vec{r})$*, and use* $dA(\vec{r})$ *both as the name of this set and as the numerical value of its area. Let the set* $dA_\lambda(\vec{r}_\lambda)$ *be defined as* $\boldsymbol{\vec{r}}_\lambda(dA(\vec{r}))$*, and denote its area also by* $dA_\lambda(\vec{r}_\lambda)$*. Then by geometric similarity*

$$dA_\lambda(\vec{r}_\lambda) = \lambda^2 dA(\vec{r}). \qquad (13.2.16)$$

Let $\hat{n}(\vec{r})$ *be the unit outward normal to* $\partial K'$ *at* $\vec{r}$*, and let* $\hat{n}_\lambda(\vec{r}_\lambda)$ *be the unit outward normal to* $\partial K'_\lambda$ *at* $\vec{r}_\lambda$*. By similarity,* $\hat{n}(\vec{r})$ *and* $\hat{n}_\lambda(\vec{r}_\lambda)$ *point in the same direction. Being unit vectors, they are equal:*

$$\hat{n}_\lambda(\vec{r}_\lambda) = \hat{n}(\vec{r}). \qquad (13.2.17)$$

Since $\vec{r}_0$ *is fixed, it follows that*

$$\int_{\partial K'_\lambda} dA_\lambda(\vec{r}_\lambda)\vec{S}(\vec{r}_0, \hat{n}_\lambda(\vec{r}_\lambda)) = \lambda^2 \int_{\partial K'} dA(\vec{r})\vec{S}(\vec{r}_0, \hat{n}(\vec{r})), \qquad (13.2.18)$$

If λ is small enough, $K' \subseteq K$. Then, by hypothesis, we have (13.2.13) *with K' and $\partial K'$ replaced by K'_λ and $\partial K'_\lambda$. Therefore*[2]

$$\int_{K'_\lambda} dV(\vec{r})\vec{f}(\vec{r}) = \int_{\partial K'_\lambda} dA_\lambda(\vec{r}_\lambda)\Big\{\vec{S}(\vec{r}_\lambda, \hat{n}_\lambda(\vec{r}_\lambda)) - \vec{S}(\vec{r}_0, \hat{n}_\lambda(\vec{r}_\lambda))\Big\} + \int_{\partial K'_\lambda} dA_\lambda(\vec{r}_\lambda)\vec{S}(\vec{r}_0, \hat{n}_\lambda(\vec{r}_\lambda)).$$

From (13.2.18) *it follows that*

$$\int_{\partial K'} dA(\vec{r})\vec{S}(\vec{r}_0, \hat{n}(\vec{r})) = \frac{1}{\lambda^2}\int_{K'_\lambda} dV(\vec{r})\vec{f}(\vec{r}) + \frac{1}{\lambda^2}\int_{\partial K'_\lambda} dA_\lambda(\vec{r}_\lambda)\Big\{\vec{S}(\vec{r}_\lambda, \hat{n}_\lambda(\vec{r}_\lambda)) - \vec{S}(\vec{r}_0, \hat{n}_\lambda(\vec{r}_\lambda))\Big\}. \qquad (13.2.19)$$

Let $m_{\vec{S}}(\lambda)$ = maximum value of $\|\vec{S}(\vec{r}_0, \hat{n}) - \vec{S}(\vec{r}, \hat{n})\|$ for all $\vec{r} \in \partial K'_\lambda$ and all $\hat{n} \in N_U$.
Let $m_{\vec{f}}(\lambda)$ = maximum value of $|\vec{f}(\vec{r})|$ for all $\vec{r} \in K'_\lambda$.
Let $|\partial K'_\lambda|$ = area of $\partial K'_\lambda$, $|\partial K'|$ = area of $\partial K'$.
Let $|K'_\lambda|$ = volume of K'_λ, $|K'|$ = volume of K'.
Then $|\partial K'_\lambda| = \lambda^2|\partial K'|$ and $|K'_\lambda| = \lambda^3|K'|$, so (10.2.3) *and* (13.2.19) *imply*

$$\left\|\int_{\partial K'} dA(\vec{r})\vec{S}(\vec{r}_0, \hat{n}(\vec{r}))\right\| \leqslant \lambda|K'|m_{\vec{f}}(\lambda) + |\partial K'|m_{\vec{S}}(\lambda). \qquad (13.2.20)$$

As $\lambda \to 0$, $m_{\vec{f}}(\lambda)$ remains bounded (in fact $\to \|f(\vec{r}_0)\|$) and $m_{\vec{S}}(\lambda) \to 0$ because $\vec{S} : K \times N_U \to V$ is continuous. Therefore, as $\lambda \to 0$, the right side of (13.2.20) $\to 0$. *Inequality* (13.2.20) *is true for all sufficiently small $\lambda > 0$, and the left side is non-negative and independent of λ. Therefore the left side must be* 0. *This proves* (13.2.15) *and hence proves lemma* 13.2.29.

So, the result of this lemma is the fundamental relation (13.2.15) for traction vectors at the given point $\vec{r}_0$, as the following:

$$\int_{\partial K'} dA_U(\vec{r})\vec{S}(\vec{r}_0, \hat{n}_U(\vec{r})) = \vec{0}_V$$

If we compare this relation with the presented similar relations in the previous sections, the lemma 13.2.29 and its proof are the improved achievements by Backus. Because:

- This relation is obtained by an exact process not by an approximate process.
- This relation is exactly valid not only for an infinitesimal volume where the volume of K' tend to zero but also, for any volume of K' in continuum media.
- In this integral relation the position vector is fixed at point $\vec{r}_0$, so the stress vector changes only by changing the unit normal vector on the surface of the mass element at a given time. This is the key character that leads to the exact validation of this relation for any volume of mass element in continuum media. In the former proofs, stress vector changes by changing both the position vector

[2] In the first integral of the right hand side, ∂ is missed in the original book.

$\vec{r}$ and the unit normal vector on surface of the mass element at a given time and this leads to limiting and approximating proofs for only the mass elements with infinitesimal volume.

Backus uses a second lemma to prove the existence of stress tensor based on the relation (13.2.15). On pages 176-180:

We also need [The second lemma:]
***Lemma* 13.2.30** *Suppose* $\vec{S} : N_U \to V$*. Suppose that for any open set* K' *with piecewise smooth boundary* $\partial K'$*,* $\vec{S}$ *satisfies*

$$\int_{\partial K'} dA(\vec{r})\vec{S}(\hat{n}(\vec{r})) = \vec{0}_V \qquad (13.2.21)$$

where $\hat{n}(\vec{r})$ *is the unit outward normal to* $\partial K'$ *as* $\vec{r} \in \partial K'$*. Suppose* $F : U \to V$ *is defined as follows:*

$$F(\vec{0}_U) = \vec{0}_V \text{ and if } \vec{u} \neq \vec{0}_U,\ F(\vec{u}) = \|\vec{u}\|\vec{S}(\frac{\vec{u}}{\|\vec{u}\|}). \qquad (13.2.22)$$

Then F *is linear.*

Proof of Lemma* 13.2.30*: *a)* $F(\vec{u}) = F(\vec{u})$ *for all* $\vec{u} \in U$*. To prove this, it suffices to prove*

$$\vec{S}(-\hat{n}) = -\vec{S}(\hat{n}) \text{ for all } \hat{n} \in N_U. \qquad (13.2.23)$$

Let K' *be the flat rectangular box shown at upper right. For this box,* (13.2.21) *gives*

$$L^2\vec{S}(\hat{n}) + L^2\vec{S}(-\hat{n}) + \epsilon L(\vec{S}(\hat{n}_1) + \vec{S}(-\hat{n}_1) + \vec{S}(\hat{n}_2) + \vec{S}(-\hat{n}_2)) = \vec{0}_V.$$

Hold L fixed and let $\epsilon \to 0$*. Then divide by* L^2 *and* (13.2.23) *is the result.*

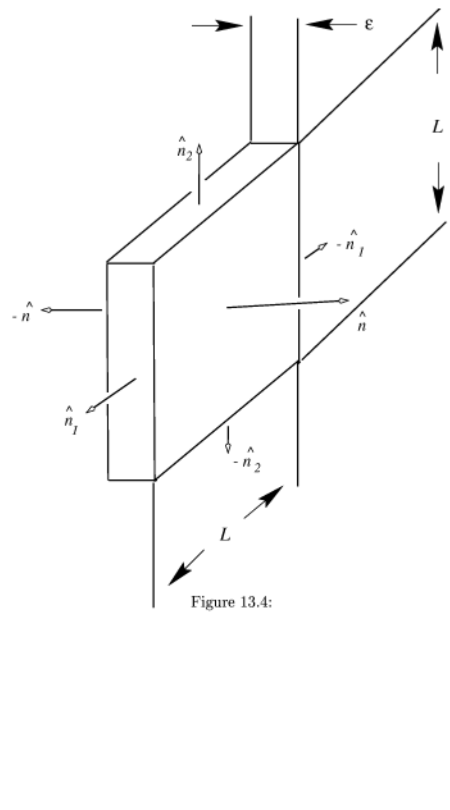

Figure 13.4:

b) If $c \in \mathcal{R}$ *and* $\vec{u} \in U$, $F(c\vec{u}) = cF(\vec{u})$.

i) *If* $c = 0$ *or* $\vec{u} = \vec{0}_U$, *this is obvious from* $F(\vec{0}_U) = \vec{0}_V$.

ii) *If* $c > 0$ *and* $\vec{u} \neq \vec{0}_U$, $F(c\vec{u}) = \|c\vec{u}\|\vec{S}(c\vec{u}/\|c\vec{u}\|) = c\|\vec{u}\|\vec{S}(c\vec{u}/c\|\vec{u}\|) = c\|\vec{u}\|\vec{S}(\vec{u}/\|\vec{u}\|) = cF(\vec{u})$.

iii) *If* $c < 0$, $F(c\vec{u}) = -F(-c\vec{u})$ *by a) above. But* $-c > 0$ *so* $F(-c\vec{u}) = -cF(\vec{u})$ *by ii). Then* $F(c\vec{u}) = -(-c)F(\vec{u}) = cF(\vec{u})$.

c) $F(\vec{u}_1 + \vec{u}_2) = F(\vec{u}_1) + F(\vec{u}_2)$ *for all* $\vec{u}_1$, $\vec{u}_2 \in U$.

i) *If* $\vec{u}_1 = \vec{0}_U$, $F(\vec{u}_1 + \vec{u}_2) = F(\vec{u}_2) = \vec{0}_V + F(\vec{u}_2) = F(\vec{u}_1) + F(\vec{u}_2)$.

ii) *If* $\vec{u}_1 \neq \vec{0}_U$ *and* $\vec{u}_2 = c\vec{u}_1$ *then* $F(\vec{u}_1 + \vec{u}_2) = F((1 + c)\vec{u}_1) = (1 + c)F(\vec{u}_1) = F(\vec{u}_1) + cF(\vec{u}_1) = F(\vec{u}_1) + F(c\vec{u}_1) = F(\vec{u}_1) + F(\vec{u}_2)$.

iii) *If* $\{\vec{u}_1, \vec{u}_2\}$ *is linearly independent, let* $\vec{u}_3 = -\vec{u}_1 - \vec{u}_2$. *We want to prove* $F(-\vec{u}_3) = F(\vec{u}_1) + F(\vec{u}_2)$, *or* $-F(\vec{u}_3) = F(\vec{u}_1) + F(\vec{u}_2)$, *or*

$$F(\vec{u}_1) + F(\vec{u}_2) + F(\vec{u}_3) = \vec{0}_V. \qquad (13.2.24)$$

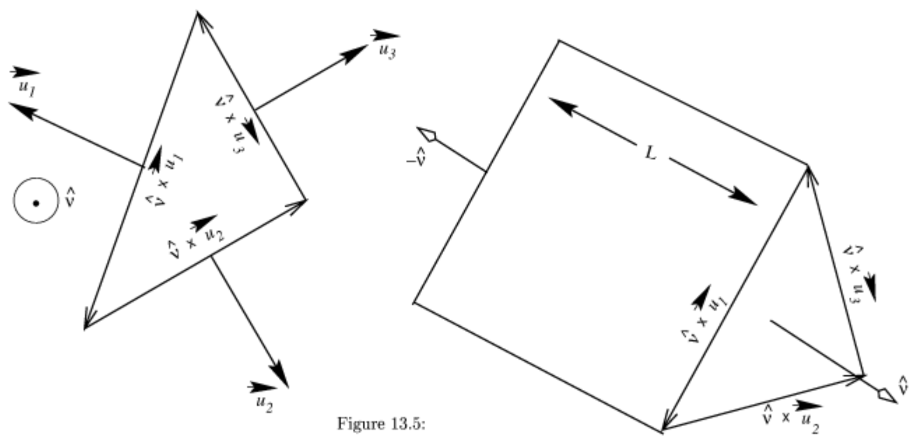

Figure 13.5:

To prove (13.2.24) *note that since* $\vec{u}_1$, $\vec{u}_2$ *are linearly independent, we can define the unit vector* $\hat{\nu} = (\vec{u}_1 \times \vec{u}_2)/\|\vec{u}_1 \times \vec{u}_2\|$. *We place the plane of this paper so that it contains* $\vec{u}_1$ *and* $\vec{u}_2$, *and* $\hat{\nu}$ *points out of the paper. The vectors* $\vec{u}_1$, $\vec{u}_2$, $\vec{u}_3$ *form the three sides of a nondegenerate triangle in the plane of the paper.* $\hat{\nu} \times \vec{u}_i$ *is obtained by rotating* $\vec{u}_i$ $90°$ *counterclockwise. If we rotate the triangle with sides* $\vec{u}_1$, $\vec{u}_2$, $\vec{u}_3$ $90°$ *counterclockwise, we obtain a triangle with sides* $\hat{\nu} \times \vec{u}_1$, $\hat{\nu} \times \vec{u}_2$, $\hat{\nu} \times \vec{u}_3$. *The length of side* $\hat{\nu} \times \vec{u}_i$ *is* $\|\hat{\nu} \times \vec{u}_i\| = \|\vec{u}_i\|$, *and* $\vec{u}_i$ *is perpendicular to that side and points out of the triangle. Let* K' *be the right cylinder whose base is the triangle with sides* $\hat{\nu} \times \vec{u}_i$, *and whose generators perpendicular to the base have length* L. *The base and top of the cylinder have area* $A = \|\vec{u}_1 \times \vec{u}_2\|/2$ *and their unit outward normals are* $\vec{\nu}$ *and* $-\vec{\nu}$. *The three rectangular faces of* K' *have areas* $L\|\vec{u}_i\|$ *and unit outward normals* $\vec{u}_{(i)}/\|\vec{u}_{(i)}\|$ *Applying* (13.2.21) *to this K' gives*

$$A\vec{S}(\hat{\nu}) + A\vec{S}(-\hat{\nu}) + \sum_{i=1}^{3} L\|\vec{u}_i\|\vec{S}(\vec{u}_i/\|\vec{u}_i\|) = \vec{0}_V.$$

But $\vec{S}(\hat{\nu}) = -\vec{S}(-\hat{\nu})$ so dividing by L and using (13.2.22) *gives* (13.2.23) *[correction* (13.2.24)*].*

***Corollary* 13.2.44** *(to Lemma* 13.2.30.*) Under the hypotheses of lemma* 13.2.30*, there is a unique $\overleftrightarrow{S}$ ... such that for all $\hat{n} \in N_U$*

$$\vec{S}(\hat{n}) = \hat{n}.\overleftrightarrow{S}. \qquad (13.2.25)$$

So, Backus uses two lemmas to prove the Cauchy's theorem. The first lemma 13.2.29 leads to the fundamental integral relation (13.2.15) for traction vectors at exact point $\vec{r}_0$, that has some important enhancements as compared with other works. In the second lemma 13.2.30 he tries to prove the existence of stress tensor based on relation (13.2.15).

In the second lemma there is a process similar to the process in the Noll's proof in [36] to prove the properties of the linear transformation for traction vectors that we have discussed it in the previous sections. But here this proof is on a different base form the Noll's proof. We saw that the Noll's proof was based on the infinitesimal volume and where the element's lengths approach zero [36], [98], [56]. But here, Backus applies the Noll's relation (13.2.22) to the obtained integral relation (13.2.15) that is valid exactly for any volume of the mass element. So, all of the related challenges to this step are removed in the Backus proof.

A challenge is related to part (a) in the proof of lemma 13.2.30, where in the equation:

$$L^2\vec{S}(\hat{n}) + L^2\vec{S}(-\hat{n}) + \epsilon L(\vec{S}(\hat{n}_1) + \vec{S}(-\hat{n}_1) + \vec{S}(\hat{n}_2) + \vec{S}(-\hat{n}_2)) = \vec{0}_V.$$

the expression *"Hold L fixed and let $\epsilon \to 0$. Then divide by L^2 and* (13.2.23) *is the result"*, may be interpreted as the result is valid only for an infinitesimal thin flat rectangular box. But if we replace this expression by:

"Hold L fixed and let ϵ change, it is not necessary that ϵ be a small value. Since the first two terms are independent of ϵ, then we must have $\vec{S}(\hat{n}) + \vec{S}(-\hat{n}) = \vec{0}_V$. So, (13.2.23) is the result."

This implies that the important relation $\vec{S}(-\hat{n}) = -\vec{S}(\hat{n})$ is independent of the volume of mass element. So, the challenges related to the infinitesimal volume are removed. Then, in parts (b) and (c), Backus proves exactly the essential properties of a linear transformation in vector space for $\vec{S}(\hat{n})$. Since a linear transformation in vector space can be shown by a second order tensor, the Backus's proof for the existence of stress tensor is completed. Also, for the derivation of the differential equation of linear momentum, Backus uses the divergence theorem.

7. Conclusion

In this article, we considered the tetrahedron arguments and the proofs for the existence of stress tensor in the literature. First, we showed the birth, importance and location of the tetrahedron argument and existence of stress tensor in the foundation of continuum mechanics. By representation of the formal tetrahedron argument in detail, that is presented in many books, we extracted some fundamental challenges and discussed the importance of them. These conceptual challenges related to the result of applying the conservation of linear momentum to any mass element in continuum media, the definition of traction vectors on the surfaces that pass through the same point, the order of the surface and volume terms in the integral equation of conservation of linear momentum, the limiting and approximating processes in derivation of stress tensor, and some others. Then, in a comprehensive review on a large number of the related books during nearly two centuries from 1823 until now, we presented the different versions of tetrahedron argument and the proofs for the existence of stress tensor, and in each one the challenges and the improvements are considered. They can be classified in two general approaches. In the first approach, that is followed in most texts, the traction vectors are not defined on the surfaces that pass through the same point, but in a limiting and approximating process when the volume of the mass element goes to zero, the traction vectors on the surfaces are regarded as the traction vectors on the surfaces that pass through the same point. In the second approach, that is followed in a few books, the traction vectors are exactly defined at the same point on the different surfaces that pass through that point. Then in a limiting and approximating process when the volume of the mass element goes to zero, a linear relation that leads to the existence of stress tensor, is obtained. By this approach some of the challenges are removed. We also presented and considered the improved works of Hamel and Backus. Most of the challenges on the existence of stress tensor are removed in the unknown and original work of Backus. We presented the main parts of this proof and stated its improvements and challenges.

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