Feshbach Resonances in Ultracold Gases

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Feshbach resonances are the essential tool to control the interaction between atoms in ultracold quantum gases. They have found numerous experimental applications, opening up the way to important breakthroughs. This Review broadly covers the phenomenon of Feshbach resonances in ultracold gases and their main applications. This includes the theoretical background and models for the description of Feshbach resonances, the experimental methods to find and characterize the resonances, a discussion of the main properties of resonances in various atomic species and mixed atomic species systems, and an overview of key experiments with atomic Bose-Einstein condensates, degenerate Fermi gases, and ultracold molecules.

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I. INTRODUCTION

A. Ultracold gases and Feshbach resonances: Scope of the Review

The great impact of ultracold atomic and molecular quantum gases on present-day physics is linked to the extraordinary degree of control that such systems offer to investigate the fundamental behavior of quantum matter under various conditions. The interest goes beyond atomic and molecular physics, reaching far into other fields, like condensed matter, few- and many-body physics. In all these applications, Feshbach resonances represent the essential tool to control the interaction between the atoms, which has been the key to many great breakthroughs.

Ultracold gases are generally produced by laser cooling (Metcalf and van der Straten, 1999) and subsequent evaporative cooling (Ketterle and van Druten, 1997). At temperatures in the nanokelvin range and typical number densities somewhere between $10^{12} \,\mathrm{cm}^{-3}$ and $10^{15} \,\mathrm{cm}^{-3}$, quantum-degenerate states of matter are formed when the atomic de-Broglie wavelength exceeds the typical interparticle distance and quantum statistics governs the behavior of the system. The attainment of Bose-Einstein condensation (BEC) in dilute ultracold gases marked the starting point of a new era in physics (Anderson et al., 1995; Bradley et al., 1995; Davis et al., 1995), and degenerate atomic Fermi gases entered the stage a few years later (DeMarco et al., 1999; Schreck et al., 2001; Truscott et al., 2001). The developments of the techniques to cool and trap atoms by laser light were recognized with the 1997 Nobel prize in physics (Chu, 1998; Cohen-Tannoudji, 1998; Phillips, 1998). Only four years later, the achievement of BEC in dilute gases of alkali atoms and early fundamental studies of the properties of the condensates led to the 2001 Nobel prize (Cornell and Wieman, 2002; Ketterle, 2002) ¹.

In this Review, we give a broad coverage of Feshbach resonances in view of the manifold applications they have found in ultracold gases. Regarding theory, we focus on the underlying two-body physics and on models to describe Feshbach resonances. In the experimental part we include applications to few- and many-body physics; we discuss typical or representative results, instead of the impossible attempt to exhaustively review all developments in this rapidly growing field. Several aspects of Feshbach resonances and related topics have already been reviewed elsewhere. An early review on Feshbach resonance theory was given by (Timmermans et al., 1999). In another theoretical review, (Duine and Stoof, 2004) focussed on atom-molecule coherence. (Köhler et al., 2006) and (Hutson and Soldán, 2006) reviewed the formation of ultracold molecules near Feshbach resonances. The closely related topic of photoassociation was reviewed by (Jones et al., 2006).

In Sec. II, we start with a presentation of the theoretical background. Then, in Sec. III, we present the various experimental methods to identify and characterize Feshbach resonances. There we also discuss the specific interaction properties of different atomic species, which can exhibit vastly different behavior. In Sec. IV, we present important applications of interaction control in experiments on atomic Bose and Fermi gases. In Sec. V, we discuss properties and applications of ultracold molecules created via Feshbach association. Finally, in Sec. VI, we discuss some related topics, like optical Feshbach resonances, interaction control in optical lattices, few-body physics, and the relation to molecular scattering resonances and cold chemistry.

B. Basic physics of a Feshbach resonance

The physical origin and the elementary properties of a Feshbach resonance can be understood from a simple picture. Here we outline the basic ideas, referring the reader to Sec. II for a more detailed theoretical discussion.

We consider two molecular potential curves $V_{\rm bg}(R)$ and $V_{\rm c}(R)$, as illustrated in Fig. 1. For large internuclear distances R, the background potential $V_{\rm bg}(R)$ asymptotically connects to two free atoms in the ultracold gas. For a collision process, having the very small energy E, this potential represents the energetically open channel, in the following referred to as the entrance channel. The other potential, $V_{\rm c}(R)$, representing the closed channel, is important as it can support bound molecular states near the threshold of the open channel.

A Feshbach resonance occurs when the bound molecular state in the closed channel energetically approaches the scattering state in the open channel. Then even weak coupling can lead to strong mixing between the two channels. The energy difference can be controlled via a magnetic field when the corresponding magnetic moments are different. This leads to a magnetically tuned Feshbach resonance. The magnetic tuning method is the common way to achieve resonant coupling and it has found numerous applications, as will be extensively discussed in this Review. Alternatively, resonant coupling can be

¹ For overviews on laser cooling and trapping, BEC, and ultracold Fermi gases see the proceedings of the Varenna summer schools in 1991, 1998, and 2006 (Arimondo et al., 1992; Inguscio et al., 2008, 1999). For reviews on the theory of degenerate quantum gases of bosons and fermions see (Dalfovo et al., 1999) and (Giorgini et al., 2008), respectively, and the textbooks by (Stringari and Pitaevskii, 2003) and (Pethick and Smith, 2008).

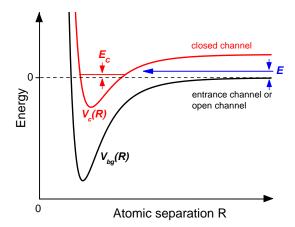


FIG. 1 Basic two-channel model for a Feshbach resonance. The phenomenon occurs when two atoms colliding at energy E in the entrance channel resonantly couple to a molecular bound state with energy E_c supported by the closed channel potential. In the ultracold domain, collisions take place near zero-energy, $E \to 0$. Resonant coupling is then conveniently realized by magnetically tuning E_c near 0, if the magnetic moments of the closed and open channel differ.

achieved by optical methods, leading to *optical Feshbach* resonances with many conceptual similarities to the magnetically tuned case; see Sec. VI.A. Such resonances are promising for cases where magnetically tunable resonances are absent.

A magnetically tuned Feshbach resonance can be described by a simple expression², introduced by (Moerdijk *et al.*, 1995), for the s-wave scattering length a as a function of the magnetic field B,

$$a(B) = a_{\text{bg}} \left(1 - \frac{\Delta}{B - B_0} \right). \tag{1}$$

Figure 2(a) illustrates this resonance expression. The background scattering length $a_{\rm bg}$, which is the scattering length associated with $V_{\rm bg}(R)$, represents the offresonant value. It is directly related to the energy of the last-bound vibrational level of $V_{\rm bg}(R)$. The parameter B_0 denotes the resonance position, where the scattering length diverges $(a \to \pm \infty)$, and the parameter Δ is the resonance width. Note that both $a_{\rm bg}$ and Δ can be positive or negative. An important point is the zero crossing of the scattering length associated with a Feshbach resonance; it occurs at a magnetic field $B = B_0 + \Delta$. Note also that we will use G as the magnetic field unit in this Review, because of its near-universal usage among groups working in this field; $1 \text{ G} = 10^{-4} \text{ T}$.

The energy of the weakly bound molecular state near the resonance position B_0 is shown in Fig. 2(b), relative

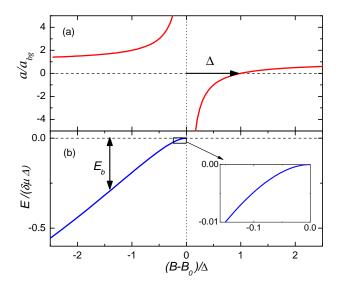


FIG. 2 Scattering length a (Panel (a)) and molecular state energy E (Panel (b)) near a magnetically tuned Feshbach resonance. The binding energy is defined to be positive, $E_b = -E$. The inset shows the universal regime near the point of resonance where a is very large and positive.

to the threshold of two free atoms with zero kinetic energy. The energy approaches threshold at E=0 on the side of the resonance where a is large and positive. Away from resonance, the energy varies linearly with B with a slope given by $\delta\mu$, the difference in magnetic moments of the open and closed channels. Near resonance the coupling between the two channels mixes in entrance-channel contributions and strongly bends the molecular state.

In the vicinity of the resonance position at B_0 , where the two channels are strongly coupled, the scattering length is very large. For large positive values of a, a "dressed" molecular state exists with a binding energy given by

$$E_b = \frac{\hbar^2}{2\mu a^2} \,, \tag{2}$$

where μ is the reduced mass of the atom pair. In this limit E_b depends quadratically on the magnetic detuning $B-B_0$ and results in the bend seen in the inset to Fig. 2. This region is of particular interest because of its universal properties; here the state can be described in terms of a single effective molecular potential having scattering length a. In this case, the wavefunction for the relative atomic motion is a quantum halo state which extends to a very large size on the order of a; the molecule is then called a halo dimer; see Sec. V.B.2.

A very useful distinction can be made between resonances that exist in various systems; see Sec. II.B.2. For narrow resonances with a width Δ typically well below 1 G (see Appendix) the universal range persist only for a very small fraction of the width. In contrast, broad resonances with a width typically much larger than 1 G tend to have a large universal range extending over a

² This simple expression applies to resonances without inelastic two-body channels. Some Feshbach resonances, especially the optical ones, feature two-body decay. A more general discussion including inelastic decay is given in Sec. II.A.3

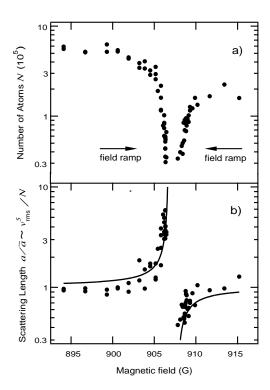


FIG. 3 Observation of a magnetically tuned Feshbach resonance in an optically trapped BEC of Na atoms. The upper panel shows a strong loss of atoms near the resonance, which is due to enhanced three-body recombination. The lower panel shows the dispersive shape of the scattering length a near the resonance, as determined from measurements of the mean-field interaction by expansion of the condensate after release from the trap; here a is normalized to the background value $a_{\rm bg}$. The magnetic field is given in G, where $1~{\rm G}=10^{-4}~{\rm T}$. Reprinted by permission from Macmillan Publishers Ltd: Nature (Inouye et~al., 1998), copyright 1998.

considerable fraction of the width. The first class of resonances is referred to as closed-channel dominated resonances, whereas the second class is called entrance-channel dominated resonances. For the distinction between both classes, the width Δ is not the only relevant parameter. Also, the background scattering length $a_{\rm bg}$ and the differential magnetic moment $\delta\mu$ need to be taken into account. Section II.B.2 discusses this important distinction in terms of a dimensionless resonance strength.

Figure 3 shows the observation of a Feshbach resonance as reported in 1998 by (Inouye et al., 1998) for an optically trapped BEC of Na atoms. This early example highlights the two most striking features of a Feshbach resonance, the tunability of the scattering length according to Eq. 1 and the fast loss of atoms in the resonance region. The latter can be attributed to strongly enhanced three-body recombination and molecule formation near a Feshbach resonance; see Sec. III.A.2.

A Feshbach resonance in an ultracold atomic gas can serve as a gateway into the molecular world and is thus strongly connected to the field of ultracold molecules; see Sec. V. Various techniques have been developed to associate molecules near Feshbach resonances. Ultracold molecules produced in this way are commonly referred to as Feshbach molecules. The meaning of this term is not precisely defined, as Feshbach molecules can be transferred to many other states near threshold or to much more deeply bound states, thus being converted to more conventional molecules. We will use the term Feshbach molecule for any molecule that exists near the threshold in an energy range set by the quantum of energy for near-threshold vibrations. The universal halo state is a special, very weakly bound case of a Feshbach molecule.

C. Historical remarks

Early investigations on phenomena arising from the coupling of a bound state to the continuum go back to the 1930s. (Rice, 1933) considered how a bound state predissociates into a continuum, (Fano, 1935; Fano et al., 2005) described asymmetric line shapes occurring in such a situation as a result of quantum interference, and (Beutler, 1935) reported on the observation of highly asymmetric lineshapes in rare gas photoionization spectra. Nuclear physicists considered basically the same situation, having nuclear scattering experiments in mind instead of atomic physics. (Breit and Wigner, 1936) considered the situation in the limit when the bound state plays a dominant role and the asymmetry disappears. Later, interference and line-shape asymmetry were taken into account by several authors (Blatt and Weisskopf, 1952).

Herman Feshbach (1917-2000) and Ugo Fano (1912-2001) developed their thorough treatments of the resonance phenomena that arise from the coupling of a discrete state to the continuum. Their work was carried out independently, using different theoretical approaches. While Feshbach's work originated in the context of nuclear physics (Feshbach, 1958, 1962), Fano approached the problem on the background of atomic physics (Fano, 1961), reformulating and extending his earlier work (Fano, 1935; Fano et al., 2005). Nowadays, the term "Feshbach resonance" is most widely used in the literature for the resonance phenomenon itself, but sometimes also the term "Fano-Feshbach resonance" appears. As a curiosity Feshbach himself considered his name being attached to a well-known resonance phenomenon as a mere atomic physics jargon (Kleppner, 2004; Rau, 2005). Fano's name is usually associated with the asymmetric lineshape of such a resonance, well known in atomic physics as a "Fano profile".

A prominent example for the observation of a Feshbach resonance in atomic physics is the experiment of (Bryant et al., 1977) on photodetachment by the negative ion of hydrogen. Near a photon energy of 11 eV two prominent resonances were seen, one of them being a Feshbach resonance and the other one a "shape resonance"; see Sec. II.A.3. Many more situations where Feshbach resonances play an important role can

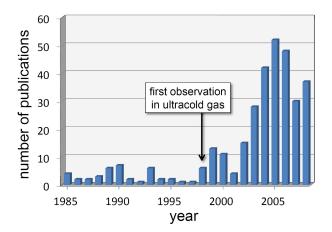


FIG. 4 Number of publications per year (from 1985 to 2008) with "Feshbach resonances" appearing in the title. Data from ISI Web of Science.

be found in atomic, molecular, and chemical physics; see (Gauyacq and Herzenberg, 1982; MacArthur et al., 1985; Nieh and Valentini, 1990; Spence and Noguchi, 1975; Weber et al., 1999) for a few examples. In such experiments, the resonances occur when the scattering energy is varied. This is in contrast to the experiments on ultracold gases, where scattering takes place in the zero-energy limit and the resonances occur when an external field tunes bound states near threshold.

In the context of quantum gases, Feshbach resonance were first considered by (Stwalley, 1976), who suggested the existence of magnetically induced Feshbach resonances in the scattering of spin-polarized hydrogen and deuterium atoms (H+D and D+D). He pointed to enhanced inelastic decay near these resonances and suggested that they should be avoided to maintain stable spin-polarized hydrogen gases. A related loss resonance in hydrogen was observed by (Reynolds et al., 1986). The positive aspect of such resonances was first pointed out by (Tiesinga et al., 1993), who showed that they can be used to change the sign and strength of the interaction between ultracold atoms. In 1998, the possibility of interaction tuning via Feshbach resonances was demonstrated by (Inouye et al., 1998) for a ²³Na BEC, as already discussed in the preceding Section. In the same year, (Courteille et al., 1998) demonstrated a Feshbach resonance in a trapped sample of ⁸⁵Rb atoms through the enhancement of photoassociative loss induced by a probe laser.

The very important role of Feshbach resonances in present-day quantum gas experiments can be highlighted by looking at the number of publications per year with "Feshbach resonances" in the title; see Fig. 4. Before 1998, one finds just a few publications with the great majority not related to ultracold atoms. Then, after 1998, a substantial increase is observed as a result of the first successful experiments with Feshbach resonances in ultracold gases. It then took a few years until Feshbach resonances had become a fully established tool and opened

up many new applications in the field. This is reflected in the steep increase of the publication rate in the period from 2002 to 2004.

II. THEORETICAL BACKGROUND

This review primarily concentrates on magnetically tunable resonances, described in detail in the next Sections, while Section VI.A discusses optical changes in scattering lengths. Here we describe the two-body physics of collision resonances, not the few-body or many-body aspects. Properties of a number of magnetic Feshbach resonances are tabulated in the Appendix; see Table IV.

A. Basic collision physics

The theory for describing 2-body collisions is described in a number of textbooks (Messiah, 1966; Mott and Massey, 1965; Taylor, 1972). Let us first consider the collision of two structureless atoms, labeled 1 and 2 with masses m_1 and m_2 interacting under the influence of the potential $V(\mathbf{R})$, where **R** is the vector between the positions of the two atoms with magnitude R. The separated atoms are prepared in a plane wave with relative kinetic energy $E=\hbar^2k^2/(2\mu)$ and relative momentum $\hbar \mathbf{k}$, where $\mu = m_1 m_2/(m_1 + m_2)$ is the reduced mass of the pair. The plane wave in turn is expanded in a standard sum over spherical harmonic functions $Y_{\ell m_{\ell}}(\hat{\mathbf{R}})$, where ℓ is the relative angular momentum, m_{ℓ} is its projection along a space-fixed z axis, and $\ddot{\mathbf{R}} = \mathbf{R}/R$ is the direction vector on the unit sphere (Messiah, 1966). This expansion is called the partial wave expansion, and the various partial waves $\ell = 0, 1, 2, \dots$ are designated s-, p-, d-, ... waves.

If the potential $V(\mathbf{R})$ is isotropic, depending only on the magnitude of \mathbf{R} , there is no coupling among partial waves, each of which is described by the solution $\psi_{\ell}(R) = \phi_{\ell}(R)/R$ to the Schrödinger equation

$$-\frac{\hbar^2}{2\mu} \frac{d^2 \phi_{\ell}(R)}{dR^2} + V_{\ell}(R)\phi_{\ell}(R) = E\phi_{\ell}(R), \qquad (3)$$

where $V_{\ell}(R) = V(R) + \hbar^2 \ell (\ell+1)/(2\mu R^2)$ includes the centrifugal potential, which is repulsive for $\ell > 0$ and vanishes for the s-wave. We assume $V(R) \to 0$ as $R \to \infty$, so that E represents the energy of the separated particles. This equation has a spectrum of N_{ℓ} bound state solutions at discrete energies $E_{n\ell}$ for E < 0 and a continuous spectrum of scattering states with E > 0. While bound states are normally labeled by vibrational quantum number $v = 0, \ldots, N_{\ell} - 1$ counting up from the bottom of the potential, we prefer to label threshold bound states by quantum number $n = -1, -2, \ldots$ counting down from the top of the potential for the last, next to last, etc., bound states. The bound state solutions $|n\ell\rangle$ are normalized to unity, $|\langle n\ell|n\ell\rangle|^2 = 1$, and $\phi_{n\ell}(R) = \langle R|n\ell\rangle \to 0$

as $R \to \infty$. The scattering solutions, representing the incident plane wave plus a scattered wave, approach

$$\phi_{\ell}(R, E) \to c \frac{\sin(kR - \pi\ell/2 + \eta_{\ell}(E))}{\sqrt{k}} e^{i\eta_{\ell}(E)}$$
 (4)

as $R \to \infty$, where $\eta_{\ell}(E)$ is the scattering phase shift and $c = \sqrt{2\mu/(\pi\hbar^2)}$ is a constant that ensures the wave function $|E\ell\rangle$ is normalized per unit energy, $\langle E\ell|E'\ell\rangle = \int_0^\infty \phi_{\ell}^*(R,E)\phi_{\ell}(R,E')dR = \delta(E-E')$. The scattering phase shift is the key parameter that incorporates the effect of the whole potential on the collision event.

(Sadeghpour et al., 2000) reviews the special properties of scattering phase shift near a collision threshold when $k \to 0$. If V(R) varies as $1/R^s$ at large R, then $\tan \eta_\ell \propto k^{2\ell+1}$ if $2\ell+1 \leq s-2$ and $\tan \eta_\ell \propto k^{s-2}$ if $2\ell+1 \geq s-2$. While Levinson's theorem shows that $\eta_\ell \to N_\ell \pi$ as $k \to 0$, we need not consider the $N_\ell \pi$ part of the phase shift in this review. For van der Waals potentials with s=6, the threshold $\tan \eta_\ell$ varies as k and k^3 for s- and p-waves, and as k^4 for all other partial waves. The properties of s-wave collisions are of primary interest for cold neutral atom collisions, where near threshold, a more precise statement of the variation of $\tan \eta_0$ with k is given by the effective range expansion,

$$k \cot \eta_0(E) = -\frac{1}{a} + \frac{1}{2}r_0k^2,$$
 (5)

where a is called the s-wave scattering length and r_0 the effective range. For practical purposes, it often suffices to retain only the scattering length term and use $\tan \eta_0(E) = -ka$. Depending on the potential, the scattering length can have any value, $-\infty < a < +\infty$.

When the scattering length is positive and sufficiently large, that is, large compared to the characteristic length scale of the molecular potential (see Section II.B.1), the last s-wave bound state of the potential, labeled by index n=-1 and $\ell=0$, is just below threshold with a binding energy $E_b = -E_{-1,0}$ given by Eq. (2) in the Introduction. The domain of universality, where scattering and bound state properties are solely characterized by the scattering length and mass, are discussed at length in recent reviews (Braaten and Hammer, 2006; Köhler et al., 2006). The universal bound state wave function takes on the form $\phi_{-1,0}(R) = \sqrt{2/a} \exp(-R/a)$ at large R. Such a state exists almost entirely at long range beyond the outer classical turning point of the potential. Such a bound state is known as a "halo state," also studied in nuclear physics (Riisager, 1994) and discussed in Sec. V.B.2.

1. Collision channels

The atoms used in cold collision experiments generally have spin structure. For each atom i = 1 or 2 in a collision the electronic orbital angular momentum \mathbf{L}_i is coupled to the total electronic spin angular momentum \mathbf{S}_i

to give a resultant \mathbf{j}_i , which in turn is coupled to the nuclear spin \mathbf{I}_i to give the total angular momentum \mathbf{f}_i . The eigenstates of each atom are designated by the composite labels q_i . At zero magnetic field these labels are $f_i m_i$, where m_i is the projection of \mathbf{f}_i . For example, alkalimetal atoms that are commonly used in Feshbach resonance experiments, have $^2\mathbf{S}_{1/2}$ electronic ground states with quantum numbers $L_i=0$ and $S_i=1/2$, for which there are only two values of $f_i=I_i-1/2$ and $I_i+1/2$ when $I_i\neq 0$. Whether f_i is an integer or half an odd integer determines whether the atom is a composite boson or fermion.

A magnetic field **B** splits these levels into a manifold of Zeeman sublevels. Only the projection m_i along the field remains a good quantum number, and B=0 levels with the same m_i but different f_i can be mixed by the field. Even at high field, where the individual f_i values no longer represent good quantum numbers, the f_i value still can be retained as a label, indicating the value at B=0 with which the level adiabatically correlates.

Figure 5 indicates the Zeeman energy levels versus B for the $^6\mathrm{Li}$ atom, a fermion, according to the classic Breit-Rabi formula (Breit and Rabi, 1931). The two f_i levels are split at B=0 by the hyperfine energy, $E_{\mathrm{hf}}/h=228$ MHz. At large fields the lower group of three levels are associated with the quantum numbers $m_S=-1/2$, while the upper group has $m_S=+1/2$. The figure also shows our standard notation for atomic Zeeman levels for any species and any field strength. We label states by lower case Roman letters a,b,c,\ldots in order of increasing energy. Some authors prefer to label the levels in order numerically as $1,2,3,\ldots$ The notation q_i can symbolically refer to the f_im_i , alphabetical, or numerical choice of labeling

The collision event between two atoms is defined by preparing the atoms in states q_1 and q_2 while they are separated by a large distance R, then allowing them to come together, interact, and afterwards separate to two atoms in states q'_1 and q'_2 . If the two final states are the same as the initial ones, $q_1, q_2 = q'_1, q'_2$, the collision is said to be elastic, and the atoms have the same relative kinetic energy E before and after the collision. If one of the final states is different from an initial state, the collision is said to be inelastic. This often results in an energy release that causes a loss of cold atoms when the energetic atoms escape from the shallow trapping potential. We will concentrate primarily on collisions where the two-body inelastic collision rate is zero or else very small in comparison to the elastic rate, since this corresponds in practice to most cases of practical experimental interest. This condition is necessary for efficient evaporative cooling or to prevent rapid decay of the cold gas. Section III.A.2 discusses how atom loss due to 3-body collisions can be used to detect the presence of 2-body resonances.

In setting up the theory for the collision of two atoms, the scattering channels are defined by the internal states of the two atoms 1 and 2 and the partial wave, $|\alpha\rangle$ =

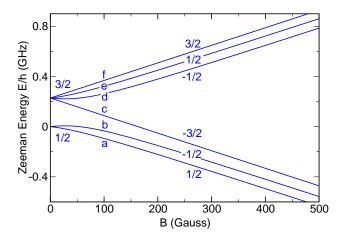


FIG. 5 Atomic energy levels of the ⁶Li atom, which has S = 1/2, I = 1, and f = 1/2 and 3/2. The figure shows both the projection m of f and the alphabetic shorthand notation $q_i = a, b, c, d, e$, and f used to label the levels in order of increasing energy.

 $|q_1q_2\rangle|\ell m_\ell\rangle$, where $\langle \hat{R}|\ell m_\ell\rangle = Y_{\ell m_\ell}(\hat{\mathbf{R}})$. Since for collisions in a magnetic field the quantum number $M=m_1+m_2+m_\ell$ is strictly conserved, a scattering channel can be conveniently labeled by specifying the set of quantum numbers $\{q_1q_2\ell M\}$. For s-waves, where $\ell=m_\ell=0$ and $M=m_1+m_2$, it is only necessary to specify the quantum numbers $\{q_1q_2\}$ to label a channel.

When the two atoms are of the same isotopic species, the wave function must be symmetric (antisymmetric) with respect to exchange of identical bosons (fermions). We assume such symmetrized and normalized functions, as described by (Stoof $et\ al.$, 1988). Exchange symmetry ensures that identical atoms in identical spin states can only collide in s-, d-, ... waves for the case of bosons and in p-, f- ... waves in the case of fermions; in all other cases, collisions in all partial waves are allowed.

The channel energy $E_{\alpha} = E(q_1) + E(q_2)$ is the internal energy of the separated atoms. Let us assume that the atoms are prepared in channel α with relative kinetic energy E so that the total energy is $E_{\rm tot} = E_{\alpha} + E$. Any channel β with $E_{\beta} \leq E_{\rm tot}$ is called an open channel, and any channel with $E_{\beta} > E_{\rm tot}$ is called a closed channel. A collision can produce atoms in an open channel after the collision, but not in a closed channel, since the atoms do not have enough energy to separate to the product atoms.

2. Collision rates

The partial collision cross section for starting in open channel α with relative kinetic energy E and ending in open channel β can be expressed in terms of the $S_{\alpha,\beta}(E)$

element of the multichannel unitary scattering matrix S. The cross section for elastic scattering at energy E in channel α is

$$\sigma_{\mathrm{el},\alpha}(E) = g_{\alpha} \frac{\pi}{L^2} |1 - S_{\alpha,\alpha}(E)|^2, \qquad (6)$$

whereas the unitarity property of **S** allows us to express the cross section for loss of atoms from channel α as

$$\sigma_{\text{loss},\alpha}(E) = g_{\alpha} \frac{\pi}{k^2} \left(1 - |S_{\alpha,\alpha}(E)|^2 \right) . \tag{7}$$

The corresponding partial elastic and inelastic rate coefficients $K_{\rm el,\alpha}(E)$ and $K_{\rm loss,\alpha}(E)$ are found by multiplying these partial cross sections by the relative collision velocity $v=\hbar k/\mu$. The factor $g_{\alpha}=1$ except for certain special cases involving identical particles. The factor $g_{\alpha}=2$ for describing thermalization or inelastic collisions in a normal Maxwellian gas of two atoms of the same species in identical spin states. Inelastic decay of a pure Bose-Einstein condensate has $g_{\alpha}=1$ (Kagan et al., 1985; Stoof et al., 1989).

If only one open channel α is present, collisions are purely elastic and $S_{\alpha,\alpha}(E) = \exp(2i\eta_{\alpha}(E))$. For s-waves the real-valued $\tan \eta_{\alpha}(E) \to -ka_{\alpha}$ as $k \to 0$ and a_{α} is the scattering length for channel α . When other open channels are present, the amplitude $|S_{\alpha,\alpha}(E)|$ is no longer unity, and for s-wave we can represent the complex phase $\eta_{\alpha}(E) \to -k\tilde{a}_{\alpha}$ for $k \to 0$ in terms of a complex scattering length (Balakrishnan et al., 1997; Bohn and Julienne, 1996)

$$\tilde{a}_{\alpha} = a_{\alpha} - ib_{\alpha} \,, \tag{8}$$

where a and b are real, and $1 - |S_{\alpha,\alpha}(E)|^2 \to 4kb_{\alpha} \ge 0$ as $k \to 0$. The threshold behavior is

$$\sigma_{\mathrm{el},\alpha}(E) = 4\pi g_{\alpha}(a_{\alpha}^2 + b_{\alpha}^2), \qquad (9)$$

for the s-wave elastic collision cross section and

$$K_{\text{loss},\alpha}(E) = \frac{2h}{\mu} g_{\alpha} b_{\alpha} \,. \tag{10}$$

for inelastic collisions that remove atoms from channel α . Both $\sigma_{\mathrm{el},\alpha}$ and $K_{\mathrm{loss},\alpha}$ approach constant values when E is sufficiently small.

The unitarity property of the S-matrix also sets an upper bound on the cross sections. Since there is a rigorous upper bound of $|S_{\alpha,\alpha}(E)| \leq 1$, we find that the elastic scattering cross section is maximum

$$\sigma_{\mathrm{el},\alpha}(E) = \frac{4\pi}{k^2} g_\alpha \,, \tag{11}$$

for any channel α (and thus any partial wave ℓ) when $S_{\alpha,\alpha}(E)=-1$. Furthermore, $\sigma_{\mathrm{loss},\alpha}(E)$, if nonvanishing, has a maximum value of $\sigma_{\mathrm{loss},\alpha}(E)=g_{\alpha}\pi/k^2$ when $S_{\alpha,\alpha}(E)=0$. These limits are called the unitarity limits of the cross sections. For s-wave collisions this limit is approached at quite low energy approximately equal to $E\approx\hbar^2/(2\mu a_{\alpha}^2)$, where $ka_{\alpha}\approx 1$.

In order to compare with experimental data the partial rate coefficients must be summed over partial waves and thermally averaged over the distribution of relative collision velocities at temperature T. This defines the total rate coefficients $K_{\mathrm{el},q_1q_2}(T)$ and $K_{\mathrm{loss},q_1q_2}(T)$ when the atoms are prepared in states q_1 and q_2 , respectively. Often the temperatures are sufficiently small that only the s-wave entrance channel contributes.

3. Resonance scattering

The idea of resonance scattering in atomic and molecular systems has been around since the earliest days of quantum physics, as described in the introduction. A conventional "resonance" occurs when the phase shift changes rapidly by $\approx \pi$ over a relatively narrow range of energy, due to the presence of a quasibound level of the system that is coupled to the scattering state of the colliding atoms. Such a resonance may be due to a quasibound level trapped behind a repulsive barrier of a single potential, or may be due to some approximate bound state which has a different symmetry and potential from that of the colliding atoms. The former is commonly known as a "shape resonance", whereas the latter is often called a "Feshbach resonance", in honor of Herman Feshbach, who developed a theory and a classification scheme for resonance scattering phenomena in the context of nuclear physics (Feshbach, We will follow here Fano's configura-1958, 1962). tion interaction treatment of resonant scattering (Fano, 1961), which is common in atomic physics. riety of treatments of the two-body physics of resonances in the context of ultracold Bose gases have been given, for example, (Duine and Stoof, 2004; Góral et al., 2004; Marcelis et al., 2004; Raoult and Mies, 2004; Timmermans et al., 1999).

We first consider the standard scattering picture away from any collision threshold defined by a two-channel Hamiltonian H. Assume that we can describe our system to a good approximation by two uncoupled "bare" channels, as schematically shown in Fig. 1. One is the open background scattering channel $|bg\rangle$ with scattering states $|E\rangle = \phi_{\rm bg}(R, E)|bg\rangle$ labeled by their collision energy E. The other is the closed channel $|c\rangle$ supporting a bound state $|C\rangle = \phi_c(R)|c\rangle$ with eigenenergy E_c . The functions $\phi_c(R)$ and $\phi_{\rm bg}(R,E)$ are the solutions to Eq. (3) for the background potential $V_{\rm bg}(R)$ and the closed channel potential $V_c(R)$ respectively. Here $\phi_c(R)$ is normalized to unity. The scattering in the open channel is characterized by a background phase shift $\eta_{\rm bg}(E)$. When the Hamiltonian coupling W(R) between the two channels is taken into account, then the two states become mixed, or "dressed", by the interaction, and the scattering phase picks up a resonant part due to the bound state embedded in the scattering continuum,

$$\eta(E) = \eta_{\text{bg}}(E) + \eta_{\text{res}}(E), \qquad (12)$$

where $\eta_{res}(E)$ takes on the standard Breit-Wigner form (Mott and Massey, 1965; Taylor, 1972):

$$\eta_{\rm res}(E) = -\tan^{-1}\left(\frac{\frac{1}{2}\Gamma(E_c)}{E - E_c - \delta E(E_c)}\right). \tag{13}$$

The interaction W(R), which vanishes at large R, determines two key features of the resonance, namely, its width

$$\Gamma(E) = 2\pi |\langle C|W(R)|E\rangle|^2, \qquad (14)$$

and its shift δE to a new position at $E_c + \delta E(E)$,

$$\delta E(E) = \mathcal{P} \int_{-\infty}^{\infty} \frac{|\langle C|W(R)|E'\rangle|^2}{E - E'} dE'$$
 (15)

where \mathcal{P} implies a principal part integral, which includes a sum over the contribution from any discrete bound states in the spectrum of the background channel. When the resonance energy is not near the channel threshold, it is normally an excellent approximation to take the width and shift as energy-independent constants, $\Gamma(E_c)$ and $\delta E(E_c)$, evaluated at the resonance energy E_c , as in Eq. (13). The resonance phase changes by $\approx \pi$ when Evaries over a range on the order of Γ from below to above resonance.

The essential difference between conventional and threshold resonance scattering is that if E_c is close to the open channel threshold at E=0, the explicit energy-dependence of the width and shift become crucial (Bohn and Julienne, 1999; Julienne and Gao, 2006; Marcelis *et al.*, 2004):

$$\eta_{\rm res}(E) = -\tan^{-1}\left(\frac{\frac{1}{2}\Gamma(E)}{E - E_c - \delta E(E)}\right). \tag{16}$$

The threshold laws for the s-wave width and shift as $k \to 0$ are

$$\frac{1}{2}\Gamma(E) \to (ka_{\rm bg})\Gamma_0. \tag{17}$$

$$E_c + \delta E(E) \rightarrow E_0$$
. (18)

where Γ_0 and E_0 are E-independent constants. Since $\Gamma(E)$ is positive definite, Γ_0 has the same sign as $a_{\rm bg}$. Combining these limits with the background phase property, $\eta_{\rm bg}(E) \to -ka_{\rm bg}$, and for the sake of generality, adding a decay rate γ/\hbar for the decay of the bound state into all available loss channels, gives in the limit of $k \to 0$,

$$\tilde{a} = a - ib = a_{\text{bg}} + \frac{a_{\text{bg}}\Gamma_0}{-E_0 + i(\gamma/2)}$$
 (19)

The unique role of scattering resonances in the ultracold domain comes from the ability to tune the threshold resonance position E_0 through zero by varying either an external magnetic field with strength B or optical field with frequency ν .

Both magnetically and optically tunable resonances are treated by the same theoretical formalism given above, although the physical mechanisms determining the coupling and tuning are quite different. In the case of a magnetically tunable resonance, the channel can often be chosen so that γ is zero or small enough to be ignored, whereas optical resonances are always accompanied by decay processes γ due to decay of the excited state. The resonance strength Γ_0 is fixed for magnetic resonances, but $\Gamma_0(I)$ for optical resonances can be turned off and on by varying the laser intensity I. It may also be possible to gain some control over Γ_0 by using a combination of electric and magnetic fields (Marcelis et al., 2008).

In the case of a magnetically tunable resonance, there is a difference $\delta \mu = \mu_{\rm atoms} - \mu_c$ between the magnetic moment $\mu_{\rm atoms}$ of the separated atoms and the magnetic moment μ_c of the bare bound state $|C\rangle$. Thus, the energy E_c of the state $|C\rangle$ relative to the channel energy of the separated atoms,

$$E_c = \delta \mu (B - B_c) \,, \tag{20}$$

can be tuned by varying the magnetic field, and E_c is zero at a magnetic field equal to B_c . Then, given that $\gamma = 0$, the scattering length takes on the simple form given in Eq. (1),

$$a(B) = a_{\rm bg} - a_{\rm bg} \frac{\Delta}{B - B_0},$$
 (21)

where

$$\Delta = \frac{\Gamma_0}{\delta \mu}$$
 and $B_0 = B_c + \delta B$ (22)

are the width and the position of the singularity in the scattering length, shifted due to the interaction between the closed and open channels by an amount $\delta B = -\delta E/\delta \mu$. Note that Δ has the same sign as $\delta \mu/a_{\rm bg}$. Figure 2 schematically illustrates the scattering length near the point of resonance B_0 .

The complex scattering length of an optically tunable resonance at laser frequency ν includes the collisional loss due to excited state decay (Bohn and Julienne, 1999; Fedichev *et al.*, 1996a),

$$\tilde{a}(\nu, I) = a_{\text{bg}} + \frac{a_{\text{bg}} \Gamma_0(I)}{h(\nu - \nu_c - \delta \nu(I)) + i(\gamma/2)}. \tag{23}$$

where the optically induced width $\Gamma_0(I)$ and shift $\delta\nu(I)$ are linear in I, and ν_c represents the frequency of the unshifted optical transition between the excited bound state and the collisional state of the two atoms at E=0.

Whenever bound state decay is present, whether for magnetically or optically tunable resonances, Eq. (19) shows that resonant control of the scattering length,

$$a = a_{\text{bg}} - a_{\text{res}} \frac{\gamma E_0}{E_0^2 + (\gamma/2)^2},$$
 (24)

is accompanied by collisional loss given by

$$b = \frac{1}{2} a_{\text{res}} \frac{\gamma^2}{E_0^2 + (\gamma/2)^2} \,. \tag{25}$$

The resonant length parameter

$$a_{\rm res} = a_{\rm bg} \frac{\Gamma_0}{\gamma} \tag{26}$$

is useful for defining the strength of an optical resonance (Bohn and Julienne, 1997; Ciuryło et al., 2005) or any other resonance with strong decay (Hutson, 2007). Figure 6 gives an example of such a resonance. The scattering length has its maximum variation of $a_{\rm bg} \pm a_{\rm res}$ at $E_0 = \pm \gamma/2$, where $b = a_{\rm res}$. Resonances with $a_{\rm res} \leq |a_{\rm bg}|$ only allow relatively small changes in scattering length, yet b remains large enough that they are typically accompanied by large inelastic rate coefficients. On the other hand, if $a_{\rm res} \gg |a_{\rm bg}|$, losses can be overcome by using large detuning, since the change in scattering length is $a - a_{\rm bg} = -a_{\rm res}(\gamma/E_0)$ when $|E_0| \gg \gamma$, whereas $b/|a - a_{\rm bg}| = \frac{1}{2}|\gamma/E_0| \ll 1$.

The resonance length formalism is quite powerful. By introducing the idea of an energy-dependent scattering length (Blume and Greene, 2002; Bolda *et al.*, 2002) it can be extended to Feshbach resonances in reduced dimensional systems such as pancake or cigar shaped optical lattice cells (Naidon and Julienne, 2006).

While this discussion has concentrated on resonant scattering properties for E>0, the near-threshold resonant properties of bound Feshbach molecules for energy E<0 are very important aspects of Feshbach physics; see Fig. 2 and (Köhler et al., 2006). In particular, as the bound state becomes more deeply bound, the closed channel character of the bound state increases and the binding energy E_b is no longer described by the universal expression in Eq. (2). The "dressed" or true molecular bound state of the system with energy $-E_b$ is a mixture of closed and background channel components,

$$|\psi_b(R)\rangle = \sqrt{Z}\phi_c(R)|c\rangle + \chi_{\rm bg}(R)|bg\rangle,$$
 (27)

where $0 \leq Z \leq 1$ represents the fraction of the eigenstate $|\psi_b(R)\rangle$ in the closed channel component (Duine and Stoof, 2003). Unit normalization of $|\psi_b(R)\rangle$ ensures that $\int |\chi_{\rm bg}(R)|^2 dR = 1 - Z$. Since the variation of the energy $-E_b$ with a parameter x of the Hamiltonian satisfies the Hellman-Feynmann theorem $\partial(-E_b)/\partial x = \langle \psi_b | \partial H/\partial x | \psi_b \rangle$, it follows from Eq. (27) that

$$Z = \frac{\partial(-E_b)}{\partial E_c} = \frac{\delta\mu_b}{\delta\mu} \,. \tag{28}$$

Here $\delta \mu_b = \partial E_b/\partial B = \mu_{\rm atoms} - \mu_b$ is the difference between the magnetic moment of the separated atoms and the magnetic moment μ_b of the "dressed" molecular eigenstate. Since $\delta \mu_b$ vanishes in the limit $B \to B_0$, where $E_b \to 0$ according to the universality condition in Eq. (2), then Z vanishes in this limit also. Section II.C.5 develops more specific properties and conditions for E_b and Z in this limit.

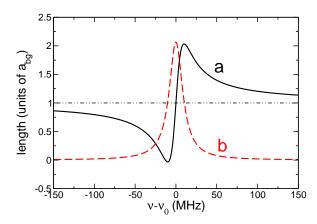


FIG. 6 Scattering length for an optically tunable Feshbach resonance as a function of laser tuning $\nu-\nu_0$. The lengths a and b are defined in Eqs. (24) and (25). Here $a_{\rm bg}=5.29$ nm, $\Gamma_0/h=21$ MHz at I=500 W/cm², $a_{\rm res}=5.47$ nm, and $\gamma/h=20$ MHz. Numerical values for the strength and spontaneous linewidth of the resonance are typical for ⁸⁷Rb and are taken from Fig. 1 of (Theis et~al., 2004).

B. Basic molecular physics

Most atoms that can be trapped at ultracold temperatures have ground S-states with zero electronic orbital angular momentum (L=0), as for alkali-metal or alkaline-earth-metal atoms. The collision between two atoms is controlled by the electronic Born-Oppenheimer interaction potential(s) between them. All potentials are isotropic for the interaction of two S-state atoms. We restrict our discussion of molecular physics to such cases. Figure 7 shows as an example the ${}^{1}\Sigma_{q}^{+}$ and ${}^{3}\Sigma_{u}^{+}$ potentials for two ground state ²S Li atoms, which are analogous to the similar potentials for the H₂ molecule or other alkali-metal atoms. The superscripts 1 and 3 refer to singlet and triplet coupling of the spins of the unpaired electrons from each atom, i.e., the total electron spin $S = S_1 + S_2$ has quantum numbers S = 0 and 1. The Σ refers to zero projection of electronic angular momentum on the interatomic axis for the S-state atoms, and g/u refers to gerade/ungerade electronic inversion symmetry with respect to the center of mass of the molecule. The q/u symmetry is absent when the two atoms are not of the same species.

The Born-Oppenheimer potentials are often available from *ab initio* or semi-empirical sources. When R is sufficiently small, typically less than $R_{\rm ex}\approx 20~{\rm a_0}\approx 1$ nm for alkali-metal atoms, electron exchange and chemical bonding effects determine the shape of the potentials. For $R\gg R_{\rm ex}$, the potentials are determined by the long-range dispersion interaction represented by a sum of second-order multipolar interaction terms.

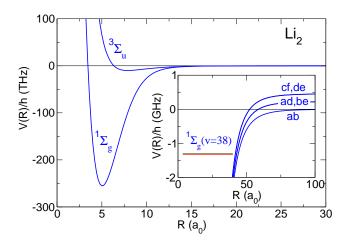


FIG. 7 Molecular potentials V(R)/h versus R of the two electronic states of Li₂ that correlate with two separated $^2\mathrm{S}$ atoms. The inset shows an expanded view of the long-range s-wave potentials of $^6\mathrm{Li}$ at B=0, indicating the five hyperfine states of the separated atoms (see Fig. 5) for which the total angular momentum has projection M=0. The inset also shows the last two nearly degenerate bound states (unresolved on the figure) of the $^6\mathrm{Li}_2$ molecule from a coupled channels calculation. It is a good approximations to label these nearly degenerate levels as the I=0 and 2 components of the total nuclear spin $\mathbf{I}=\mathbf{I}_1+\mathbf{I}_2$ of the last v=38 vibrational level of the $^1\Sigma_q^+$ potential.

1. Van der Waals bound states and scattering

Many aspects of ultracold neutral atom interactions and of Feshbach resonances, in particular, can be understood qualitatively and even quantitatively from the scattering and bound state properties of the long range van der Waals potential. The properties of this potential relevant for ultracold photoassociation spectroscopy have been reviewed in (Jones et al., 2006). Its analytic properties are discussed by (Gao, 1998b, 2000; Gribakin and Flambaum, 1993; Mott and Massey, 1965).

In the case of S-state atoms, the lead term in the longrange part of all Born-Oppenheimer potentials for a given atom pair has the same van der Waals potential characterized by a single C_6 coefficient for the pair. Consequently, all q_1q_2 spin combinations have the long-range potential

$$V_{\ell}(R) = -\frac{C_6}{R^6} + \frac{\hbar^2}{2\mu} \frac{\ell(\ell+1)}{R^2} \,. \tag{29}$$

A straightforward consideration of the units in Eq. (29) suggests that it is useful to define length and energy scales

$$R_{\text{vdw}} = \frac{1}{2} \left(\frac{2\mu C_6}{\hbar^2} \right)^{1/4} \text{ and } E_{\text{vdw}} = \frac{\hbar^2}{2\mu} \frac{1}{R_{\text{vdw}}^2}.$$
 (30)

(Gribakin and Flambaum, 1993) defined an alternative van der Waals length scale which they called the mean scattering length:

$$\bar{a} = 4\pi/\Gamma(1/4)^2 R_{\text{vdw}} = 0.955978 \dots R_{\text{vdw}},$$
 (31)

where $\Gamma(x)$ is the Gamma function. A corresponding energy scale is $\bar{E} = \hbar^2/(2\mu\bar{a}^2) = 1.09422\dots E_{\rm vdw}$. The parameter \bar{a} occurs frequently in formulas based on the van der Waals potential. Table I gives the values of $R_{\rm vdw}$ and $E_{\rm vdw}$ for several cases. Values of C_6 for other systems are tabulated in (Derevianko *et al.*, 1999; Porsev and Derevianko, 2006; Tang *et al.*, 1976).

The van der Waals energy and length scales permit a simple physical interpretation (Julienne and Mies, 1989). A key property for ultracold collisions is that C_6/R^6 becomes large compared to the collision energy E when $R < R_{\rm vdw}$. Thus, the wave function for any partial wave oscillates rapidly with R when $R < R_{\text{vdw}}$, since the local momentum $\hbar k(R) = \sqrt{2\mu(E - V(R))}$ becomes large compared to the asymptotic $\hbar k$. On the other hand, when $R > R_{\text{vdw}}$, the wavefunction approaches its asymptotic form with oscillations on the scale determined by the long de Broglie wavelength of the ultracold collision. The energy scale E_{vdw} determines the nature of the connection between the long- and short-range forms of the wavefunction. The de Broglie wavelength $\lambda = 2\pi (R_{\text{vdw}})$ for $E = E_{\text{vdw}}$. When $E \ll E_{\text{vdw}}$ so that $\lambda \gg R_{\text{vdw}}$, a WKB connection can not be made near R_{vdw} between the asymptotic s-wave and the short-range wavefunction (see Fig. 15 of (Jones et al., 2006)). Consequently, the quantum properties of the collision are manifest for $E < E_{\text{vdw}}$.

The van der Waals length also characterizes the extent of vibrational motion for near-threshold bound state. The outer turning point for classical motion for all low ℓ bound states is on the order of $R_{\rm vdw}$. The wave function for $\ell=0$ oscillates rapidly for $R < R_{\rm vdw}$ and decays exponentially as $e^{-k_b R}$ for $R \gg R_{\rm vdw}$, where $\hbar^2 k_b^2/(2\mu)$ is the binding energy. The only case where the wave function extends far beyond $R_{\rm vdw}$ is that of the last s wave bound state for the case of the universal halo molecule, where $a \gg R_{\rm vdw}$; see Secs. II.A and V.B.2.

The van der Waals potential determines the interaction over a wide zone between $R_{\rm vdw}$ and the much smaller $R_{\rm ex}$ where chemical forces become important. Thus, nearthreshold bound and scattering state properties are determined to a large extent by the long-range van der Waals potential. The effect of short-range is then contained within the phase of the wave function, or equivalently, the log derivative (Moerdijk and Verhaar, 1994; Vogels et al., 2000). More precisely, for any R_z satisfying $R_{\rm ex} < R_z \ll R_{\rm vdw}$ so that $k(R_z) \gg k$, the wave function phase is nearly independent of E and almost the same for all near-threshold bound or scattering states. In fact, the phase is nearly independent of partial wave ℓ as well, since the centrifugal potential is typically small compared to the van der Waals potential for such an R_z . Using this phase as a boundary condition for propagating the wavefunction to large R in the asymptotic domain determines

the energy-dependent scattering phase $\eta_{\ell}(E)$ and bound state energies. In fact the phase of the wave function in the zone $R_{\rm ex} < R_z \ll R_{\rm vdw}$ is uniquely related to the s-wave scattering length (Gao, 1998a). Consequently, to a good approximation the near-threshold bound states and scattering properties for all low partial waves are determined by the s-wave scattering length, the C_6 constant, and the reduced mass (Gao, 2001).

(Gao, 2000) has worked out the energies $E_{n,\ell}$ of the bound states of all partial waves for a van der Waals potential as a function of the s-wave scattering length, where $n = -1, -2, \dots$ is the vibrational quantum number and ℓ is the rotational quantum number of the bound state. He showed that the energies of weakly bound states have a $\Delta \ell = 4$ periodicity. Figure 8 shows bound state energies as function of ℓ for two values of a. In the left panel $a = \pm \infty$ so there is a s-wave bound state with E=0. The figure shows that for $\ell=4$ there also is bound state with $E/E_{\rm vdw}=0$. In fact (Gao, 2000) showed for $\ell = 8, 12, \ldots$ there will be a bound state at zero energy as well. The right panel in Fig. 8 shows that when $a = \bar{a}$ there is a bound state at zero energy for $\ell=2$. There also will be a bound state at zero energy for $\ell = 6, 10, ...$

Figure 8 can also be used to define the concept of "energy bins" in which, regardless of the value of a, there must be a bound state. Bins are most easily defined by starting from a case with a bound state at zero binding energy. By changing the short-range log derivative its binding energy can be increased, or its energy lowered, and at some point the binding energy is so large that a *new* bound state appears at zero binding energy. This is exactly the situation depicted in Fig. 8a for sand g waves. In other words, for s waves there must be a n = -1 bound state between $-39.5 E_{\text{vdw}}$ and $0 E_{\text{vdw}}$, while for g waves there must be a n = -1 bound state between $-191 E_{\text{vdw}}$ and $0 E_{\text{vdw}}$. The n = -2 s-wave bound state appears between $-272 E_{\text{vdw}}$ and $-39.5 E_{\text{vdw}}$. Figure 8b can similarly be used to define the bins for other waves.

When the scattering length is large compared to \bar{a} and positive, a simple expression for the van der Waals correction to the binding energy of the last s-wave bound state can be worked out (Gribakin and Flambaum, 1993):

$$E_{-1,0} = -\frac{\hbar^2}{2\mu(a-\bar{a})^2} \,. \tag{32}$$

The universal formula in Eq. (2) only applies in the limit that $a \gg \bar{a}$ and $|E_{-1,0}| \ll E_{\rm vdw}$. (Gao, 2004) has worked out higher order corrections to the binding energy due to the van der Waals potential, which can be recast as

$$E_{-1,0} = -\frac{\hbar^2}{2\mu(a-\bar{a})^2} \left[1 + \frac{g_1\bar{a}}{a-\bar{a}} + \frac{g_2\bar{a}^2}{(a-\bar{a})^2} + \dots\right]. (33)$$

Here $g_1 = \Gamma(1/4)^4/6\pi^2 - 2 = 0.9179...$, $g_2 = (5/4)g_1^2 - 2 = -0.9468...$ are constants.

TABLE I Characteristic van der Waals scales R_{vdw} and E_{vdw} for several atomic species. (1 amu = 1/12 mass of a ^{12}C atom, 1 au= 1 $E_h a_0^6$ where E_h is a hartree and 1 a_0 = 0.0529177... nm)

| Species | mass | C ₆ | $R_{ m vdw}$ | $E_{\rm vdw}/k_B$ | $E_{\rm vdw}/h$ |
|---------------------|------------|----------------|--------------|-------------------|-----------------|
| | (amu) | (au) | (a_0) | (mK) | (MHz) |
| ⁶ Li | 6.0151223 | 1393.39^{a} | 31.26 | 29.47 | 614.1 |
| 23 Na | 22.9897680 | 1556^{b} | 44.93 | 3.732 | 77.77 |
| $^{40}\mathrm{K}$ | 39.9639987 | 3897^{b} | 64.90 | 1.029 | 21.44 |
| $^{40}\mathrm{Ca}$ | 39.962591 | 2221^c | 56.39 | 1.363 | 28.40 |
| $^{87}{ m Rb}$ | 86.909187 | 4698^{d} | 82.58 | 0.2922 | 6.089 |
| $^{88}\mathrm{Sr}$ | 87.905616 | 3170^{c} | 75.06 | 0.3497 | 7.287 |
| $^{133}\mathrm{Cs}$ | 132.905429 | 6860^{e} | 101.0 | 0.1279 | 2.666 |

a. (Yan et al., 1996)
 b. (Derevianko et al., 1999)
 c. (Porsey and Derevianko, 2002)

d. (van Kempen et al., 2002) e. (Chin et al., 2004b)

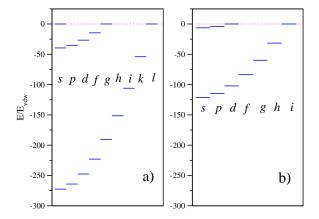


FIG. 8 Bound-state energies of the last vibrational levels of two atoms interacting via a van der Waals potential as a function of partial wave ℓ . The zero of energy is at two free atoms with zero collision energy ("at threshold"). The lowest partial waves are shown. Panel a) shows the bound state structure up to $\ell=8$ when the scattering length of the colliding atoms is infinite or equivalently that there is an s-wave bound state with zero binding energy. Panel b) shows the bound state structure up to $\ell=6$ when the scattering length is $a=0.956\dots R_{\rm vdw}=\bar{a}$, or equivalently there is a d-wave bound state with zero binding energy. The length $R_{\rm vdw}$ and energy $E_{\rm vdw}$ are defined in the text. Adapted from (Gao, 2000).

Similarly, the effective range of the potential in Eq. (5) is also determined from the van der Waals potential, given the s-wave scattering length (Flambaum et al., 1999; Gao, 1998a):

$$r_0 = \frac{\Gamma(1/4)^4}{6\pi} \bar{a} \left(1 - 2\frac{\bar{a}}{a} + 2\left(\frac{\bar{a}}{a}\right)^2 \right)$$
 (34)

where $\Gamma(1/4)^4/(6\pi) \approx 2.9179$. When $a \gg \bar{a}$, this simplifies to $r_0 = 2.9179\bar{a}$. Note that r_0 diverges as $a \to 0$.

The energy levels of the van der Waals potential are

not exact due to the slight influence from the actual short-range potential and extremely long-range retardation corrections. They are nevertheless relatively accurate guides to the expected energy spectrum for real molecules. For example when the scattering length is slightly larger than \bar{a} , which corresponds to Fig. 8b with all bound states shifted to slightly more positive energies, the d-wave bound state becomes a shape resonance, that is, a decaying quasibound state with E > 0 trapped behind the d-wave centrifugal barrier. For ²³Na and ⁸⁷Rb the experimentally observed scattering length is 10%-20% larger than \bar{a} and, indeed, in both cases a d-wave shape resonance has been observed under various circumstances (Boesten et al., 1997; Buggle et al., 2004; Samuelis et al., 2000; Thomas et al., 2004). Similarly, a p-wave shape resonance occurs when a is slightly larger than $2\bar{a}$, as for 40 K (DeMarco et al., 1999) and 171 Yb (Kitagawa et al., 2008). In addition (Kitagawa et al., 2008) show how the scattering length and binding energies of the last few bound states for the single potential of the Yb + Yb interaction are related as the reduced mass is changed by using different isotopic combinations of Yb atoms. The scattering length and binding energies can be "tuned" over a wide range by choosing different pairs of atoms among the seven stable isotopes of Yb.

The $\delta\ell=4$ characteristic of van der Waals potentials also has practical consequences for ultra-cold scattering. For $^{85}{\rm Rb}$ the scattering length has been found to be large compared to $R_{\rm vdw}$ and a g-wave shape resonance has been observed (Boesten et~al., 1996a). For $^{133}{\rm Cs}$ the scattering length is large compared to $R_{\rm vdw}$, and numerous g-wave bound states with binding energies much smaller than $E_{\rm vdw}$ were observed by (Chin et~al., 2004b) at low magnetic field. In fact, some of these bound states appear as magnetic Feshbach resonances in the collision of two Cs atoms. Recently, a weakly-bound l=8 or l-wave state has been observed as well (Mark et~al., 2007a).

2. Entrance- and closed-channel dominated resonances: Resonance strength

The van der Waals theory is very useful for characterizing and classifying the basic properties of the resonances discussed in Section II.A.3 by expressing lengths in units of \bar{a} and energies in units of \bar{E} ; see Eq. (31). The numerator of the resonant term in Eq. (19) defines a resonance strength parameter to be $a_{\rm bg}\Gamma_0$, where $\Gamma_0 = \delta\mu\Delta$; see Eq. (22). It is helpful to define a dimensionless resonance strength parameter $s_{\rm res}$ to be

$$s_{\rm res} = r_{\rm bg} \frac{\Gamma_0}{\bar{E}} = \frac{a_{\rm bg}}{\bar{a}} \frac{\delta \mu \Delta}{\bar{E}} \,.$$
 (35)

where $r_{\rm bg} = a_{\rm bg}/\bar{a}$ is the dimensionless background scattering length. The sign of $s_{\rm res}$ is always positive. The resonance phase in Eq. (16) is determined by the tunable

resonance position and the resonance width and shift. In the limit $E \to 0$, both the width

$$\frac{1}{2}\Gamma(E) = (k\bar{a})(\bar{E}s_{\rm res}) \tag{36}$$

and the shift (Góral et al., 2004; Julienne and Gao, 2006)

$$\delta E = \frac{1 - r_{\text{bg}}}{1 + (1 - r_{\text{bg}})^2} (\bar{E}s_{\text{res}}), \qquad (37)$$

are proportional to $Es_{\rm res}$. Section II.B.5 describes widths and shifts for some typical resonances. Sections II.C.3, II.C.4, and II.C.5 give additional analytic properties of threshold scattering and bound states associated with Feshbach resonances and show how Eq. (37) can be derived.

The strength parameter $s_{\rm res}$ allows us to classify Feshbach resonances into two limiting cases. (Köhler et~al., 2006; Stoll and Köhler, 2005) used $\eta=1/s_{\rm res}$ to do this. When $s_{\rm res}\gg 1$, the resonance is called an entrance channel dominated resonance. Here, the near-threshold scattering and bound states have the spin character of the entrance channel for detuning E_0 over a large fraction of the width Γ_0 and thus for $B-B_0$ over a large fraction of the resonance width Δ . In this regime, the resonance can be well modeled by the B-dependent scattering length of Eq. (1). The bound state is universal with $Z\ll 1$ (see Eq. (27)) over this large detuning range and with a binding energy well-approximated by Eq. (2). Resonances of this type have the largest resonance width Δ and are conventionally called "broad resonances".

Resonances with $s_{\rm res} \ll 1$ are called closed channel dominated resonances. Here, the near-threshold scattering and bound states have the spin character of the entrance channel only over a small fraction of the width Γ_0 near $E_0=0$ and thus over a small fraction of the resonance width Δ near $B=B_0$. A universal bound state only exists over this small detuning range. Thus, the closed channel fraction Z is only small near $B=B_0$ and is near unity over a wide detuning range away from $B=B_0$. Such resonances need to be modeled by a coupled channels description. Resonances of this type often have a small width Δ and are conventionally called "narrow resonances."

It should be emphasized that the conventional use of "broad" or "narrow" resonances referring to those that can or cannot be modelled by single channel model is not rigorously defined. Exceptions exist where resonances with apparently broad widths are actually closed channel dominated. The terms introduced here, entrance and closed channel dominance, better reflect the nature of the near threshold states over a detuning range on the order of the width Δ and can be unambiguously assigned to a resonance by evaluating $s_{\rm res}$.

Section II.B.5 illustrates the differences between entrance and closed channel dominated resonances by giving specific examples of such resonances. Section II.C.5 develops a simple model for the bound states for any

type of resonance, and shows that the norm Z of the closed channel part vanishes in the limit that $E_0 \to 0$ near the point of resonance B_0 , even for closed channel dominated resonances. (Szymanska et al., 2005) discuss in detail the implication of the distinction between open and closed channel dominance for the modeling of manybody systems, a topic that is beyond the scope of this review.

Figure 49 in the Appendix illustrates the wide range of resonance strengths $s_{\rm res}$ and widths Δ observed for various alkali atom resonances. Broad resonances with Δ larger than ~ 1 G tend to have $s_{\rm res} > 1$ and thus be entrance channel dominated ones. Narrow resonances with Δ smaller than ~ 1 G tend to have $s_{\rm res} < 1$ and thus be closed channel dominated ones. A notable exceptions is the ⁷Li 737 G resonance with $s_{\rm res} < 1$ that is very broad yet tends towards being closed channel dominated; see Section II.B.5.

Equation (36) allows us to address the question whether a sharp resonance feature appears at small but finite collision energy above threshold. A condition for having a sharp resonance is that the width $\frac{1}{2}\Gamma(E)$ should be smaller than the collision energy E. It is convenient to rewrite Eq. (36) as $\frac{1}{2}\Gamma(E) = (s_{res}/(k\bar{a}))E$. For an entrance channel dominated resonance with $s_{\rm res} \gg 1$ and $k\bar{a} < 1$ or $E < \bar{E}$, it follows that $\frac{1}{2}\Gamma(E) > E$. Thus, there can be no sharp resonance features evident in the above-threshold phase $\eta(E,B)$ of an entrance channel dominated resonance when $E < \bar{E}$. A sharp resonance feature can only appear when $E \gg \bar{E}$. (Nygaard et al., 2006) illustrate this case for a resonance involving ⁴⁰K atoms. On the other hand, for a closed channel dominated resonance with $s_{\rm res} \ll 1$ a sharp resonance feature in $\eta(E,B)$ with $\frac{1}{2}\Gamma(E) \ll E$ can appear immediately above threshold.

3. Coupled channels picture of molecular interactions

While many insights can be gained from the properties of the long range van der Waals potential, actual calculations require taking into account the full molecular Hamiltonian, including not only the full range of the Born-Oppenheimer potentials but also the various spin-dependent couplings among them. In general, the potential should be viewed as a spin-dependent potential matrix, the elements of which account for the interaction among the various spin states of the atoms. The wave function for atoms prepared in channel α can be written as a coupled channels expansion in the separated atom spin basis described in Section II.A.1 (Gao, 1996; Hutson $et\ al.$, 2008; Mies $et\ al.$, 2000; Mott and Massey, 1965; Stoof $et\ al.$, 1988):

$$|\psi_{\alpha}(R,E)\rangle = \sum_{\beta} |\beta\rangle\phi_{\beta\alpha}(R,E)/R.$$
 (38)

The allowed states $|\beta\rangle$ in this expansion are those that have the same projection quantum number $M=m_1+$

 $m_2 + m_\ell$ (see Section II.A.1). In addition parity conservation implies that all partial waves ℓ_β in the expansion are even if ℓ_α is even and odd if ℓ_α is odd.

Substituting Eq. (38) in the Schrödinger equation gives the coupled channels equations for the vector $\phi(R, E)$:

$$-\frac{\hbar^2}{2\mu}\frac{d^2\phi(R,E)}{dR^2} + \mathbf{V}(R)\phi(E,R) = E\phi(E,R).$$
 (39)

The solutions to these coupled equations give the bound and scattering states of the interacting atoms. The potential matrix $\mathbf{V}(R)$ gives the matrix elements of the Hamiltonian between the channel basis sets. It takes on the following form in the asymptotic spin basis:

$$V_{\alpha\beta} = \left(E_{\alpha} + \frac{\hbar^{2}\ell_{\alpha}(\ell_{\alpha} + 1)}{2\mu r^{2}}\right)\delta_{\alpha\beta} + V_{\text{int},\alpha\beta}(R) \quad (40)$$

The interaction matrix $V_{\text{int}}(R)$ contains the electronic Born-Oppenheimer potentials discussed in Section II.B and the relativistic electron spin-dependent interactions:

$$\mathbf{V}_{\text{int}}(R) = \mathbf{V}_{\text{el}}(R) + \mathbf{V}_{\text{ss}}(R) \tag{41}$$

For collisions of S state atoms the term $\mathbf{V}_{\mathrm{el}}(R)$ represents the strong isotropic electronic interaction that is diagonal in ℓ and m_{ℓ} but off-diagonal in the atomic channel quantum numbers q_1q_2 . The diagonal elements $V_{\mathrm{el},\alpha\alpha}$ vary at long range as the van der Waals potential (see Fig. 7). It has normally been unnecessary to include small retardation or nonadiabatic corrections to long range molecular potentials in order to fit experimental data on ground state collisions within their experimental error; see for example (Kitagawa et al., 2008). The off-diagonal elements $V_{\mathrm{el},\alpha\beta}$, where $\beta \neq \alpha$, decrease exponentially at large R as the exchange potential and become small compared to the atomic hyperfine splitting for $R > R_{\rm ex}$. The $V_{\rm el}$ coupling is responsible for elastic scattering and inelastic spin-exchange collisions and gives rise to the largest resonance strengths.

The term $\mathbf{V}_{\rm ss}(r)$ in Eq. (41) represents weak relativistic spin-dependent interactions. These include the spin-spin dipole interaction (Moerdijk et al., 1995; Stoof et al., 1988) and the second-order spin-orbit interaction (Kotochigova et al., 2000), important for heavy atoms (Leo et al., 2000). The two contributions are both anisotropic and are off-diagonal in both q_1q_2 and ℓ . Thus, $\mathbf{V}_{\rm ss}(r)$ couples different partial waves. At long range $\mathbf{V}_{\rm ss}(r)$ is proportional to α^2/R^3 , where $\alpha=1/137.0426$ is the fine structure constant. This anisotropic potential only contributes diagonal terms for partial waves $\ell \geq 1$, and does not contribute to the potential $V_{\rm int}, \alpha\alpha(R)$ when α represents an s-wave channel. The $\mathbf{V}_{\rm ss}(r)$ coupling is responsible for weak inelastic relaxation and normally gives rise to small resonance strengths.

The Born-Oppenheimer potentials are normally never known with sufficient accuracy to permit accurate calculations of threshold scattering properties. Consequently, it is usually necessary to vary the short range potentials over some range of $R < R_{\rm ex}$ to calibrate theoretical models so they reproduce measured threshold bound state or scattering data. In some cases the van der Waals coefficients are accurately known, whereas in other cases they need to be varied to fit the data as well. Once this is done, coupled channels theoretical models typically are robust and predictive of near-threshold collision and bound state properties. Some examples of high quality theoretical models based on fitting Feshbach resonance data are given by (van Abeelen and Verhaar, 1999a) for ²³Na, (Chin et al., 2000) and (Leo et al., 2000) for $^{133}\mathrm{Cs},$ (Marte et~al.,~2002) for $^{87}\mathrm{Rb},$ (Bartenstein et~al.,2005) for ${}^{6}\text{Li}$, (Werner et al., 2005) for ${}^{52}\text{Cr}$, and (Ferlaino et al., 2006; Pashov et al., 2007) for $^{40}\mathrm{K}^{87}\mathrm{Rb}$. Information on other models can be found in the references listed in Sections III.B and III.C.

4. Classification and molecular physics of Feshbach resonances

The previous Sections have laid the groundwork for classifying and understanding the properties of Feshbach resonance states in ultracold collisions of ground S-state atoms. This classification can be made according to the quantum numbers $\{q_1q_2\ell M|q_c\ell_c\}$, where $\{q_1q_2\ell M\}$ characterize the entrance channel (see Section II.A.1) and $\{q_c\ell_c\}$ characterize the "bare" closed channel bound state that gives rise to the resonance. Such a bound state has the same M as the entrance channel. Some possible choices for quantum numbers comprising the composite q_c are given below.

It is important to note that ℓ_c need not be the same as the entrance channel partial wave ℓ . Parity conservation ensures that $|\ell - \ell_c|$ is even. In the case of two $\mathbf{L} = 0$ atoms the $\mathbf{V}_{\rm el}$ term is isotropic and only gives rise to nonzero matrix elements when $\ell_c = \ell$. On the other hand, ℓ_c can be different from ℓ for the anisotropic $\mathbf{V}_{\rm ss}$ term. We are primarily concerned with entrance channel s-waves, although some resonances with p-wave (e.g., $^6\mathrm{Li}$, $^{40}\mathrm{K}$, $^{133}\mathrm{Cs}$) or d-wave ($^{52}\mathrm{Cr}$) entrance channels are known in the $\mu\mathrm{K}$ domain.

We find it is convenient to designate resonances according to the value of the closed channel bound state quantum number ℓ_c , as shown in Table II. If ℓ_c is even (odd), we assume an s- (p-) wave entrance channel unless otherwise stated. The strongest resonances with the largest widths Δ are s-wave resonances with $\ell = \ell_c = 0$, and are due to the $V_{\rm el}$ term in the Hamiltonian. A number of weak resonances with small Δ are known where the s-wave entrance channel is coupled through the V_{ss} term to bound states with even ℓ_c such as 2 or 4. Following Table II the latter are designated as d-wave or g-wave resonances, respectively. For example, d-wave resonances are known for $^{87}\mathrm{Rb}$ (Marte et al., 2002) and g-wave resonances for Cs (Chin et al., 2004b). For s-wave entrance channels the g-wave resonances are only possible due to second-order coupling in \mathbf{V}_{ss} . Entrance channel p-waves can be coupled to resonant bound states of odd $\ell_c = 1$ or

3, although the latter would tend to be quite weak and rarely observed.

TABLE II Classification of magnetic Feshbach resonances in collisions of ultracold atoms. The type of the resonance is labeled by the partial wave ℓ_c of the closed channel bound state rather than the entrance channel partial wave ℓ . Almost all cases known experimentally have $\ell=0$ or 1. Note that identical bosons (fermions) in identical spin states can only interact with even (odd) partial waves. All other cases permit both even and odd partial waves.

| Type | ℓ | $\ell_{ m c}$ |
|------------------|--------------|---------------|
| s-wave resonance | 0, 2 | 0 |
| p-wave resonance | 1, 3 | 1 |
| d-wave resonance | $0, 2 \dots$ | 2 |
| f-wave resonance | 1, 3 | 3 |
| g-wave resonance | 0, 2 | 4 |

The long-range potential $V_{el,\alpha\beta}(R)$ is diagonal for the interaction of two ground state alkali metal atoms for all combinations q_1q_2 of Zeeman sublevels, and all channels have the same van der Waals coefficient C_6 . Consequently each channel will have a spectrum of vibrational and rotational levels for a van der Waals potential as described in Fig. 8. The $n = -1, -2, \dots$ levels associated with closed spin channels α' can become scattering resonances for entrance channel α if they exist near energy E_{α} . The value of n can be one of the values comprising the set of approximate quantum numbers q_c . Some examples of the use of n in resonance classification are given by (Marte et al., 2002) for ${}^{87}\mathrm{Rb}$ (see Fig. 14) or (Köhler et al., 2006) for ⁸⁵Rb. The vibrational quantum number can be either n, counting down from the top, or v, counting up from the bottom of the well.

The approximate spin quantum numbers in q_c are determined by whatever set of quantum numbers blocks the Hamiltonian matrix into nearly diagonal parts. This will depend on the nature of the coupling among the various angular momenta of the problem so that no unique general scheme can be given. For alkali-metal dimers the spacing between vibrational levels, which is on the order of tens of E_{vdw} as seen from Fig. 8, must be compared to the spacing between the channel energies E_{α} . For example, in a light molecule like $^6\mathrm{Li}_2$ or $^{23}\mathrm{Na}_2$ they are large compared to the atomic hyperfine splitting $E_{\rm hf} =$ $|E_{I+1/2} - E_{I-1/2}|$ and Zeeman interactions. In this case, the vibrational levels are to a good approximation classified according to the electronic spin coupling, S=0 or 1 of the respective ${}^{1}\Sigma_{q}^{+}$ and ${}^{3}\Sigma_{u}^{+}$ Born-Oppenheimer potentials, with additional classification according to their nuclear spin substructure. (van Abeelen and Verhaar, 1999a; Laue et al., 2002) give an example of such a classification for ²³Na₂, and (Simonucci et al., 2005) give an example for ⁶Li₂.

In contrast to light species, heavy species like Rb₂ or Cs₂ have vibrational spacings that are smaller than $E_{\rm hf}$, so that near-threshold bound states of the $^1\Sigma_q^+$ and

 ${}^{3}\Sigma_{n}^{+}$ potentials are strongly mixed by the hyperfine interaction. The near-threshold molecular states do not correspond to either S=0 or 1, but often can be characterized by the approximate quantum number f_c , where $\mathbf{f} = \mathbf{f}_1 + \mathbf{f}_2$. As with the f_1 or f_2 atomic quantum numbers, f_c is not a good quantum number at large B but can be used as a label according to the low field state with which it adiabatically correlates. (Marte et al., 2002) give examples of such resonance classification for ⁸⁷Rb. and (Chin et al., 2004b) and (Köhler et al., 2006) do so for ¹³³Cs and ⁸⁵Rb respectively. (Hutson et al., 2008) describe improved computational methods for calculating the coupling between bound state levels and characterize a number of experimentally observed avoided crossings (Mark et al., 2007a) between Cs₂ levels having different approximate quantum numbers.

TABLE III Separated atom channel labels for the five s-wave M=0 channels of ⁶Li. The (f_1f_2) quantum numbers are only exact at B=0.

| α | $(f_1 f_2)$ | m_{f_1}, m_{f_2} |
|----|---------------------------------------|---------------------------------|
| ab | $\left(\frac{1}{2}\frac{1}{2}\right)$ | $+\frac{1}{2}, -\frac{1}{2}$ |
| ad | $\left(\frac{1}{2}\frac{3}{2}\right)$ | $+\frac{1}{2}, -\frac{1}{2}$ |
| be | $(\frac{1}{2}\frac{3}{2})$ | $-\frac{1}{2}$, $+\frac{1}{2}$ |
| cf | $\left(\frac{3}{2}\frac{3}{2}\right)$ | $-\frac{3}{2}$, $+\frac{3}{2}$ |
| de | $\left(\frac{3}{2}\frac{3}{2}\right)$ | $+\frac{1}{2}, -\frac{1}{2}$ |

5. Some examples of resonance properties

We will use the fermionic species $^6\mathrm{Li}$ to illustrate some basic features of Feshbach resonances. Figure 5 shows the atomic Zeeman levels. The inset to Fig. 7 shows the 5 channels and the potentials $V_{\alpha\alpha}(R)$ at long range needed to describe the s-wave collision of an $q_1=a$ atom with a $q_2=b$ atom. These 5 channels summarized in Table III have the same van der Waals C_6 coefficient and the same projection M=0. Due to the light mass, the last bound state of the van der Waals potential must lie in a "bin" that is $39.5E_{\mathrm{vdw}}/h=24.3$ GHz deep.

Figure 7 shows that the last two M=0 coupled channels s-wave bound states for B=0 have the character of the n=-1 or v=38 level of the $^1\Sigma_g^+$ potential. They have a binding energy of ≈ 1.38 GHz relative to the separated atom energy E_{ab} , associated with the positive scattering length of a=45.17 a_0 of the S=0 singlet potential (Bartenstein et al., 2005). The two levels have total nuclear spin I=0 or 2 and projection $m_I=0$, where $\mathbf{I}=\mathbf{I_1}+\mathbf{I_2}$. The next bound states below threshold are three M=0 spin components of the v=9 level of the $^3\Sigma_u^+$ potential, far below threshold with binding energies ≈ 24 GHz near the bottom of the n=-1 "bin". These deeply bound levels are associated with the large negative scattering length of -2140 a_0 for the S=1 potential (Abraham et al., 1997; Bartenstein et al., 2005).

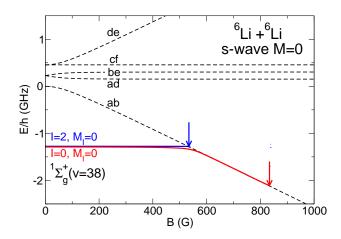


FIG. 9 Last coupled channels bound states of the $^6\mathrm{Li}_2$ dimer with M=0. The arrows indicate the locations of the 543 G and 834 G Feshbach resonances, where the binding energy of a threshold bound state equals 0. While the low B field I=2 $^1\Sigma_g^+(v=38)$ level retains its spin character as it crosses threshold near 543 G, the I=0 level mixes with the entrance channel and switches near ≈ 550 G to a bound level with ab spin character, eventually disappearing as a bound state when it crosses threshold at 834 G.

Figure 9 shows how the channel energies and the energies of the last two M=0 s-wave bound states of the $^6\mathrm{Li}_2$ molecule vary with magnetic field. (Simonucci et al., 2005) give a detailed description of the molecular physics of these multichannel bound states. At high B field, E_{ab} varies linearly with B with a slope of nearly $dE_{ab}/dB =$ $-2\mu_B$, where μ_B is a Bohr magneton. Since both bound states have S=0 character near B=0, their magnetic moment vanishes, i.e, $dE_c/dB = 0$ near B = 0. The I=2 state crosses E_{ab} near B=543 G, where it interacts weakly with the ab entrance channel and makes a very narrow Feshbach resonance, shown in Fig. 11. On the other hand, the energy of the strongly interacting I=0 bound state changes dramatically above about 540 G and becomes nearly parallel to the energy of the ab entrance channel. This state switches to S=1 character near the 540 G crossing region, and transforms into the v=10 level of the ${}^3\Sigma_u^+$ potential at higher B. This level becomes a very weakly bound "universal" halo state of dominantly entrance channel character above around 650 G, and does not disappear until it reaches the E_{ab} threshold near 834 G (Bartenstein et al., 2005), where it makes a very broad Feshbach resonance, shown in Fig. 10.

Figure 10 shows the near-threshold bound and scattering state properties of the $^6\mathrm{Li}~ab$ channel between 500 G and 1200 G, while Fig. 11 shows an expanded view of the narrow resonance near 543 G. The energy range is typical of the ultracold domain. The two figures illustrate two extremes of resonance behavior. The 834 G resonance is strongly open channel dominated with $s_{\rm res}=59$

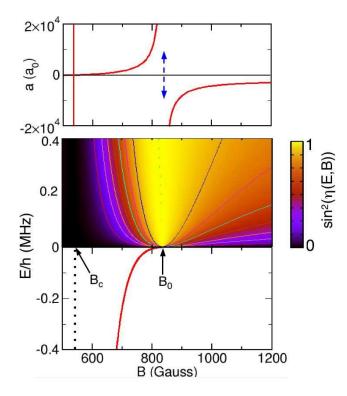


FIG. 10 Near-threshold bound and scattering state properties of the ⁶Li ab channel. The upper panel shows the coupled channels scattering length versus magnetic field B using the model of (Bartenstein et al., 2005). The double-headed arrow indicates the point of singularity B_0 for the broad resonance near 834 G, which is an entrance channel dominated resonance with width $\Delta=300$ G and $s_{\rm res}=59$ (see Sections II.A.3 and II.B.2). There also is a narrow resonance with a singularity near 543 G. The lower panel shows for E < 0 the energy of the bound state (solid line) that merges with the continuum at B_0 . The zero of energy at each B is the ab channel energy $E_{ab}(B)$. An energy of E/h = 0.4 MHz is equivalent to $E/k_B = 19 \mu K$. The universal bound state energy from Eq. (2) is indistinguishable on the scale of this graph from the coupled channels bound state energy. The nearly vertical dotted line shows the energy of the "bare" bound state $E_c(B)$ of ${}^1\Sigma_q^+(v=38,I=0)$ character that crosses threshold at B_c near 540 G. The shaded contour plot for E > 0shows $\sin^2 \eta(E,B)$. The broad light-colored region near the point of resonance indicates the region where $\sin^2 \eta \approx 1$ and the cross section is near its maximum value limited by the unitarity property of the S-matrix. Since $s_{res} \gg 1$ the width $\Gamma(E)$ in Eq. (36) is larger than E in the near-threshold region so that there is no above-threshold "resonance" feature in the collision cross section versus E.

(see Eq. (35)) and is well represented by a universal halo bound state of entrance channel character over a large fraction of its width Δ . The 543.2 G resonance is strongly closed channel dominated with $s_{\rm res}=0.001$. It exhibits open channel character and universal behavior only over a negligible detuning range spanning at most a few μ G when B is tuned near B_0 .

It is instructive to examine the wave functions for the

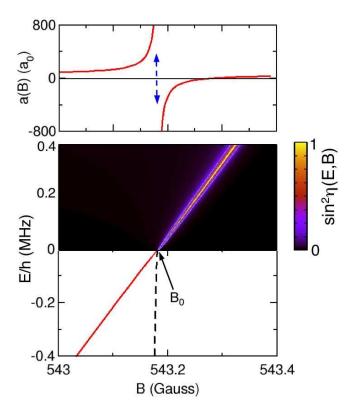


FIG. 11 Expanded view near 543 G of Fig. 10. The upper panel shows the coupled channels scattering length versus magnetic field strength B, where the double-headed arrow indicates the calculated point of singularity B_0 for the narrow resonance at 543.18 G with a width of $\Delta = 0.10$ G in excellent agreement with the measured resonance at 543.26(10) G (Strecker et al., 2003). This is a closed channel dominated resonance with $s_{\rm res} = 0.001$. The lower panel shows for E < 0the energy of the coupled channels bound state that merges with the continuum at B_0 . The dashed line shows the universal bound state energy from Eq. (2). Universality does not apply for detunings over most of the width of the resonance but is only applicable in an extremely narrow range very close to B_0 . The shaded contour plot of $\sin^2 \eta(E, B)$ for E > 0 shows a very narrow and sharp resonance emerging above threshold with a width $\Gamma(E) \ll E$, with a linear variation of position with B, and with a very small domain of unitarity of the S-matrix.

coupled channels bound states with the same binding energy near each resonance. For example, the binding energies in Figs. 10 and 11 are ≈ 200 kHz near 700 G and 543.1 G respectively. We calculate that the projection Z on the closed channel components, $Z=1-n_{\alpha}$ are 0.002 and 0.98 for these respective cases, where $n_{\alpha}=\int_{0}^{\infty}|\phi_{ab,ab}(R)|^{2}dR$ is the norm of the entrance channel component $\phi_{ab,ab}(R)$ of the bound state from the coupled channels expansion in Eq. (38). The small projection Z at 700 G is in good agreement with the value measured by (Partridge et~al., 2005); see Fig. 30. These projections for levels with the same near-threshold binding energy illustrate the very different character of entrance and closed channel dominated resonances.

The width Δ itself does not determine whether a resonance is entrance or closed channel dominated. Rather, it is necessary to apply the criterion in Eq. (35). A good example of this is provided by the bosonic ⁷Li system, which has a very broad resonance in the aa channel near 737 G with a width of 192 G (Junker et al., 2008; Khaykovich et al., 2002; Pollack et al., 2009), where a represents the state which correlates with the |f=1, m=1) state at B=0. Because the background scattering length is nearly two orders of magnitude smaller for this 7 Li case than for the 843 G 6 Li resonance, the s_{res} parameter for the ⁷Li aa resonance is only 0.80 instead of 59. Consequently, this broad ⁷Li resonance is tending towards closed channel dominance according to our classification scheme and only has a small region of universality spanning a few G when B is tuned near B_0 . Figure 12 shows the pronounced differences between the ⁶Li ab and ⁷Li aa resonances, in spite of the similar magnitudes of their widths; see also Section II.C.5.

In order to illustrate the difference between the "bare" and "dressed" resonance states introduced in Section II.A.3, Fig. 13 shows the coupled channel bound state energies and scattering phases in the near-threshold region for the 40 K ab channel. The a and b states correlate at B=0 with the $|f=\frac{9}{2}, m=-\frac{9}{2}\rangle$ and $|f=\frac{9}{2}, m=-\frac{7}{2}\rangle$ atomic states of fermionic 40 K. This resonance was observed by (Loftus et al., 2002; Regal et al., 2003a) and has additionally been characterized by (Szymanska et al., 2005) and (Nygaard et al., 2006). The actual eigenstates of the "dressed" system (solid lines) result from the avoided crossing of the ramping closed channel "bare" state energy $E_c = \delta \mu (B - B_c)$ and the last "bare" bound state at E_{-1} of the background potential (dashed lines). The shift in the location of the singularity in a(B) at B_0 from the threshold crossing of the "bare" state at B_c is given from Eq. (37):

$$B_0 - B_c = \Delta \frac{r_{\text{bg}}(1 - r_{\text{bg}})}{1 + (1 - r_{\text{bg}})^2}.$$
 (42)

This same formula predicts the large difference between B_0 and B_c evident in Fig. 10 for the broad ⁶Li ab resonance.

Finally, we give an example of resonances for a heavy species ⁸⁷Rb where classification of near-threshold bound states using the electronic spin S = 0 and 1 quantum numbers is not possible. The bin size $\approx 0.240\,\mathrm{GHz}$ for the last bound state is much less than the ⁸⁷Rb ground state hyperfine splitting $E_{hf}/h = 6.835$ GHz so that the last few bound states of the ⁸⁷Rb₂ molecule are mixed by the hyperfine interaction. Fig. 14 shows the coupled channels s-wave bound states calculated by (Marte et al., 2002) for channels with $M = m_1 + m_2 = 2$, 1, and 0. Unlike the ⁶Li case in Fig. 9, there are a number of bound states within a few GHz of threshold. The levels are labeled at B=0 by the spin quantum numbers (f_1f_2) of the separated atoms and the vibrational quantum number n counting down from the separated atom dissociation limit. The figure shows the last 3 vibrational

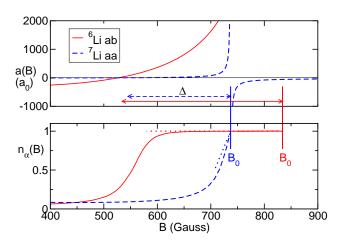


FIG. 12 Scattering length and entrance channel fraction for the ⁶Li 834 G open channel dominated resonance (solid lines) and the ⁷Li 737 G resonance (dashed lines), which is tending towards closed channel dominance. The narrow ⁶Li resonance is not shown. The upper panel shows the scattering length a(B) versus magnetic field strength B, whereas the lower panel shows the norm $n_{\alpha}(B)$ of the entrance channel spin component of the coupled channels wave function. The vertical lines indicate the location B_0 of each resonance, and the horizontal double arrows indicate the width Δ of each. The dotted lines on the lower panel indicates the slope of $n_{\alpha}(B) = 1 - Z(B)$ predicted near the resonance position B_0 by Eq. (51) of Section II.C.5. While both resonances have quite large Δ , the ⁶Li 834 G one is clearly "open channel dominated" with 1 - Z(B) remaining near unity when $|B-B_0|$ ranges over half of its width. On the other hand, the 737 G ⁷Li resonance has $s_{\rm res} = 0.80$ and is tending towards being "closed channel dominated," since 1 - Z(B) drops off rapidly from unity as $|B - B_0|$ increases from resonance, with $n_{\alpha}(B) > 0.5$ only for $|(B - B_0)/\Delta| < 0.11$. Furthermore, this resonance has a universal bound state (not shown) only over a relatively small fraction of its width, with the calculated binding energy departing from Eq. (2) by 10 percent when $|(B-B_0)/\Delta| = 0.06.$

levels of the lowest $(f_1f_2) = (11)$ separated atom limit. The B=0 energy of the (12) separated atom limit is $E_{hf}=6.835$ GHz and only the n=-4 vibrational level appears in this range. Similarly only the n=-5 level appears for the (22) separated atom limit. The closed channel dominated resonance $(s_{\rm res}=0.17)$ near 1007 G in the (11) M=2 channel has been used to make molecules in atomic gases (Dürr $et\ al.$, 2004a) and lattices (Thalhammer $et\ al.$, 2006).

C. Simplified models of resonance scattering

While coupled channels models are very valuable for understanding the near threshold molecular physics of scattering resonances and for highly quantitative predic-

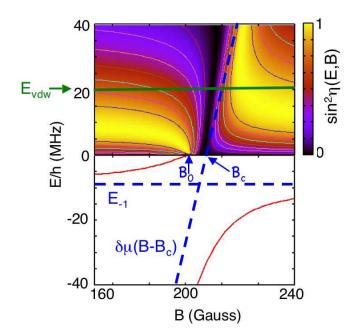


FIG. 13 Bound states and scattering phase near the $^{40}{\rm K}~ab$ 202 G resonance. The energy is between $E/h = \pm 40 \text{ MHz}$ $(E/k_B = \pm 1.9 \text{ mK})$ where the zero of energy is taken to be the separated atom energy of an a and a b atom. The horizontal solid line shows $E_{\rm vdw}/h=21$ MHz. The horizontal dashed line shows the last "bare" bound state energy E_{-1} of the background potential and the sloping dashed line shows the "bare" resonance level with energy $\delta\mu(B-B_c)$, where $\delta\mu/h=2.35$ MHz/G. The resonance width $\Delta = 7.8$ G (Greiner et al., 2003). The solid lines for E < 0 indicate the coupled channels "dressed" energies of the 40 K₂ molecule. Away from resonance these approach the "bare" energies. The strong avoided crossing between the two "bare" states leads to the shift in the point of singularity B_0 from the "bare" crossing at B_c . For positive energies the interference between the background and resonant phases is evident. Since this is a open channel dominated resonance with $s_{res} = 2$, no sharp resonant feature appears in $\sin^2 \eta(E, B)$ versus E for $0 < E < E_{\text{vdw}}$. A sharp resonance feature only emerges when $E \gg E_{\rm vdw}$.

tive calculations for a range of B field and multiple spin channels, they can be quite complicated to set up and use. Consequently, it is highly desirable to have simplified models that are accessible to experimental and theoretical researchers. Fortunately, a variety of high quality models are available, each valid over a limited domain of energy.

The key to practical approximations for the near threshold bound and scattering states for ultracold neutral atom interactions is the separation of the length and energy scales associated with the separated atoms on the one hand and the molecular interactions on the other. The molecular interactions are characterized by various energy scales associated with the van der Waals potential, the potential at $R_{\rm ex}$, or the minimum of the potential. This scale should be compared with the hyperfine, Zeeman, and kinetic energies of the ultracold

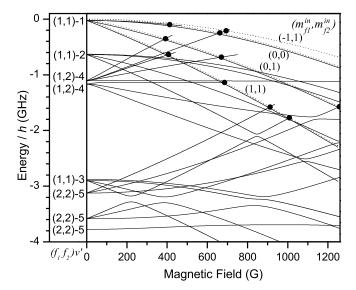


FIG. 14 Coupled channels ⁸⁷Rb s-wave bound states. The dotted lines show the channel energies for four different entrance channels with $f_1 = f_2 = 1$, labeled by the projection quantum numbers (m_1, m_2) . The bound-state energies (solid lines) for these channels are shown as a function of magnetic field B with quantum numbers $(f_1f_2)n$ assigned at B = 0, where n is the vibrational quantum number in the f_1f_2 channel counting down from the separated atom limit in that channel. A Feshbach resonance occurs when a bound state with quantum number $M = m_1 + m_2$ crosses a dissociation threshold having the same M (solid dots). From (Marte $et\ al.$, 2002).

atoms. For ranges of internuclear separation R where the molecular energy scale is much larger than the atomic one, the phases and amplitudes of the coupled channels wave function components $\phi_{\alpha\beta}$ are nearly independent of energy and partial wave over energy range on the order of the atomic scale. In effect, the short range wave function provides an energy-independent boundary condition for connecting to the near threshold asymptotic bound or scattering states, which are strongly energy dependent. While this separation of scales can be made explicit in methods based on long range coupled channels calculations (van Abeelen and Verhaar, 1999a; Tsai et al., 1997) or multichannel quantum defect methods (Burke et al., 1998; Julienne and Gao, 2006; Julienne and Mies, 1989; Raoult and Mies, 2004; Vogels et al., 1998), it remains implicit as the basis for many other approximation schemes.

A variety of simplified treatments show that it is sufficient to use the basic framework in Section II.A.3 to parameterize threshold resonances in ultracold atoms. Thus, resonances are characterized by a reduced mass μ , a background scattering length $a_{\rm bg}$, a tunable position E_0 selected by an external field, and an energy- and tuning-insensitive width Γ_0 . Resonances that decay, whether by emission of light or by relaxation to lower energy open channels, can readily be treated by introducing the decay width γ (Bohn and Julienne, 1999; Fedichev et al.,

1996b; Hutson, 2007; Köhler *et al.*, 2005). The review by (Köhler *et al.*, 2006) describes how two-channel models can be especially effective when the resonance parameters are already known.

1. Contact potential model

The simplest approximation for resonance scattering is to use the Fermi pseudopotential (Huang and Lee, 1957)

$$V(\mathbf{R}) = \frac{2\pi\hbar^2}{\mu} a(B)\delta(\mathbf{R}) \frac{\partial}{\partial R} R. , \qquad (43)$$

with a strength proportional to the scattering length a(B). This zero-range delta-function pseudopotential is an excellent approximation for the full molecular interaction when $k|a(B)| \ll 1$ and $k\bar{a} \ll 1$ and becomes exact in the limit $E \to 0$. It can be used for positive or negative a, and its phase shift is $\tan \eta(E,B) = -ka(B)$. For a>0 it has a bound state given by the universal energy of Eq. (2).

If the resonance parameters $a_{\rm bg}$, Δ , and B_0 are known, the effect of tuning near a resonance can then be fully incorporated using a(B) from Eq. (1). For an entrance channel dominated resonance with $s_{\rm res} \gg 1$, so that the universal binding energy in Eq. (2) applies (see Eq. (32)), the scattering length is the only parameter needed to treat near-threshold bound states and scattering (Köhler et al., 2006). However, more robust approximations are needed, since universality will only apply for detunings that at most span a range on the order of the width Δ and can be much less, depending on $s_{\rm res}$.

2. Other approximations

Although the underlying molecular physics often involves a number of coupled channels, many resonances are isolated in energy and magnetic field. Then the properties of the "bare" resonance level are determined by energy scales large compared to the small kinetic energies of the ultracold domain, and the level can be accurately approximated as coming from a single bound state channel, as in the Fano treatment summarized in Section II.A.3. A number of groups have developed a variety of simplified methods for characterizing the properties of ultracold scattering resonances, but we can not review this work exhaustively or in detail. For example, (Moerdijk et al., 1995), (Timmermans et al., 1999) and (Kokkelmans et al., 2002) introduce the standard Feshbach formalism of separating the system into bound and scattering subspaces, Q and P, to characterize magnetically tunable resonances for ground state alkali metal atoms. (Góral et al., 2004) use a Green's function formalism and introduce a separable potential that is chosen to accurately represent the two-body scattering and bound states of the background channel. (Marcelis et al., 2004) are especially interested in representing the case of a large negative $a_{\rm bg}$, which is relevant to the ⁸⁵Rb system. (Mies *et al.*, 2000) use the resonances of two ²³Na atoms to show how to reduce a coupled 5-channel problem to an effective 2-channel problem using a Lennard-Jones pseudopotential with the correct van der Waals coefficient. (Nygaard *et al.*, 2006) illustrates this method for the ⁴⁰K system.

One model that shows great promise for practical and accurate fitting of resonance data is the asymptotic bound state (ABM) model based on the work of (Moerdijk et al., 1995). Rather than solving for the bound states of a set of coupled equations in order to locate resonance positions, it uses an expansion in the last bound states of the Born-Oppenheimer potentials. The model is far less computationally demanding than full coupled channels calculations. (Stan et al., 2004) used a simplified version of this model to characterize measured resonances due to the triplet molecular state in the $^6\text{Li}+^{23}\text{Na}$ system. Recently (Wille et al., 2008) used this model to quantitatively characterize a number of resonances of $^6\text{Li}+^{40}\text{K}$ that involved strong mixing of the singlet and triplet molecular states.

3. van der Waals resonance model

By introducing the van der Waals C_6 coefficient as an additional model parameter, the properties of bound and scattering states can be extended away from their very near-threshold domain into the domain where $k\bar{a} \gg 1$ and the binding energy is much larger than E_{vdw} or \bar{E} . The reason is that there is a large range of R, namely $R_{\text{ex}} < R < R_{\text{vdw}}$, where the potential is accurately represented as $-C_6/R^6$ and is much larger in magnitude than E_{vdw} . The properties of the van der Waals potential have been discussed in Section II.B.1.

Feshbach resonances are characterized by a width $\Gamma(E)$ and shift $\delta E(E)$. These are given in the $E \to 0$ limit by Eqs. (36) and (37), which depend on the dimensionless resonance strength $s_{\rm res}$ and $r_{\rm bg}$. The $E \to 0$ result can be generalized to finite energy by introducing two standard functions $C_{\rm bg}(E)^{-2}$ and $\tan \lambda_{\rm bg}(E)$ of multichannel quantum defect theory (MQDT) (Julienne and Mies, 1989; Mies and Raoult, 2000; Raoult and Mies, 2004)

$$\frac{1}{2}\Gamma(E) = \frac{\bar{\Gamma}}{2} C_{\rm bg}(E)^{-2} \tag{44}$$

$$\delta E(E) = \frac{\bar{\Gamma}}{2} \tan \lambda_{\rm bg}(E). \tag{45}$$

where for the van der Waals background potential (Julienne and Gao, 2006)

$$\frac{\bar{\Gamma}}{2} = (\bar{E}s_{\rm res}) \frac{1}{1 + (1 - r_{\rm bg})^2} = \Gamma_0 \frac{r_{\rm bg}}{1 + (1 - r_{\rm bg})^2}$$
(46)

is proportional to $s_{\rm res}$ and is independent of energy. The MQDT functions have the following limiting form as $E \to 0$: $C_{\rm bg}(E)^{-2} = k\bar{a}(1+(1-r_{\rm bg})^2)$ and $\tan\lambda_{\rm bg}(E) = 1$

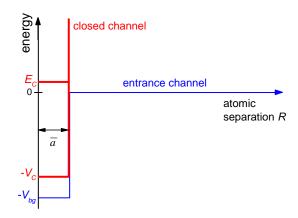


FIG. 15 Two-channel square well model of a magnetic Feshbach resonance. The potential for the "bare" entrance background channel (blue line) has a well depth $-V_{\rm bg}$. The spatial width is chosen to be \bar{a} so as to simulate the length scale of a van der Waals potential. The potential for the "bare" closed channel (red line) has a well depth $-V_c$ relative to the separated atoms and is infinite for $R > \bar{a}$. The value of $V_{\rm bg}$ is chosen so that the background scattering length is $a_{\rm bg}$. The value of V_c is chosen so that there is a bound state with energy E_c near E=0. We assume that this energy varies with magnetic field as $E_c=\delta\mu(B-B_c)$, where $\delta\mu$ is a relative magnetic moment and $E_c=0$ at $B=B_c$.

 $r_{\rm bg}$. When $E \gg \bar{E}$, $C_{\rm bg}(E)^{-2} \to 1$ and $\tan \lambda_{\rm bg}(E) \to 0$. Consequently, $\Gamma(E) = \bar{\Gamma}$ and $\delta E(E)$ vanishes when E becomes large compared to \bar{E} . If $|r_{\rm bg}| \gg 1$, the $C_{\rm bg}(E)^{-2}$ function has a maximum and $|\tan \lambda_{\rm bg}(E)|$ has decreased to half its E = 0 value at $E \approx \hbar^2/(2\mu \left(a_{\rm bg} - \bar{a}\right)^2)$.

The functions $C_{\text{bg}}(E)$ and $\tan \lambda_{\text{bg}}(E)$, as well as $\eta_{\text{bg}}(E)$, depend on only three parameters, C_6 , μ , and a_{bg} (Julienne and Gao, 2006). The near-threshold phase $\eta(E)$ in Eq. (16) can be evaluated over a wide range of energy on the order of \bar{E} and larger from a knowledge of these three parameters plus s_{res} , the magnetic moment difference $\delta\mu$, and the resonant position B_0 . The $\sin^2 \eta(E,B)$ function evaluated using Eqs. (44) and (45) are virtually indistinguishable from the coupled channels results shown in Figs. 10-13.

4. Analytic 2-channel square well model

A very simple square well model, because it is analytically solvable, can capture much of the physics of nearthreshold bound and scattering states. (Bethe, 1935) used such a model to successfully explain the threshold scattering of cold neutrons from atomic nuclei, where the neutron de Broglie wavelength was very large compared to the size of the nucleus. (Kokkelmans et al., 2002) and (Duine and Stoof, 2004) have introduced 2-channel square well models to represent Feshbach resonances in ultracold atom scattering.

Figure 15 shows the "bare" background and closed channel potentials for a square well model where for convenience of analysis we take the width of the wells to be the van der Waals length \bar{a} . The background entrance channel and the closed channel are designated by $|bg\rangle$ and $|c\rangle$ respectively. Using a two-state coupled channels expansion as in Eq. (38), $|\psi(R,E)\rangle = |c\rangle\phi_{\rm c}(R,E)/R + |bg\rangle\phi_{\rm bg}(R,E)/R$, the potential matrix in Eq. (40) is

$$\mathbf{V} = \begin{pmatrix} -V_{\rm c} & W \\ W & -V_{\rm bg} \end{pmatrix} \text{ for } R < \bar{a}$$

$$= \begin{pmatrix} \infty & 0 \\ 0 & 0 \end{pmatrix} \text{ for } R > \bar{a},$$

$$(47)$$

The off-diagonal matrix element W describes the weak coupling between the two channels.

In order to simulate a magnetically tuned Feshbach resonance, the model parameters need to be chosen so as to give the correct parameters for that resonance. The well depth $V_{\rm bg}$ is chosen so that the background channel scattering length is $a_{\rm bg}$. The well depth V_c is chosen so that the well has a "bare" bound state at E_c . The tuning of the bound state as $E_c = \delta \mu (B - B_c)$ can be simulated by varying V_c linearly with the external magnetic field B. Finally, weak coupling requires $|W| \ll |V_{\rm bg} - V_c|$. The coupling parameter W can then be chosen to give the right resonance width $\Gamma(E) = 2ka_{\rm bg}\delta\mu\Delta$ at low energies (see Eq. (17)), using the known resonance width Δ . Analytically calculating the matrix element defining $\Gamma(E)$ in Eq. (14) relates W to Δ as follows,

$$\frac{2V_{\rm c}W^2}{(V_{\rm bg} - V_{\rm c})^2} = \frac{r_{\rm bg}}{(1 - r_{\rm bg})^2} \delta\mu\Delta.$$
 (48)

With the chosen parameters, the square well model yields analytic form for the scattering phase shift as in Eq. (16) and the scattering length as in Eq. (1).

The square well model also permits an analytic evaluation of the weakly bound state below the continuum. Assuming an eigenstate $|\psi_b\rangle$ exists at energy $-E_b = -\hbar^2 k_b^2/(2\mu) < 0$ and $|a_{\rm bg}| \gg \bar{a}$, we get

$$k_b = \frac{1}{a_{bg} - \bar{a}} + \frac{\Gamma_{\text{sq}}/2}{\bar{a}(E_b + E_c)},$$
 (49)

where $\Gamma_{\rm sq}/2 = \delta \mu \Delta r_{\rm bg} (1-r_{\rm bg})^{-2}$. (Marcelis et al., 2004) derived a similar result for a contact potential. Note that when the coupling term $W \to 0$ so that $\Gamma_{\rm sq} \to 0$, the solutions $E_b = -E_c$ and $E_b = \hbar^2/[2\mu(a_{bg} - \bar{a})^2]$ correspond to the bare states of the square well in the closed and open channel (for $a_{bg} > \bar{a}$), as expected. Since the resonant singularity in the scattering length occurs when $E_b \to 0$, taking this limit of Eq. (49) allows us to calculate the resonance energy shift $\delta E = \delta \mu (B_0 - B_c)$ as

$$\delta E = \frac{\Gamma_{sq}}{2} (1 - r_{\rm bg}). \tag{50}$$

Both Eq. (50) and Eq. (49) can also be derived from the van der Waals model with Γ_{sq} replaced by $\bar{\Gamma} = \Gamma_{sq}[1 +$

 $(r_{\rm bg}-1)^{-2}$]. Note that Γ_{sq} and $\bar{\Gamma}$ are nearly the same for $|r_{bg}|\gg 1$. The modified version of Eq. (50) is equivalent to Eq. (37) and Eq. (42), derived from the van der Waals model.

(Lange et al., 2009) extends the above model to precisely determine the scattering length and the resonance parameters in the magnetic field regime where multiple Feshbach resonances overlap.

5. Properties of Feshbach molecules

A variety of properties of Feshbach molecules can be calculated by solving Eq. (49) for the binding energy E_b . For example, the closed channel fraction Z of the eigenstate can be found by differentiating E_b with respect to E_c ; see Eq. (28). In the limit $B \to B_0$ where E_b vanishes and $a \to +\infty$, we have

$$Z = \frac{1}{\zeta} \left| \frac{B - B_0}{\Delta} \right| . \tag{51}$$

where the dimensionless proportionality constant

$$\zeta = \frac{1}{2} s_{\text{res}} |r_{\text{bg}}| = \frac{r_{bg}^2}{2} \frac{|\delta \mu \Delta|}{\bar{E}}$$
 (52)

determines the rate at which the Feshbach molecular state deviates from the entrance channel dominated regime or, equivalently, the halo molecule regime, when B is tuned away from B_0 . Eq. (51) shows that having a small closed channel fraction $Z \ll 1$ requires the magnetic field to be close to resonance, $|B-B_0| \ll \zeta |\Delta|$. Figure 12 of Section II.B.5 compares 1-Z from Eq. (51) for the respective open and closed channel dominated ⁶Li 834 G and ⁷Li 737 G resonances.

For open channel dominated resonances, where $s_{\rm res}\gg 1$, it is usually true that $|r_{\rm bg}|\geq 1$ and $\zeta\gg 1$, and Z remains small over a large fraction of the resonance width Δ . The bound state wave function takes on primarily entrance channel character over this range. See the examples of the ⁶Li 834 G or ⁴⁰K 202 G resonances in Fig. 16. Closed channel dominated resonances have $s_{\rm res}\ll 1$ and small $\zeta\ll 1$. Consequently, Z remains small only over a small range of the resonance; see the example of the ⁸⁷Rb 1007 G or ⁶Li 543 G resonances in Fig. 16. A small $\zeta_{Rb}=0.09$ for the Rb 1007 G resonance implies that the molecular state is entrance channel dominated only within $\sim 9\%$ of the resonance width. Table IV in the Appendix lists ζ for several other resonances.

Expanding Eq. (49) at small binding energies, the molecular binding energy has the following form in the threshold limit:

$$E_b = \frac{\hbar^2}{2\mu(a - \bar{a} + R^*)^2},\tag{53}$$

where $R^* = \bar{a}/s_{\rm res}$. This expression applies in the limit that $a \gg \bar{a}$ and $a \gg 4R^*$. The binding energy E_b shows two corrections to the universal $1/a^2$ threshold law in

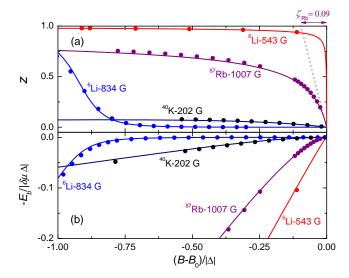


FIG. 16 (Color) Closed channel fraction Z (panel a) and energy $-E_b$ (panel b) of weakly-bound molecular states as a function of magnetic field for selected Feshbach resonances. The open circles come from coupled channels calculations as described in Section II.B.3. The solid curves are calculated based on Eq. (49), Eq. (28), and the parameters in Table IV. The slope of Z at $B=B_0$ defines the ζ parameter; an example for the $^{87}{\rm Rb}$ 1007 G resonance is shown as the dashed line with $\zeta_{Rb}=0.09$. This $^{87}{\rm Rb}$ resonance and the $^{6}{\rm Li}$ 543 G one are closed channel dominated and have $s_{red}, \zeta < 1$. The 6 Li 834 G and the $^{40}{\rm K}$ 202 G resonances are open channel dominated resonances with $s_{res}, \zeta > 1$.

Eq. (2). One is the finite range correction \bar{a} shown in Eq. (32); the other one, R^* , introduced by (Petrov, 2004), is unique to Feshbach resonances. The correction R^* is negligibly small for open channel dominated resonances with $s_{\rm res}\gg 1$. Closed closed channel dominated resonances can have large $R^*\gg \bar{a}$. Such resonances only have a regime with a bound state of predominant entrance channel character near B_0 where $a\gg 4R^*$. This condition is consistent with the one given in the previous paragraphs, namely, $|B-B_0|\ll \zeta |\Delta|$.

III. FINDING AND CHARACTERIZING FESHBACH RESONANCES

Magnetically tunable Feshbach resonances have been experimentally observed in essentially all alkali-metal species, in some mixtures of different alkali-metal atoms, as well as for Cr atoms. Experimental data on the resonance positions have in many cases enabled the construction of accurate models to describe the near-threshold behavior, including scattering properties and molecular states. We will first review the various experimental approaches to identify and characterize Feshbach resonances in Sec. III.A and then discuss observations of Feshbach resonances for alkali-metal atoms in Sec. III.B and various other systems in Sec. III.C.

A. Experimental methods

Experimental approaches to detect magnetic Feshbach resonances can be classified into several types. After some general considerations in Sec. III.A.1, we will discuss detection by inelastic collisional trap loss in Sec. III.A.2, by elastic collision properties in Sec. III.A.3, and loss in the presence of optical radiation in Sec. III.A.4. Finally, Section III.A.5 discusses precision radio-frequency spectroscopy of Feshbach molecules.

1. General considerations

a. What is the magnetic field range to be explored? The typical spacing between two Feshbach resonances can be estimated from the ratio of the energy splitting between closed channel molecular levels and the relative magnetic moment $\delta\mu$ between the entrance channel and the closed channel. The vibrational energy splitting between nearthreshold bound states is determined by the long-range van der Waals potential to be on the order of $100E_{\rm vdw}$; see Sec. II.B.1. For alkali-metal atoms $\delta\mu$ is on the order of two Bohr magneton $2\mu_B = 2.8 \,\mathrm{MHz/G}$ and 6 times larger for ⁵²Cr. For atoms with a small hyperfine splitting compared to $100E_{\rm vdw}$, Feshbach resonances are induced by the last bound states. This leads to a typical s-wave Feshbach resonance separation of $\sim 10,000 \text{ G}$ for 6 Li and ~ 100 G for 52 Cr. For atoms with hyperfine splittings much larger than $100E_{\rm vdw}$, resonances can be induced by much deeper bound states in the closed channel (see Fig. 14 for ⁸⁷Rb), and the expected spacings can be estimated accordingly.

The density of Feshbach resonances increases when higher partial-wave scattering and multiple closed hyperfine channels, defined in Sec. II.B.3, are included. The relevant number of channels is determined by the angular momentum dependence of the molecular potentials and identical particle statistics. For alkali-metal atoms there are on the order of 10 closed channels for the lowest partial wave $\ell=0$. The number of channels increases rapidly for higher partial waves. The ultralow temperature usually limits scattering to s- and sometimes p-wave entrance channels, but coupling to molecular states with up to $\ell_{\rm c}=4$ has been observed; see Tables II and IV.

For species without unpaired electrons, e.g., Sr and Yb, one expects no or very limited magnetic tunability because there is no electron contribution to the magnetic moment and the nuclear contribution is very small. In these systems, Feshbach resonances can possibly be optically induced; see Sec. VI.A.

b. What is the required magnetic field resolution? The width of the resonance generally determines the magnetic field resolution required for detection. Many s-wave Feshbach resonances have widths larger than 1 G; see Table IV in the Appendix. High partial wave Feshbach resonances are typically much narrower because of the weaker Feshbach coupling strength. Usually, a resolution

in the milligauss range is required to detect d- or g-wave Feshbach resonances.

c. How to trap atoms for collision studies? Optical dipole traps, reviewed by (Grimm et al., 2000), are the main tool to confine cold atoms for collision studies related to Feshbach resonances. Optical potentials trap atoms in any sub-level of the electronic ground state and permit investigation of collisions in any corresponding spin channel. For many experimental applications, the lowest atomic state is of particular interest, which is a high-field seeking state and can therefore not be trapped magnetically. Optical dipole traps allow for the application of arbitrary homogenous magnetic fields without affecting the trapping potential. In contrast, magnetic traps can only confine atoms in low-field seeking states, and the application of a magnetic bias field for Feshbach tuning can strongly influence the trap parameters. This limits the application of Feshbach tuning in magnetic traps to very few situations.

d. How low a temperature is needed to observe the resonances? In most experiments, a temperature of a few μK is sufficiently low to observe a clear resonant structure in inelastic collisional loss. Collision studies can be performed with thermal samples, BECs or degenerate Fermi gases. Elastic collision measurements are more complex. Enhancement of elastic collision rates near Feshbach resonances is more prominent at lower temperatures $< 1 \mu K$. On the other hand, suppression of elastic collision rates due to a zero crossing of the scattering length can be easily seen well above $1 \mu K$; see Sec. III.A.3.

2. Inelastic loss spectroscopy

Resonant losses are the most frequently observed signatures of Feshbach resonances in cold-atom experiments. These losses can be induced by two-body or three-body processes. Loss occurs because of the release of internal energy into the motion when colliding atoms end up in a lower internal state or when a molecule is formed. The gain in kinetic energy is on the order of the Zeeman energy, the hyperfine energy, or the molecular vibrational energy, depending on the inelastic channel, and is generally so large that all atoms involved in the collisions are lost. Near a Feshbach resonance, inelastic loss is strongly enhanced because the Feshbach bound states have strong couplings to inelastic outgoing channels.

Two-body and three-body collision loss can be quantified based on the evolution of the atom number N(t), which for a single species satisfies

$$\dot{N}(t) = -\frac{N(t)}{\tau} - \int \left[L_2 n^2(\mathbf{r}, t) + L_3 n^3(\mathbf{r}, t) \right] d^3 r, \quad (54)$$

where τ is the one-body lifetime, typically determined by background gas collisions, $n(\mathbf{r},t)$ is the position- and time-dependent atomic density distribution and L_2 (L_3) is the thermally averaged two-body (three-body) loss coefficient.

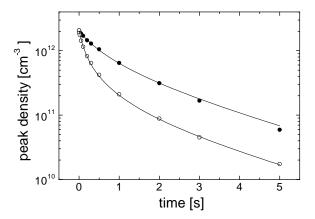


FIG. 17 Time evolution of the peak atomic density in a cloud of 133 Cs atoms in the $|f=4,m=-4\rangle$ state at $T=5.3\,\mu\text{K}$. The solid circles show the off-resonant evolution at $B=140\,\text{G}$ whereas the open circles show the on-resonant evolution at $B=205\,\text{G}$ (open circles), where a Feshbach resonance is located. The fit is based on Eq. (55) with $L_3=0$, and the peak density corresponds to $2\sqrt{2}\bar{n}$. From (Chin et al., 2004b).

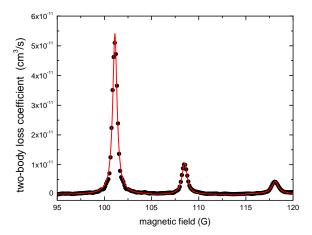


FIG. 18 Two-body inelastic loss coefficient of cesium atoms in the $|f=3,m=-3\rangle$ state as a function of magnetic field. The loss coefficient is extracted from the atomic density evolution, as shown in Fig 17. Three resonances are identified here. The solid line is a Lorentzian fit. From (Chin *et al.*, 2004b).

The loss equation can be further simplified under the assumption that thermalization is much faster than inelastic loss. For example, for a thermal cloud with temperature T in a 3D harmonic trap, one finds

$$\dot{\bar{n}}(t) = -\frac{\bar{n}(t)}{\tau} - L_2 \bar{n}(t)^2 - (4/3)^{3/2} L_3 \bar{n}(t)^3, (55)$$

where $\bar{n} = N\bar{\omega}^3 (4\pi k_B T/m)^{-3/2}$ is the mean density, m the atomic mass, and $\bar{\omega}$ the geometric mean of the three trap vibrational frequencies. Examples of density-dependent loss curves are shown in Fig. 17.

The trap loss coefficient L_2 is related to the inelastic loss coefficient in Sec. II.A.2 by $L_2 = K_{loss}(T)$, where we have assumed both atoms are lost in one collision event. Near a Feshbach resonance, L_2 is enhanced and has a

Lorentzian profile at low temperatures; see Sec. II.A.3 for more details on inelastic scattering resonances. Two-body collision loss has been observed in many cold atom system; see Fig. 18 for an example.

Three-body loss, as described by the loss coefficient L_3 in Eq. (54), is also strongly enhanced near Feshbach resonances. In many experiments cold atoms are polarized in the lowest ground state and two-body inelastic collisions (in the aa channel) do not occur so that three-body recombination loss is the dominant trap loss process. Three-body recombination occurs when three atoms interact and form a diatomic molecule and a free atom. In this process, the molecular binding energy is released into the kinetic energies of the outgoing molecule and the third atom, which except for very small molecular binding energies leads to immediate trap loss.

In the first experimental report on atomic Feshbach resonances, (Inouye et al., 1998) observed very fast trap loss of a sodium BEC near a Feshbach resonance. In this experiment, three-body recombination is the leading trap loss process. Recombination losses induced by Feshbach resonances have been observed and studied in numerous later experiments, for example, those by (Roberts et al., 2000) on ⁸⁵Rb, by (Marte et al., 2002; Smirne et al., 2007) on ⁸⁷Rb and by (Weber et al., 2003a) on ¹³³Cs.

For bosonic atoms with large scattering length $a\gg \bar{a}$ and low temperatures, L_3 scales generally as a^4 (Braaten and Hammer, 2006; Esry et al., 1999; Fedichev et al., 1996b; Nielsen and Macek, 1999), but with additional quantum features (resonance and interference effects) as discussed in Sec. VI.C on Efimov physics. For fermionic atoms the situation is more complicated because of Pauli suppression effects; see Sec. IV.B, but generally a loss feature accompanies a Feshbach resonance.

3. Elastic collisions

Elastic collisions refer to scattering processes in which the colliding atoms only change their motional state, but not the internal state. The cross section $\sigma_{\rm el}$ for elastic s-wave collisions follows from Eq. (6). Neglecting the effective range correction ($r_0 = 0$ in Eq. (5)), one obtains the simple expression

$$\sigma_{\rm el}(E) = g \frac{4\pi a^2}{1 + k^2 a^2},\tag{56}$$

where a depends on magnetic field B and g is the symmetry factor introduced in Sec. II.A.2. Near a Feshbach resonance, the scattering length becomes very large. The elastic s-wave cross section is g $4\pi a^2$ at very low energy as $k \to 0$ and approaches its upper bound $g 4\pi/k^2$ in the unitarity limit at finite k where $ka \gtrsim 1$. The latter can be reached in the μK regime if a becomes very large.

A strong enhancement of the thermally averaged elastic collision rate $n\langle \sigma_{\rm el} v \rangle$ can indicate the occurrence of a Feshbach resonance. One experimental approach is to

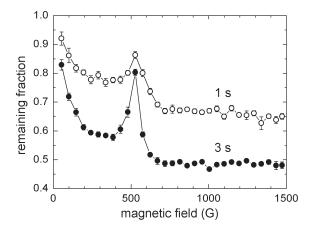


FIG. 19 Fraction of trapped atoms remaining after evaporation of $^6\mathrm{Li}$ atoms in an optical dipole trap after 1 s (open circles) and 3 s (closed circles) hold times. The peak near 530 G indicates a minimum of evaporative loss. This minimum results from the zero crossing of the scattering length induced by the broad 834-G Feshbach resonance. From (Jochim *et al.*, 2002).

measure the thermalization rate, which is proportional to the elastic collision rate as $\kappa n \langle \sigma_{\rm el} v \rangle$. Here κ is numerically calculated as 2.7 in the low temperature $k \to 0$ limit (Monroe et al., 1993), 10.5 in the unitarity limit (Arndt et al., 1997). Similarly, (DeMarco et al., 1999) found κ =4.1 for p-wave collisions in Fermi gases. Finding Feshbach resonances based on analyzing thermalization rates was reported by (Vuletić et al., 1999) on ¹³³Cs atoms and by (Loftus et al., 2002) on ⁴⁰K.

Near a resonance the thermalization rate can be limited under hydrodynamic conditions, which are reached when the cross section is so large that the collision rate in the trap exceeds the trap frequency (Vuletić et al., 1999). The maximum collision rate is also bounded by the unitarity limit when a is large. In both cases resonance structure that is due to elastic scattering can become less evident near B_0 .

Another efficient method to identify Feshbach resonances based on elastic collisions is to locate the zero crossing of the scattering length, that is, the magnetic field for which the scattering length vanishes near resonance where $B=B_0+\Delta$; see Eq. (1). This can be monitored by measuring atom loss resulting from elastic collisions during the process of evaporation. Thermalization and evaporation loss are suppressed at the zero crossing. Schemes to locate the zero crossing have been applied to ⁸⁵Rb by (Roberts et al., 1998), ¹³³Cs by (Chin et al., 2000), ⁴⁰K by (Loftus et al., 2002), ⁶Li by (Jochim et al., 2002; O'Hara et al., 2002b) and to a ⁴⁰K-⁸⁷Rb mixture by (Zaccanti et al., 2006). An example is shown in Fig. 19. While the zero crossing is evident, no resonance feature is seen near $B_0=834$ G.

Finally, resonant changes of elastic scattering can also be revealed through the detection of collision shifts in atomic clock experiments (Marion *et al.*, 2004) and by

measurements of the mean-field interaction in Bose-Einstein condensates (Cornish *et al.*, 2000; Inouye *et al.*, 1998; Regal *et al.*, 2003b); see discussion in Sec. IV.A.2.

4. Radiative Feshbach spectroscopy

Radiative Feshbach spectroscopy makes use of red- or blue-detuned light to detect the variation of the collisional wave function near a Feshbach resonance. The amplitudes of both the open and closed channel components of the wave function are strongly modified for distances on the order of or less than \bar{a} when B is tuned near resonance. This is evident for the closed channel component since the outer turning point will be on the order of \bar{a} ; see Sec. II.B.1. The optical transition induces loss of atoms by excitation of atom pairs at such separations.

(Courteille et al., 1998) adopted the idea of radiative spectroscopy to identify a Feshbach resonance in 85 Rb. In this experiment, a photoassociation laser beam (Jones et al., 2006) is held at a fixed frequency to the red of the strong $S \to P$ atomic transition and serves as a sensitive probe to measure the resonance position B_0 . The light excites the colliding pair of atoms to an excited molecular level in a state with an attractive potential. The excited level decays by spontaneous emission, giving rise to atom loss. The experiment monitors the atom loss as B is varied near B_0 , thus locating the resonance.

In contrast, (Chin et al., 2003) applied a laser with far blue detuning to detect Feshbach resonances in cesium samples. The blue-detuned light excites a molecular state with a repulsive potential so that the atoms are repelled, accelerated from one another, and lost from the trap. This method requires less detailed knowledge of molecular structure than the previous method. In this experiment, multiple narrow d- and g-wave resonances were identified in two different collision channels of cesium atoms; see Fig. 20.

5. Binding energy measurements

The detection methods discussed in the previous subsections provide good ways to determine the existence of resonances and their positions. Measurements of the magnetic-field dependent binding energies of near-threshold Feshbach molecules can yield information to precisely determine the scattering properties near a specific resonance; see Fig. 2 and Sec. II.C.5.

(Bartenstein et al., 2005; Regal et al., 2003a) employed radio-frequency (rf) spectroscopy on Feshbach molecules to measure very small molecular binding energies. An example of the rf spectroscopy is shown in Fig. 21. In this experiment, weakly bound molecules are first prepared near the Feshbach resonance; see Sec. V.A. An rf field is then applied to drive either a "bound-free" transition, which dissociates the molecules, or a "bound-

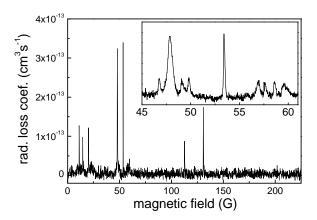


FIG. 20 Radiatively induced atom loss versus magnetic field due to a far blue-detuned laser beam applied to a sample of $^{133}\mathrm{Cs}$ atoms confined in an optical dipole trap at a temperature of 3 $\mu\mathrm{K}$. The laser wavelength is tuned 5 nm above the free atomic transition to excite the colliding atoms to a repulsive molecular state. Once excited, the atoms quickly dissociate and are lost from the trap. Multiple narrow d-and g-wave Feshbach resonances were observed. The inset shows an expanded view with more resonances resolved. From (Chin et al., 2003).

bound" transition, which converts them into a different molecular state. Based on the lineshape functions calculated by (Chin and Julienne, 2005), binding energy of Feshbach molecules can be measured to 1 kHz. This kind of precise data can be combined with theoretical modeling to determine the position and the width of the resonance.

Other efforts to spectroscopically probe weakly bound states include oscillating magnetic field spectroscopy employed by (Papp and Wieman, 2006; Thompson et al., 2005a), and rf and microwave techniques by (Mark et al., 2007a; Zirbel et al., 2008b). Using the theoretical models described in Sec. II.C.4, (Lange et al., 2009) show how Feshbach resonance parameters can be extracted from molecular binding energy measurements.

B. Homonuclear alkali-metal systems

Feshbach resonances have been found and characterized in essentially all single-species alkali-metal systems. The scattering properties show vast differences between the various species and also between different isotopes of the same species. Each system is unique and has particular properties. Here, we give a brief account for each species or isotope, ranging from first observations to the best current knowledge, and we discuss the characteristic properties. A table of important resonances can be found in the Appendix (Table IV).

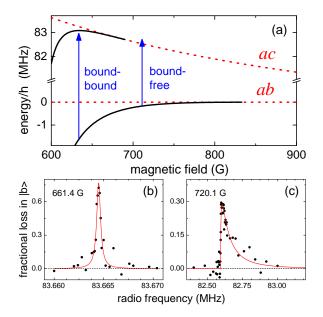


FIG. 21 Investigation of Feshbach resonances in the lowest two scattering channels in ⁶Li with radio-frequency (rf) spectroscopy on ultracold molecules. In this experiment, 10^5 molecules, initially prepared in the ab channel and confined in a dipole trap, are radiatively excited to the ac channel. The rf excitation lineshape shows a markedly different behavior below and above 690 G. Panel (a) shows the molecular bound state (solid) and channel energies (dotted) for the two channels. Below 690 G, a weakly bound state exists in the ac channel, and a bound-bound transition is possible (blue arrow). Above 690 G. no weakly bound state exists in the ac channel, and only a bound-free transition that dissociates the molecules is possible (blue arrow). Panel (b) shows the observed narrow and symmetric bound-bound excitation lineshape at 661.4 G. Panel (c) shows the observed broad and asymmetric bound-free lineshape at 720.1 G. The disappearance of a bound state in ac indicates that a Feshbach resonance occurs in this channel. Adapted from (Bartenstein et al., 2005).

1. Lithium

Lithium has two stable isotopes, one fermionic (6 Li) and one bosonic (7 Li).

 $^6\mathrm{Li}$ – In 1998 (Houbiers et al., 1998) predicted Feshbach resonances in cold collisions of fermionic $^6\mathrm{Li}$. Experimental evidence for a prominent resonance was established by monitoring inelastic decay (Dieckmann et al., 2002) and elastic collision properties (Jochim et al., 2002; O'Hara et al., 2002b). These experiments showed large variations in loss and thermalization rates as a function of the magnetic field strength for equal mixtures of the lowest two hyperfine ground states a and b prepared in an optical dipole trap. Because of the fermionic nature of $^6\mathrm{Li}$ even partial-wave scattering and, in particular, s-wave scattering only occurs between atoms in unlike hyperfine states. Consequently, if the gas is sufficiently cold such that the $\ell > 0$ centrifugal barriers are higher than the temperature of the gas, s-wave collisions in the ab chan-

nel represent the essential thermalization mechanism. (Jochim et al., 2002; O'Hara et al., 2002b) reported a strongly suppressed thermalization rate near 530 G, indicating the zero crossing of the scattering length in the ab channel. Most recently, (Du et al., 2008) located the zero crossing to 527.5 ± 0.2 G. (Dieckmann et al., 2002) observed enhanced inelastic loss near 680 G, about 150 G below the actual resonance location ($B_0 = 834$ G). Note that this particular resonance is extraordinarily broad ($\Delta \approx -300$ G). Further experiments by (Bourdel et al., 2003; Gupta et al., 2003) provided confirmation of this Feshbach resonance.

In order to further pinpoint the resonance position, (Bartenstein $et\ al.$, 2005) conducted radio-frequency spectroscopy on Feshbach molecules in the ab channel as described in Sec. III.A.4. The resonance could be located with an uncertainty of 1.5 G, more than two orders of magnitude smaller than its width. Broad resonances in two other entrance channels (ac and bc) were reported as well.

(Strecker et al., 2003) identified a narrow s-wave resonance at 543.25 G in the ab channel, and (Schunck et al., 2005; Zhang et al., 2004) observed three p-wave resonances near 200 G, one in each of the channels aa, ab, and bb. All of these observed resonances are induced by s- and p-wave rotational levels of the v=38 bound state of the singlet $X^1\Sigma_g^+$ potential, as shown in Fig. 9 for s-wave resonances.

The two s-wave resonances in the ab channel illustrate the concept of resonance strength, as described in Sec. II.B.2. The broad resonance at 834 G is strongly entrance-channel dominated, while the narrow resonance at 543 G is an extreme case of a closed-channel dominated resonance. The former has an extraordinarily large magnetic field range of universal behavior, while that for the latter is vanishingly small (detunings of a few μ G or less); see Sec. II.C.5 and Fig. 16. This has important consequences for molecules formed near these resonances.

The very broad resonance at $834~\mathrm{G}$ in the ab channel is used to create molecular Bose-Einstein condensates and fermionic superfluids. This will be extensively discussed in Sec. IV.B.

 $^7{\rm Li}$ – (Moerdijk and Verhaar, 1994) predicted Feshbach resonances in collisions of bosonic $^7{\rm Li}$ atoms in $|f=1,m=1\rangle$ (aa channel). (Strecker et al., 2002) identified a zero crossing of the scattering length induced by a broad Feshbach resonance. This resonance was used to study the formation of bright solitons by (Khaykovich et al., 2002; Strecker et al., 2002) at small and negative scattering lengths near the zero crossing; see also Sec. IV.A.3. In these early experiments, the resonance position was estimated to 720 G. Later measurements by (Junker et al., 2008) and (Pollack et al., 2009) accurately pinpointed the resonance position and the zero crossing to 736.8(2) G and 543.6(1) G, respectively.

(Gross and Khaykovich, 2008) observed two resonances in the state $|f = 1, m = 0\rangle$ (bb channel). A narrower one was found at 831(4) G with a width of 7 G,

while a broader one (width 34 G) was located at $\sim 884\,\rm G$. In between these two resonances, a zero crossing was found at $836(4)\,\rm G.$

2. Sodium

(Inouye et al., 1998) pioneered experimental research in locating Feshbach resonances. In an optically trapped BEC of 23 Na (the only stable isotope) with all atoms in the lowest hyperfine state $|f=1,m=1\rangle$, they identified resonances at 853 G and 907 G. A third resonance in a different channel at 1195 G was later reported by (Stenger et al., 1999). All three resonances are narrow and s wave in nature. The 1998 experiment showed both strongly enhanced trap loss and the dispersive tuning of the scattering length near the 907-G resonance; see Fig. 3.

The experimental determination of the resonances has enabled detailed models of the interaction potentials between ultracold Na atom (van Abeelen and Verhaar, 1999a). These models were further refined by (Samuelis *et al.*, 2000) based on conventional molecular spectroscopy in combination with photoassociation data.

The Feshbach resonance at 907 G was used to create ultracold Na₂ molecules (Xu *et al.*, 2003) and to demonstrate coherent molecule optics (Abo-Shaeer *et al.*, 2005); see also Sec. V.A.1.

3. Potassium

Potassium has three stable isotopes, two of them are bosonic (39 K and 41 K) and one is fermionic (40 K).

 39 K – In 1996 (Boesten *et al.*, 1996b) predicted Feshbach resonance locations in collisions between 39 K atoms. Their results were based on spectroscopic data of binding energies of ro-vibrational states of the $X^1\Sigma_g^+$ and $^3\Sigma_u^+$ potentials. The data, however, were not sufficiently complete to give quantitative resonance locations, but it did show that the likelihood of resonances was large. Another early prediction of resonance locations was made by (Bohn *et al.*, 1999).

Feshbach resonances in ³⁹K were observed at 402 G and analyzed by (D'Errico et al., 2007), and applied to create a tunable BEC of this species (Roati et al., 2007). The zero crossing of the scattering length near the broad 402-G resonance has found intriguing applications for atom interferometry with non-interacting condensates (Fattori et al., 2008a); see also Sec. IV.A.4. This resonance has an intermediate character between that of an entrance channel and a closed channel dominated resonance.

 40 K – Early predictions on Feshbach resonances in 40 K were made by (Bohn *et al.*, 1999). The first experimental observation was reported by (Loftus *et al.*, 2002), who demonstrated resonant control of elastic collisions via the 202-G resonance in a mixture of the lowest two spin states

(ab channel). One year later, the same group reported on a p-wave resonance at 199 G (Regal et al., 2003b); they measured the resonantly enhanced elastic collision rate of atoms in the second-lowest hyperfine state (bb channel) at a temperature of 3 μ K. At an even lower temperature and by monitoring the collision-induced heating rate, (Ticknor et al., 2004) found that this resonance is actually a doublet. This doublet structure in the p-wave resonance is due to a small energy splitting between the $|\ell=1,m_{\ell}=0\rangle$ and $|\ell=1,m_{\ell}=\pm 1\rangle$ molecular states. The anisotropic nature of the p-wave resonances has found interesting applications in low-dimensional traps (Günter et al., 2005).

In spin mixtures with s-wave interactions, (Regal and Jin, 2003; Regal $et\ al.$, 2003b) identified 10 G wide s-wave Feshbach resonances in the ab and ac channels at 201.6(6) G and 224.21(5) G, respectively, by monitoring the thermalization rate and mean-field shifts. These resonances provide a convenient tool to study strongly interacting Fermi gases and fermionic condensates; see Secs.IV.B.

⁴¹K – A Feshbach resonance was recently observed by (Kishimoto *et al.*, 2009) at 51.4 G in the *cc* channel. The observation confirmed a theoretical prediction by (D'Errico *et al.*, 2007), which was based on experimental data available for the other potassium isotopes.

4. Rubidium

 85 Rb – (Courteille et al., 1998) reported a Feshbach resonance in a magnetically trapped thermal sample by observing enhanced photoassociative loss; see Sec. III.A.4. This result confirmed the prediction by (Vogels et al., 1997). (van Abeelen et al., 1998) suggested using photoassociation as a probe to identify Feshbach resonances. By monitoring inelastic loss, (Roberts et al., 1998) determined the position of this 10-G wide resonance to be at 155 G. (Claussen et al., 2003) then used a BEC to perform a high-precision spectroscopic measurement of the molecular binding energy and determined the resonance parameters B_0 and Δ within 20 mG.

Attainment of BEC in ⁸⁵Rb crucially depended on the existence of the 155-G resonance; see Sec. IV.A.1. Only in a 10-G window near the resonance the scattering length is positive and (Cornish *et al.*, 2000) were able to Bose condense ⁸⁵Rb. As the first available BEC with widely tunable interactions the system has received considerable attention; see Sec. IV.A.3. Coherent atom-molecule coupling (Donley *et al.*, 2002) and the formation of ultracold ⁸⁵Rb₂ molecules (Thompson *et al.*, 2005a) were reported based on this Feshbach resonance.

 $^{87}{\rm Rb}$ – (Marte et al., 2002) conducted a systematic search for Feshbach resonances in bosonic $^{87}{\rm Rb}$. For atoms polarized in various combinations of magnetic sublevels in the f=1 hyperfine manifold, more than 40 resonances were observed between 300 G and 1200 G by

monitoring atom loss in an optical dipole trap. These resonances are induced by s- and d-wave bound states and are all very narrow. For the s-wave states, the underlying molecular structure is shown in Fig. 14. The widest and most often used resonance is located at 1007 G and has a width of 0.2 G. In different experiments, (Erhard $et\ al.$, 2004; Widera $et\ al.$, 2004) observed a low-field resonance near 9 G in the $ae\$ channel.

Several resonances in ⁸⁷Rb have been used to form ultracold Feshbach molecules. They are formed by ramping the magnetic field through the resonance (Dürr *et al.*, 2004a); see Sec. V.A.1. The great potential of combining Feshbach resonances with optical lattices has been demonstrated in a series of experiments with ⁸⁷Rb (Syassen *et al.*, 2007; Thalhammer *et al.*, 2006; Volz *et al.*, 2006; Winkler *et al.*, 2006); see Sec. VI.B.

5. Cesium

Cesium, for which the isotope ¹³³Cs is the only stable one, was proposed as the first alkali-metal species in which Feshbach resonances could be observed (Tiesinga *et al.*, 1993). With the limited state of knowledge of the interaction potentials no quantitative predictions could be made. The first observation of Feshbach resonances in cesium collisions was published seven years later (Vuletić *et al.*, 1999).

(Chin et al., 2000, 2004b; Vuletić et al., 1999) reported on more than 60 Feshbach resonances of various types using several detection schemes. These include s-, p-, d-, f- and g-wave resonances in 10 different scattering channels. (Vuletić et al., 1999) used cross-axis thermalization rates to identify resonances in the lowest aa channel and used trap loss measurements in the gg channel. (Chin et al., 2000) reported many more resonances by preparing the atoms in other internal states and by monitoring the evaporation rates to more efficiently measure the elastic cross section. The resonances provided (Leo et al., 2000) with the essential information to precisely determine the interaction potentials of ultracold cesium.

Narrow resonances were observed by (Chin et al., 2003, 2004b) using radiative Feshbach spectroscopy; see Sec. III.A.4. In these experiments ¹³³Cs atoms were illuminated by a far blue-detuned laser beam, whose wavelength was optimized to only remove atoms near a resonance. These resonances are induced by g-wave bound states and are only strong enough to be observed due to the large second-order spin-orbit coupling of cesium atoms. The observed resonances are at low magnetic fields, which can be understood as a consequence of the large background scattering length in the aa collision channel; see Sec. II.B.1. In addition, l-wave Feshbach molecules have been produced by (Knoop et al., 2008; Mark et al., 2007a); their coupling to the s-wave continuum is too weak to lead to observable resonances in collision experiments. Finally, (Lee et al., 2007) predicted a broad s-wave resonance at ~ 800 G in the aa channel.

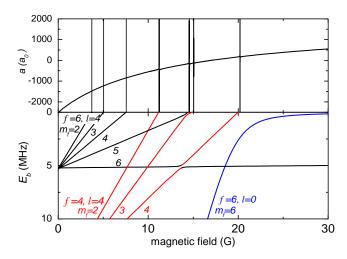


FIG. 22 Scattering length and bound state energies for cesium atoms in the lowest internal state as a function of magnetic field. From (Chin *et al.*, 2004b).

This is a magnetic field regime that has not been experimentally explored yet.

Figure 22 shows the scattering length in the aa channel. It has a zero crossing at 17.1 G (Chin $et\ al.$, 2004b; Gustavsson $et\ al.$, 2008) and multiple narrow resonances below 50 G. The gradual change in a from $-2500\ a_0$ to 500 a_0 across 30 G in the Figure is actually the tail of a broad resonance with $B_0=-12$ G; see Appendix and Table IV. The negative B_0 follows from fitting to Eq. (1). Both the narrow and broad resonances have provided favorable conditions for many exciting experiments. This included the attainment of BEC with Cs atoms, a Bose condensate (Weber $et\ al.$, 2003a), the formation of Cs₂ (Chin $et\ al.$, 2003; Herbig $et\ al.$, 2003), the observation of resonances between ultracold molecules (Chin $et\ al.$, 2005), and studies on Efimov physics; see Sec. VI.C.

C. Heteronuclear and other systems

Most of the experimental and theoretical attention so far has been focused on locating and using magnetic Feshbach resonances in single-species alkali-metal atom gases. Over the last five years, however, considerable progress has also been made in locating Feshbach resonance in other atomic species and in mixtures of alkalimetal atoms. These systems are investigated for various reasons. Of particular interest is the promise of more exotic quantum many-body behavior (Bloch et al., 2008; Menotti et al., 2008; Micheli et al., 2006).

Heteronuclear systems provide the path to prepare mixtures of bosonic and fermionic quantum degenerate gases. Intriguing applications include the creation of fermionic molecules in an atomic Bose-Fermi mixture (Ospelkaus et al., 2006a; Zirbel et al., 2008a) and novel quantum phases of fermions with unequal masses (Petrov et al., 2007). Feshbach resonances provide means to tune the interactions between different

species in order to explore quantum phases in various regimes.

Atoms with magnetic moments interact via the long-range magnetic dipole-dipole interaction V_{ss} in addition to the van der Waals and more short-range interactions. For alkali-metal atoms the effect of this dipole-dipole interaction on collective behavior is small. In atomic species with much larger magnetic moments, however, the dipole-dipole interaction can have a significant impact on the many-body behavior of the gas (Goral et al., 2000; Santos and Pfau, 2006; Santos et al., 2003). In an atomic species like chromium (Sec. III.C.1), magnetic Feshbach resonances can be used to tune the relative strength of the short-ranged interactions and the long-range dipole-dipole interaction (Lahaye et al., 2007; Yi and You, 2002).

Another way to create exotic many-body systems is by pairing different atomic species into Feshbach molecules, which can then be converted into deeply bound molecules with a large electric dipole moment. Such a moment gives rise to large dipolar interactions, orders of magnitude larger than possible with magnetic dipole moments. These molecules, which are either bosonic or fermionic, have many applications in dipolar molecular quantum gases (Büchler et al., 2007; Micheli et al., 2006) and quantum computation (DeMille, 2002).

1. Chromium

In 2005 (Griesmaier $et\ al.$, 2005) announced BEC of $^{52}\mathrm{Cr}$ atoms. This species has a magnetic moment that is six times larger than that for alkali-metal atoms. Subsequently, (Stuhler $et\ al.$, 2005) showed that the expansion of a cigar-shaped chromium condensate depended on the relative orientation of the magnetic moment and the elongated condensate, which for the first time showed the effects of the dipole-dipole interaction in a quantum-degenerate gas.

(Werner et al., 2005) measured fourteen magnetic Feshbach resonances in ⁵²Cr in the energetically lowest magnetic sublevel. Feshbach resonances were detected by measuring, after a fixed hold time, the number of remaining atoms as a function of magnetic field. Atom loss could only have occurred by enhanced three-body recombination near the resonance. Figure 23 shows the s-wave scattering length as a function of magnetic field derived from the observed locations of the Feshbach resonances and a multi-channel scattering model of the collision. Most of the resonances are g-wave resonances with the exception of one of the two nearly-degenerate resonances at 29 mT and those at 50 mT and 59 mT. These three resonances have d-wave character. Moreover, (Werner et al., 2005) also assigned two resonances as originating from d-wave collisions and coupling to an s-wave closed channel, i.e. $\ell=2$ and $\ell_c=0$ according to the notation discussed in Sec. II.B.4. Note that these resonances do not show up in the s-wave scattering length as displayed in Fig. 23.

One of these unusual resonances was investigated in some detail in (Beaufils *et al.*, 2009).

2. Mixed species

 $\rm K+Rb$ – The first mixed system to receive a detailed effort to locate Feshbach resonances was a mixture fermionic $^{40}\rm K$ and bosonic $^{87}\rm Rb$ atoms (Simoni et~al., 2003). Based on experimentally determined inelastic and elastic rate coefficients at zero magnetic field they predicted the location of fifteen Feshbach resonances with an uncertainty ranging from 10 G to 100 G.

(Inouye et al., 2004) presented the first direct determination of the magnetic field location of three Feshbach resonances. The two atomic species, each in their energetically lowest hyperfine state, were optically trapped. As shown in Fig. 24 the Feshbach resonances were detected by measuring, after a fixed hold time, the number of remaining atoms as a function of magnetic field. The atom loss is due to three-body recombination.

(Ferlaino et al., 2006) confirmed the positions of these three resonances and found nine additional resonances. Theoretical modeling uniquely assigned each resonance and determined the scattering lengths of the $X^1\Sigma^+$ and $a^3\Sigma^+$ Born-Oppenheimer potentials to about 2%. The difference between the experimental Feshbach locations and those of the best-fit theory was less than 1 G for all resonances. The experimental uncertainty in the resonance locations was 0.2 G.

(Ferlaino et al., 2006) also predicted the location of resonances of several isotopic combinations. For the bosonic ³⁹K and ⁸⁷Rb collision with both atoms in their lowest hyperfine state (Roati et al., 2007) confirmed the location of one such resonance. It was found at 317.9 G well within the uncertainties quoted by (Ferlaino et al., 2006). (Klempt et al., 2007) observed a number of new Feshbach resonances, constructed an accurate potential model, and predicted resonances in other isotopic KRb combinations. (Simoni et al., 2008) presented a refined near-threshold model for scattering and bound-state calculations for all isotopic combinations of K and Rb.

Feshbach resonances in mixtures of 40 K and 87 Rb have been applied to the creation of Feshbach molecules in optical traps (Zirbel et~al., 2008a,b) and in single sites of an optical lattice (Ospelkaus et~al., 2006a); see Sec. VI.B.1. Recently these fermionic molecules have been transferred to more deeply bound levels (Ospelkaus et~al., 2008); see Sec. V.B.4. Furthermore, interspecies interaction tuning has been exploited to study the collective behavior of a 40 K+ 87 Rb Bose-Fermi mixture (Ospelkaus et~al., 2006b) and to realize a tunable double species BEC in a 41 K+ 87 Rb Bose-Bose mixture (Thalhammer et~al., 2008).

 ${\rm Li+Na-(Stan~\it et~\it al.,~2004)}$ have observed three magnetic Feshbach resonances in the interaction between a degenerate fermionic ${}^6{\rm Li}$ gas and a Bose-Einstein condensate of Na. The resonances were observed by detect-

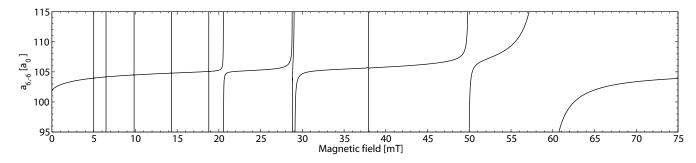


FIG. 23 Scattering length of 52 Cr atoms versus magnetic field. The feature near 29 mT is a pair of nearly-degenerate Feshbach resonances. From (Werner *et al.*, 2005).

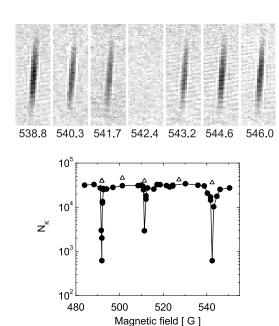


FIG. 24 Observation of Feshbach resonances in $^{40}{\rm K}+^{87}{\rm Rb}.$ The top panel shows in-trap absorption images of $^{40}{\rm K}$ atoms after a fixed hold time of ≈ 1 s at various magnetic fields. The label on each image gives the magnetic field in Gauss. No $^{40}{\rm K}$ atoms could be seen at 542.2 G. The bottom panel shows the number of remaining $^{40}{\rm K}$ atoms after the fixed hold time as a function of magnetic field. Narrow features are observed at 492 G, 512 G, and 543 G. Adapted from (Inouye et~al.,~2004).

ing atom loss when sweeping the magnetic field at constant rate through the resonances. Atom loss could only have occurred from three-body recombination or from molecule formation during the sweep. The observed resonances at 746.0 G, 759.6 G, and 795.6 G are s-wave resonances. In a recent theoretical study based on this experimental input data, (Gacesa et al., 2008) derive precise values of the triplet and singlet scattering lengths for both the ⁶Li-Na and the ⁷Li-Na combination. Moreover they predict a variety of additional Feshbach resonances within an experimentally attainable field range.

⁶Li+⁴⁰K – (Wille *et al.*, 2008) described the observation of thirteen Feshbach resonances in fermionic ⁶Li and ⁴⁰K in various hyperfine states. Their theoretical analysis, which relies on the model developed by (Stan *et al.*, 2004) and discussed in Sec. II.C.2, indicated that the resonances were either *s*- or *p*-wave resonances. This isotopic combination is a prime candidate for the study of strongly interacting Fermi-Fermi mixtures.

Na+Rb – Predictions of Feshbach resonance locations based on analysis of high-resolution Fourier spectroscopy of the molecular $\mathbf{X}^1\Sigma_g^+$ and $\mathbf{a}^3\Sigma_u^+$ states in a 600 K beam of NaRb molecules are described in (Bhattacharya et al., 2004) and (Pashov et al., 2005). For example (Pashov et al., 2005) predict for the ultracold collision between ²³Na and ⁸⁵Rb, both in the energetically lowest hyperfine state, s-wave Feshbach resonances at 170 G and 430 G with an uncertainty of about 50 G. This uncertainty is sufficiently small that the predictions will be helpful for planning experiments which can accurately locate the resonances.

Initial experiments on other combinations of mixed atomic species have been performed. Feshbach resonances have been reported in the ⁶Li+⁸⁷Rb system by (Deh *et al.*, 2008), in the ⁷Li+⁸⁷Rb system by (Marzok *et al.*, 2009), and in the ⁸⁷Rb+¹³³Cs system by (Pilch *et al.*, 2009).

3. Isotopic mixtures

A special case of a mixed system is that where different isotopes of the same element are combined. In particular, isotopic mixtures of Rb, K, and Li have been studied. Isotopic mixtures of K or Li are of particular interest as both fermionic and bosonic isotopes exist.

Feshbach resonances in isotopic mixtures of rubidium have recently been observed. (Papp and Wieman, 2006) found two s-wave Feshbach resonances in the collision of $^{85}\mathrm{Rb}$ and $^{87}\mathrm{Rb}$ when both isotopes are in their lowest hyperfine state. Their magnetic field locations of 265.44(0.15) G and 372.4(1.3) G are consistent with the predictions of (Burke et~al., 1998).

(van Kempen et al., 2004) have predicted the location of $^6\mathrm{Li}+^7\mathrm{Li}$ Feshbach resonances. When both isotopes are in the lowest hyperfine state resonances occur between 200 G and 250 G as well as between 550 G and

560 G. Four of these resonances have been observed by (Zhang *et al.*, 2005).

In predictions for isotopic mixtures "mass scaling" is often used. As the interatomic potentials are to good approximation independent of the isotopic composition of the dimer, the only thing that changes is the (reduced) mass of the dimer. Corrections are due to the breakdown of the Born-Oppenheimer approximation. (van Kempen et al., 2002; Seto et al., 2000) showed that for Rb the experimental data on ⁸⁵Rb⁸⁵Rb, ⁸⁵Rb, ⁸⁶Rb and ⁸⁷Rb⁸⁷Rb are consistent with mass scaling. A new analysis that includes the 2006 observations of (Papp and Wieman, 2006) is needed.

For the lithium system (van Kempen et al., 2004) showed that mass scaling is insufficient to explain the observed data for the homonuclear ⁶Li+⁶Li and ⁷Li+⁷Li systems. Consequently, (van Kempen et al., 2004) quote a 1 G uncertainty for the location of ⁶Li+⁷Li Feshbach resonances from the breakdown of mass scaling.

IV. CONTROL OF ATOMIC QUANTUM GASES

Tuning two-body interactions via Feshbach resonances is the experimental key to control collective phenomena in degenerate quantum gases. This has found numerous applications, both with atomic Bose-Einstein condensates (Cornell and Wieman, 2002; Ketterle, 2002) and with degenerate Fermi gases (Inguscio et al., 2008).

The different decay properties of Bose and Fermi gases near Feshbach resonances play a crucial role for the experiments. For Bose gases, resonant two-body scattering in general leads to rapid decay via three-body collisions, as we have discussed in context with loss spectroscopy on Feshbach resonances in Sec. III.A.2. Three-body decay limits the practical applicability of Feshbach tuning to Bose gases, restricting the experiments to the dilute gas regime where the scattering length is small compared to typical interparticle separations. In contrast, Fermi gases can be remarkably stable near s-wave Feshbach resonances (Petrov et al., 2004).

For atomic Bose-Einstein condensates in the dilute gas regime (Sec. IV.A), the collective behavior can be described in a mean-field approach. In strongly interacting Fermi gases (Sec. IV.B), the role of the scattering length is much more complex. Here Feshbach tuning can be used to control the nature of fermionic pairing in different superfluid regimes.

A. Bose-Einstein condensates

In experiments on atomic BEC, the role of Feshbach tuning can be divided into two parts. Firstly, the control of collision properties can be essential for the attainment of BEC. Secondly, the possibility to control the mean-field interaction opens up a variety of interesting applications.

1. Attainment of BEC

Some atomic species offer favorable collision properties for the attainment of BEC without any necessity of interaction tuning; ⁸⁷Rb and ²³Na are the most prominent examples (Cornell and Wieman, 2002; Ketterle, 2002). In other cases, however, Feshbach tuning is essential either to produce large condensates (⁷Li), or to achieve BEC at all (⁸⁵Rb, ¹³³Cs, ³⁹K).

Let us first consider the general question, what is a "good" scattering length a for making a BEC. First, a should be positive, because condensates undergo collapse at negative scattering length when the number of condensed atoms exceeds a relatively small critical value (Bradley et al., 1997). Moreover, a should not be too small, because a sufficiently large elastic collision rate is required for evaporative cooling towards BEC; the cross section for elastic collisions between identical bosons is $8\pi a^2$. Finally, the scattering length should not be too large to avoid rapid decay by three-body collisions (Roberts et al., 2000; Weber et al., 2003b) as threebody decay scales as a^4 (Braaten and Hammer, 2006; Fedichev et al., 1996b). In practice these conditions result in typical values for a "good" value of a between a few ten and a few hundred times the Bohr radius a_0 . In detail, the optimum value for a depends on the confinement properties of the trap and behavior of inelastic

For ⁷Li, early magnetic trapping experiments showed BEC in the internal state f = 2, m = 2, where the scattering length $a = -27 a_0$ is small and negative (Bradley et al., 1997, 1995); here the number of condensate atoms was limited through collapse to only a few hundred. Later experiments (Khaykovich et al., 2002; Strecker et al., 2002) on optically trapped ⁷Li in the internal state f = 1, m = 1 exploited the 737-G Feshbach resonance to tune the scattering length from its very small background value $(a_{\rm bg} \approx +5 a_0)$ to sufficiently large positive values, typically in a range between $+40 a_0$ and $+200 a_0$. Evaporative cooling then resulted in condensates with up to 3×10^5 atoms. More recently, (Gross and Khaykovich, 2008) exploited Feshbach tuning in the state f = 1, m = 0 for the all-optical production of a BEC. They obtained favorable conditions for efficient evaporative cooling at 866 G, where the scattering length is about $+300 a_0$.

Bose-Einstein condensation of 85 Rb was achieved by evaporative cooling in a magnetic trap (Cornish *et al.*, 2000) exploiting the broad resonance at 155 G (Sec. III.B.4) in the state f=2, m=-2 to tune the scattering length to positive values. The large negative background scattering length $a_{\rm bg}=-443\,a_0$ would limit the number of condensate atoms to less than one hundred. Two stages of cooling were performed. The first stage used a magnetic field of 250 G, where the scattering length a is close to its background value; the second was close to the resonance at $^{162.3}$ G, where $a=+170\,a_0$. This procedure optimized the ratio of

elastic to inelastic collision rates (Roberts *et al.*, 2000) for the temperatures occurring during these stages.

Feshbach tuning played a crucial role for the attainment of BEC in ¹³³Cs (Kraemer et al., 2004; Weber et al., 2003a). The condensate was produced in an optical trap in the state f = 3, m = 3. In this lowest internal state, two-body decay is energetically forbidden, and the scattering length a(B) shows a large variation at low magnetic fields (Fig. 22), which is due to a broad Feshbach resonance at -12 G. In a first evaporative cooling stage a shallow large-volume optical trap was employed, and a large scattering length of $a = +1200 a_0$ at $B = 73 \,\mathrm{G}$ provided a sufficiently large elastic collisions rate at rather low atomic densities. The second cooling stage employed a much tighter trap. Here, at much higher densities, an optimum magnetic field of 21 G was found, where $a = +210 a_0$. Highly efficient evaporation led to the attainment of BEC. Later experiments revealed a minimum of three-body decay in this magnetic field region (Kraemer et al., 2006); see also Sec. VI.C.2.

For the attainment of BÉC in 39 K, (Roati et al., 2007) employed a mixture with 87 Rb atoms with both species being in their internal ground states. First, an interspecies Feshbach resonance near 318 G was used to optimize sympathetic cooling; the interspecies scattering length was tuned to $+150 a_0$ by choosing a magnetic field of 316 G. Then, after the removal of the Rb atoms, final evaporative cooling towards BEC was performed near the 402-G resonance of 39 K (Sec. III.B.3) with the scattering length tuned to a positive value of $+180 a_0$.

2. Condensate mean field

Trapped atomic BECs in the dilute-gas regime are commonly described (Dalfovo *et al.*, 1999) by the Gross-Pitaevskii equation for the condensate wave function Φ ,

$$i\hbar \frac{\partial}{\partial t} \Phi = \left(-\frac{\hbar^2 \nabla^2}{2m} + V_{\text{ext}} + V_{\text{mf}} \right) \Phi,$$
 (57)

where $V_{\rm ext}$ is the external trapping potential. Interactions are taken into account by the mean-field potential

$$V_{\rm mf} = \frac{4\pi\hbar^2 a}{m} \, n \,, \tag{58}$$

where the atomic number density n is related to Φ by $n=|\Phi|^2$. This mean field enters the Gross-Pitaevskii equation as a nonlinearity and leads to many interesting phenomena.

In the Thomas-Fermi regime of large condensates with a>0 one can neglect the kinetic energy term and obtain the equilibrium density distribution of a BEC

$$n = \frac{m}{4\pi\hbar^2 a} \left(\mu - V_{\text{ext}}\right) \,, \tag{59}$$

which applies for $\mu > V_{\rm ext}$; otherwise n = 0. For a given particle number N, the chemical potential μ follows from the normalization condition $N = \int n d^3 r$.

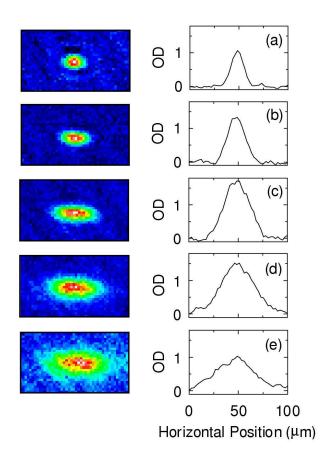


FIG. 25 In-situ images and density profiles showing the variation of the equilibrium size of a magnetically trapped ⁸⁵Rb BEC close to a Feshbach resonance. With decreasing magnetic field (a-e), the scattering length increases from a very small positive value at $165.2 \,\mathrm{G}$ (zero crossing at $165.8 \,\mathrm{G}$) to a very large value of $a \approx +6000 \,a_0$ at $156.4 \,\mathrm{G}$ (resonance position at $155.0 \,\mathrm{G}$). From (Cornish *et al.*, 2000).

In many cases of practical relevance, stable condensates with positive a are confined in harmonic traps and are in the Thomas-Fermi regime. The condensate is then characterized by the Thomas-Fermi radius $r_{\rm TF}$, given as the radius at which the external trapping potential equals the chemical potential, and the peak number density n_0 . These two quantities follow the scaling laws

$$r_{\rm TF} \propto a^{1/5}, \ n_0 \propto a^{-3/5}.$$
 (60)

Figure 25 illustrates how the size of a trapped 85 Rb condensate increases when the Feshbach resonance at 155 G is approached. The *in-situ* measurements were used to experimentally determine a(B) (Cornish *et al.*, 2000). The results are in good agreement with later measurements of the molecular binding energy, which allowed for a more precise determination of the scattering properties near the Feshbach resonance (Claussen *et al.*, 2003).

The mean-field approach is valid for scattering lengths which are small compared to the typical interparticle distance. The prospect to observe beyond-mean-field effects in BECs (Dalfovo *et al.*, 1999) has been an im-

portant motivation for experiments near Feshbach resonances at large a. In atomic Bose gases, however, the fast inelastic decay makes it very difficult to observe such phenomena. (Papp $et\ al.$, 2008) finally demonstrated beyond-mean-field behavior by Bragg spectroscopy on a $^{85}{\rm Rb}$ BEC. In molecular BECs created in atomic Fermi gases (Sec. IV.B.1), the collisional stability facilitates the observation of beyond-mean-field behavior by simpler means. For example, (Altmeyer $et\ al.$, 2007) observed beyond-mean-field behavior by studying collective oscillations of a $^6{\rm Li}_2$ molecular BEC.

3. Controlled collapse and bright solitons

For negative scattering lengths the condensate mean field is attractive. The resulting nonlinearity can then lead to a condensate collapse and to the formation of bright matter-wave solitons. To study phenomena of this kind by Feshbach tuning, the general experimental strategy is to first produce a stable BEC at positive a. Then, the attractive interaction is introduced by changing a to negative values. Experiments of this class have been performed with ⁸⁵Rb (Cornish et al., 2006; Donley et al., 2001; Roberts et al., 2001), ⁷Li (Khaykovich et al., 2002; Strecker et al., 2002), and ¹³³Cs (Rychtarik et al., 2004; Weber et al., 2003a).

Exploiting the 155-G Feshbach resonance in ⁸⁵Rb, (Roberts *et al.*, 2001) investigated the stability of a BEC with attractive interactions. They first produced the condensate in a magnetic trap at a moderate positive scattering length. They then slowly changed the atomatom interaction from repulsion to attraction by ramping the magnetic field into the region of negative scattering length. With increasing attractive interaction they observed an abrupt transition in which atoms were ejected from the condensate. These measurements of the onset of condensate collapse provided a quantitative test of the stability criterion for a BEC with attractive interactions.

The controlled collapse in ⁸⁵Rb following a sudden change of a led to the spectacular observation of a "Bosenova," a condensate implosion with fascinating and unexpected properties (Donley et al., 2001). An anisotropic burst of atoms was observed that exploded from the condensate during the early stage of collapse (Fig. 26), leaving behind a highly excited long-lived remnant condensate. Strikingly, the number of atoms in the remnant BEC was significantly larger than the critical number for a collapse. The surprising fact that the remnant BEC did not undergo further collapse was later explained by its fragmentation into bright solitons (Cornish et al., 2006).

Condensate collapse experiments were also used to detect the presence of a small BEC of Cs atoms in an optical surface trap (Rychtarik *et al.*, 2004). While a thermal gas did not show loss when the scattering length was suddenly switched to negative values, the condensate showed up in the sudden onset of collapse-induced loss.

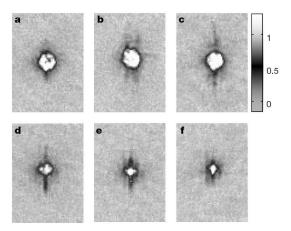


FIG. 26 Striking phenomena have been observed in the controlled collapse of a ⁸⁵Rb BEC. The images show the formation of 'jets', where streams of atoms with highly anisotropic velocities are ejected by the collapsing condensate. Reprinted by permission from Macmillan Publishers Ltd: Nature (Donley *et al.*, 2001), copyright 2001.

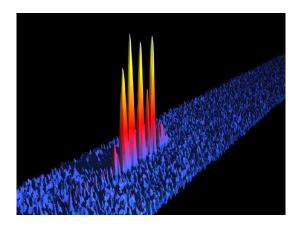


FIG. 27 A train of matter-wave solitons created from an optically trapped BEC of ⁷Li atoms (Strecker *et al.*, 2002). The individual solitons contain up to about 5000 atoms. Figure courtesy of Randall Hulet.

Bright solitons were observed in experiments on ⁷Li atoms near the broad 736-G Feshbach resonance. (Khaykovich et al., 2002) produced a BEC by evaporative cooling at $a \approx 40 a_0$ in an optical trap. They then released the BEC into a one-dimensional optical waveguide and studied the propagation of the resulting matterwave packet for an ideal gas (a = 0) and a gas with a small attractive mean-field interaction $(a = -4 a_0)$. In the latter case, they observed the dispersion-free propagation that is characteristic for a soliton. In a similar experiment (Strecker et al., 2002) created a train of solitons, see Fig. 27, from an optically trapped ⁷Li BEC by abruptly switching the scattering length from $200 a_0$ to $-3 a_0$. They also observed the propagation of the solitons in the trap and their mutual repulsion. These spectacular experiments on bright solitons highlight the analogy between bright matter-wave solitons and optical solitons

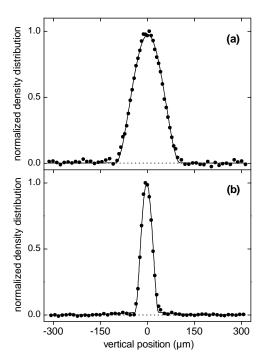


FIG. 28 Density profiles of a Cs BEC after 100 ms of expansion at (a) $a=210\,a_0$ and (b) a=0. The expansion energy of the non-interacting condensate is as low as $k_B\times 50\,\mathrm{pK}$. From (Kraemer *et al.*, 2004).

in fibers and thus the intimate connection between atom optics with BECs and light optics.

4. Non-interacting condensates

The zero crossing of the scattering length near a Feshbach resonance can be used to realize non-interacting ideal-gas condensates. BECs of ^7Li , ^{39}K , ^{85}Rb , and ^{133}Cs are good candidates to reduce |a| to very small values on the order of a_0 or smaller.

To explore non-interacting condensates with $^{133}\mathrm{Cs}$ (Kraemer et al., 2004; Weber et al., 2003a) exploited the zero crossing near 17 G and studied expansion after release from the trap. The scattering length was abruptly changed to zero when the condensate was released from the trap. An extremely slow expansion was observed with a release energy as low as $k_B \times 50 \,\mathrm{pK}$; see Fig. 28. The surprising observation that the release energy is a factor of five below the kinetic energy associated with the motional ground state of the trap is explained by the fact that the initial size of the expanding matter-wave packet is determined by the repulsive condensate self-interaction before release, which is larger than the bare ground state of the trap. The momentum spread is thus significantly smaller. In contrast, a slow change of a to zero before release would have ideally resulted in a wavepacket with position and momentum spread corresponding to the bare ground state.

Besides the small equilibrium size of the condensate,

a vanishing scattering length has profound consequences for the collective behavior of a BEC. The sound velocity ($\propto na^{1/2}$) is vanishingly small so that all excitations will become particle-like, and not phonon-like. Moreover, the healing length ($\propto na^{-1/2}$) becomes very large, which may be applied in experiments on rotating condensates (Madison *et al.*, 2000) to increase the core size of vortices.

Non-interacting condensates are promising for the observation that are masked by interacting effects, as e.g. phenomena on a lower energy scale. In atom interferometry the mean-field interaction of a condensate is a substantial systematic error source, as (Gupta et al., 2002) observed in the context of photon recoil measurements. (Roati et al., 2004) have studied Bloch oscillations in an optical lattice under the influence of gravity. They showed that, for interacting bosons, the oscillations lost contrast much faster than for identical fermions without s-wave interaction. A non-interacting BEC combines the advantages of an ultralow momentum spread with very long observation times. Two recent experiments have reported on long-lived Bloch oscillations with BECs of ¹³³Cs (Gustavsson et al., 2008) and ³⁹K (Fattori et al., 2008a), which is an important advance towards highprecision atom interferometry. This could, for example, open up new possibilities for precision measurements of gravitational effects (Carusotto et al., 2005).

Another intriguing application of the zero crossing of a Feshbach resonance was demonstrated by (Lahaye et al., 2007) with a BEC of ⁵²Cr atoms. This species exhibits a very large magnetic dipole-dipole interaction because of its magnetic moment of $6 \mu_B$. When the isotropic contact interaction is reduced by tuning the scattering length close to zero, the magnetic dipole interaction dominates. In ⁵²Cr this was achieved near the 589-G Feshbach resonance (Fig. 23). The resulting dipolar quantum gas represents a model system for a "quantum ferrofluid", the anisotropic properties of which have been attracting considerable interest (Menotti et al., 2008). In further work on ⁵²Cr BEC near the zero of the scattering length, (Lahaye et al., 2008) investigated the controlled collapse of the system and demonstrated its complex dynamics and (Koch et al., 2008) studied the stability of the dipolar condensate depending on the trap geome-The effect of the magnetic dipole interaction has also been observed in non-interacting condensates made of alkali atoms with magnetic moments of the order of $1\mu_B$, for ³⁹K by (Fattori *et al.*, 2008b) and for ⁷Li by (Pollack et al., 2009). The recent observation of Anderson localization of matter waves in a disordered optical potential by (Roati et al., 2008) represents a further exiting application of a non-interacting condensate.

B. Degenerate Fermi gases

In experiments on ultracold Fermi gases (Inguscio *et al.*, 2008), Feshbach resonances serve

as a key to explore many-body physics in the strongly interacting regime (Bloch et al., 2008). This regime is realized when the scattering length exceeds the interparticle spacing and connects the field of ultracold atoms to fundamental questions in various fields of physics, like high T_c -superconductors, nuclear matter, neutron stars, and the quark-gluon plasma. The first Feshbach resonance in a Fermi gas was observed by (Loftus et al., 2002). (O'Hara et al., 2002a) produced the first strongly interacting Fermi gas. Since then the research field has undergone rapid developments with many exciting achievements (Inguscio et al., 2008).

The decay properties of ultracold Fermi gases are strongly influenced by Pauli's exclusion principle (Esry et al., 2001; Petrov, 2003; Petrov et al., 2004; Suno et al., 2003). Three-body recombination processes in one- and two-component Fermi gases necessarily involve identical particles. This generally leads to a suppression of loss as compared to Bose gases or systems with three nonidentical particles. The majority of recent experiments on Fermi gases (Inguscio et al., 2008) has focussed on two-component spin mixtures of ⁶Li or ⁴⁰K with resonant s-wave interactions, realized near broad, entrance-channel dominated resonances. In such systems, it is possible to realize a resonant s-wave interaction $(a \to \pm \infty)$ practically without any decay. Nevertheless, these resonances are accompanied by subtle loss features (Dieckmann et al., 2002), which do not appear at the resonance center, but at side where the scattering length is positive. In contrast to the remarkable stability near this special s-wave scenario, three-body collisions near a p-wave Feshbach resonance usually lead to significant loss (Regal et al., 2003b; Schunck et al., 2005; Zhang et al., 2004).

Here, as prominent examples for the application of Feshbach tuning, we review the attainment of BEC of molecules (Sec. IV.B.1) and studies of the BEC-BCS crossover and the observation of fermion superfluidity (Sec. IV.B.2).

1. BEC of molecules

Bose-Einstein condensation of molecules were created in atomic Fermi gases of ⁶Li (Bourdel et al., 2004; Jochim et al., 2003b; Zwierlein et al., 2003) and ⁴⁰K (Greiner et al., 2003). The molecules are very weakly bound dimers at the side of an entrance-channel dominated s-wave Feshbach resonance where the scattering length is positive and very large. These dimers are formed in a halo state (Sec. V.B.2), which is stable against inelastic decay in atom-dimer and dimer-dimer collisions (Petrov et al., 2004). This stability originates from basically the same Pauli suppression effect that also affects three-body decay in an atomic Fermi gas. Such weakly bound molecules can be detected by converting them back to atoms or by direct absorption imaging; see Sec. V.B.1.

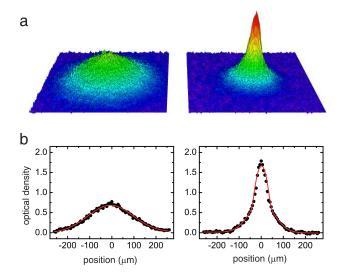


FIG. 29 Emergence of a molecular BEC in an ultracold Fermi gas of ⁴⁰K atoms, observed in time-of-flight absorption images. The density distribution on the left-hand side (upper graph, 2D surface plot; lower graph, 1D cross section) was taken for a weakly interacting Fermi gas which was cooled down to 19% of the Fermi temperature. After ramping across the Feshbach resonance no BEC was observed as the sample was too hot. The density distribution on the right-hand side was observed for a colder sample at 6% of the Fermi temperature. Here the ramp across the Feshbach resonance resulted in a bimodal distribution, revealing the presence of a molecular BEC with a condensate fraction of 12%. Reprinted by permission from Macmillan Publishers Ltd: Nature (Greiner et al., 2003), copyright 2003.

In a spin mixture of ⁶Li in the lowest two internal states, the route to molecular BEC is particularly simple (Jochim et al., 2003b). Evaporative cooling towards BEC can be performed in an optical dipole trap at a constant magnetic field of about 764G near the broad 834-G Feshbach resonance; here $a = +4500a_0$ and $E_b = k_B \times 1.5 \,\mu\text{K}$. In the initial stage of evaporative cooling the gas is purely atomic and a is the relevant scattering length for elastic collisions between the atoms in different spin states. With decreasing temperature the atom-molecule equilibrium (Sec. V.A.3) favors the formation of molecules and, in the final evaporation stage, a purely molecular sample is cooled down to BEC. The large atom-dimer and dimer-dimer scattering lengths of 1.2 a and 0.6 a along with strongly suppressed loss (Sec. V.B.3) facilitate an efficient evaporation process. In this way, molecular BECs are achieved with a condensate fraction exceeding 90%.

The experiments in 40 K followed a different approach to achieve molecular BEC (Greiner *et al.*, 2003). For 40 K the weakly bound dimers are less stable because of less favorable short-range three-body interaction properties. Therefore the sample is first cooled above the 202-G Feshbach resonance, where a is large and negative, to achieve a deeply degenerate atomic Fermi gas. A sweep across the Feshbach resonance then converts the sample into a

partially condensed cloud of molecules. Figure 29 demonstrates the emergence of the molecular BEC in $^{40}{\rm K}.$

The molecular BEC can be described in the mean-field approach outlined in Sec. IV.A.2 by simply replacing the atomic with the molecular mass $(m \to 2m)$, and the atomic with the molecular scattering length $(a \to 0.6a)$. The mean field of the molecular condensate was experimentally studied in (Bartenstein et al., 2004b; Bourdel et al., 2004). However, because of the large scattering length, molecular BECs show considerable beyond-mean-field effects (Altmeyer et al., 2007).

2. BEC-BCS crossover and fermion superfluidity

At a Feshbach resonance in a two-component Fermi gas, different regimes of fermion pairing and superfluidity can be experimentally realized. Pairing on the side with a > 0 can be understood in terms of molecule formation, and superfluidity results from molecular Bose-Einstein condensation. On the other side of the resonance (a < 0), pairing is a many-body effect and the ground state of the system at zero temperature is a fermionic superfluid. In the limit of weak interactions, this regime can be understood in the framework of the well-established Bardeen-Cooper-Schrieffer (BCS) theory, developed in the 1950s to describe superconductivity. Both limits, BEC and BCS, are smoothly connected by a crossover through a regime where the gas is strongly interacting. This BEC-BCS crossover has attracted considerable attention in many-body quantum physics for more than two decades, and has recently reviewed in (Chen et al., 2005; Giorgini et al., 2008; Inguscio et al., 2008). A theoretical description of this challenging problem is very difficult and various approaches have been developed. With tunable Fermi gases, a unique testing ground has become available to quantitatively investigate the crossover problem.

The interaction regime can be characterized by a dimensionless parameter $1/(k_F a)$, where k_F is the Fermi wave number of a non-interacting gas, related to the Fermi energy by $E_F = \hbar^2 k_F^2/(2m)$. For $1/(k_F a) \gg 1$, the molecular BEC regime is realized. For $1/(k_F a) \ll -1$, the system is in the BCS regime. In between $(1/|k_F a| \lesssim 1)$, the Fermi gas is strongly interacting. In the experiments, $E_F/k_B = 1~\mu \text{K}$ gives a typical value for the Fermi energy and $1/k_F \approx 4000~a_0$ sets the typical length scale. The realization of a strongly interacting gas thus requires $|a| \gtrsim 4000~a_0$, which for the particularly broad ⁶Li resonance (Fig. 10) is obtained over a more than 100-G wide magnetic field range.

A particularly interesting situation is the exact resonance case, where $1/(k_F a) = 0$. Here, a is no longer a relevant quantity to describe the problem and scattering is only limited by unitarity (Sec. II.A.2). Consequently, k_F^{-1} and E_F define the relevant scales for length and energy, and the Fermi gas acquires universal properties (Giorgini *et al.*, 2008; Inguscio *et al.*, 2008). For

example, the size and shape of a harmonically trapped "universal Fermi gas" can be obtained just as a rescaled version of a non-interacting Fermi gas.

Experimentally, various properties of strongly interacting Fermi gases have been explored in the BEC-BCS crossover (Inguscio et al., 2008). All these experiments have been performed on two-component spin mixtures of ⁶Li near the 834-G resonance or of ⁴⁰K near the 202-G resonance. In both cases, the resonances have entrance-channel dominated character, where the two-body interaction can be modeled in terms of a single scattering channel and universality applies (Sec. II.B.2). This condition is particularly well fulfilled for the ⁶Li resonance, where the resonance is exceptionally strong.

(Regal et al., 2004) introduced fast magnetic-field sweeps to observe the condensed fraction of pairs in the crossover. Starting with an ultracold ⁴⁰K Fermi gas in the strongly interacting regime, they performed fast Feshbach ramps into the BEC regime. The ramps were fast compared to the time scale of establishing a thermal atom-molecule equilibrium by collisions; see Sec. V.A.3. However, the Feshbach ramps were slow enough to adiabatically convert fermion pairs formed in the strongly interacting regime into molecules. After the ramp, the observed molecular condensate reflected the fermion condensate before the ramp. The fast-ramp method was applied by (Zwierlein et al., 2004) to observe fermion condensates in ⁶Li.

For ⁶Li (Bartenstein *et al.*, 2004b) showed that *slow* Feshbach ramps allow conversion of the gas in a reversible way from the molecular BEC to the BCS regime. Here the gas adiabatically follows and stays in thermal equilibrium. They also observed in-situ profiles of the trapped, strongly interacting gas and measured its changing size for variable interaction strength.

Collective modes in the BEC-BCS crossover were studied in 6 Li gases. (Kinast et~al., 2004) reported on ultraslow damping in a universal Fermi gas with resonant interactions, providing evidence for superfluidity. (Bartenstein et~al., 2004a) measured how the frequencies of collective modes in the crossover changed with variable interaction parameter $1/(k_Fa)$. They also observed a breakdown of hydrodynamic behavior on the BCS side of the resonance, which marks a transition from the superfluid to the normal phase. Precision measurements of collective modes also revealed beyond-mean-field effects in the molecular BEC regime (Altmeyer et~al., 2007).

(Chin et al., 2004a) performed spectroscopy on fermion pairs by using a radio-frequency technique. They measured the binding energy of the pairs in the crossover. They showed how an effective pairing gap continuously evolved from the molecular regime, where it simply reflects the dimer binding energy, into a many-body regime of pairing; see also more recent work by (Schunck et al., 2008).

(Partridge et al., 2005) used an optical molecularprobe technique based on photoassociation to measure the closed-channel contribution of pairs in the crossover.

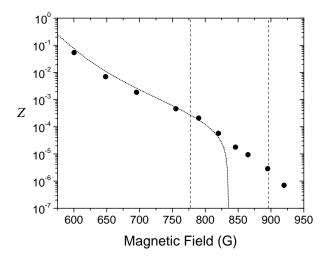


FIG. 30 Measurements of the closed-channel fraction Z for pairs in the BEC-BCS crossover. The experiment uses the 834-G resonance in 6 Li and a photoassociation probe that only couples to closed-channel singlet component. The vertical dashed lines indicate the boundaries of the strongly interacting regime, $k_F|a|>1$. The dotted line shows the result of a coupled channels calculation for molecules; see Sec. II.C.5 and Fig. 16. Comparison with the experimental data shows that two-body physics describes the situation well up to close to the 834-G resonance. For higher fields, Z shows strong manybody effects. Above the resonance, where two-body pairs cannot exist, the many-body system shows the closed-channel admixture of the many-body pairs. From (Partridge $et\ al.$, 2005).

Their results, displayed in Fig. 30, show that this fraction is very small in the strongly interacting regime. These observations strongly support single-channel descriptions for the crossover along with the concept of universality. The results also demonstrate that fermionic pairing reaches from the strongly interacting regime well into the weakly interacting BCS regime.

Superfluidity of a ⁶Li Fermi gas in the BEC-BCS crossover was observed by (Zwierlein *et al.*, 2005). They produced a rotating Fermi gas and observed the formation of vortex arrays. Here Feshbach tuning was applied not only to explore different regimes in the crossover, but also to increase the vortex cores in the expanding Fermi gas after release from the trap; the latter was essential to observe the vortices by optical imaging.

Currently, there is considerable interest in exploring novel regimes of pairing and superfluidity. (Zwierlein et al., 2006) and (Partridge et al., 2006) performed experiments with unbalanced mixtures of two spin states, i.e. polarized Fermi gases. This led to deeper insight into phenomena like phase separation (Partridge et al., 2006; il Shin et al., 2008). Ultracold Fermi-Fermi mixtures of different species, like ⁶Li and ⁴⁰K, have recently become available (Taglieber et al., 2008; Voigt et al., 2008; Wille et al., 2008), adding the mass ratio and independent control of the trapping potentials for both components to our tool box to explore

the broad physics of strongly interacting fermions.

V. ULTRACOLD FESHBACH MOLECULES

Cold molecules are at the center of a rapidly developing research field (Doyle et al., 2004; Hutson and Soldán, 2006), offering many new opportunities for cold chemistry, precision measurements, many-body physics, and quantum information. The coldest attainable molecules, at temperatures in the nanokelvin range, are created by Feshbach association in ultracold atomic gases. Here Feshbach resonances serve as the experimental key to couple pairs of colliding atoms into molecules.

In 2002, (Donley et al., 2002) observed coherent oscillations between atom pairs and Feshbach molecules in a BEC of ⁸⁵Rb atoms. The oscillation frequency reflected the small binding energy of the dimer in a weakly bound state and provided indirect evidence for the creation of molecules. In 2003, several groups reported on more direct observations of Feshbach molecules in Fermi gases of ⁴⁰K (Regal et al., 2003a) and ⁶Li (Cubizolles et al., 2003; Jochim et al., 2003a; Strecker et al., 2003) and BECs of Cs (Herbig et al., 2003), ⁸⁷Rb (Dürr et al., 2004a), and Na (Xu et al., 2003). This rapid development culminated in the attainment of molecular BEC at the end of 2003 (Sec. IV.B.1), and has paved the way for numerous applications.

A comprehensive review of Feshbach molecules and their theoretical background has been given by (Köhler et al., 2006). We do not consider optical methods of making molecules, since this was recently reviewed by (Jones et al., 2006). Here we discuss various formation methods based on magnetic Feshbach resonances (Sec. V.A) and the main properties of Feshbach molecules (Sec. V.B).

A. Formation

Various schemes to create ultracold molecules near Feshbach resonances have been developed in the last few years, most of them relying on the application of time-varying magnetic fields. Section V.A.1 describes the use of magnetic field ramps while Sec. V.A.2 discusses the application of oscillatory fields. These schemes have been applied to a variety of bosonic and fermionic atom gases. Section V.A.3 describes the formation of collisionally stable molecules of fermionic atoms through three-body recombination and thermalization.

Here we restrict our discussion to ultracold gases confined in macroscopic traps. The microscopic trapping sites of an optical lattice, where atom pairs can be tightly confined, constitute a special environment for molecule formation. This will be reviewed separately in Sec. VI.B.

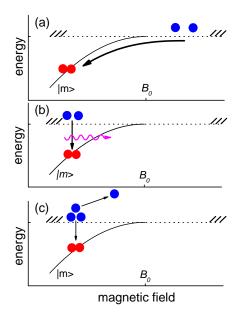


FIG. 31 Illustration of experimental schemes to create ultracold molecules. The solid line marks the weakly bound molecular state $|m\rangle$, which dissociates into the continuum (indicated by the dotted line) at resonance $B=B_0$. In (a), the magnetic field is ramped across the resonance, which adiabatically converts two interacting atoms into one molecule; in (b), an oscillatory magnetic field drives the transition from the scattering state to the molecular state; in (c), three-body recombination results in molecule formation.

1. Feshbach ramps

Ramping an external magnetic field across a Feshbach resonance is the most commonly adopted scheme to form Feshbach molecules. This scheme, usually referred to as a Feshbach ramp, was proposed by (van Abeelen and Verhaar, 1999b; Mies et al., 2000; Timmermans et al., 1999). In a simplified picture, illustrated in Fig. 31(a), the resonant coupling between the scattering state and the molecular state opens up a way to adiabatically convert interacting atom pairs into molecules. The atomic gas is prepared at a field B away from resonance where the two atoms do not have a weakly bound state. In Fig. 31(a), this corresponds to $B > B_0$. The field is then ramped to a final $B < B_0$ to make a Feshbach molecule.

(Regal et al., 2003a) created ultracold molecules in a degenerate Fermi gas of 40 K, exploiting the resonance at 224 G in the ac channel. In the 10-G ramp across the resonance with a ramp speed of 25 G/ms, about 50% of the atoms were converted to molecules. The experimental signatures, shown in Fig. 32(a), were a disappearance of atoms when the field was ramped below 224 G and a recovery of the atoms when the field was ramped back. Since the Feshbach bound state exists below 224 G, their observation strongly suggests formation and dissociation of Feshbach molecules below and above 224 G, respectively. For the 202-G resonance in the ab channel of 40 K,

(Hodby *et al.*, 2005) reported conversion efficiencies of up to 80%. This resonance was also used for the formation of a molecular BEC; see Sec. IV.B.1.

Feshbach ramps were also applied to degenerate Fermi gases of $^6\mathrm{Li}$ in the lowest two spin states (ab channel), using both the narrow resonance at 543 G (Strecker et al., 2003) and the broad resonance at 834 G (Cubizolles et al., 2003). These two resonances in $^6\mathrm{Li}$ are extensively discussed in Sec. II.B.5. Both experiments revealed a remarkable collisional stability of the molecules, see Sec. V.B.3.

In bosonic gases, an efficient atom-molecule conversion by a Feshbach ramp has to overcome inelastic collision loss during the ramp process; see Sec. V.B.3. In general, ramps applied to bosonic atom samples do not only create weakly bound Feshbach molecules, but they also lead to atom loss that cannot be recovered by a reverse field ramp. These atoms are lost presumably to deeply bound molecular states as a result of atom-molecule inelastic collisions. To optimize the Feshbach molecule fraction, which corresponds to the recoverable fraction, it is crucial to optimize the ramp speed. Furthermore, a fast separation of the molecules from the remaining atoms is essential. The latter can, for example, be achieved by Stern-Gerlach or optical methods.

(Herbig et al., 2003) created a pure sample of about $3000 \, q$ -wave molecules from a BEC of 133 Cs atoms by a Feshbach ramp across the narrow resonance at 20 G. The presence of an inhomogeneous magnetic field during the ramp facilitated an immediate Stern-Gerlach separation of the molecules from the atomic cloud. The Feshbach ramp removed about 60% of the atoms from the BEC, but only 20% were found in the weakly bound molecular state and thus could be recovered by a reverse field ramp. Expansion measurements on the Cs₂ cloud showed temperatures below 5 nK and suggested phase-space densities close to or exceeding unity. In later experiments on ¹³³Cs, (Chin et al., 2005; Knoop et al., 2008; Mark et al., 2007b) produced Feshbach molecules in a variety of different internal states and partial waves including s- to l-wave ($\ell = 8$) states.

(Dürr et al., 2004a) applied a similar scheme to a ⁸⁷Rb BEC and formed 7000 s-wave molecules near the 1007-G resonance with a 7% conversion efficiency. A magnetic gradient was applied right after the Feshbach ramp to separate atoms and molecules. The experiments showed remarkable oscillations of the molecular cloud in the gradient field (Fig. 33), which appear as a result of changing magnetic moment near an avoided crossing between two molecular states. The creation of ⁸⁷Rb Feshbach molecules has led to fascinating applications in optical lattices; see Sec. VI.B.1.

(Xu et al., 2003) demonstrated molecule formation with a 4% conversion efficiency in a large BEC of 23 Na atoms by a Feshbach ramp across the $907\,\mathrm{G}$ resonance . A large number of more than 10^5 Feshbach molecules was observed. Expansion measurements on the Na₂ cloud showed that it was in the quantum-degenerate

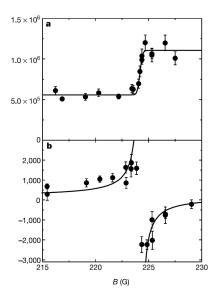


FIG. 32 Molecule formation in a 40 K gas near the 224-G Feshbach resonance. (a) The number of atoms measured after a ramp across the resonance as a function of the final magnetic field shows the disappearance of atoms having formed molecules. For comparison (b) shows the magnetic-field dependence of the s-wave scattering length in units of Bohr radius, as measured by radio-frequency spectroscopy. From (Regal $et\ al.$, 2003a).

regime. By diffraction off an optical standing wave, (Abo-Shaeer $et\ al.$, 2005) demonstrated the matter-wave coherence of Na₂ molecules.

(Hodby et al., 2005) presented a model of the atom-molecule conversion efficiency that is valid for both near-degenerate fermionic and bosonic atoms. Figure 34 shows a comparison of this model with data from experiments on $^{40}{\rm K}$ and $^{85}{\rm Rb}.$ The atom-molecule conversion efficiency follows a Landau-Zener like behavior

$$\frac{P}{P_{\text{max}}} = 1 - \exp\left(-\alpha n \frac{\hbar}{m} \left| \frac{\Delta a_{\text{bg}}}{\dot{B}} \right| \right), \quad (61)$$

where $P_{\rm max}$ is the maximum conversion efficiency, solely determined by the phase-space density of the atomic cloud. This expression reveals a simple dependence on the resonance parameters Δ and $a_{\rm bg}$, the atomic number density n, and the ramp speed \dot{B} ; the dimensionless prefactor α is discussed in (Köhler et~al., 2006).

Improved conversion techniques have been reported for atomic BECs. For the case of the narrow 20-G g-wave resonance of 133 Cs, (Mark et~al., 2005) successfully converted 30% of the atoms into Feshbach molecules. Their scheme relied on fast switching of the magnetic field from 21 G right on the resonance, followed by a hold time of $\sim 10 \,\mathrm{ms}$ and a further switch to 18 G. This scheme was found to be superior to using an optimized linear ramp.

Feshbach ramps have also been applied to create p-wave molecules in ultracold Fermi gases. Formation of p-wave molecules in a $^6\mathrm{Li}$ gas was reported by (Zhang et~al., 2004) based on the 185-G resonance in the

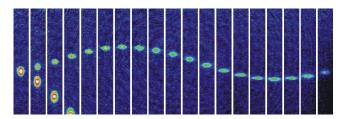


FIG. 33 Motion of $^{87}\mathrm{Rb}_2$ molecules in a magnetic field gradient. Right after the Feshbach ramp, the molecules are separated from the atoms because of the different magnetic moments. While the atom cloud leaves the observation region (see first four images) the molecules undergo an oscillatory motion, which is due to a changing magnetic moment caused by an avoiding level crossing in the molecular states. The images correspond to steps of 1 ms, and the field of view of each image is $0.24 \times 1.7\,\mathrm{mm}$. From (Dürr et~al.,~2004a).

ab channel. (Fuchs $et\ al.$, 2008) measured the binding energies of such p-wave Feshbach molecules in three channels $(aa,\ ab,\ bb)$ using an oscillating magnetic field; see Sec. III.A.5. (Inada $et\ al.$, 2008) studied the collisional properties of these molecules in all three channels. (Gaebler $et\ al.$, 2007) created p-wave molecules in a 40 K gas by fast switching of the magnetic field to the p-wave Feshbach resonances at 198.4 and 198.8 G (both in the bb channel) and studied the lifetimes of the molecules.

Similarly, Feshbach ramps have also been applied to ultracold atomic mixtures to create heteronuclear molecules, such as $^{40}\mathrm{K}^{87}\mathrm{Rb}$ (Ni et al., 2008) and $^6\mathrm{Li}^{40}\mathrm{K}$ (Voigt et al., 2008). An isotopic rubidium mixture was used to associate $^{85}\mathrm{Rb}^{87}\mathrm{Rb}$ molecules by (Papp and Wieman, 2006). A variety of other heteronuclear molecule systems, are currently under investigation in different laboratories. Feshbach ramps have become a standard approach to create ultracold molecules and serve as a starting point to investigate the dynamics and the interaction properties of Feshbach molecules.

2. Oscillatory fields

Another powerful method to produce ultracold Feshbach molecules is based on a modulation of the magnetic field strength (Hanna et~al., 2007; Thompson et~al., 2005b). The oscillating field induces a stimulated transition of two colliding atoms into a bound molecular state; see Fig. 31(b). Heating and atom loss are reduced since association occurs at a bias field B away from the resonance position B_0 . For a $^{85}{\rm Rb}$ BEC near the 155-G resonance, (Thompson et~al., 2005b) report on high conversion efficiencies for molecules with binding energies on the order of 10 kHz; the molecule formation was inferred from the observation of a resonant loss signal. (Lange et~al., 2009) used the same method to explore weakly bound molecular states of Cs atoms in the aa channel in an energy range of up to 300 kHz.

In a ⁴⁰K spin mixture, p-wave molecules were pro-

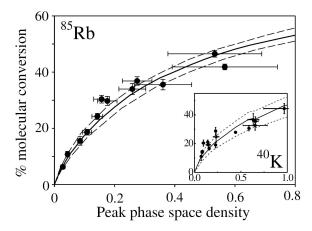


FIG. 34 Illustration of the dependence of the atom-molecule conversion efficiency on the atomic phase-space density. The main figure shows the results obtained in a near-degenerate bosonic gas of $^{85}{\rm Rb}$ atoms, while the inset shows corresponding results on a fermionic gas of $^{40}{\rm K}$ atoms. The lines refer to a theoretical model that is based on the phase-space overlap of atom pairs in the trapped gases. From (Hodby et~al., 2005).

duced with an oscillating magnetic field near the 198-G resonance doublet in the bb channel (Gaebler et~al., 2007). The method was also applied to produce heteronuclear s-wave molecules in mixtures of the two Rb isotopes (Papp and Wieman, 2006) and of 41 K and 87 Rb atoms (Weber et~al., 2008).

In contrast to the magnetic-field modulation method, radio-frequency transitions in the range of tens of MHz that involve a change of spin channel can also be used to associate two atoms to make a Feshbach molecule. This is the inverse of the dissociation process described in Sec. III.A.5. In this way (Klempt $et\ al.$, 2008; Zirbel $et\ al.$, 2008b) achieved association of $^{87}{\rm Rb}$ and $^{40}{\rm K}$ in a dipole trap. Similar association experiments in an optical lattice are described in Sec. VI.B.1.

3. Atom-molecule thermalization

A particular situation for molecule formation arises in a spin mixture of ⁶Li near the 834-G Feshbach resonance. At the low-field side of the resonance there is a broad field range, where the s-wave scattering is large and positive. Here a weakly bound state exists with a pronounced halo character. The molecular state shows an extraordinary stability against inelastic decay, which opened the way to efficiently create molecular BECs by straightforward evaporative cooling at a constant magnetic field near 764 G (Jochim et al., 2003b; Zwierlein et al., 2003).

The formation of molecules in this region can be understood in terms of a chemical atom-molecule equilibrium (Chin and Grimm, 2004; Kokkelmans *et al.*, 2004), where exoergic three-body recombination events compete with endoergic two-body dissociation processes. From a

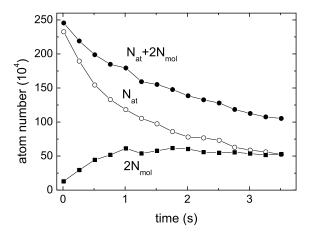


FIG. 35 An ultracold $^6\mathrm{Li}$ gas approaches a chemical atom-molecule equilibrium on the molecular side of the 834-G Feshbach resonance. The experiment starts with a non-degenerate, purely atomic gas at a temperature of $2.5\,\mu\mathrm{K}$ and a peak atomic phase-space density of 0.04. The magnetic field is set to 690 G, where the molecular binding energy corresponds to $15\,\mu\mathrm{K}$. $N_{\rm at}$ and $N_{\rm mol}$ denote the number of unbound atoms and the number of molecules, respectively. The total number of unbound and bound atoms $2N_{\rm mol}+N_{\rm at}$ slowly decreases because inelastic loss is not fully suppressed. From (Jochim et~al., 2003a).

balance of these processes one can intuitively understand that molecule formation is favored at low temperatures and high number densities, i.e. at high phase-space densities. Indeed, for a non-degenerate gas, the atom-molecule equilibrium follows a simple relation (Chin and Grimm, 2004)

$$\phi_{\rm mol} = \phi_{\rm at}^2 \exp\left(\frac{E_b}{k_{\rm B}T}\right),$$
 (62)

where ϕ_{mol} and ϕ_{at} denote the molecular and atomic phase-space densities, respectively. The Boltzmann factor, determined by the ratio of the molecular binding energy E_b and the thermal energy k_BT , enhances the fraction of molecules and can partially compensate for a low atomic phase-space density.

The thermal atom-molecule equilibrium was experimentally investigated by (Jochim $et\ al.$, 2003a) in a non-degenerate gas of $^6{\rm Li}$ atoms. Figure 35 illustrates how the initially pure atomic gas tends to an atom-molecule equilibrium. The observation that more than 50% of the atoms form molecules at a phase-space density of 0.04 highlights the role of the Boltzmann factor. The chemical atom-molecule equilibrium also played an essential role in the experiment by (Cubizolles $et\ al.$, 2003), where a slow Feshbach ramp, which kept the sample in thermal equilibrium, led to a conversion efficiency of 85%.

B. Properties

1. Dissociation and detection

A general way to detect Feshbach molecules is their controlled dissociation through reverse magnetic-field sweeps, followed by imaging of the resulting cloud of atoms. If the image is taken immediately after the forced dissociation, it just reflects the spatial distribution of the molecules before the onset of dissociation. However, if the image is taken after a certain time of flight, it will be strongly affected by a release of kinetic energy. The image then contains additional information on the dissociation process.

A reverse Feshbach ramp brings the molecule into a quasi-bound state above the dissociation threshold, from which it decays into two atoms in the continuum. The decay rate $\Gamma(E)/\hbar$ depends on the energy E above threshold and can be calculated from Eqs. (14), (17), and (22),

$$\Gamma(E) = 2ka_{\rm bg}\delta\mu\Delta = 2k\bar{a}\bar{E}s_{\rm res}.$$
 (63)

(Mukaiyama et al., 2004) gave the energy spectrum of hot atoms by using this Fermi's golden rule expression for a linear ramp and showed that it agreed well with measurements with ²³Na₂ Feshbach molecules. (Góral et al., 2004) verified the golden rule theory with a full quantum dynamics calculation of Feshbach molecule dissociation.

The mean kinetic energy released in a reverse Feshbach ramp corresponds to the typical energy that the molecules can reach in the quasi-bound level before the dissociative decay takes place. In experiments on $^{87}\text{Rb}_2$, (Dürr et al., 2004b) studied the dependence of the energy release on the ramp rate and on the resonance width. They demonstrated kinetic energy measurements after reverse Feshbach ramps as a powerful indirect tool to determine the widths of weak resonances with Δ in the mG range, where direct methods are impractical. They also demonstrated the production of a monoenergetic spherical wave of atoms by rapidly switching the magnetic field instead of ramping it.

The dissociation properties of Feshbach molecules can provide additional spectroscopic information. (Volz et al., 2005) observed interesting dissociation patterns of 87 Rb₂, when the molecular state was brought high above the threshold with fast jumps of the magnetic field. The patterns, shown in Fig. 36, reveal a d-wave shape resonance. The dissociation of 133 Cs₂ molecules in l-wave states was observed by (Mark et al., 2007b). The dissociation pattern showed a strikingly different behavior from molecules in a g-wave state and allowed to clearly distinguish between these two types of molecules.

We note that direct imaging of Feshbach molecules is not feasible in most situations because of the absence of cycling optical transitions. An exception, however, is the direct imaging of atoms in halo states as demonstrated for ⁶Li (Bartenstein *et al.*, 2004b; Zwierlein *et al.*, 2003). In this special case, the extremely weakly bound dimer absorbs resonant light essentially like free atoms.

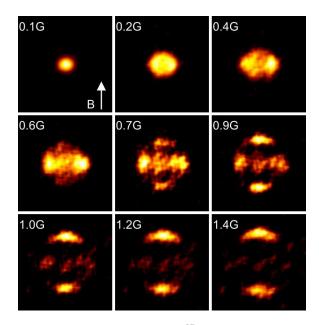


FIG. 36 Dissociation patterns of $^{87}\text{Rb}_2$ molecules, showing the interference of s- and d- partial waves. At small magnetic field offsets $B-B_0$ (values given in the upper left corners), the s-wave pattern dominates; at large offsets, d-waves are strongly enhanced due to a d-wave shape resonance. From (Volz et al., 2005).

2. Halo dimers

Broad, entrance-channel dominated s-wave resonances feature a considerable region where Feshbach molecules acquire universal properties; see Sec. II and (Köhler et al., 2006). "Universality" means that details of the interaction become irrelevant and that all properties of the dimer are characterized by a single parameter, the s-wave scattering length a or alternatively the binding energy $E_b = \hbar^2/(ma^2)$. The reason for this great simplification is the fact that the wavefunction extends far out of the classical interaction range of the potential. States of this kind have been coined "quantum halos". They have attracted considerable attention in nuclear physics and, more recently, in molecular physics and have been extensively reviewed in (Jensen et al., 2004). An early example is the deuteron, where the neutron and proton are likely to be found outside of the classically allowed region. Before the advent of Feshbach molecules, the most extended halo system experimentally accessible was the helium dimer (⁴He₂), which is about ten times larger than typical diatomic molecules. An extreme example is given by Bose-condensed ⁶Li₂ Feshbach dimers with a size of $a/2 \simeq 2000 a_0$, which exceeds the van der Waals length $R_{\rm vdw} \simeq 30 \, a_0$ by almost two orders of magnitude.

First experiments on halo dimers have been conducted with bosonic ⁸⁵Rb (Donley *et al.*, 2002) and ¹³³Cs (Mark *et al.*, 2007a), and fermionic ⁴⁰K and ⁶Li; see Sec. IV.B.1. For ⁸⁵Rb, the dimers are not formed from atoms in the lowest internal states and thus have open

decay channels. This leads to spontaneous dissociation without the presence of other atoms or molecules. (Thompson et~al., 2005a) have observed an a^{-3} scaling of the dissociation rate, which can be understood as a direct consequence of universality through wavefunction overlap arguments (Köhler et~al., 2005). For halo dimers of $^{133}\mathrm{Cs}_2$ created from atoms in their lowest internal states there are no open dissociation channels. These molecules cannot decay spontaneously but decay through collisions with other atoms or molecules (Ferlaino et~al., 2008; Knoop et~al., 2009).

A future promising direction with Feshbach molecules in halo states is the experimental investigation of universal few-body physics; see Sec. VI.C.

3. Collision properties

Fast collisional loss is usually observed in trapped samples of ultracold molecules. This has been seen in experiments with bosonic ²³Na₂ (Mukaiyama et al., 2004), ⁸⁷Rb₂ (Syassen et al., 2006; Wynar et al., 2000) and 133Cs₂ (Chin et al., 2005). (Zirbel et al., 2008b) measured large rate coefficients for fermionic 40K87Rb Feshbach molecules due to collisions with ⁸⁷Rb or ⁴⁰K. Both atom-dimer and dimer-dimer collisions are generally found to cause strong inelastic loss, as demonstrated by the example in Fig. 37. Vibrational relaxation is the dominant mechanism, which leads to large loss rate coefficients of the order of 10^{-10} cm³/s. Rate coefficients of such magnitude result in molecular lifetimes on the order of a few ms or less for densities characteristic of ultracold gases. Rearrangement reactions, such as trimer formation, may also play a significant role in limiting molecular lifetimes.

Inelastic collision rates of Feshbach molecules in the very highest bound state, when they are not universal "halo" states, are not significantly different from rates for more deeply bound states, for which fast inelastic loss has been observed (Staanum et al., 2006; Zahzam et al., 2006) and predicted (Cvitas et al., 2007; Quéméner et al., 2005). (Hudson et al., 2008) measured large inelastic collision rate coefficients for vibrationally excited triplet ⁸⁷Rb¹³³Cs molecules colliding with ¹³³Cs or ⁸⁷Rb atoms. They also used a simple model to help understand why such large rate constants are typical for atom-molecule vibrational relaxation. Assuming a probability near unity for inelastic loss when the collision partners approach one another in the short-range region of chemical bonding, the overall collision rate coefficient is then determined from the threshold scattering of the long range van der Waals potential. If effect, the rate constant is given by Eq. (10), where the length b turns out to be similar in magnitude to the van der Waals length $R_{\rm vdw}$. Such a simple model gives the typical order of magnitude of 10^{-10} cm³/s for the vibrational relaxation rate constant, nearly independent of the vibrational level, found for the ⁸⁷Rb¹³³Cs system.

Halo molecules comprised of two unlike fermions bound in an s-wave state offer an exception to the rule of fast inelastic dimer-dimer and atom-dimer collisions. This has allowed stable molecular samples and even molecular Bose-Einstein condensation; see Sec. IV.B. (Petrov et al., 2004) showed that a combination of two effects explains this stability. The first effect is a small wavefunction overlap of a halo dimer with more deeply bound dimer states, and the second one is Pauli suppression in the few-body process. For inelastic dimer-dimer collisions, (Petrov et al., 2004) predicted the rate coefficient for inelastic loss to scale as $(a/R_{\rm vdw})^{-2.55}$ whereas, for the elastic part, they obtained a dimer-dimer scattering length of 0.6 a. For the atom-dimer interaction, the predicted scaling of inelastic loss is $(a/R_{\rm vdw})^{-3.33}$ and the scattering length is $1.2\,a$.

An interesting case is the observation of stable $^6\mathrm{Li}_2$ molecules created near the closed-channel dominated resonance at 543 G (Strecker *et al.*, 2003). These are not halo molecules and would be expected to have a large collisional loss rate coefficient similar to molecules comprised of bosons. The collision properties of these molecules still await detailed investigation.

A possible way to overcome harmful inelastic collisional loss is the application of an optical lattice (Thalhammer et al., 2006). Here a pair of atoms or a single molecule can be trapped in an individual lattice site, which offers shielding from collisions with other molecules or atoms. Many experiments on ultracold molecules are now being performed with Feshbach resonances and molecules in an optical lattice; see discussion in Sec. VI.B. Another way to prevent inelastic collisions of ultracold molecules is to transfer them to their lowest energy ground state, where they do not undergo vibrational, rotational, or spin relaxation. However, reactive collisions may still be possible.

4. Internal state transfer

A Feshbach resonance can serve as an "entrance gate" into the rich variety of molecular states below threshold, allowing preservation of the ultralow temperature of the atomic gas that is used as a starting point. The magnetic association technique (Sec. V.A.1) produces a molecule in a specific weakly bound molecular state, i.e. the particular molecular state that represents the closed scattering channel of the resonance. This leads to the question how a Feshbach molecule can be transferred to other states with specific properties of interest or, ultimately, to the absolute ro-vibrational ground state. Various methods have been developed for a controlled internal state transfer based on magnetic field ramps, radio-frequency or microwave radiation, or optical Raman excitations.

When a Feshbach ramp after initially associating the molecules is continued over a wider magnetic field range, the molecule will perform a passage through many level crossings; see e.g. Fig. 14. The ramp speed controls

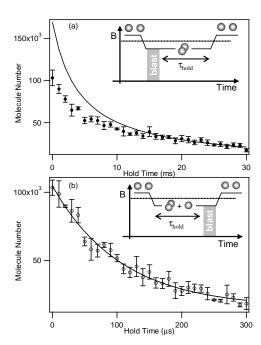


FIG. 37 Fast collisional decay of ²³Na₂ Feshbach molecules. The molecules are trapped alone (a) or together with atoms (b). From (Mukaiyama *et al.*, 2004).

whether crossings are traversed diabatically (fast ramp) or whether they are followed adiabatically (slow ramp). (Mark et al., 2007a) demonstrated the controlled transfer of ¹³³Cs₂ molecules into different states by elaborate magnetic field ramps. In this way, they could populate various states from s- up to l-waves with binding energies of up to $\sim 10\,\mathrm{MHz}$. In practice, finite ramp speeds limit this method to rather weak crossings with energy splittings of up to typically 200 kHz. (Lang et al., 2008a) showed how this problem can be overcome with the help of radio-frequency excitation. They demonstrated the transfer of ⁸⁷Rb₂ molecules over nine level crossings when the magnetic field was ramped down from the 1007-G resonance to zero field in 100 ms. This produced molecules having a binding energy $E_b = h \times 3.6 \,\mathrm{GHz}$ with a total transfer efficiency of about 50%.

More deeply bound states can be reached by twophoton Raman transitions, as implemented in a very efficient way by stimulated Raman adiabatic passage (STIRAP) (Bergmann et al., 1998). Fig. 38 illustrates STIRAP between the two highest vibrational levels in ⁸⁷Rb₂ with binding energies corresponding to 24 MHz and 637 MHz, as demonstrated in a proof-of-principle experiment by (Winkler et al., 2007). The experiment also highlighted the great potential of the approach to combine Feshbach association with stimulated Raman optical transitions, as originally suggested by (Kokkelmans et al., 2001) to create deeply bound molecules.

In 2008, enormous experimental progress was made in applications of STIRAP to transfer both homo- and heteronuclear Feshbach molecules into deeply bound

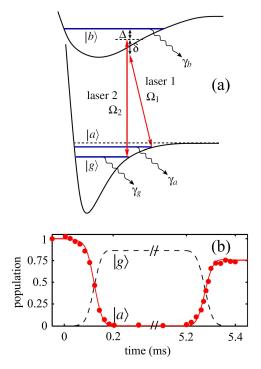


FIG. 38 STIRAP with Feshbach molecules. (a) After creation of Feshbach molecules in a weakly bound state $|a\rangle$, a two-photon transition is used to transfer the molecules into a more deeply bound state $|g\rangle$. In the counterintuitive pulse sequence of STIRAP, laser field 2 is switched on before laser field 1, and the molecule is always kept in a dark superposition state of $|a\rangle$ and $|g\rangle$ during the transfer sequence. This suppresses the unwanted electronic excitation of state $|b\rangle$ and leads to a very high transfer efficiency. The measurements shown in (b) refer to an experiment in ${}^{87}\text{Rb}_2$, where $|a\rangle$ and $|g\rangle$ correspond to the last and second-to-last bound vibrational level. The data show the measured population in $|a\rangle$ during a 0.2-ms transfer sequence and a reverse transfer sequence after a 5-s hold time in state $|q\rangle$. The observed efficiency of 75% for a double passage corresponds to single-passage transfer efficiency of 87%. Adapted from (Winkler et al., 2007).

states. (Danzl et al., 2008) explored ¹³³Cs₂ molecules and demonstrated large binding energies corresponding to 31.8 THz. (Lang et al., 2008b) reached the rovibrational ground state in the triplet potential of ⁸⁷Rb₂, the binding energy of which corresponds to 7.0 THz. The heteronuclear case was successfully explored with ⁴⁰K⁸⁷Rb. Initial experiments by (Ospelkaus et al., 2008) demonstrated the transfer to states with a binding energy corresponding to 10.5 GHz. Only shortly afterwards, the same group (Ni et al., 2008) demonstrated polar molecules in both the triplet and the singlet rovibrational ground state, where the binding energies correspond to 7.2 THz and 125 THz, respectively. These experiments opened up a promising new research field related to the exciting interaction properties of ground-state molecular quantum gases.

All the above methods for controlled state transfer rely on coherent processes. Therefore they can also be applied to produce coherent superpositions of molecular states. This can, for example, be used for precise interferometric measurements of the molecular structure. (Mark et al., 2007b) and (Lang et al., 2008a) investigated molecular level crossings in this way. Using STI-RAP, (Winkler et al., 2007) created quantum superpositions between neighboring vibrational states and tested their coherence interferometrically.

VI. RELATED TOPICS

A. Optical Feshbach resonances

Magnetic fields have proven to be a powerful tool to change the interaction strength or scattering length between ultracold atoms. As discussed at length in this Review this has been made possible by the presence of a molecular bound state that is resonantly coupled to the colliding atom pair. The width of the resonance (Δ in Eq. (1)), however, is governed by the interatomic forces between the two atoms. Optical Feshbach resonances promise control of both the resonance location and its width.

1. Analogies

Figure 39 shows a schematic diagram of an optical Feshbach resonance. As first proposed by (Fedichev et al., 1996a) laser light nearly resonant with a transition from a colliding atom pair and a ro-vibrational level of an excited electronic state induces a Feshbach resonance and modifies the scattering length of the two atoms. Excited electronic states dissociate to one ground- and one electronically-excited atom for large interatomic separations. For many kinds of atoms the photon needed to reach such states is in the visible or optical domain and, hence, the term "optical Feshbach resonance" has been adopted.

The location and strength of an optical Feshbach resonance is determined by the laser frequency ν and intensity I, respectively. Both can be controlled experimentally. There is, however, a crucial difference between magnetic and optical Feshbach resonances. For optical Feshbach resonances the resonant state has a finite energy width γ and thus lifetime \hbar/γ due to spontaneous emission. Hence the scattering length becomes a complex number.

Note that by changing the laser frequency and simultaneously detecting the population in the excited electronic potentials the ro-vibrational level structure of these potentials can be studied. This is called ultracold photoassociative spectroscopy and has been reviewed in (Jones $et\ al.$, 2006).

(Bohn and Julienne, 1996, 1999) obtained expressions for the complex scattering length in Eqs. (24)-(25). Resonances are characterized by a width $\Gamma(E)$ and shift $h\delta\nu_c$,

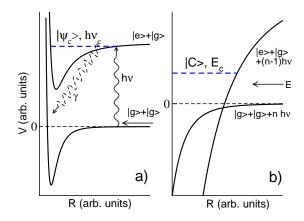


FIG. 39 Schematic of an optical Feshbach resonance. Panel (a) refers to two colliding ground-state atoms, $|g\rangle$ that are coupled through a laser at frequency ν to a vibrational level $|\psi_c\rangle$ of an excited electronic state that dissociates to one excited-state $|e\rangle$ and one ground-state $|g\rangle$ atom. The energy $h\nu_c$ is the energy of the vibrational level relative to two colliding atoms at zero collision energy. The excited bound state $|\psi_c\rangle$ irreversibly decays with an (energy) width γ . Panel (b) shows the dressed state picture corresponding to panel (a). The initial state is an atom pair with n photons and the closed channel state $|C\rangle$ represents $|\psi_c\rangle$ with n-1 photons. The atom pair has a relative kinetic energy E and $E_c = h\nu_c - h\nu$. In the dressed state picture the ground and excited interatomic potentials cross at a point called the Condon point.

where
$$E_0 = h\nu_c - h\nu - h\delta\nu_c$$
. The width is
$$\Gamma(E) = 2\pi |\langle C|\vec{d} \cdot \vec{\mathcal{E}}|E\rangle|^2, \tag{64}$$

where $\vec{\mathcal{E}}$ is the electric field of the laser and \vec{d} is the molecular electronic transition dipole moment. Both the width and the shift of the resonance are proportional to I. As in magnetic resonances $|E\rangle$ is the scattering wavefunction at collision energy E in the entrance channel.

Figure 6 in Section II.A.3 shows the real and imaginary part of the scattering length a-ib for an optical Feshbach resonance. The numbers are based on an analysis of the strength and lifetime of an experimentally-observed optical Feshbach resonance in ⁸⁷Rb (Theis et al., 2004). For the intensity used in the figure the optical length $a_{\rm res}$ defined by Eq. (26) is 5.47 nm and $\Gamma_0/h=21$ MHz. That is, $a_{\rm res}\approx a_{\rm bg}$ and $\Gamma_0\approx \gamma$. Since $ka_{\rm bg}\ll 1$, the width $\Gamma(E)\ll \gamma$ so there is negligible power broadening. Unlike for a magnetic Feshbach resonance, the real part of the scattering length is now finite for any detuning with a peak to peak variation of $2a_{\rm res}$. The length b peaks at zero detuning. The maximum value is $2a_{\rm res}$ and the full-width half-maximum is γ .

In order for an optical Feshbach resonance to be practical it is necessary that the change in the real part of the scattering length $a-a_{\rm bg}$ is large compared to b. This requires that the detuning $h\nu-h\nu_c$ is large compared to

 γ . On the other hand we would also like to ensure that $a-a_{\rm bg}$ is at least on the order of $a_{\rm bg}$ at such a detuning. In order to satisfy these requirements simultaneously we need $a_{\rm res}\gg a_{\rm bg}$ or equivalently $\Gamma_0\gg\gamma$. For the parameters used in Fig. 6 this is not true. Section VI.A.3 discusses how for alkaline-earth atoms it is possible to satisfy $a_{\rm res}\gg a_{\rm bg}$.

By adding extra laser fields at different frequencies the scattering length can be further manipulated. Of particular interest is the case where a second field is nearly resonant with the excited bound state $|\psi_c\rangle$ and a second molecular bound state. If this second bound state is in the electronic ground state, then the situation corresponds to a Raman transition. The analytic expression for the scattering length (Bohn and Julienne, 1999; Thalhammer et al., 2005) is

$$a - ib = a_{\text{bg}} + \frac{1}{2k} \frac{\Gamma(E)}{h\nu - h\nu_0 + \Omega^2/\Delta_2 + i(\gamma/2)}$$
 (65)

where $a_{\rm bg}$, Γ , and γ are defined as before, and $h\nu_0 = h\nu_c + h\delta\nu_c$. The two-photon detuning Δ_2 is zero when the absolute value of the frequency difference of the two lasers equals the absolute value of binding energy of the ground bound state relative to two free atoms at rest. It is positive when the absolute value of the frequency difference is larger than the absolute value of the binding energy. The quantity Ω is the coupling matrix element between the bound levels in the ground and excited states and is proportional to the square root of the intensity of the second laser.

2. Observations in alkali systems

In a magneto-optical trap filled with cold (< 1 mK) atomic sodium (Fatemi et al., 2000) confirmed the predictions of (Fedichev et al., 1996a). They observed the changing scattering length by detecting the corresponding change in the scattering wavefunction. Weak detection lasers, with frequencies that differ from those used for the optical Feshbach resonance, induce a molecular ion signal that probes the change in the scattering wave function. For the molecular level used in the experiment (Fatemi et al., 2000) were able to deduce the strength of the resonance as well as the light shift δv_c . They found an $a_{\rm res}$ of around 2 nm and Γ_0/h of around 20 MHz for the maximum reported laser intensity of $I=100~{\rm W/cm^2}$. For Na the background scattering length is $a_{\rm bg}=2.8~{\rm nm}$.

(Theis et al., 2004) tuned the scattering length by optical means in a Bose-Einstein condensate. In a ⁸⁷Rb condensate they were able to change the scattering length over one order of magnitude from 0.5 nm to 10 nm. The parameters of the optical resonance are given in the caption of Fig. 6. The scattering length was measured using Bragg spectroscopy, where a change in condensate mean field energy proportional to the scattering length is measured by a change in the frequency that determines the Bragg condition.

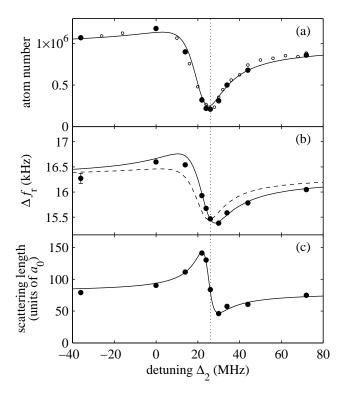


FIG. 40 Optical Feshbach resonance using a two-color Raman transition in an ⁸⁷Rb BEC. Panel (a) shows the measured atom number after a 100 μ s Raman pulse as a function of the two-photon detuning Δ_2 . Panel (b) shows the measured Bragg resonance frequency and Panel (c) shows the scattering length, as determined from Panels (a) and (b). The open and filled circles correspond to the data. As indicated by the error bar in Panel (b) the frequency uncertainty in the Bragg spectroscopy is smaller than 100 Hz. The solid lines have been obtained from a fit to Eq. (65) adapted to include the finite linewidth of the ground-bound state. The dashed line in Panel (b) shows the expected signal if there was only loss in atom number but no change in scattering length. The vertical line indicates the location of maximal loss in Panel (a) and helps to compare the relative positions of the three curves. From (Thalhammer et al., 2005).

In (Thalhammer et al., 2005) the scattering length was modified by a two-color Raman transition. As in (Theis et al., 2004) the change in scattering length was observed in an ⁸⁷Rb condensate and detected by Bragg spectroscopy. In fitting to Eq. (65) a complication arose. (Thalhammer et al., 2005) could only explain their observations if they assumed that the target ground state had a finite linewidth. It turned out that, even though this state cannot be lost by spontaneous emission, it can absorb a photon from the (strong) laser that photoassociates the scattering atom pair. This process gave rise to a linewidth of 2 MHz. The results are shown in Fig. 40.

3. Prospects in alkaline-earth systems

So far, the scattering length has been experimentally modified by optical means in ultracold alkali-metal atom collisions. In these experiments the optical length $a_{\rm res}$ was of the same order of magnitude as the background scattering length $a_{\rm bg}$ so that changes in scattering length were accompanied by large atom losses. (Ciuryło et al., 2005) showed that optical Feshbach resonances in ultracold alkaline-earth atom collisions can have $a_{\rm res} \gg a_{\rm bg}$. The presence of intercombination lines in alkaline-earth systems make this possible. Atomic intercombination lines are transitions between the ground $^1{\rm S}_0$ state and the excited $^3{\rm P}_1$ state. The transition is only weakly allowed by virtue of relativistic mixing with the $^1{\rm P}_1$ state. For example, for Sr isotopes $\gamma/h=7.5$ kHz is much smaller than for alkali atoms.

When γ is very small, it is possible to use excited bound levels that are very close to the excited state dissociation threshold, while simultaneously maintaining the large detunings that are necessary to suppress losses. Using levels close to threshold allow a very large value of the ratio Γ_0/γ , and consequently $a_{\rm res}$ can be orders of magnitude larger than $a_{\rm bg}$. (Ciurylo *et al.*, 2005) illustrate this by model calculations of optical lengths of ultracold calcium, for which $\gamma/h=0.7$ kHz. Using a model that assumes a level with a binding energy on the order of 100 MHz, they predict that $a_{\rm res}$ could be as large as 100 nm at the relatively low intensity of $I=1~{\rm W/cm^2}$.

(Zelevinsky et al., 2006) obtained photoassociation spectra near the intercombination line of ⁸⁸Sr and measured the strength of various transitions. They found that the last bound state of the excited potential had an optical length $a_{\rm res}=24~\mu{\rm m}$ at $I=1~{\rm W/cm^2}$. The very large value of $a_{\rm res}$ implies that practical changes in the scattering length should be feasible in this species. Similar photoassociation spectra have been observed for two different isotopes of Ytterbium (Tojo et al., 2006), which has electronic structure like that of the alkaline earth atoms. Optical control of both bosonic and fermionic isotopic species may become possible with alkaline earth or Yb atoms. (Enomoto et al., 2008) have demonstrated optically induced changes in scattering length for ¹⁷²Yb and ¹⁷⁶Yb.

B. Feshbach resonances in optical lattices

Ultracold atoms in optical lattices are of great interest because of the exciting prospects to simulate a variety of condensed matter phenomena, to realize large scale quantum information processing (Bloch, 2005), and to form ultracold molecules in individual lattice sites to avoid detrimental collision instability. In all these research directions, Feshbach resonances will provide excellent tools to control the interaction of the constituent atoms and to explore the transition between different quantum regime and quantum phases. In the following sections we review

atom-atom scattering in optical lattices and describe the role of Feshbach resonances therein.

Optical lattices are realized by standing-wave laser fields, which result in spatially-periodic potentials for the atoms. The atoms are confined in the individual potential minima or sites of the lattice potential. One-, two-, and three-dimensional lattices can be created in this way. In experiments with three-dimensional configurations the optical lattice can be filled with only one or two atoms per site.

Section VI.B.1 discusses that when two atoms are held in a single lattice site a Feshbach resonance can be used to very efficiently produce stable molecules. One advantage of lattice confinement is that such molecules are protected from harmful collisional losses with a third body. Section VI.B.2 discusses the possibility that confinement in one spatial direction can induce resonant behavior in scattering along the remaining directions. Finally, Section VI.B.3 describes uses of Feshbach resonances in optical lattices where tunneling between lattices sites is important.

1. Atom pairs and molecules

When an atom pair is trapped in a single site of a threedimensional optical lattice the motion is fully quantized. For deep optical lattices the confining potential is harmonic, and the center-of-mass and relative motion of the atom pair separate. In other words the six-dimensional wavefunction of the two atoms becomes a product of a center-of-mass and relative wavefunction. The center-ofmass motion is harmonic and is solved trivially. The relative motion is determined by a potential that is the sum of the atom-atom interaction potentials and a harmonic potential.

The atom-atom interactions between alkali-metal atoms are independent of the relative orientation of the atoms when very weak spin-dependent interactions, $V_{\rm ss}$, are ignored. Consequently, for a spherically symmetric harmonic trapping potential the three-dimensional relative motion can be further simplified. The angular motion can be solved analytically and only a radial Schrödinger equation for the atom-atom interaction potential plus $\mu\omega^2R^2/2$ needs to be solved. Here, μ is the reduced mass and ω is the oscillation frequency in the trap.

Figure 41 shows eigenenergies for two Na atoms with zero relative orbital angular momentum ($\ell=0$) in a spherically-symmetric harmonic trap as a function of magnetic field (Tiesinga et al., 2000). The atoms are in their lowest hyperfine state and the energies are obtained from coupled-channels calculations. For these atomic states there is a Feshbach resonance near 910 G (91 mT). The zero of the vertical axis corresponds to zero relative kinetic energy in the absence of a trapping potential. Hence, positive energies correspond to atoms in the trap and negative energies correspond to molecules bound in

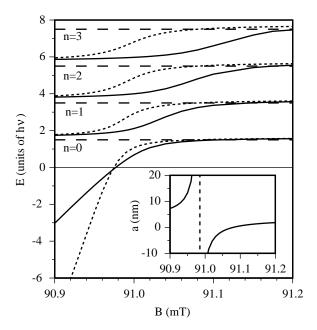


FIG. 41 The energy in the relative motion of two trapped interacting Na atoms in their lowest hyperfine state $|a\rangle$, as a function of magnetic field. The trapping frequency $\nu=\omega/2\pi$ is 1 MHz. The full lines correspond to energies obtained from exact numerical calculations. The dotted lines correspond to eigenenergies for trapped Na atoms interacting via a regularized delta-function potential with a magnetic field dependent scattering length given by the inset. This inset shows the exact scattering length for two freely scattering $|a\rangle$ states near a Feshbach resonance. In the theoretical model $B_0=90.985$ mT, whereas experimentally $B_0=90.7$ mT; see Section III.B.2. The long-dashed lines correspond to energies of the $\ell=0$, n=0, 1, 2, and 3 harmonic oscillator states. From (Tiesinga et al., 2000).

the atom-atom interaction potential. Avoided crossings between the closed channel Feshbach state and the trap levels are clearly visible.

An analytic solution for two interacting atoms trapped in a harmonic trap was found by (Busch et al., 1998). The atom-atom interaction was modeled by a so-called regularized delta-function potential, $4\pi\hbar^2/(2\mu)a\delta(\vec{R})(d/dR)R$, where a is the scattering length. This approximation of the actual interaction potential is valid in the Wigner threshold regime. In particular, they showed that the eigenenergies for the relative motion are given by

$$\frac{a}{\sigma} = \frac{\Gamma(-E/2 + 1/4)}{2\Gamma(-E/2 + 3/4)},\tag{66}$$

where Γ is the Gamma function, the energy E is in units of $\hbar\omega$, and $\sigma = \sqrt{\hbar/(\mu\omega)}$ is the harmonic oscillator length for the relative motion.

Near a magnetic Feshbach resonance the scattering length is a rapidly-changing function of B, as for example shown in the inset of Fig. 41. The dotted lines in Fig. 41 are the eigenenergies found from combining Eq. (66) and the a(B) for the resonance. The exact

and model calculations do not agree where the scattering length is large. As shown by (Blume and Greene, 2002; Bolda et al., 2003) this is due to the breakdown of the Wigner threshold regime at the finite zero-point energy of the atoms near the resonance. (Blume and Greene, 2002) introduced an energy-dependent scattering length, based on the effective range theory. (Bolda et al., 2002) found that an energy-dependent regularized delta-function potential reproduces well the exact results in Fig. 41.

(Dickerscheid et al., 2005; Gubbels et al., 2006) developed an analytic approach extending the theory of (Busch et al., 1998) to the states of two trapped atoms in a single lattice site that interact strongly through a Feshbach resonance. They applied their two-body theory to the broad ⁶Li resonance near 834 G. They also showed how to incorporate this theory into a many-body Hubbard model that treats the tunneling of atoms between lattice sites.

(Mies et al., 2000) have theoretically shown that by varying the magnetic field in time two atoms in a single lattice site can be converted into a molecule with near 100% efficiency. The idea is to prepare the atoms in the lowest trap level at a magnetic field where the Feshbach state has a higher energy. In Fig. 41 this corresponds to the nominally n=0 state at for example B=912 G. The magnetic field is then varied. If the ramp is sufficiently slow the atom pair will be adiabatically converted into a molecule. (Mies et al., 2000) also showed that this process can be described by a Landau-Zener curve crossing model. A more recent derivation is given in (Julienne et al., 2004).

(Widera et al., 2004) used a magnetic Feshbach resonance to entangle two ⁸⁷Rb atoms in a site of an optical lattice. In a Ramsey-type interferometer a sequence of microwave pulses manipulate and control the superposition between the $|0\rangle \equiv |F=1,m_F=1\rangle$ and $|1\rangle \equiv |F=2,m_F=-1\rangle$ hyperfine states of each ⁸⁷Rb atom. Initially the atoms are in state $|0\rangle$. The population in the two hyperfine states after the pulses will depend on the atomatom interaction, which entangles the two atoms. By controling the scattering length by the magnetic field and the hold time between the pulses, (Widera et al., 2004) were able to create maximally-entangled Bell states.

(Stöferle et al., 2006) spectroscopically mapped the avoided crossing between the Feshbach and the lowest harmonic oscillator state as a function a magnetic field. They prepared fermionic ⁴⁰K atoms in a three-dimensional optical lattice configured such that the bottom of the lattice sites are spherically symmetric. The two atoms in each lattice site are in different hyperfine states. They confirm that the model of trapped atoms interacting via an energy-dependent delta-function potential agrees with the experimental observations.

(Thalhammer et al., 2006) showed in an experiment with $^{87}\mathrm{Rb}$ in its lowest hyperfine state that the Landau-Zener model for a time-dependent sweep of the magnetic field through the Feshbach resonance, as developed by (Julienne et al., 2004; Mies et al., 2000), is valid. The

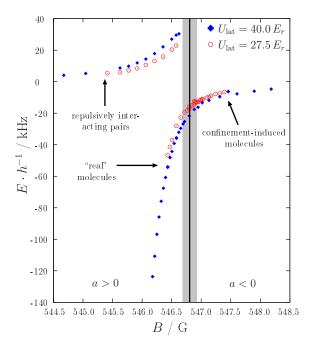


FIG. 42 Energy of heteronuclear 40 K - 87 Rb molecules versus B in an optical lattice for two different lattice depths $U_{\rm lat}$ in units of the 87 Rb recoil energy $E_r = \hbar^2 k^2/2m_{\rm Rb}$ ($E_r/h \approx 3.7$ kHz). The center of the Feshbach resonance is located at $B_0 = 546.8(1)$ G. The zero of energy corresponds to two atoms in the lattice cell with B far from B_0 . Molecules which are stable in free space are observed for $B < B_0$ where a > 0. Confinement-induced molecules are observed for $B > B_0$ where a < 0 and no threshold bound state exists in free space. In addition, repulsively interacting pairs are observed for $B < B_0$. From (Ospelkaus et al., 2006a).

Feshbach resonance near 1007 G was used. In the experiment the ramp speed dB/dt was varied over four orders of magnitude and for the slowest ramp speed of 2×10^3 G/s a 95% conversion efficiency was observed. The experiment demonstrated a dramatic increase in the lifetime of the trapped molecules, where the lattice protected them from harmful collisions. Molecular lifetimes up to 700 ms were observed.

(Ospelkaus et al., 2006a) were able to make heteronuclear molecules by associating atoms of two different species, 40 K and 87 Rb, trapped on the same lattice site. They used a rf association technique both to form the molecule and to measure its binding energy. Figure 42 shows the energy of the near-threshold states of the atom pair as a function of B and illustrates an avoided crossing similar to that shown in Fig. 41. (Deuretzbacher et al., 2008) developed a theoretical model to account for anharmonic corrections, which couple center-of-mass and relative motion of the atoms in the trap.

2. Reduced dimensional scattering

Optical lattices that confine atoms in only one or two directions in combination with magnetic Feshbach resonances lead to controllable quasi-2D or quasi-1D scattering, respectively. (Yurovsky et al., 2008) have recently reviewed such reduced dimensional scattering. By integrating out the confined spatial direction, effective one- and two-dimensional atom-atom potentials can be derived. Their strength is related to the magnetic-field dependent scattering length for free scattering.

(Bergeman et al., 2003; Olshanii, 1998) derived the effective atom-atom potential for quasi one-dimensional scattering. As in (Busch et al., 1998) the starting point is a regularized three-dimensional delta-function potential for the atom-atom interaction potential. The trapping potential along the two confined dimensions is the same and harmonic with frequency ω_{\perp} . They find that for an atom pair in the lowest harmonic oscillator state of the confined directions the atoms interact via a one-dimensional delta-function potential $g_{1D}\delta(z)$, with coupling constant

$$g_{\rm 1D} = 2 \frac{\hbar^2}{\mu} \frac{a}{\sigma_{\perp}^2} \frac{1}{1 - Ca/\sigma_{\perp}} \,,$$

where $C=1.4602\ldots$ and $\sigma_{\perp}=\sqrt{\hbar/(\mu\omega_{\perp})}$. The coupling constant is singular when $a=\sigma_{\perp}/C$ and approaches the negative finite value $g_{\infty}=-2\hbar^2/(\mu C\sigma_{\perp})$ for $a\to\pm\infty$. (Olshanii, 1998) has called the singularity a confinement-induced resonance. In practice, the resonance condition can be fulfilled by changing a with a magnetic Feshbach resonance. For fermionic atoms in quasi one-dimensional confinement the effective atomatom potential has been derived by (Granger and Blume, 2004).

(Petrov et al., 2000; Petrov and Shlyapnikov, 2001) derived a similar coupling constant for one-dimensional confinement or two-dimensional scattering. In this case the resonance location not only depends on a but also logarithmically on the relative wavenumber between the atoms along the two free spatial directions. (Naidon et al., 2007) describe how these reduced dimensional treatments can be extended to much tighter confinements than previously thought and made more accurate by using the energy-dependent scattering length of (Blume and Greene, 2002) with the effective range expansion for the scattering phase.

(Moritz et al., 2005) presented experimental evidence for a confinement-induced bound state in a one-dimensional system. They confirmed the existence of the bound state of the one-dimensional Hamiltonian $H_{1\mathrm{D}}=-\hbar^2/(2\mu)d^2/dz^2+g_{1\mathrm{D}}\delta(z)$ by changing both a and ω_\perp . The experiment was performed by employing an array of 1D tubes, each containing about 300 $^{40}\mathrm{K}$ atoms, equally divided between the $|f=9/2, m=-9/2\rangle$ and $|f=9/2, m=-7/2\rangle$ hyperfine state, were held in a two-dimensional harmonic trap with frequency $\omega_\perp/(2\pi)=69$ kHz. The scattering length for the collision between these

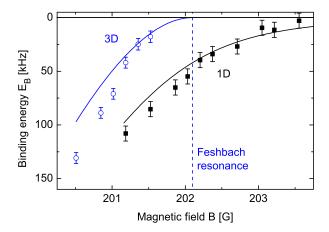


FIG. 43 Measured molecular bound state binding energy versus magnetic field B near the 202.1-G resonance between the two lowest spin states of 40 K. The upper curve labeled 3D is for a free space gas, whereas the lower curve labeled 1D is for a quasi-1D gas with tight confinement in two dimensions. The points show measured binding energies and the lines show theoretical predictions. From (Moritz $et\ al.$, 2005).

two hyperfine state was varied using the magnetic Feshbach resonance at $B_0=202~\mathrm{G}$. Figure 43 shows that the measured energy of the tightly confined atom pair varies as predicted by theory. Confinement-induced molecules exist in reduced dimension for $B>B_0$, where they do not exist in free space. (Dickerscheid and Stoof, 2005) develop an analytical nonperturbative two-channel theory of the binding energy that is in excellent agreement with the data.

In (Günter et al., 2005) fermionic 40 K atoms were prepared in a single hyperfine state and held in either a one- or two-dimensional optical lattice. By virtue of Fermi statistics the atoms can only collide via odd partial waves, which for ultracold collision energies have very small cross sections. One might expect the atomic gas to be a noninteracting Fermi gas. Nevertheless, (Günter et al., 2005) could observe a Feshbach resonance in the p-wave collision by the losses it induced. The losses were even sensitive to the orientation of the magnetic field relative to the principal axis of the trap. Moreover, they showed that the magnetic-field location of the resonance is modified by the confinement.

(Nygaard et al., 2008a) investigate the effect of tuning a narrow Feshbach resonance across the Bloch band of a one-dimensional optical lattice for the case when the resonance width is small compared to the width of the band, such as the 414 G ⁸⁷Rb resonance studied by (Syassen et al., 2007). They investigate the changes in scattering and bound states due to the band structure in the periodic structure and characterize the time-dependent dynamics of sweeping the resonance across the band. (Nygaard et al., 2008b) extend this work, develop the concept of a generalized scattering length at the band edges, and show the existence of a "universal" bound

state near the top and bottom band edges at the field strength where the resonance emerges from the band.

3. Scattering in shallow lattices

The previous two Sections VI.B.1 and VI.B.2 discussed deep optical lattices where the tunneling between lattices sites could safely be neglected. For weaker optical lattices, atoms tunnel from site to site and then interact with all other atoms. This leads to many-body systems that can be described by either a mean-field Gross-Pitaevskii equation or a Bose- or Fermi-Hubbard Hamiltonian (Bloch, 2005). The presence of Feshbach resonances has added and continues to add new twists to these kinds of Hamiltonians.

We will discuss the more simple situation where only two atoms scatter in an optical lattice. (Fedichev et al., 2004) studied the case of two atoms scattering in a weak three-dimensional optical lattice of cubic symmetry and interacting with the regularized delta-function potential. They predict the presence of a geometrical resonance in the 3D lattice based on a derivation of an effective atomatom interaction between the atoms, which by virtue of the periodic potential have an effective mass m^* that is much larger than their atomic mass m. Figure 44 shows the results of their calculation. The resonance occurs at a scattering length $a = l^* \equiv -(\pi/(2 \ln 2))(m/m^*)(\sigma/d)^2 \sigma$, where σ is the atomic harmonic oscillator length for motion in a single lattice site and d is the lattice period. Note that, in practice, $l^* \ll \sigma$. (Orso et al., 2005) presented a similar analysis for a one-dimensional optical lattice. (Grupp et al., 2007) studied the effect of a very-narrow Feshbach resonance in scattering in an onedimensional lattice.

C. Efimov states and universal few-body physics

Feshbach resonances provide experimental access to systems with very large values of the scattering length. Such systems are governed by "universal" physics, i.e. their low-energy observables are independent of details of the interaction (Braaten and Hammer, 2006). Universality appears as a consequence of the quantum-halo character of the wave function carrying its dominant part far out of the classically allowed region. In this case, details of the interaction potential become irrelevant and the system can be described by a few global parameters. Halo dimers (Sec. V.B.2) are the most simple example. For addressing universal physics with ultracold gases, Feshbach resonances that are strongly entrancechannel dominated $(s_{res} \gg 1)$ are of particular interest, as they allow a description in terms of a single-channel model with a large range of universal behavior; see discussion in Sections II.B.2 and II.C.5.

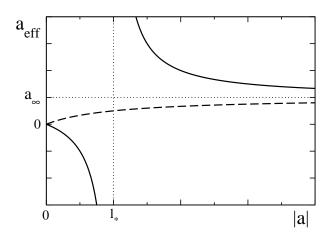


FIG. 44 The dependence of the effective scattering length $a_{\rm eff}$ on the absolute value of the microscopic scattering length |a|. The dashed line corresponds to a repulsive interaction potential (a>0). The solid line corresponds to an attractive potential (a<0) with a geometric resonance at $|a|=l_*$. From (Fedichev *et al.*, 2004).

1. Efimov's scenario

Efimov quantum states in a system of three identical bosons (Efimov, 1970, 1971) are a paradigm for universal few-body physics. These states have attracted considerable interest, fueled by their bizarre and counter-intuitive properties and by the fact that they had been elusive to experimentalists for more than 35 years. In 2006, (Kraemer et al., 2006) reported on experimental evidence for Efimov states in an ultracold gas of cesium atoms. By Feshbach tuning they could identify a pronounced three-body resonance, which occurs as a fingerprint of an Efimov state at the three-body scattering threshold. Two years later, (Knoop et al., 2009) presented additional evidence for Efimov-like trimer states, reporting on the observation of a decay resonance in atom-dimer scattering.

Efimov's scenario is illustrated in Fig. 45, showing the energy spectrum of the three-body system as a function of the inverse scattering length 1/a. For a < 0, the natural zero of energy is the three-body dissociation threshold for three atoms at rest. States below are trimer states and states above are continuum states of three free atoms. For a > 0, the dissociation threshold is given by $-E_b = -\hbar^2/(ma^2)$ where E_b is the universal binding energy of the weakly bound halo dimer; at this threshold a trimer dissociates into a dimer and an atom. All states below threshold are necessarily three-body bound states. Efimov predicted that in the limit $a \to \pm \infty$ there would be an infinite sequence of weakly bound trimer states with a universal scaling behavior. Each successive Efimov state is larger in size by a universal scaling factor $e^{\pi/s_0} \approx 22.7 \ (s_0 = 1.00624)$ and has a weaker binding

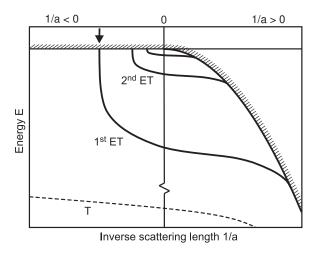


FIG. 45 Efimov's scenario: Appearance of an infinite series of weakly bound Efimov trimer states (ET) for resonant two-body interaction. The binding energy is plotted as a function of the inverse two-body scattering length 1/a. The shaded region indicates the scattering continuum for three atoms (a < 0) and for an atom and a dimer (a > 0). The arrow marks the intersection of the first Efimov trimer with the three-atom threshold. To illustrate the series of Efimov states, the universal scaling factor is artificially reduced from 22.7 to 2. For comparison, the dashed line indicates a more tightly bound non-Efimov trimer (T) which does not cross the scattering continuum. Adapted by permission from Macmillan Publishers Ltd: Nature (Kraemer et al., 2006), copyright 2006.

energy by a factor of $(22.7)^2 \approx 515$.

Efimov states exist on both sides of a resonance, and Fig. 45 shows the adiabatic connection between both sides. For a>0, an Efimov state near the atom-dimer dissociation threshold can be regarded as a weakly bound state of an atom and a dimer with a size set not by a but by the even larger atom-dimer scattering length (Braaten and Hammer, 2006). For a<0, Efimov states are "Borromean" states (Jensen $et\ al.$, 2004), which means that a weakly bound three-body state exists in the absence of a weakly bound two-body state. This property that three quantum objects stay together without pairwise binding is part of the bizarre nature of Efimov states.

Resonant scattering phenomena arise as a natural consequence of this scenario (Efimov, 1979), and they are closely related to the basic idea of a Feshbach resonance. When an Efimov state intersects with the continuum threshold for a < 0, three free atoms resonantly couple to a trimer. This results in a "triatomic Efimov resonance". When an Efimov state intersects with the atom-dimer threshold for a > 0, the result is an "atom-dimer Efimov resonance" (Nielsen et al., 2002).

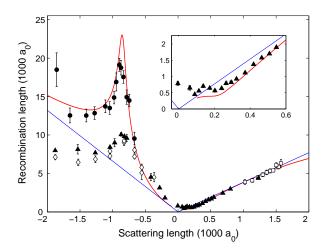


FIG. 46 Observation of an Efimov resonance in three-body decay of an ultracold gas of cesium atoms. The data are presented in terms of a recombination length $\rho_3 = [2m/(\sqrt{3}\hbar) L_3]^{1/4}$ (Esry et al., 1999). The general a^4 -scaling of L_3 corresponds to a linear behavior in $\rho_3(a)$ (straight lines). The filled circles represent measurements taken at temperatures around 10 nK, whereas the filled triangles and open diamonds refer to measurements in the range of 200 nK - 250 nK. The solid line is a fit to the low-temperature data based on effective-field theory (Braaten and Hammer, 2006). The inset shows an expanded view of the region of positive scattering lengths up to 600 a_0 . Reprinted by permission from Macmillan Publishers Ltd: Nature (Kraemer et al., 2006), copyright 2006.

2. Observations in ultracold cesium

In an ultracold atomic gas with resonant interactions, Efimov physics manifests itself in three-body decay properties (Bedaque et al., 2000; Braaten and Hammer, 2001, 2006; Esry et al., 1999; Nielsen and Macek, 1999). The three-body loss coefficient L_3 (Sec. III.A.2) can be conveniently expressed in the form $L_3 = 3 C(a) \hbar a^4/m$, which separates an overall a^4 -scaling from an additional dependence C(a). Efimov physics is reflected in a logarithmically periodic behavior C(22.7a) = C(a), corresponding to the scaling of the infinite series of weakly bound trimer states. A triatomic Efimov resonance leads to giant recombination loss (Braaten and Hammer, 2001; Esry et al., 1999), as the resonant coupling of three atoms to an Efimov state opens up fast decay channels into deeply bound dimer states plus a free atom.

(Kraemer et al., 2006) observed a triatomic Efimov resonance in an ultracold, thermal gas of Cs atoms. They made use of the strong variation in the low-field region (Fig. 22). This tunability results from a strongly entrance-channel dominated resonance at $-12\,\mathrm{G}$ with $s_{\mathrm{res}}=566$ (Table IV), which provides a broad range of universal behavior. By applying magnetic fields between 0 G and 150 G, (Kraemer et al., 2006) varied the s-wave scattering length a between $-2500\,a_0$ and $1600\,a_0$, large enough to study the universal regime, which re-

quires $|a| \gg R_{\rm vdw} \simeq 100\,a_0$. The occurrence of one triatomic Efimov resonance could be expected in the accessible negative-a region. The position, however, could not be predicted from knowledge of the scattering length alone as, for a three-body process, a second parameter is required to characterize the universal properties (Braaten and Hammer, 2006).

Figure 46 shows the results of (Kraemer et al., 2006). The three-body loss resonance was found at a magnetic field of $7.5 \,\mathrm{G}$, corresponding to a scattering length of $-850 \,a_0$. The behavior of loss at temperatures around $10 \,\mathrm{nK}$ closely resembles the theoretical predictions of (Esry et al., 1999), who numerically solved the three-body Schrödinger equation for a generic two-body model potential. The observed behavior is also well fit with a universal analytic expression obtained in the framework of effective-field theory (Braaten and Hammer, 2006). Experimental data taken at higher temperatures demonstrated the unitarity limitation of three-body loss (D'Incao et al., 2004) and showed how the Efimov resonance evolved into a triatomic continuum resonance (Bringas et al., 2004).

For positive scattering lengths, theory predicts a variation of C(a) between very small values and a maximum of about 70 (Bedaque et al., 2000; Esry et al., 1999; Nielsen and Macek, 1999). The results in Fig. 46 are consistent with the upper loss limit, represented by the straight line for a > 0. For a below $600 a_0$, the experimentally determined recombination length significantly drops below this limit, as seen in the inset of the figure. Further measurements of (Kraemer et al., 2006) revealed the existence of a loss minimum at B = 21 G, where $a = +210 a_0$. It is interesting to note that earlier experiments by the same group (Kraemer et al., 2004) had identified 21 G as an optimum magnetic field for evaporative cooling of cesium and attainment of BEC; see Sec. IV.A.1. The nature of the minimum may be interpreted in the framework of universal physics, following theoretical predictions of an interference effect between two different recombination pathways (Esry et al., 1999; Nielsen and Macek, 1999). However, as the minimum occurs at a scattering length which is only a factor of two larger than $R_{\rm vdw} \simeq 100 \, a_0$ (Table I) the application of universal theory to describe this feature is questionable. (Massignan and Stoof, 2008) presented an alternative theoretical approach, which reproduced both this minimum and the maximum observed for negative a on the basis of the two-body physics of the particular Feshbach resonance.

In a pure sample of trapped atoms, as discussed so far, three-body recombination is the only probe for Efimov physics. Mixtures of atoms and dimers can provide complementary information on Efimov states through resonances in inelastic atom-dimer collisions (Braaten and Hammer, 2007; Nielsen et al., 2002). In a recent experiment, (Knoop et al., 2009) prepared an optically trapped mixture of Cs atoms and Cs₂ halo dimers. Their measurements revealed an atom-dimer scattering

resonance, which is centered at a large value of the two-body scattering length, $a \simeq +390\,a_0$ at a magnetic field of 25 G. This observation provides strong evidence for a trimer state approaching the atom-dimer threshold. The situation is close to the atom-dimer resonance in Efimov's scenario, but it probably remains a semantic question whether, at $a \simeq 4R_{\rm vdw}$, the underlying trimer state may be called an Efimov state.

In the Cs experiments described in this Section, the Efimov state that causes the observed triatomic resonance at 7.5 G does not connect to the state that causes the atom-dimer resonance at 25 G when the magnetic field is varied. This is because these two cases are separated by a zero crossing in the scattering length (Fig. 22) and not by the pole as in Efimov's scenario in Fig. 45. A universal relation between these regimes may nevertheless exist (Kraemer et al., 2006). (Lee et al., 2007) provided a further interpretation of these observations in terms of the underlying Cs₃ states and point out the analogies to trimer states of helium (Schöllkopf and Toennies, 1994).

3. Prospects in few-body physics

Ultracold gases with resonantly tuned interactions offer many opportunities to study universal, Efimov-related few-body physics. Cesium alone has much more to offer than the experiments could explore so far. Moreover, several other systems with broad Feshbach resonances promise new insight into this field.

In cesium, a predicted broad Feshbach resonance near 800 G (Lee et al., 2007) in the aa channel offers similar properties as the low-field region explored in previous experiments, but overcomes the disadvantage that only the tail of the resonance is accessible at low fields. The broad 155-G resonance in ⁸⁵Rb (Sec. III.B.4) might be another interesting candidate, but experiments may suffer from strong two-body decay which is absent for the discussed cesium resonances. A further interesting candidate is the 402-G resonance in ³⁹K (Sec. III.B.3); (Zaccanti et al., 2008) have studied three-body decay near this resonance and found features strongly indicative of Efimov physics.

Many more opportunities for studying Efimov-related physics in ultracold gases with resonant interactions are offered by mixtures of different spin states or different species. In ⁶Li, all three combinations of the lowest three spin states (channels ab, ac, and bc) have broad Feshbach resonances (Bartenstein et al., 2005) that overlap in a magnetic-field range between 650 G and 850 G. For such a three-component fermionic spin mixture (Luu and Schwenk, 2007) predicted a novel Borromean three-body state. (Ottenstein et al., 2008) and (Huckans et al., 2009) experimentally investigated the stability of a ⁶Li three-component spin mixture and found evidence for a three-body resonance at 130 G. Its interpretation in terms of coupling to a three-body bound state is supported by several theoretical

studies (Braaten et al., 2008; Naidon and Ueda, 2008; Schmidt et al., 2008). A particularly interesting situation arises in mixtures of atoms with different masses. With increasing mass ratio the Efimov factor substantially decreases from its value of 22.7 at equal masses to values as low as 4.9 for the mass ratio of ¹³³Cs combined with ⁶Li. (D'Incao and Esry, 2006) pointed out that this will substantially enhance the observability of the Efimov effect in terms of the logarithmically periodic variation of the three-body loss coefficient with increasing two-body scattering length.

Four-body processes at large values of the s-wave scattering length a represent a logical next step in understanding universal few-body physics. Theoretical studies (Hammer and Platter, 2007; von Stecher et~al., 2009; Wang and Esry, 2009) predict the existence of universal four-body states, and consider the process of atomic four-body recombination. A first experimental step into this field was made by (Ferlaino et~al., 2008), who studied collisions of Cs₂ halo dimers at large positive a. They observed a loss minimum in the same region where atom-dimer scattering shows a maximum ($a \simeq 500~a_0$ at 30 G), which may be related to a universal connection between four- and three-body physics (Hammer and Platter, 2007; von Stecher et~al., 2009).

Optical lattices have proven a powerful tool for the manipulation of ultracold Feshbach molecules, see Sec. VI.B.1 and may also open up new possibilities for the creation of Efimov trimers and, more generally, for the controlled production of few-body quantum states (Luu and Schwenk, 2007; Stoll and Köhler, 2005).

D. Molecular resonances and cold chemistry

While we have reviewed the formation of cold molecules from ultracold atoms, parallel progress has been made by other techniques for preparing samples of cold molecules that extend the range far beyond alkalimetal species. These advances have been made possible by Stark deceleration of molecules such as ND₃, OH, and formaldehyde (van de Meerakker et al., 2006) or by buffer gas cooling with liquid helium (DeCarvalho et al., 1999). In contrast to the association of cold atoms to make Feshbach molecules, which have a high level of vibrational excitation, these methods can produce cold molecules in the rotational and vibrational ground state. See (Doyle et al., 2004; Hutson and Soldán, 2007; Krems, 2005) for overviews of the issues involved in trapping, cooling, and colliding such molecules.

Resonances will play a very prominent role in atommolecule and molecule-molecule collisions. The complexity of these systems will increase the number of closed channels and lead to numerous resonances with diverse properties. Their presence will make it possible to change molecular scattering properties as well as to create more complex molecules. Both static magnetic and electric fields can be used to tune the molecule-molecule resonance and provide control over collisions, as many molecules not only have a magnetic moment but an electric dipole moment as well. Here we will briefly review some of this work.

(Forrey et al., 1998) pointed out that Feshbach resonances occur in ultracold atom-diatom scattering, giving an example from collisions of H₂ with He. The effect of resonance states in chemical reactions has been studied by (Balakrishnan and Dalgarno, 2001) for F+H₂→FH+H and (Weck and Balakrishnan, 2005) for Li+HF→H+LiF reactions. Recent coupled channels models of Rb+OH (Lara et al., 2007) and He+NH (González-Martinez and Hutson, 2007) collisions have been developed to give a realistic assessment of atom-molecule scattering. The latter study demonstrated the effect of magnetic tuning of a decaying resonance across a threshold, using the resonance length formalism described in Sec. II.A.3.

In 2002 (Bohn et al., 2002) realized that, unlike for atomic systems where Feshbach resonances originate from the hyperfine structure of the atoms, for molecules the resonances can also be due to rotational states. For many molecules the rotational spacing for low-lying rotational levels is of the same order of magnitude as hyperfine interactions in atoms. Based on the rotational splittings and a potential energy surface (Bohn et al., 2002) estimated the mean spacing and widths of the resonances and found for collisions between oxygen molecules as many as 30 resonances for collision energies below $E/k_B=1~{\rm K}.$

(Chin et al., 2005) observed magnetic Feshbach-like resonances between two weakly bound $^{133}\mathrm{Cs}_2$ molecules that temporarily form a tetramer during a collision. Their data are shown in Fig. 47. The $^{133}\mathrm{Cs}_2$ molecules in this experiment are bound by no more than $E/h{=}5$ MHz and have a temperature and peak density of 250 nK and 5×10^{10} cm $^{-3}$, respectively. As the magnetic field is varied near B =13 G the lifetime of the molecules rapidly changes, indicating two resonances.

Heteronuclear molecules can be manipulated by static electric fields in addition to magnetic fields. The electric field Stark shifts the rotational levels of the molecule. These level shifts can then give rise to electric Feshbach resonances (Avdeenkov and Bohn, 2002). In atoms the levels can also be sufficiently Stark shifted to induce collisional resonances but rather large fields are required (Marinescu and You, 1998).

Figure 48 shows the results of a calculation on reaction rate coefficients of formaldehyde $\rm H_2CO$ reacting with OH to yield HCO and $\rm H_2O$ (Hudson et al., 2006). Both $\rm H_2CO$ and OH are in their lowest vibrational state of their ground electronic configuration. Multiple resonances occur for electric fields up to 2 kV/cm. More recently, (Tscherbul and Krems, 2008) have studied reaction rates of LiF with H in the presence of electric fields.

We can expect atom-molecule and molecule-molecule collisions to exhibit a rich variety of resonance phenom-

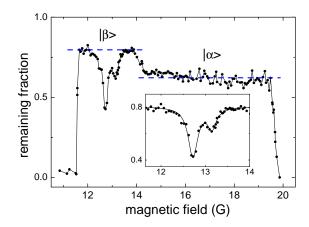


FIG. 47 Observation of Feshbach-like resonances in the collisions of $^{133}\mathrm{Cs}_2$ molecules. The figure shows the number of remaining $^{133}\mathrm{Cs}$ molecules after fixed storage time in an optical trap as a function of external magnetic field. The two features near 13 G shown magnified in the inset have been interpreted as tetramer resonances. From (Chin *et al.*, 2005).

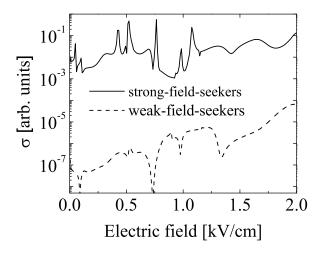


FIG. 48 Calculated chemical reaction cross section of ${\rm H_2CO} + {\rm OH} \rightarrow {\rm HCO} + {\rm H_2O}$ at a collision energy of $E/k_B = 1$ mK as a function of applied electric field. Strong-/weak-field seekers correspond to states of ${\rm H_2CO}$ and OH that can be held in an electric trap where the center of the trap has the larger/smallest electric field strength, respectively. Figure and model are from (Hudson et al., 2006).

ena in elastic, inelastic, and reactive collisions. Such phenomena are likely to become progressively more important to understand as sources of trap loss and for coherent control of molecular ensembles.

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Over the past decade, many people have contributed to advancing our knowledge on Feshbach resonances in ultracold gases. For valuable discussions and insights related to this exciting field, we particularly acknowledge B. Esry, F. Ferlaino, C. Greene, J. Hecker Denschlag, J. Hutson, S. Knoop, H.-C. Nägerl, W. Phillips, F. Schreck, and G. Shlyapnikov. We thank E. Braaten, S. Dürr, T. Esslinger, M. Gustavsson, L. Khaykovich, H. Moritz, W. Ketterle, S. Kokkelmans, T. Pfau, H. Stoof, B. Verhaar, J. Walraven, M. Zaccanti, C. Zimmermann, and in particular I. Spielman and S. Jochim for helpful comments on the manuscript.

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APPENDIX: Tables of selected resonances

Table IV lists positions and properties of resonances for various species. The data is a combination of experimentally determined as well as theoretically derived values. Most of the magnetic field locations are experimentally determined. Most of the widths Δ and background scattering lengths $a_{\rm bg}$ are determined from theoretical calculations. Where unavailable from the literature we calculated values based on the best Born-Oppenheimer potentials obtained from the literature. For a complete list as well as error bars on the resonance locations the reader is referred to the literature. The notation defined in Secs. II.A.3 and II.B.4 has been used in the table.

The table shows a richness in the kinds of resonances available for magnetic field values that are relatively easily created in laboratories. Some of the resonances are very narrow with Δ on the order of a mG. Others are very broad with Δ larger than 100 G. The background scattering length can be either negative or positive, its absolute value ranging from a few tens to several thousands Bohr radii. The magnetic moment of the resonance state is always on the order of the Bohr magneton, which reflects the form of the Zeeman interaction. The partial wave of the resonance states ranges from zero to four ($\ell_c = 0...4$). Finally, the resonances are characterized in terms of their background scattering lengths $a_{\rm bg}$, their strengths $s_{\rm res}$ (Sec. II.B.2), and the parameter ζ (Sec. II.C.5).

For atomic cesium, a resonance location is given with a negative magnetic field value. This is not an experimental value. Here B_0 is determined from a fit of Eq. (1) to the slowly varying scattering length as shown in Fig. 22. (Vogels et al., 1998) give the physical interpretation of a negative B_0 , namely, taking B < 0 corresponds to the case for B > 0 with the spin projections of each atom reversed in sign. For the case of cesium, a negative magnetic field in the aa channel corresponds to a positive field in the gg channel.

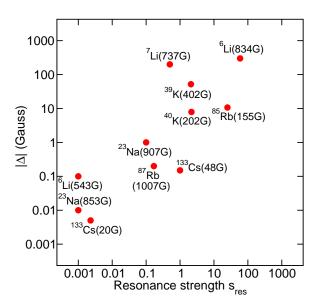


FIG. 49 Overview of selected Feshbach resonances in terms of widths Δ and strengths $s_{\rm res}$.

Figure 49 illustrates the rich variety of Feshbach resonances in terms of their widths Δ and strengths $s_{\rm res}$. Both parameters change over six orders of magnitude. Resonances with $\Delta > 1\,\rm G$ tend to be entrance-channel dominated ($s_{\rm res} > 1$). A notable exception is the ⁷Li 737-G resonance mentioned in Sec. II.B.5

References

van Abeelen, F. A., D. J. Heinzen, and B. J. Verhaar, 1998, "Photoassociation as a probe of Feshbach resonances in cold-atom scattering," Phys. Rev. A 57, R4102–R4105.

van Abeelen, F. A., and B. J. Verhaar, 1999a, "Determination of collisional properties of cold Na atoms from analysis of bound-state photoassociation and Feshbach resonance field data," Phys. Rev. A **59**, 578–584.

van Abeelen, F. A., and B. J. Verhaar, 1999b, "Time-dependent Feshbach resonance scattering and anomalous decay of a Na Bose-Einstein condensate," Phys. Rev. Lett. 83, 1550–1553.

Abo-Shaeer, J. R., D. E. Miller, J. K. Chin, K. Xu, T. Mukaiyama, and W. Ketterle, 2005, "Coherent molecular optics using ultracold sodium dimers," Phys. Rev. Lett. 94, 040405.

Abraham, E. R. I., W. I. McAlexander, J. M. Gerton, R. G. Hulet, R. Côté, and A. Dalgarno, 1997, "Triplet s-wave resonance in ⁶Li collisions and scattering lengths of ⁶Li and ⁷Li," Phys. Rev. A **55**, R3299–R3002.

Altmeyer, A., S. Riedl, C. Kohstall, M. Wright, R. Geursen, M. Bartenstein, C. Chin, J. Hecker Denschlag, and R. Grimm, 2007, "Precision measurements of collective modes in the BEC-BCS crossover," Phys. Rev. Lett. 98, 040401.

Anderson, M. H., J. R. Ensher, M. R. Matthews, C. E. Wieman, and E. A. Cornell, 1995, "Observation of Bose-

TABLE IV Properties of selected Feshbach resonances. The first column describes the atomic species and isotope. The next three columns characterize the scattering and resonance state, which includes the incoming scattering channel (ch.), partial wave ℓ , and the angular momentum of the resonance state $\ell_{\rm c}$. This is followed by the resonance location B_0 , the width Δ , the background scattering length $a_{\rm bg}$, the differential magnetic moment $\delta\mu$, the dimensionless resonance strength $s_{\rm res}$, the background scattering length in van der Waals units $r_{\rm bg} = a_{\rm bg}/\bar{a}$, and the bound state parameter ζ from Eq. (52). Here a_0 is the Bohr radius and μ_B is the Bohr magneton. Definitions are given in Sec. II. The last column gives the source. A string "na" indicates that the corresponding property is not defined. For example $a_{\rm bg}$ is not defined for p-wave scattering.

| | 1 | 0 | 0 | D (C) | A (CI) | | 5 / | | | | C |
|--|-----------------|--------|---------------|----------|-------------|------------------|-------|--------|--------------|--------|--|
| atom | ch. | ℓ | | $B_0(G)$ | $\Delta(G)$ | $a_{\rm bg}/a_0$ | | Sres | $r_{\rm bg}$ | 1.400 | reference |
| ⁶ Li | ab | s | s | 834.1 | -300 | -1405 | 2.0 | 59 | -47 | 1400 | (Bartenstein et al., 2005) |
| | ac | s | s | 690.4 | -122.3 | -1727 | 2.0 | 29 | -58 | 850 | (Bartenstein et al., 2005) |
| | bc | s | s | 811.2 | -222.3 | -1490 | 2.0 | 46 | -50 | 1200 | (Bartenstein et al., 2005) |
| | ab | s | s | 543.25 | 0.1 | 60 | 2.0 | 0.001 | 2.0 | 0.001 | (Strecker <i>et al.</i> , 2003) |
| | aa | p | p | 159.14 | na | na | 2.0 | na | na | na | (Schunck et al., 2005; Zhang et al., 2004) |
| | ab | p | p | 185.09 | na | na | 2.0 | na | na | na | (Schunck et al., 2005; Zhang et al., 2004) |
| 7 | bb | p | p | 214.94 | na | na | 2.0 | na | na | na | (Schunck et al., 2005; Zhang et al., 2004) |
| ⁷ Li | aa | s | s | 736.8 | -192.3 | -25 | 2.0 | 0.80 | -0.79 | 0.31 | (Pollack et al., 2009; Strecker et al., 2002) ^a |
| ²³ Na | cc | s | s | 1195 | 1.4 | 62 | -0.15 | 0.0050 | 1.4 | 0.004 | (Inouye <i>et al.</i> , 1998; Stenger <i>et al.</i> , 1999) ^a |
| | aa | s | s | 907 | 1 | 63 | 3.8 | 0.09 | 1.5 | 0.07 | (Inouye et al., 1998; Stenger et al., 1999) a |
| | aa | s | s | 853 | 0.0025 | 63 | 3.8 | 0.0002 | 1.5 | 0.0002 | (Inouye <i>et al.</i> , 1998; Stenger <i>et al.</i> , 1999) ^a |
| $^{39}{ m K}$ | aa | s | s | 402.4 | -52 | -29 | 1.5 | 2.1 | -0.47 | 0.49 | (D'Errico <i>et al.</i> , 2007) |
| ⁴⁰ K | bb | p | p | 198.4 | na | na | 0.134 | na | na | na | (Regal et al., 2003b; Ticknor et al., 2004) ^a |
| | bb | p | p | 198.8 | na | na | 0.134 | na | na | na | (Regal et al., 2003b; Ticknor et al., 2004) a |
| | ab | s | s | 202.1 | 8.0 | 174 | 1.68 | 2.2 | 2.8 | 3.1 | (Regal $et~al.,~2004$) ^a |
| | ac | s | s | 224.2 | 9.7 | 174 | 1.68 | 2.7 | 2.8 | 3.8 | (Regal and Jin, 2003) a |
| $^{85}\mathrm{Rb}$ | ee | s | s | 155.04 | 10.7 | -443 | -2.33 | 28 | -5.6 | 80 | (Claussen et al., 2003) |
| ⁸⁷ Rb | aa | s | s | 1007.4 | 0.21 | 100 | 2.79 | 0.13 | 1.27 | 0.08 | (Dürr et al., 2004b; Volz et al., 2003) a |
| | aa | s | s | 911.7 | 0.0013 | 100 | 2.71 | 0.001 | 1.27 | 0.0006 | (Marte $et~al.,~2002$) ^a |
| | aa | s | s | 685.4 | 0.006 | 100 | 1.34 | 0.006 | 1.27 | 0.004 | (Dürr et al., 2004b; Marte et al., 2002) a |
| | aa | s | s | 406.2 | 0.0004 | 100 | 2.01 | 0.0002 | 1.27 | 0.0001 | (Marte $et \ al., 2002$) ^a |
| | ae | s | s | 9.13 | 0.015 | 99.8 | 2.00 | 0.008 | 1.27 | 0.005 | (Widera et al., 2004) |
| ¹³³ Cs | aa | s | s | -11.7 | 28.7 | 1720 | 2.30 | 560 | 17.8 | 5030 | (Chin et al., 2004b; Lange et al., 2009) a |
| | aa | s | d | 47.97 | 0.12 | 926 | 1.21 | 0.67 | 9.60 | 3.2 | (Chin et al., 2004b; Lange et al., 2009) ^a |
| | aa | s | g | 19.84 | 0.005 | 160 | 0.57 | 0.002 | 1.66 | 0.002 | (Chin et al., 2004b) ^a |
| | aa | s | g | 53.5 | 0.0025 | 995 | 1.52 | 0.019 | 10.3 | 0.1 | (Chin et al., 2004b; Lange et al., 2009) ^a |
| | aa | s | s | 547 | 7.5 | 2500 | 1.79 | 170 | 26 | 2200 | a |
| | aa | s | s | 800 | 87.5 | 1940 | 1.75 | 1470 | 20 | 15000 | a |
| ⁵² Cr | aa | s | d | 589.1 | 1.7 | 105 | 2.00 | 0.31 | 2.45 | 0.38 | (Werner <i>et al.</i> , 2005) ^a |
| | aa | s | d | 499.9 | 0.08 | 107 | 4.00 | 0.03 | 2.49 | 0.04 | (Werner et al., 2005) a |
| ⁶ Li ²³ Na | aa | s | s | 746 | 0.44 | 14.0 | | | | | (Stan et al., 2004), (Gacesa et al., 2008) |
| | aa | s | s | 795.6 | 2.177 | 13.0 | | | | | (Stan et al., 2004), (Gacesa et al., 2008) |
| ⁶ Li ⁴⁰ K | aa | s | s | 157.6 | 0.25 | 10.0 | | | | | (Wille et al., 2008) |
| | aa | s | s | 168.2 | 0.15 | | | | | | (Wille et al., 2008) |
| ⁶ Li ⁸⁷ Rb | $\frac{aa}{aa}$ | p | $\frac{b}{p}$ | 882 | na | na | | na | na | na | (Deh et al., 2008; Li et al., 2008) |
| | | | _ | 1067 | 10.62 | 11a | | 11a | па | 11a | (Deh et al., 2008; Li et al., 2008) |
| $^7\mathrm{Li}^{87}\mathrm{Rb}$ | aa | s | s | 649 | -70 | -36 | | | | | (Marzok et al., 2009) |
| ³⁹ K ⁸⁷ Rb | aa | s | s | | | | 2.0 | 0.74 | 0.50 | 0.19 | |
| $^{40}\mathrm{K}^{87}\mathrm{Rb}$ | aa | s | s | 317.9 | 7.6 | 34 | 2.0 | | | 0.18 | (Simoni et al., 2008) |
| 41K ⁸⁷ Rb | aa | s | s | 546.9 | -3.10 | -189 | 2.30 | -1.96 | -2.75 | 2.70 | (Pashov et al., 2007; Simoni et al., 2008) |
| | aa | s | s | 39 | 37 | 284 | 1.65 | 25.8 | 4.11 | 53.0 | (Simoni et al., 2008; Thalhammer et al., 2008) |
| $\frac{^{41}\text{K}^{87}\text{Rb}}{^{85}\text{Rb}^{87}\text{Rb}}$ | aa | s | s | 79 | 1.2 | 284 | 1.59 | 0.81 | 4.11 | 1.66 | (Simoni et al., 2008; Thalhammer et al., 2008) |
| | ec | s | s | 265.4 | 5.8 | 213 | | | | | (Papp and Wieman, 2006) |
| 85 Rb 87 Rb | ec | s | s | 372 | 1 | 213 | | | | | (Papp and Wieman, 2006) |

^a Table entries partially based on unpublished calculations by the authors.

- Einstein condensation in dilute atomic vapor," Science 269, 198-201.
- Arimondo, E., W. D. Phillips, and F. Strumia (eds.), 1992, Laser Manipulation of Atoms and Ions (North Holland, Amsterdam), Proceedings of the International School of Physics "Enrico Fermi", Course CXVIII, Varenna, 9-19 July 1991.
- Arndt, M., M. B. Dahan, D. Guéry-Odelin, M. W. Reynolds, and J. Dalibard, 1997, "Observation of a zero-energy resonance in Cs-Cs collisions," Phys. Rev. Lett. 79, 625–628.
- Avdeenkov, A. V., and J. L. Bohn, 2002, "Collisional dynamics of ultracold OH molecules in an electrostatic field," Phys. Rev. A 66, 052718.
- Balakrishnan, N., and A. Dalgarno, 2001, "Chemistry at ultracold temperatures," Chem. Phys. Lett. **341**, 652–656.
- Balakrishnan, N., V. Kharchenko, R. Forrey, and A. Dalgarno, 1997, "Complex scattering lengths in multi-channel atom-molecule collisions," Chem. Phys. Lett. 280, 5–9.
- Bartenstein, M., A. Altmeyer, S. Riedl, R. Geursen, S. Jochim, C. Chin, J. Hecker Denschlag, R. Grimm, A. Simoni, E. Tiesinga, C. J. Williams, and P. S. Julienne, 2005, "Precise determination of ⁶Li cold collision parameters by radio-frequency spectroscopy on weakly bound molecules," Phys. Rev. Lett. 94, 103201.
- Bartenstein, M., A. Altmeyer, S. Riedl, S. Jochim, C. Chin, J. Hecker Denschlag, and R. Grimm, 2004a, "Collective excitations of a degenerate gas at the BEC-BCS crossover," Phys. Rev. Lett. **92**, 203201.
- Bartenstein, M., A. Altmeyer, S. Riedl, S. Jochim, C. Chin, J. Hecker Denschlag, and R. Grimm, 2004b, "Crossover from a molecular Bose-Einstein condensate to a degenerate Fermi gas," Phys. Rev. Lett. 92, 120401.
- Beaufils, Q., A. Crubellier, T. Zanon, B. Laburthe-Tolra, E. Maréchal, L. Vernac, and O. Gorceix, 2009, "Feshbach resonance in d-wave collisions," Phys. Rev. A 79, 032706.
- Bedaque, P. F., E. Braaten, and H.-W. Hammer, 2000, "Three-body recombination in bose gases with large scattering length," Phys. Rev. Lett. 85, 908–911.
- Bergeman, T., M. G. Moore, and M. Olshanii, 2003, "Atomatom scattering under cylindrical harmonic confinement: Numerical and analytic studies of the confinement induced resonance," Phys. Rev. Lett. 91, 163201.
- Bergmann, K., H. Theuer, and B. W. Shore, 1998, "Coherent population transfer among quantum states of atoms and molecules," Rev. Mod. Phys. **70**, 1003–1025.
- Bethe, H., 1935, "Theory of disintegration of nuclei by neutrons," Phys. Rev. 47, 747–759.
- Beutler, H., 1935, "Über Absorptionsserien von Argon, Krypton und Xenon zu Termen zwischen den beiden Ionisierungsgrenzen $^2P_3^{2/0}$ und $^2P_1^{2/0}$," Z. Phys. **93**, 177–196.
- Bhattacharya, M., L. O. Baksmaty, S. B. Weiss, and N. P. Bigelow, 2004, "Feshbach resonances in ²³Na-⁸⁷Rb," Eur. Phys. J. D **31**, 301–306.
- Blatt, J. M., and V. F. Weisskopf, 1952, Theoretical nuclear physics (Wiley, New York).
- Bloch, I., 2005, "Ultracold quantum gases in optical lattices," Nature Phys. 1, 23–30.
- Bloch, I., J. Dalibard, and W. Zwerger, 2008, "Many-body physics with ultracold gases," Rev. Mod. Phys. 80, 885.
- Blume, D., and C. H. Greene, 2002, "Fermi pseudopotential approximation: two particles under external confinement," Phys. Rev. A 65, 043613.
- Boesten, H. M. J. M., C. C. Tsai, J. R. Gardner, D. J.

- Heinzen, and B. J. Verhaar, 1997, "Observation of a shape resonance in the collision of two cold ⁸⁷Rb atoms," Phys. Rev. A **55**, 636–640.
- Boesten, H. M. J. M., C. C. Tsai, B. J. Verhaar, and D. J. Heinzen, 1996a, "Observation of a shape resonance in coldatom scattering by pulsed photoassociation," Phys. Rev. Lett. 77, 5194–5197.
- Boesten, H. M. J. M., J. M. Vogels, J. G. C. Tempelaars, and B. J. Verhaar, 1996b, "Properties of cold collisions of ³⁹K atoms and of ⁴¹K atoms in relation to Bose-Einstein condensation," Phys. Rev. A **54**, R3726–R3729.
- Bohn, J., and P. S. Julienne, 1997, "Prospects for influencing scattering lengths with far-off-resonant light," Phys. Rev. A 56, 1486–1491.
- Bohn, J. L., A. V. Avdeenkov, and M. P. Deskevich, 2002, "Rotational Feshbach resonances in ultracold molecular collisions," Phys. Rev. Lett. 89, 203202.
- Bohn, J. L., J. P. Burke, C. H. Greene, H. Wang, P. L. Gould, and W. C. Stwalley, 1999, "Collisional properties of ultracold potassium: Consequences for degenerate Bose and Fermi gases," Phys. Rev. A 59, 3660–3664.
- Bohn, J. L., and P. S. Julienne, 1996, "Semianalytic treatment of two-color photoassociation spectroscopy and control of cold atoms," Phys. Rev. A 54, R4637–R4640.
- Bohn, J. L., and P. S. Julienne, 1999, "Semianalytic theory of laser-assisted resonant cold collisions," Phys. Rev. A 60, 414–425.
- Bolda, E. L., E. Tiesinga, and P. S. Julienne, 2002, "Effective-scattering-length model of ultracold atomic collisions and Feshbach resonances in tight harmonic traps," Phys. Rev. A 66, 013403.
- Bolda, E. L., E. Tiesinga, and P. S. Julienne, 2003, "Pseudopotential model of ultracold atomic collisions in quasione- and two-dimensional traps," Phys. Rev. A 68, 032702.
- Bourdel, T., J. Cubizolles, L. Khaykovich, K. M. F. Magalhães, S. J. J. M. F. Kokkelmans, G. V. Shlyapnikov, and C. Salomon, 2003, "Measurement of the interaction energy near a Feshbach resonance in a ⁶Li Fermi gas," Phys. Rev. Lett. 91, 020402.
- Bourdel, T., L. Khaykovich, J. Cubizolles, J. Zhang, F. Chevy, M. Teichmann, L. Tarruell, S. J. J. M. F. Kokkelmans, and C. Salomon, 2004, "Experimental study of the BEC-BCS crossover region in lithium-6," Phys. Rev. Lett. 93, 050401.
- Braaten, E., and H.-W. Hammer, 2001, "Three-body recombination into deep bound states in a Bose gas with large scattering length," Phys. Rev. Lett. 87, 160407.
- Braaten, E., and H.-W. Hammer, 2006, "Universality in few-body systems with large scattering length," Phys. Rep. 428, 259–390.
- Braaten, E., and H.-W. Hammer, 2007, "Resonant dimer relaxation in cold atoms with a large scattering length," Phys. Rev. A 75, 052710.
- Braaten, E., H.-W. Hammer, D. Kang, and L. Platter, 2008, "Three-body recombination of fermionic atoms with large scattering lengths," arXiv:0811.3578.
- Bradley, C. C., C. A. Sacket, and R. G. Hulet, 1997, "Bose-Einstein condensation of lithium: Observation of limited condensate number," Phys. Rev. Lett. **78**, 985–989.
- Bradley, C. C., C. A. Sackett, J. J. Tollett, and R. G. Hulet, 1995, "Evidence of Bose-Einstein condensation in an atomic gas with attractive interactions," Phys. Rev. Lett. 75, 1687–1690.
- Breit, G., and I. I. Rabi, 1931, "Measurement of nuclear spin," Phys. Rev. 38, 2082–2083.

- Breit, G., and E. Wigner, 1936, "Capture of slow neutrons," Phys. Rev. 49, 519–531.
- Bringas, F., M. T. Yamashita, and T. Frederico, 2004, "Triatomic continuum resonances for large negative scattering lengths," Phys. Rev. A 69, 040702(R).
- Bryant, H. C., B. D. Dieterle, J. Donahue, H. Sharifian, H. Tootoonchi, D. M. Wolfe, P. A. M. Gram, and M. A. Yates-Williams, 1977, "Observation of resonances near 11 eV in the photodetachment cross section of the H- ion," Phys. Rev. Lett. **38**, 228–230.
- Büchler, H. P., E. Demler, M. Lukin, A. Micheli, N. Prokof'ev, G. Pupillo, and P. Zoller, 2007, "Strongly correlated 2D quantum phases with cold polar molecules: Controlling the shape of the interaction potential," Phys. Rev. Lett. 98, 060404.
- Buggle, C., J. Léonard, W. von Klitzing, and J. T. M. Walraven, 2004, "Interferometric determination of the s and d-Wave scattering amplitudes in ⁸⁷Rb," Phys. Rev. Lett. 93, 173202.
- Burke, J. P., J. L. Bohn, B. D. Esry, and C. H. Greene, 1998, "Prospects for mixed-isotope Bose-Einstein condensates in rubidium," Phys. Rev. Lett. 80, 2097–2100.
- Busch, T., B.-G. Englert, K. Rzažewski, and M. Wilkens, 1998, "Two cold atoms in a harmonic trap," Found. of Phys. 28, 549–559.
- Carusotto, I., L. Pitaevskii, S. Stringari, G. Modugno, and M. Inguscio, 2005, "Sensitive measurements of forces at the micron scale using Bloch oscillations of ultracold atoms," Phys. Rev. Lett. 95, 093202.
- Chen, Q., J. Stajic, S. Tan, and K. Levin, 2005, "BCS-BEC crossover: From high-temperature superconductors to ultracold superfluids," Phys. Rep. 412, 1–88.
- Chin, C., M. Bartenstein, A. Altmeyer, S. Riedl, S. Jochim, J. Hecker Denschlag, and R. Grimm, 2004a, "Observation of the pairing gap in a strongly interacting Fermi gas," Science 305, 1128–1130.
- Chin, C., and R. Grimm, 2004, "Thermal equilibrium and efficient evaporation of an ultracold atom-molecule mixture," Phys. Rev. A 69, 033612.
- Chin, C., and P. S. Julienne, 2005, "Radio-frequency transitions on weakly bound ultracold molecules," Phys. Rev. A 71, 012713.
- Chin, C., A. J. Kerman, V. Vuletić, and S. Chu, 2003, "Sensitive detection of cold cesium molecules formed on Feshbach resonances," Phys. Rev. Lett. 90, 033201.
- Chin, C., T. Kraemer, M. Mark, J. Herbig, P. Waldburger, H.-C. Nägerl, and R. Grimm, 2005, "Observation of Feshbachlike resonances in collisions between ultracold molecules," Phys. Rev. Lett. 94, 123201.
- Chin, C., V. Vuletić, A. J. Kerman, and Chu, 2000, "High resolution Feshbach spectroscopy of cesium," Phys. Rev. Lett. 85, 2717–2720.
- Chin, C., V. Vuletić, A. J. Kerman, S. Chu, E. Tiesinga, P. J. Leo, and C. J. Williams, 2004b, "Precision Feshbach spectroscopy of ultracold Cs₂," Phys. Rev. A 70, 032701.
- Chu, S., 1998, "Nobel Lecture: The manipulation of neutral particles," Rev. Mod. Phys. 70, 685–706.
- Ciurylo, R., E. Tiesinga, and P. S. Julienne, 2005, "Optical tuning of the scattering length of cold alkaline earth atoms," Phys. Rev. A 71, 030701.
- Claussen, N. R., S. J. J. M. F. Kokkelmans, S. T. Thompson, E. A. Donley, E. Hodby, and C. E. Wieman, 2003, "Very-high-precision bound-state spectroscopy near a ⁸⁵Rb Feshbach resonance," Phys. Rev. A 67, 060701(R).

- Cohen-Tannoudji, C. N., 1998, "Nobel Lecture: Manipulating atoms with photons," Rev. Mod. Phys. **70**, 707–719.
- Cornell, E. A., and C. E. Wieman, 2002, "Nobel Lecture: Bose-Einstein condensation in a dilute gas, the first 70 years and some recent experiments," Rev. Mod. Phys. **74**, 875–893.
- Cornish, S. L., N. R. Claussen, J. L. Roberts, E. A. Cornell, and C. E. Wieman, 2000, "Stable ⁸⁵Rb Bose-Einstein condensates with widely tunable interactions," Phys. Rev. Lett. 85, 1795–1798.
- Cornish, S. L., S. T. Thompson, and C. E. Wieman, 2006, "Formation of bright matter-wave solitions during the collapse of Attractive Bose-Einstein condensates," Phys. Rev. Lett. 96, 170401.
- Courteille, P., R. S. Freeland, D. J. Heinzen, F. A. van Abeelen, and B. J. Verhaar, 1998, "Observation of a Feshbach resonance in cold atom scattering," Phys. Rev. Lett. 81, 69–72.
- Cubizolles, J., T. Bourdel, S. J. J. M. F. Kokkelmans, G. V. Shlyapnikov, and C. Salomon, 2003, "Production of long-lived ultracold Li₂ molecules from a Fermi gas," Phys. Rev. Lett. 91, 240401.
- Cvitas, M. T., P. Soldan, J. M. Hutson, P. Honvault, and J.-M. Launay, 2007, "Interactions and dynamics in Li+Li₂ ultracold collisions," J. Chem. Phys. **127**, 074302.
- Dalfovo, F., S. Giorgini, L. P. Pitaevskii, and S. Stringari, 1999, "Theory of Bose-Einstein condensation in trapped gases," Rev. Mod. Phys. 71, 463–512.
- Danzl, J. G., E. Haller, M. Gustavsson, M. J. Mark, R. Hart, N. Bouloufa, O. Dulieu, H. Ritsch, and H.-C. Nägerl, 2008, "Quantum gas of deeply bound ground state molecules," Science 321, 1062–1066.
- Davis, K. B., M. O. Mewes, M. R. Andrews, N. J. van Druten,
 D. S. Durfee, D. M. Kurn, and W. Ketterle, 1995, "Bose-Einstein condensation in a gas of sodium atoms," Phys. Rev. Lett. 75, 3969–3973.
- DeCarvalho, R., J. M. Doyle, B. Friedrich, T. Guillet, J. Kim, D. Patterson, and J. D. Weinstein, 1999, "Buffer-gas loaded magnetic traps for atoms and molecules: A primer," Eur. Phys. J. D 7, 289–309.
- Deh, B., C. Marzok, C. Zimmermann, and P. W. Courteille, 2008, "Feshbach resonances in mixtures of ultracold $^6{\rm Li}$ and $^{87}{\rm Rb}$ gases," Phys. Rev. A 77, 010701(R).
- DeMarco, B., J. L. Bohn, J. P. Burke, Jr., M. Holland, and D. S. Jin, 1999, "Measurement of *p*-wave threshold law using evaporatively cooled fermionic atoms," Phys. Rev. Lett. **82**, 4208–4211.
- DeMille, D., 2002, "Quantum computation with trapped polar molecules," Phys. Rev. Lett. 88, 067901.
- Derevianko, A., W. R. Johnson, M. S. Safronova, and J. F. Babb, 1999, "High-precision calculations of dispersion coefficients, static dipole polarizabilities, and atom-wall interaction constants for alkali-metal atoms," Phys. Rev. Lett. 82, 3589–3592.
- D'Errico, C., M. Zaccanti, M. Fattori, G. Roati, M. Inguscio, G. Modugno, and A. Simoni, 2007, "Feshbach resonances in ultracold ³⁹K," New J. Phys. 9, 223.
- Deuretzbacher, F., K. Plassmeier, D. Pfannkuche, F. Werner, C. Ospelkaus, S. Ospelkaus, K. Sengstock, and K. Bongs, 2008, "Heteronuclear molecules in an optical lattice: Theory and experiment," Phys. Rev. A 77, 032726.
- Dickerscheid, D. B. M., U. A. Khawaja, D. van Oosten, and H. T. C. Stoof, 2005, "Feshbach resonances in an optical lattice," Phys. Rev. A 71, 043604.

- Dickerscheid, D. B. M., and H. T. C. Stoof, 2005, "Feshbach molecules in a one-dimensional Fermi gas," Phys. Rev. A 72, 053625.
- Dieckmann, K., C. A. Stan, S. Gupta, Z. Hadzibabic, C. H. Schunck, and W. Ketterle, 2002, "Decay of an ultracold fermionic lithium gas near a Feshbach resonance," Phys. Rev. Lett. 89, 203201.
- D'Incao, J. P., and B. D. Esry, 2006, "Enhancing the observability of the Efimov effect in ultracold atomic gas mixtures," Phys. Rev. A 73, 030703 (pages 4).
- D'Incao, J. P., H. Suno, and B. D. Esry, 2004, "Limits on universality in ultracold three-boson recombination," Phys. Rev. Lett. 93, 123201.
- Donley, E. A., N. R. Clausen, S. L. Cornish, J. L. Roberts, E. A. Cornell, and C. E. Wieman, 2001, "Dynamics of collapsing and exploding Bose-Einstein condensates," Nature 412, 295–299.
- Donley, E. A., N. R. Clausen, S. T. Thompson, and C. E. Wieman, 2002, "Atom-molecule coherence in a Bose-Einstein condensate," Nature 417, 529–533.
- Doyle, J., B. Friedrich, R. V. Krems, and F. Masnou Seeuws, 2004, "Ultracold polar molecules: Formation and collision," Eur. Phys. J. D 31, 149–446 special issue on polar molecules.
- Du, X., L. Luo, B. Clancy, and J. E. Thomas, 2008, "Observation of anomalous spin segregation in a trapped Fermi gas," Phys. Rev. Lett. 101, 150401.
- Duine, R. A., and H. T. C. Stoof, 2003, "Microscopic many-body theory of atomic Bose gases near a Feshbach resonance," J. Opt. B: Quantum Semiclass. Opt. 5, S212.
- Duine, R. A., and H. T. C. Stoof, 2004, "Atom-molecule coherence in Bose gases," Phys. Rep. **396**, 115–195.
- Dürr, S., T. Volz, A. Marte, and G. Rempe, 2004a, "Observation of molecules produced from a Bose-Einstein condensate," Phys. Rev. Lett. 92, 020406.
- Dürr, S., T. Volz, and G. Rempe, 2004b, "Dissociation of ultracold molecules with Feshbach resonances," Phys. Rev. A 70, 031601(R).
- Efimov, V., 1970, "Energy levels arising from resonant two-body forces in a three-body system," Phys. Lett. B 33, 563–564.
- Efimov, V., 1971, "Weakly-bound states of three resonantly-interacting particles," Sov. J. Nucl. Phys. 12, 589–595.
- Efimov, V., 1979, "Low-energy properties of three resonantly interacting particles," Sov. J. Nucl. Phys. 29, 546–553.
- Enomoto, K., K. Kasa, M. Kitagawa, and Y. Takahashi, 2008, "Optical Feshbach resonance using the intercombination transition," Phys. Rev. Lett. 101, 203201.
- Erhard, M., H. Schmaljohann, J. Kronjäger, K. Bongs, and K. Sengstock, 2004, "Measurement of a mixed-spin-channel Feshbach resonance in ⁸⁷Rb," Phys. Rev. A 69, 032705.
- Esry, B. D., C. H. Greene, and J. P. Burke, 1999, "Recombination of three atoms in the ultracold limit," Phys. Rev. Lett. 83, 1751–1754.
- Esry, B. D., C. H. Greene, and H. Suno, 2001, "Threshold laws for three-body recombination," Phys. Rev. A 65, 010705.
- Fano, U., 1935, "Sullo spettro di assorbimento dei gas nobili presso il limite dello spettro darco," Nuovo Cimento 12, 154–161.
- Fano, U., 1961, "Effects of configuration interaction on intensities and phase shifts," Phys. Rev. A 124, 1866–1878.
- Fano, U., G. Pupillo, A. Zannoni, and C. W. Clark, 2005, "On the absorption spectrum of noble gases at the arc spectrum limit," J. Res. Natl. Inst. Stand. Technol. 110, 583–587.

- Fatemi, F. K., K. M. Jones, and P. D. Lett, 2000, "Observation of optically induced Feshbach resonances in collisions of cold atoms," Phys. Rev. Lett. 85, 4462–4465.
- Fattori, M., C. D'Errico, G. Roati, M. Zaccanti, M. Jona-Lasinio, M. Modugno, M. Inguscio, and G. Modugno, 2008a, "Atom interferometry with a weakly-interacting Bose Einstein condensate," Phys. Rev. Lett. 100, 080405.
- Fattori, M., G. Roati, B. Deissler, C. D'Errico, M. Zaccanti, M. Jona-Lasinio, L. Santos, M. Inguscio, and G. Modugno, 2008b, "Magnetic dipolar interaction in a Bose-Einstein condensate atomic interferometer," Phys. Rev. Lett. 101, 190405.
- Fedichev, P. O., M. J. Bijlsma, and P. Zoller, 2004, "Extended molecules and geometric scattering resonances in optical lattices," Phys. Rev. Lett. 92, 080401.
- Fedichev, P. O., Y. Kagan, G. V. Shlyapnikov, and J. T. M. Walraven, 1996a, "Influence of nearly resonant light on the scattering length in low-temperature atomic gases," Phys. Rev. Lett. 77, 2913–2916.
- Fedichev, P. O., M. W. Reynolds, and G. V. Shlyapnikov, 1996b, "Three-body recombination of ultracold atoms to a weakly bound s level," Phys. Rev. Lett. 77, 2921–2924.
- Ferlaino, F., C. D'Errico, G. Roati, M. Zaccanti, M. Inguscio, G. Modugno, and A. Simoni, 2006, "Feshbach spectroscopy of a K-Rb atomic mixture," Phys. Rev. A 73, 040702(R), ibid 74, 039903(E).
- Ferlaino, F., S. Knoop, M. Mark, M. Berninger, H. Schöbel, H.-C. Nägerl, and R. Grimm, 2008, "Collisions between tunable halo dimers: exploring an elementary four-body process with identical bosons," Phys. Rev. Lett. 101, 023201.
- Feshbach, H., 1958, "Unified theory of nuclear reactions," Ann. Phys. (NY) 5, 357–390.
- Feshbach, H., 1962, "Unified theory of nuclear reactions II," Ann. Phys. (NY) 19, 287–313.
- Flambaum, V. V., G. F. Gribakin, and C. Harabti, 1999, "Analytic calculation of cold-atom scattering," Phys. Rev. A 59, 1998–2005.
- Forrey, R. C., N. Balakrishnan, V. Kharchenko, and A. Dalgarno, 1998, "Feshbach resonances in ultracold atomdiatom scattering," Phys. Rev. A 58, R2645–R2647.
- Fuchs, J., C. Ticknor, P. Dyke, G. Veeravalli, E. Kuhnle, W. Rowlands, P. Hannaford, and C. J. Vale, 2008, "Binding energies of ⁶Li p-wave Feshbach molecules," Phys. Rev. A 77, 053616.
- Gacesa, M., P. Pellegrini, and R. Côté, 2008, "Feshbach resonances in ultracold 6,7 Li+ 23 Na atomic mixtures," Phys. Rev. A **78**, 010701(R).
- Gaebler, J. P., J. T. Stewart, J. L. Bohn, and D. S. Jin, 2007, "p-wave Feshbach molecules," Phys. Rev. Lett. 98, 200403.
- Gao, B., 1996, "Theory of slow-atom collisions," Phys. Rev. A $\mathbf{54}$, 2022–2039.
- Gao, B., 1998a, "Quantum-defect theory of atomic collisions and molecular vibrational spectra," Phys. Rev. A 58, 4222– 4225.
- Gao, B., 1998b, "Solutions of the Schrödinger equation for an attractive $1/r^6$ potential," Phys. Rev. A **58**, 1728–1734.
- Gao, B., 2000, "Zero-energy bound or quasibound states and their implications for diatomic systems with an asymptotic van der Waals interaction," Phys. Rev. A 62, 050702.
- Gao, B., 2001, "Angular-momentum-insensitive quantum-defect theory for diatomic systems," Phys. Rev. A 64, 010701.
- Gao, B., 2004, "Binding energy and scattering length for di-

- atomic systems," J. Phys. B 37, 4273-4279.
- Gauyacq, J. P., and A. Herzenberg, 1982, "Nuclear-excited Feshbach resonances in e+HCl scattering," Phys. Rev. A 25, 2959–2967.
- Giorgini, S., L. P. Pitaevskii, and S. Stringari, 2008, "Theory of ultracold Fermi gases," Rev. Mod. Phys. 80, 1215–1274.
- González-Martinez, M. L., and J. M. Hutson, 2007, "Ultracold atom-molecule collisions and bound states in magnetic fields: Tuning zero-energy Feshbach resonances in He-NH ($^{3}\Sigma^{-}$)," Phys. Rev. A **75**, 022702.
- Góral, K., T. Köhler, S. A. Gardiner, E. Tiesinga, and P. S. Julienne, 2004, "Adiabatic association of ultracold molecules via magnetic-field tunable interactions," J. Phys. B 37, 3457–3500.
- Goral, K., K. Rzazewski, and T. Pfau, 2000, "Bose-Einstein condensation with magnetic dipole-dipole forces," Phys. Rev. A 61, 051601.
- Granger, B. E., and D. Blume, 2004, "Tuning the interactions of spin-polarized fermions using quasi-one-dimensional confinement," Phys. Rev. Lett. **92**, 133202.
- Greiner, M., C. A. Regal, and D. S. Jin, 2003, "Emergence of a molecular Bose-Einstein condensate from a Fermi gas," Nature 412, 537–540.
- Gribakin, G. F., and V. V. Flambaum, 1993, "Calculation of the scattering length in atomic collisions using the semiclassical approximation," Phys. Rev. A 48, 546–553.
- Griesmaier, A., J. Werner, S. Hensler, J. Stuhler, and T. Pfau, 2005, "Bose-Einstein condensation of chromium," Phys. Rev. Lett. 94, 160401.
- Grimm, R., M. Weidemüller, and Y. B. Ovchinnikov, 2000, "Optical dipole traps for neutral atoms," Adv. At. Mol. Opt. Phys. 42, 95–170.
- Gross, N., and L. Khaykovich, 2008, "All-optical production of ⁷Li Bose-Einstein condensation using Feshbach resonances," Phys. Rev. A 77, 023604.
- Grupp, M., R. Walser, W. P. Schleich, M. A., and M. Weitz, 2007, "Resonant Feshbach scattering of fermions in onedimensional optical lattices," J. Phys. B: At. Mol. Opt. Phys. 40, 2703–2718.
- Gubbels, K. B., D. B. M. Dickerscheid, and H. T. C. Stoof, 2006, "Dressed molecules in an optical lattice," New J. Phys 8, 151.
- Günter, K., T. Stöferle, H. Moritz, M. Köhl, and T. Esslinger, 2005, "p-Wave interactions in low-dimensional fermionic gases," Phys. Rev. Lett. 95, 230401.
- Gupta, S., K. Dieckmann, Z. Hadzibabic, and D. E. Pritchard, 2002, "Contrast interferometry using Bose-Einstein condensates to measure h/m and α ," Phys. Rev. Lett. 89, 140401.
- Gupta, S., Z. Hadzibabic, M. W. Zwierlein, C. A. Stan, K. Dieckmann, C. H. Schunck, E. G. M. van Kempen, B. J. Verhaar, and W. Ketterle, 2003, "Radio-frequency spectroscopy of ultracold fermions," Science 300, 1723–1726.
- Gustavsson, M., E. Haller, M. J. Mark, J. G. Danzl, G. Rojas-Kopeinig, and H.-C. Nägerl, 2008, "Control of interactioninduced dephasing of Bloch oscillations," Phys. Rev. Lett. 100, 080404.
- Hammer, H.-W., and L. Platter, 2007, "Universal properties of the four-body system with large scattering length," Eur. Phys. J. A **32**, 113–120.
- Hanna, T. M., T. Köhler, and K. Burnett, 2007, "Association of molecules using a resonantly modulated magnetic field," Phys. Rev. A 75, 013606.
- Herbig, J., T. Kraemer, M. Mark, T. Weber, C. Chin, H.-

- C. Nägerl, and R. Grimm, 2003, "Preparation of a pure molecular quantum gas," Science **301**, 1510–1513.
- Hodby, E., S. T. Thompson, C. A. Regal, M. Greiner, A. C. Wilson, D. S. Jin, and E. A. Cornell, 2005, "Production efficiency of ultra-cold Feshbach molecules in bosonic and fermionic systems," Phys. Rev. Lett. 94, 120402.
- Houbiers, M., H. T. C. Stoof, W. I. McAlexander, and R. G. Hulet, 1998, "Elastic and inelastic collisions of ⁶Li atoms in magnetic and optical traps," Phys. Rev. A 57, R1497.
- Huang, K., and T. D. Lee, 1957, "Quantum mechanical manybody problem with hard sphere interaction," Phys. Rev. 105, 767–775.
- Huckans, J. H., J. R. Williams, E. L. Hazlett, R. W. Stites, and K. M. O'Hara, 2009, "Three-body recombination in a three-state Fermi gas with widely tunable interactions," Phys. Rev. Lett. 102, 165302.
- Hudson, E. R., N. B. Gilfoy, S. Kotochigova, J. M. Sage, and D. DeMille, 2008, "Inelastic collisions of ultracold heteronuclear molecules in an optical trap," Phys. Rev.Lett. 100, 203201.
- Hudson, E. R., C. Ticknor, B. C. Sawyer, C. A. Taatjes, H. J. Lewandowski, J. R. Bochinski, J. L. Bohn, and Y. Jun, 2006, "Production of cold formaldehyde molecules for study and control of chemical reaction dynamics with hydroxyl radicals," Phys. Rev. A 73, 063404.
- Hutson, J., and P. Soldán, 2006, "Molecule formation in ultracold atomic gases," Int. Rev. Phys. Chem. 25, 497–526.
- Hutson, J. M., 2007, "Feshbach resonances in the presence of inelastic scattering: threshold behavior and suppression of poles in scattering lengths," New J. Phys. 9, 152.
- Hutson, J. M., and P. Soldán, 2007, "Molecular collisions in ultracold atomic gases," Int. Rev. Phys. Chem. 26, 1–28.
- Hutson, J. M., E. Tiesinga, and P. S. Julienne, 2008, "Avoided crossings between bound states of ultracold cesium dimers," Phys. Rev. A 78, 052703.
- Inada, Y., M. Horikoshi, S. Nakajima, M. Kuwata-Gonokami, M. Ueda, and T. Mukaiyama, 2008, "Collisional Properties of p-Wave Feshbach Molecules," Phys. Rev. Lett. 101, 100401
- Inguscio, M., W. Ketterle, and C. Salomon (eds.), 2008, Ultracold Fermi Gases (IOS Press, Amsterdam), Proceedings of the International School of Physics "Enrico Fermi", Course CLXIV, Varenna, 20-30 June 2006.
- Inguscio, M., S. Stringari, and C. E. Wieman (eds.), 1999, Bose-Einstein Condensation in Atomic Gases (IOS Press, Amsterdam), Proceedings of the International School of Physics "Enrico Fermi", Course CXL, Varenna, 7-17 July 1998.
- Inouye, S., M. R. Andrews, J. Stenger, H.-J. Miesner, D. M. Stamper-Kurn, and W. Ketterle, 1998, "Observation of Feshbach resonances in a Bose-Einstein condensate," Nature 392, 151–154.
- Inouye, S., J. Goldwin, M. L. Olsen, C. Ticknor, J. L. Bohn, and D. S. Jin, 2004, "Observation of heteronuclear Feshbach resonances in a mixture of bosons and fermions," Phys. Rev. Lett. 93, 183201.
- Jensen, A. S., K. Riisager, D. V. Fedorov, and E. Garrido, 2004, "Structure and reactions of quantum halos," Rev. Mod. Phys. 76, 215–261.
- Jochim, S., M. Bartenstein, A. Altmeyer, G. Hendl, C. Chin, J. Hecker Denschlag, and R. Grimm, 2003a, "Pure gas of optically trapped molecules created from fermionic atoms," Phys. Rev. Lett 91, 240402.
- Jochim, S., M. Bartenstein, A. Altmeyer, G. Hendl, S. Riedl,

- C. Chin, J. Hecker Denschlag, and R. Grimm, 2003b, "Bose-Einstein condensation of molecules," Science **302**, 2101–2103.
- Jochim, S., M. Bartenstein, G. Hendl, J. Hecker Denschlag, R. Grimm, A. Mosk, and W. Weidemüller, 2002, "Magnetic field control of elastic scattering in a cold gas of fermionic lithium atoms," Phys. Rev. Lett 89, 273202.
- Jones, K. M., E. Tiesinga, P. D. Lett, and P. S. Julienne, 2006, "Photoassociation spectroscopy of ultracold atoms: long-range molecules and atomic scattering," Rev. Mod. Phys. 78, 483–535.
- Julienne, P. S., and B. Gao, 2006, in Atomic Physics 20, edited by C. Roos, H. Häffner, and R. Blatt (AIP, Melville, New York), 261–268.
- Julienne, P. S., and F. H. Mies, 1989, "Collisions of ultracold trapped atoms," J. Opt. Soc. Am. B 6, 2257.
- Julienne, P. S., E. Tiesinga, and T. Köhler, 2004, "Making cold molecules by time-dependent Feshbach resonances," J. Mod. Optics 51, 1787–1806.
- Junker, M., D. Dries, C. Welford, J. Hitchcock, Y. P. Chen, and R. G. Hulet, 2008, "Photoassociation of a Bose-Einstein condensate near a Feshbach resonance," Phys. Rev. Lett. 101, 060406.
- Kagan, Y., B. V. Svistunov, and G. V. Shlyapnikov, 1985, "Effect of Bose condensation on inelastic processes in gases," Pis'ma Zh. Eksp. Teor. Fiz. 42, 169–172 [JETP Lett. 42, 209-212 (1985)].
- van Kempen, E. G. M., S. J. J. M. F. Kokkelmans, D. J. Heinzen, and B. J. Verhaar, 2002, "Interisotope determination of ultracold rubidium interactions from three high-precision experiments," Phys. Rev. Lett. 88, 093201.
- van Kempen, E. G. M., B. Marcelis, and S. J. J. M. F. Kokkelmans, 2004, "Formation of fermionic molecules via interisotope Feshbach resonances." Phys. Rev. A 70, 050701.
- Ketterle, W., 2002, "Nobel lecture: When atoms behave as waves: Bose-Einstein condensation and the atom laser," Rev. Mod. Phys. 74, 1131–1151.
- Ketterle, W., and N. J. van Druten, 1997, "Evaporative cooling of trapped atoms," Adv. At. Mol. Opt. Phys. 96, 181–236
- Khaykovich, L., F. Schreck, G. Ferrari, T. Bourdel, J. Cu-bizolles, L. D. Carr, Y. Castin, and C. Salomon, 2002, "Formation of a matter-wave bright soliton," Science 296, 1290–1293.
- Kinast, J., S. L. Hemmer, M. E. Gehm, A. Turlapov, and J. E. Thomas, 2004, "Evidence for superfluidity in a resonantly interacting Fermi gas," Phys. Rev. Lett. 92, 150402.
- Kishimoto, T., J. Kobayashi, K. Noda, K. Aikawa, M. Ueda, and S. Inouye, 2009, "Direct evaporative cooling of ⁴¹K into a Bose-Einstein condensate," Phys. Rev. A 79, 031602.
- Kitagawa, M., K. Enomoto, K. Kasa, Y. Takahashi, R. Ciurylo, P. Naidon, and P. S. Julienne, 2008, "Two-color photoassociation spectroscopy of ytterbium atoms and the precise determinations of s-wave scattering lengths," Phys. Rev. A 77, 012719.
- Klempt, C., T. Henninger, O. Topic, M. Scherer, L. Kattner, E. Tiemann, W. Ertmer, and J. J. Arlt, 2008, "Radio-frequency association of heteronuclear Feshbach molecules," Phys. Rev. A 78, 061602(R).
 Klempt, C., T. Henninger, O. Topic, J. Will, W. Ertmer,
- Klempt, C., T. Henninger, O. Topic, J. Will, W. Ertmer, E. Tiemann, and J. Arlt, 2007, "⁴⁰K-⁸⁷Rb Feshbach resonances: Modeling the interatomic potential," Phys. Rev. A 76, 020701(R).
- Kleppner, D., 2004, "Professor Feshbach and his resonance,"

- Phys. Today **57**(8), 12–13.
- Knoop, S., F. Ferlaino, M. Mark, J. G. Danzl, T. Kraemer, H.-C. Nägerl, and R. Grimm, 2009, "Observation of an Efimov-like trimer resonance in ultracold atom-dimer scattering," Nature Phys. 5, 227–230.
- Knoop, S., M. Mark, F. Ferlaino, J. G. Danzl, T. Kraemer, H.-C. Nägerl, and R. Grimm, 2008, "Metastable Feshbach molecules in high rotational states," Phys. Rev. Lett. 100, 083002.
- Koch, T., T. Lahaye, J. Metz, B. Fröhlich, A. Griesmaier, and T. Pfau, 2008, "Stabilization of a purely dipolar quantum gas against collapse," Nature Phys. 4, 218–212.
- Köhler, T., K. Góral, and P. S. Julienne, 2006, "Production of cold molecules via magnetically tunable Feshbach resonances," Rev. Mod. Phys. 78, 1311–1361.
- Köhler, T., E. Tiesinga, and P. S. Julienne, 2005, "Spontaneous dissociation of long-range Feshbach molecules," Phys. Rev. Lett. **94**, 020402.
- Kokkelmans, S. J. J. M. F., J. N. Milstein, M. L. Chiofalo, and R. W. M. J. Holland, 2002, "Resonance superfluidity: Renormalization of resonance scattering theory," Phys. Rev. A 65, 053617.
- Kokkelmans, S. J. J. M. F., G. V. Shlyapnikov, and C. Salomon, 2004, "Degenerate atom-molecule mixture in a cold Fermi gas," Phys. Rev. A 69, 031602.
- Kokkelmans, S. J. J. M. F., H. M. J. Vissers, and B. J. Verhaar, 2001, "Formation of a Bose condensate of stable molecules via a Feshbach resonance," Phys. Rev. A 63, 031601(R).
- Kotochigova, S., E. Tiesinga, and P. S. Julienne, 2000, "Relativistic ab initio treatment of the second-order spin-orbit splitting of the a ${}^{3}\Sigma_{u}^{+}$ potential of rubidium and cesium dimers," Phys. Rev. A **63**, 12517.
- Kraemer, T., J. Herbig, M. Mark, T. Weber, C. Chin, H.-C. Nägerl, and R. Grimm, 2004, "Optimized production of a cesium Bose-Einstein condensate," Appl. Phys. B 79, 1013–1019
- Kraemer, T., M. Mark, P. Waldburger, J. G. Danzl, C. Chin, B. Engeser, A. D. Lange, K. Pilch, A. Jaakkola, H.-C. Nägerl, and R. Grimm, 2006, "Evidence for Efimov quantum states in an ultracold gas of cesium atoms," Nature 440, 315–318.
- Krems, R. V., 2005, "Molecules near absolute zero and external field control of atomic and molecular dynamics," Int. Rev. Phys. Chem. 24, 99–118.
- Lahaye, T., T. Koch, B. Fröhlich, M. Fattori, J. Metz, A. Griesmaier, S. Giovanazzi, and T. Pfau, 2007, "Strong dipolar effects in a quantum ferrofluid," Nature 448, 672– 676.
- Lahaye, T., J. Metz, B. Fröhlich, T. Koch, M. Meister, A. Griesmaier, T. Pfau, H. Saito, Y. Kawaguchi, and M. Ueda, 2008, "d-wave collapse and explosion of a dipolar Bose-Einstein condensate," Phys. Rev. Lett. 101, 080401.
- Lang, F., P. v. d. Straten, B. Brandstätter, G. Thalhammer, K. Winkler, P. S. Julienne, R. Grimm, and J. Hecker Denschlag, 2008a, "Cruising through molecular bound state manifolds with radio frequency," Nature Phys. 4, 223.
- Lang, F., K. Winkler, C. Strauss, R. Grimm, and J. Hecker Denschlag, 2008b, "Ultracold triplet molecules in the rovibrational ground state," Phys. Rev. Lett. 101, 133005.
- Lange, A. D., K. Pilch, A. Prantner, F. Ferlaino, B. Engeser, H.-C. Nägerl, R. Grimm, and C. Chin, 2009, "Determination of atomic scattering lengths from measurements of molecular binding energies near Feshbach resonances,"

- Phys. Rev. A 79, 013622.
- Lara, M., J. L. Bohn, D. E. Potter, P. Soldán, and J. M. Hutson, 2007, "Cold collisions between OH and Rb: The field-free case," Phys. Rev. A 75, 012704.
- Laue, T., E. Tiesinga, C. Samuelis, H. Knöckel, and E. Tiemann, 2002, "Magnetic-field imaging of weakly bound levels of the ground-state Na₂ dimer," Phys. Rev. A 65, 023412.
- Lee, M. D., T. Köhler, and P. S. Julienne, 2007, "Excited Thomas-Efimov levels in ultracold gases," Phys. Rev. A 76, 012720.
- Leo, P. J., C. J. Williams, and P. S. Julienne, 2000, "Collision properties of ultracold ¹³³Cs atoms," Phys. Rev. Lett. 85, 2721–2724.
- Li, Z., S. Singh, T. V. Tscherbul, and K. W. Madison, 2008, "Feshbach resonances in ultracold ⁸⁵Rb-⁸⁷Rb and ⁶Li-⁸⁷Rb mixtures," Phys. Rev. A 78, 022710.
- Loftus, T., C. A. Regal, C. Ticknor, J. L. Bohn, and D. S. Jin, 2002, "Resonant control of elastic collisions in an optically trapped Fermi gas of atoms," Phys. Rev. Lett. 88, 173201.
- Luu, T., and A. Schwenk, 2007, "Three-fermion problems in optical lattices," Phys. Rev. Lett. 98, 103202.
- MacArthur, D. W., K. B. Butterfield, D. A. Clark, J. B. Donahue, P. A. M. Gram, H. C. Bryant, C. J. Harvey, W. W. Smith, and G. Comtet, 1985, "Energy measurement of the lowest ¹P⁰ Feshbach resonance in H⁻," Phys. Rev. A 32, 1921–1923.
- Madison, K. W., F. Chevy, W. Wohlleben, and J. Dalibard, 2000, "Vortex formation in a stirred Bose-Einstein condensate," Phys. Rev. Lett. 84, 806–809.
- Marcelis, B., E. G. M. van Kempen, B. J. Verhaar, and S. J. J. M. F. Kokkelmans, 2004, "Feshbach resonances with large background scattering length: Interplay with open-channel resonances," Phys. Rev. A 70, 012701.
- Marcelis, B., B. Verhaar, and S. Kokkelmans, 2008, "Total control over ultracold interactions via electric and magnetic fields," Phys. Rev. Lett. 100, 153201.
- Marinescu, M., and L. You, 1998, "Controlling atom-atom interaction at ultralow temperatures by dc electric fields," Phys. Rev. Lett. 81, 4596.
- Marion, H., S. Bize, L. Cacciapuoti, D. Chambon, F. P. D. Santos, G. Santarelli, P. Wolf, A. Clairon, A. Luiten, M. Tobar, S. Kokkelmans, and C. Salomon, 2004, "First observation of Feshbach resonances at very low magnetic field in a ¹³³Cs fountain," arXiv:physics/0407064.
- Mark, M., F. Ferlaino, S. Knoop, J. G. Danzl, T. Kraemer, C. Chin, H.-C. Nägerl, and R. Grimm, 2007a, "Spectroscopy of ultracold, trapped cesium Feshbach molecules," Phys. Rev. A 76, 042514.
- Mark, M., T. Kraemer, J. Herbig, C. Chin, H.-C. Nägerl, and R. Grimm, 2005, "Efficient creation of molecules from a cesium Bose-Einstein condensate," Europhys. Lett. 69, 706-712
- Mark, M., T. Kraemer, P. Waldburger, J. Herbig, C. Chin, H.-C. Nägerl, and R. Grimm, 2007b, "Stückelberg interferometry with ultracold molecules," Phys. Rev. Lett. 99, 113201.
- Marte, A., T. Volz, J. Schuster, S. Dürr, G. Rempe, E. G. M. van Kempen, and B. J. Verhaar, 2002, "Feshbach resonances in rubidium 87: Precision measurement and analysis," Phys. Rev. Lett. 89, 283202.
- Marzok, C., B. Deh, C. Zimmermann, P. W. Courteille, E. Tiemann, Y. V. Vanne, and A. Saenz, 2009, "Feshbach resonances in an ultracold ⁷Li and ⁸⁷Rb mixture," Phys.

- Rev. A 79, 012717.
- Massignan, P., and H. T. C. Stoof, 2008, "Efimov states near a Feshbach resonance," Phys. Rev. A 78, 030701(R).
- van de Meerakker, S. Y. T., N. Vanhaecke, and G. Meijer, 2006, "Stark deceleration and trapping of OH radicals," Annu. Rev. Phys. Chem. 57, 159–190.
- Menotti, C., M. Lewenstein, T. Lahaye, and T. Pfau, 2008, in *Proceedings of the Workshop "Dynamics and Thermodynamics of Systems with Long Range Interactions" (Assisi, July 2007)*, AIP Conference Proceedings.
- Messiah, A., 1966, *Quantum Mechanics* (John Wiley and Sons, Inc., New York).
- Metcalf, H. J., and P. van der Straten, 1999, Laser Cooling and Trapping (Springer, New York).
- Micheli, A., G. K. Brennen, and P. Zoller, 2006, "A toolbox for lattice-spin models with polar molecules," Nature Phys. 2, 341.
- Mies, F. H., and M. Raoult, 2000, "Analysis of threshold effects in ultracold atomic collisions," Phys. Rev. A 62, 012708.
- Mies, F. H., E. Tiesinga, and P. S. Julienne, 2000, "Manipulation of Feshbach resonances in ultracold atomic collisions using time-dependent magnetic fields," Phys. Rev. A 61, 022721.
- Moerdijk, A. J., and B. J. Verhaar, 1994, "Prospects for Bose-Einstein condensation in atomic ⁷Li and ²³Na," Phys. Rev. Lett. **73**, 518–521.
- Moerdijk, A. J., B. J. Verhaar, and A. Axelsson, 1995, "Resonances in ultracold collisions of ⁶Li, ⁷Li, and ²³Na," Phys. Rev. A **51**, 4852–4861.
- Monroe, C. R., E. A. Cornell, C. A. Sackett, C. J. Myatt, and C. E. Wieman, 1993, "Measurement of Cs-Cs elastic scattering at $T = 30\mu\text{K}$." Phys. Rev. Lett. **70**, 414–417.
- Moritz, H., T. Stöferle, K. Günter, M. Köhl, and T. Esslinger, 2005, "Confinement induced molecules in a 1D Fermi gas," Phys. Rev. Lett. **94**, 210401.
- Mott, N. F., and H. S. W. Massey, 1965, *Theory of atomic collisions* (Oxford University Press, London), 3rd edition.
- Mukaiyama, T., J. R. Abo-Shaeer, K. Xu, J. K. Chin, and W. Ketterle, 2004, "Dissociation and decay of ultracold sodium molecules," Phys. Rev. Lett. **92**, 180402.
- Naidon, P., and P. S. Julienne, 2006, "Optical Feshbach resonances of alkaline-earth-metal atoms in a one- or two-dimensional optical lattice," Phys. Rev. A 74, 062713.
- Naidon, P., E. Tiesinga, W. F. Mitchell, and P. S. Julienne, 2007, "Effective-range description of a Bose gas under strong one- or two-dimensional confinement," New J. Phys. 9, 19.
- Naidon, P., and M. Ueda, 2008, "A possible Efimov trimer state in three-component lithium-6," arXiv:0812.4086 .
- Ni, K.-K., S. Ospelkaus, M. H. G. de Miranda, A. Pe'er, B. Neyenhuis, J. J. Zirbel, S. Kotochigova, P. S. Julienne, D. S. Jin, and J. Ye, 2008, "A high phase-space-density gas of polar molecules," Science 322, 231–235.
- Nieh, J.-C., and J. J. Valentini, 1990, "State-to-state dynamics of the H+p-H₂→o,p-H₂+H reaction: Feshbach resonances and vibrational spectroscopy of the transition state," J. Chem. Phys. **92**, 1083–1097.
- Nielsen, E., and J. H. Macek, 1999, "Low-energy recombination of identical bosons by three-body collisions," Phys. Rev. Lett. 83, 1751–1754.
- Nielsen, E., H. Suno, and B. D. Esry, 2002, "Efimov resonances in atom-diatom scattering," Phys. Rev. A 66, 012705.

- Nygaard, N., R. Piil, and K. Mølmer, 2008a, "Feshbach molecules in a one-dimensional optical lattice," Phys. Rev. A 77, 021601(R).
- Nygaard, N., R. Piil, and K. Mølmer, 2008b, "Two-channel Feshbach physics in a structured continuum," Phys. Rev. A 78, 023617.
- Nygaard, N., B. I. Schneider, and P. S. Julienne, 2006, "A two-channel R-matrix analysis of magnetic field induced Feshbach resonances," Phys. Rev. A 73, 042705.
- O'Hara, K., S. L. Lemmer, M. E. Gehm, S. R. Granade, and J. E. Thomas, 2002a, "Observation of a strongly interacting degenerate Fermi gas," Science 298, 2179–2182.
- O'Hara, K. M., S. L. Hemmer, S. R. Granade, M. E. Gehm, J. E. Thomas, V. Venturi, E. Tiesinga, and C. J. Williams, 2002b, "Measurement of the zero crossing in a Feshbach resonance of fermionic ⁶Li," Phys. Rev. A **66**, 041401.
- Olshanii, M., 1998, "Atomic scattering in the presence of an external confinement and a gas of impenetrable bosons," Phys. Rev. Lett. 81, 000938.
- Orso, G., L. P. Pitaevskii, S. Stringari, and M. Wouters, 2005, "Formation of molecules near a Feshbach resonance in a 1D optical lattice," Phys. Rev. Lett. **95**, 060402.
- Ospelkaus, C., S. Ospelkaus, L. Humbert, P. Ernst, K. Sengstock, and K. Bongs, 2006a, "Ultracold heteronuclear molecules in a 3D optical lattice," Phys. Rev. Lett. 97, 120402.
- Ospelkaus, S., C. Ospelkaus, L. Humbert, K. Sengstock, and K. Bongs, 2006b, "Tuning of heteronuclear interactions in a degenerate Fermi-Bose mixture," Phys. Rev. Lett. 97, 120403.
- Ospelkaus, S., A. Pe'er, K.-K. Ni, J. J. Zirbel, B. Neyenhuis, S. Kotochigova, P. S. Julienne, J. Ye, and D. S. Jin, 2008, "Efficient state transfer in an ultracold dense gas of heteronuclear molecules." Nature Phys. 4, 622–626.
- Ottenstein, T. B., T. Lompe, M. Kohnen, A. N. Wenz, and S. Jochim, 2008, "Collisional stability of a three-component degenerate Fermi gas," Phys. Rev. Lett. 101, 203202.
- Papp, S. B., J. M. Pino, R. J. Wild, S. Ronen, C. E. Wieman, D. S. Jin, and E. A. Cornell, 2008, "Bragg spectroscopy of a strongly interacting ⁸⁵Rb Bose-Einstein condensate," Phys. Rev. Lett. 101, 135301.
- Papp, S. B., and C. E. Wieman, 2006, "Observation of heteronuclear Feshbach molecules from a ⁸⁵Rb–⁸⁷Rb gas," Phys. Rev. Lett. 97, 180404.
- Partridge, G. B., W. Li, R. I. Kamar, Y. an Liao, and R. G. Hulet, 2006, "Pairing and phase separation in a polarized Fermi gas," Science **311**, 503–505.
- Partridge, G. B., K. E. Strecker, R. I. Kamar, M. W. Jack, and R. G. Hulet, 2005, "Molecular probe of pairing in the BEC-BCS crossover," Phys. Rev. Lett. 95, 020404.
- Pashov, A., O. Docenko, M. Tamanis, R. Ferber, H. Knöckel, and E. Tiemann, 2005, "Potentials for modeling cold collisions between Na (3S) and Rb (5S) atoms," Phys. Rev. A 72, 062505.
- Pashov, A., O. Docenko, M. Tamanis, R. Ferber, H. Knöckel, and E. Tiemann, 2007, "Coupling of the $X^1\Sigma^+$ and a $^3\Sigma^+$ states of KRb," Phys. Rev. A **76**, 022511.
- Pethick, C. J., and H. Smith, 2008, Bose-Einstein condensation in dilute gases (Cambridge University Press).
- Petrov, D. S., 2003, "Three-body problem in Fermi gases with short-range interparticle interaction," Phys. Rev. A 67, 010703.
- Petrov, D. S., 2004, "Three-boson problem near a narrow Feshbach resonance," Phys. Rev. Lett. 93, 143201.

- Petrov, D. S., G. E. Astrakharchik, D. J. Papoular, C. Salomon, and G. V. Shlyapnikov, 2007, "Crystalline phase of strongly interacting Fermi mixtures," Phys. Rev. Lett. 99, 130407.
- Petrov, D. S., M. Holzmann, and G. V. Shlyapnikov, 2000, "Bose-Einstein condensation in quasi-2D trapped gases," Phys. Rev. Lett. 84, 2551–2554.
- Petrov, D. S., C. Salomon, and G. V. Shlyapnikov, 2004, "Weakly bound molecules of fermionic atoms," Phys. Rev. Lett. 93, 090404.
- Petrov, D. S., and G. V. Shlyapnikov, 2001, "Interatomic collisions in a tightly confined Bose gas," Phys. Rev. A 64, 012706.
- Phillips, W. D., 1998, "Nobel Lecture: Laser cooling and trapping of neutral atoms," Rev. Mod. Phys. **70**, 721–741.
- Pilch, K., A. D. Lange, A. Prantner, G. Kerner, F. Ferlaino, H.-C. Nägerl, and R. Grimm, 2009, "Observation of interspecies Feshbach resonances in an ultracold Rb-Cs mixture," Phys. Rev. A 79, 042718.
- Pollack, S. E., D. Dries, M. Junker, Y. P. Chen, T. A. Corcovilos, and R. G. Hulet, 2009, "Extreme tunability of interactions in a 7Li Bose-Einstein condensate," Phys. Rev. Lett. 102, 090402.
- Porsev, S. G., and A. Derevianko, 2002, "High-accuracy relativistic many-body calculations of van der Waals coefficients C_6 for alkaline-earth-metal atoms," Phys. Rev. A **65**, 020701(R).
- Porsev, S. G., and A. Derevianko, 2006, "High-accuracy calculations of dipole, quadrupole, and octupole electric dynamic polarizabilities and van der Waals coefficients C_6 , C_8 , and C_{10} for alkaline-earth dimers," JETP **102**, 195–205 [Pis'ma Zh. Eksp. Teor. Fiz., **129**, 227–238 (2006)].
- Quéméner, G., P. Honvault, J.-M. Launay, P. Soldán, D. E. Potter, and J. M. Hutson, 2005, "Ultracold quantum dynamics: Spin-polarized K+K₂ collisions with three identical bosons or fermions," Phys. Rev. A 71, 032722.
- Raoult, M., and F. H. Mies, 2004, "Feshbach resonance in atomic binary collisions in the Wigner threshold law regime," Phys. Rev. A 70, 012710.
- Rau, A. R. P., 2005, "Historical notes on Feshbach resonances," Phys. Today 58(2), 13–13.
- Regal, C. A., M. Greiner, and D. S. Jin, 2004, "Observation of resonance condensation of fermionic atom pairs," Phys. Rev. Lett. 92, 040403.
- Regal, C. A., and D. S. Jin, 2003, "Measurement of positive and negative scattering lengths in a Fermi gas of atoms," Phys. Rev. Lett. **90**, 230404.
- Regal, C. A., C. Ticknor, J. L. Bohn, and D. S. Jin, 2003a, "Creation of ultracold molecules from a Fermi gas of atoms," Nature 424, 47–50.
- Regal, C. A., C. Ticknor, J. L. Bohn, and D. S. Jin, 2003b, "Tuning p-wave interactions in an ultracold Fermi gas of atoms," Phys. Rev. Lett. 90, 053201.
- Reynolds, M. W., I. Shinkoda, R. W. Cline, and W. N. Hardy, 1986, "Observation of inverse predissocation of spin-polarized atomic hydrogen at low temperatures," Phys. Rev. B 34, 4912–4915.
- Rice, O. K., 1933, "Predissociation and the crossing of molecular potential energy curves," J. Chem. Phys. 1, 375–389.
- Riisager, K., 1994, "Nuclear halo states," Rev. Mod. Phys. 66, 1106–1116.
- Roati, G., G. de Mirandes, F. Ferlaino, G. Modugno, and M. Inguscio, 2004, "Atom interferometry with trapped Fermi gases," Phys. Rev. Lett. **92**, 230402.

- Roati, G., C. D'Errico, F. L, M. Fattori, C. Fort, M. Zaccanti, G. Modugno, M. Modugno, and M. Inguscio, 2008, "Anderson localization of a non-interacting BoseEinstein condensate," Nature 453, 895–898.
- Roati, G., M. Zaccanti, C. D'Errico, J. Catani, M. Modugno, A. Simoni, M. Inguscio, and G. Modugno, 2007, "39 K Bose-Einstein condensate with tunable interactions," Phys. Rev. Lett. 99, 010403.
- Roberts, J. L., N. R. Clausen, J. P. Burke Jr., C. H. Greene, E. A. Cornell, and C. E. Wieman, 1998, "Resonant magnetic field control of elastic scattering in cold ⁸⁵Rb," Phys. Rev. Lett. 81, 5109–5112.
- Roberts, J. L., N. R. Clausen, S. L. Cornish, E. A. Donley, E. A. Cornell, and C. E. Wieman, 2001, "Controlled collapse of a Bose-Einstein condensate," Phys. Rev. Lett. 86, 4211–4214.
- Roberts, J. L., N. R. Claussen, S. L. Cornish, and C. E. Wieman, 2000, "Magnetic field dependence of ultracold inelastic collisions near a Feshbach resonance," Phys. Rev. Lett. 85, 728–731.
- Rychtarik, D., B. Engeser, H.-C. Nägerl, and R. Grimm, 2004, "Two-dimensional Bose-Einstein condensate in an optical surface trap," Phys. Rev. Lett. 92, 173003.
- Sadeghpour, H., J. Bohn, M. Cavagnero, B. Esry, I. Fabrikant, J. Macek, and A. Rau, 2000, "Collisions near threshold in atomic and molecular physics," J. Phys. B 33, R93–R140.
- Samuelis, C., E. Tiesinga, T. Laue, M. Elbs, H. Knöckel, and E. Tiemann, 2000, "Cold atomic collisions studied by molecular spectroscopy," Phys. Rev. A 63, 012710.
- Santos, L., and T. Pfau, 2006, "Spin-3 chromium Bose-Einstein condensates," Phys. Rev. Lett. **96**, 190404.
- Santos, L., G. V. Shlyapnikov, and M. Lewenstein, 2003, "Roton-maxon spectrum and stability of trapped dipolar Bose-Einstein condensates," Phys. Rev. Lett. 90, 250403.
- Schmidt, R., S. Floerchinger, and C. Wetterich, 2008, "Three-body loss in lithium from functional renormalization," arXiv:0812.1191.
- Schöllkopf, W., and J. Toennies, 1994, "Nondestructive mass selection of small van der Waals clusters," Science **266**, 1345–1348.
- Schreck, F., L. Khaykovich, K. L. Corwin, G. Ferrari, T. Bourdel, J. Cubizolles, and C. Salomon, 2001, "Quasipure Bose-Einstein condensate immersed in a Fermi sea," Phys. Rev. Lett. 87, 080403.
- Schunck, C. H., Y. il Shin, A. Schirotzek, and W. Ketterle, 2008, "Determination of the fermion pair size in a resonantly interacting superfluid," Nature 454, 739–744.
- Schunck, C. H., M. W. Zwierlein, C. A. Stan, S. M. F. Raupach, W. Ketterle, A. Simoni, E. Tiesinga, C. J. Williams, and P. S. Julienne, 2005, "Feshbach resonances in fermionic ⁶Li," Phys. Rev. A 71, 045601.
- Seto, J. Y., R. J. LeRoy, J. Vergés, and C. Amiot, 2000, "Direct potential fit analysis of the $X^1\Sigma_g^+$ state of Rb₂: Nothing else will do!," J. Chem. Phys. **113**, 3067–3076.
- il Shin, Y., C. H. Schunck, A. Schirotzek, and W. Ketterle, 2008, "Phase diagram of a two-component Fermi gas with resonant interactions," Nature 451, 689.
- Simoni, A., F. Ferlaino, G. Roati, G. Modugno, and M. Inguscio, 2003, "Magnetic control of the interaction in ultracold K-Rb mixtures," Phys. Rev. Lett. 90, 163202.
- Simoni, A., M. Zaccanti, C. D'Errico, M. Fattori, G. Roati, M. Inguscio, and G. Modugno, 2008, "Near-threshold model for ultracold KRb dimers from interisotope Feshbach spectroscopy," Phys. Rev. A 77, 052705.

- Simonucci, S., P. Pieri, and G. C. Strinati, 2005, "Broad vs. narrow Fano-Feshbach resonances in the BCS-BEC crossover with trapped Fermi atoms," Europhys. Lett. **69**, 713–718.
- Smirne, G., R. M. Godun, D. Cassettari, V. Boyer, C. J. Foot, T. Volz, N. Syassen, S. Dürr, G. Rempe, M. D. Lee, K. Góral, and T. Köhler, 2007, "Collisional relaxation of Feshbach molecules and three-body recombination in ⁸⁷Rb Bose-Einstein condensates," Phys. Rev. A 75, 020702.
- Spence, D., and T. Noguchi, 1975, "Feshbach resonances associated with Rydberg states of the hydrogen halides," J. Chem. Phys. **63**, 505–514.
- Staanum, P., S. D. Kraft, J. Lange, R. Wester, and M. Weidemüller, 2006, "Experimental investigation of ultracold atom-molecule collisions," Phys. Rev. Lett. 96, 023201.
- Stan, C. A., M. W. Zwierlein, C. H. Schunck, S. M. F. Raupach, and W. Ketterle, 2004, "Observation of Feshbach resonances between two different atomic species," Phys. Rev. Lett. 93, 143001.
- Stenger, J., S. Inouye, M. R. Andrews, H.-J. Miesner, D. M. Stamper-Kurn, and W. Ketterle, 1999, "Strongly enhanced inelastic collisions in a Bose-Einstein condensate near Feshbach resonances," Phys. Rev. Lett. 82, 2422–2425.
- Stöferle, T., H. Moritz, K. Günter, M. Köhl, and T. Esslinger, 2006, "Molecules of fermionic atoms in an optical lattice," Phys. Rev. Lett. 96, 030401.
- Stoll, M., and T. Köhler, 2005, "Production of three-body Efimov molecules in an optical lattice," Phys. Rev. A 72, 022714
- Stoof, H. T. C., A. M. L. Jansen, J. M. V. A. Koelman, and B. J. Verhaar, 1989, "Decay of spin-polarized atomic hydrogen in the presence of a Bose condensate," Phys. Rev. A 39, 3157–3169.
- Stoof, H. T. C., J. M. V. A. Koelman, and B. J. Verhaar, 1988, "Spin-exchange and dipole relaxation rates in atomic hydrogen: rigorous and simplified calculations," Phys. Rev. B 38, 4688–4697.
- Strecker, K. E., G. B. Partridge, and R. G. Hulet, 2003, "Conversion of an atomic Fermi gas to a long-lived molecular Bose gas," Phys. Rev. Lett. **91**, 080406.
- Strecker, K. E., G. B. Partridge, A. G. Truscott, and R. G. Hulet, 2002, "Formation and propagation of matter-wave soliton trains," Nature 417, 150–153.
- Stringari, S., and L. Pitaevskii, 2003, Bose-Einstein condensation (Oxford University Press, London).
- Stuhler, J., A. Griesmaier, T. Koch, M. Fattori, T. Pfau, S. Giovanazzi, P. Pedri, and L. Santos, 2005, "Observation of dipole-dipole interaction in a degenerate quantum gas," Phys. Rev. Lett. 95, 150406.
- Stwalley, W. C., 1976, "Stability of spin-aligned hydrogen at low temperatures and high magnetic fields: New field-dependent scattering resonances and predissociations," Phys. Rev. Lett. 37, 1628–1631.
- Suno, H., B. D. Esry, and C. H. Greene, 2003, "Recombination of three ultracold fermionic atoms," Phys. Rev. Lett. 90, 053202.
- Syassen, N., D. M. Bauer, M. Lettner, D. Dietze, T. Volz, S. Dürr, and G. Rempe, 2007, "Atom-molecule Rabi oscillations in a Mott insulator," Phys. Rev. Lett. 99, 033201.
- Syassen, N., T. Volz, S. Teichmann, S. Dürr, and G. Rempe, 2006, "Collisional decay of ⁸⁷Rb Feshbach molecules at 1005.8 G," Phys. Rev. A 74, 062706.
- Szymanska, M. H., K. Góral, T. Köhler, and K. Burnett, 2005, "Conventional character of the BCS-BEC crossover

- in ultracold gases of ⁴⁰K," Phys. Rev. A **72**, 013610.
- Taglieber, M., A.-C. Voigt, T. Aoki, T. W. Hänsch, and K. Dieckmann, 2008, "Quantum degenerate two-species Fermi-Fermi mixture coexisting with a Bose-Einstein condensate," Phys. Rev. Lett. 100, 010401.
- Tang, K. T., J. M. Norbeck, and P. R. Certain, 1976, "Upper and lower bounds of two- and three-body dipole, quadrupole, and octupole van der Waals coefficients for hydrogen, noble gas, and alkali atom interactions," J. Chem. Phys. 64, 3063–3074.
- Taylor, J. R., 1972, Scattering theory (John Wiley and Sons, Inc., New York).
- Thalhammer, G., G. Barontini, L. D. Sarlo, J. Catani, F. Minardi, and M. Inguscio, 2008, "Double species Bose-Einstein condensate with tunable interspecies interactions," Phys. Rev. Lett. 100, 210402.
- Thalhammer, G., M. Theis, K. Winkler, R. Grimm, and J. Hecker Denschlag, 2005, "Inducing an optical Feshbach resonance via stimulated Raman coupling," Phys. Rev. A 71, 033403.
- Thalhammer, G., K. Winkler, F. Lang, S. Schmid, R. Grimm, and J. Hecker Denschlag, 2006, "Long-lived Feshbach molecules in a 3D optical lattice," Phys. Rev. Lett. 96, 050402.
- Theis, M., G. Thalhammer, K. Winkler, M. Hellwig, G. Ruff, R. Grimm, and J. Hecker Denschlag, 2004, "Tuning the scattering length with an optically induced Feshbach resonance," Phys. Rev. Lett. 93, 123001.
- Thomas, N. R., N. Kjærgaard, P. S. Julienne, and A. C. Wilson, 2004, "Imaging of s and d partial-wave interference in quantum scattering of identical bosonic atoms," Phys. Rev. Lett. **93**, 173201.
- Thompson, S. T., E. Hodby, and C. E. Wieman, 2005a, "Spontaneous dissociation of ⁸⁵Rb Feshbach molecules," Phys. Rev. Lett. 94, 020401.
- Thompson, S. T., E. Hodby, and C. E. Wieman, 2005b, "Ultracold molecule production via a resonant oscillating magnetic field," Phys. Rev. Lett. 95, 190404.
- Ticknor, C., C. A. Regal, D. S. Jin, and J. L. Bohn, 2004, "Multiplet structure of Feshbach resonances in non-zero partial waves," Phys. Rev. A 69, 042712.
- Tiesinga, E., B. J. Verhaar, and H. T. C. Stoof, 1993, "Threshold and resonance phenomena in ultracold groundstate collisions," Phys. Rev. A 47, 4114–4122.
- Tiesinga, E., C. J. Williams, F. H. Mies, and P. S. Julienne, 2000, "Interacting atoms under strong quantum confinement," Phys. Rev. A 61, 063416.
- Timmermans, E., P. Tommasini, M. Hussein, and A. Kerman, 1999, "Feshbach resonances in atomic Bose-Einstein condensates," Phys. Rep. 315, 199–230.
- Tojo, S., M. Kitagawa, K. Enomoto, Y. Kato, Y. Takasu, M. Kumakura, and Y. Takahashi, 2006, "High-resolution photoassociation spectroscopy of ultracold ytterbium atoms by using the intercombination transition," Phys. Rev. Lett. 96, 153201.
- Truscott, A. G., K. E. Strecker, W. I. McAlexander, G. B. Partridge, and R. G. Hulet, 2001, "Observation of Fermi pressure in a gas of trapped atoms," Science 291, 2570–2572.
- Tsai, C. C., R. S. Freeland, J. M. Vogels, H. M. J. M. Boesten, B. J. Verhaar, and D. J. Heinzen, 1997, "Two-color photoassociation spectroscopy of ground State Rb₂," Phys. Rev. Lett. 79, 1245–1248.
- Tscherbul, T. V., and R. V. Krems, 2008, "Quantum the-

- ory of chemical reactions in the presence of electromagnetic fields," J. Chem. Phys. **129**, 034112.
- Vogels, J. M., R. S. Freeland, C. C. Tsai, B. J. Verhaar, and D. J. Heinzen, 2000, "Coupled singlet-triplet analysis of two-color cold-atom photoassociation spectra," Phys. Rev. A 61, 043407.
- Vogels, J. M., C. C. Tsai, R. S. Freeland, S. J. J. M. F. Kokkelmans, B. J. Verhaar, and D. J. Heinzen, 1997, "Prediction of Feshbach resonances in collisions of ultracold rubidium atoms." Phys. Rev. A 56, R1067–R1070.
- Vogels, J. M., B. J. Verhaar, and R. H. Blok, 1998, "Diabatic models for weakly bound states and cold collisions of ground-state alkali-metal atoms," Phys. Rev. A 57, 4049–4052.
- Voigt, A.-C., M. Taglieber, L. Costa, T. Aoki, W. Wieser, T. W. Hänsch, and K. Dieckmann, 2008, "Ultracold heteronuclear Fermi-Fermi molecules," Phys. Rev. Lett. 102, 020405.
- Volz, T., S. Dürr, S. Ernst, A. Marte, and G. Rempe, 2003, "Characterization of electic scattering near a Feshbach resonance in ⁸⁷Rb," Phys. Rev. A 68, 010702.
- Volz, T., S. Dürr, N. Syassen, G. Rempe, E. van Kempen, and S. Kokkelmans, 2005, "Feshbach spectroscopy of a shape resonance," Phys. Rev. A 72, 010704.
- Volz, T., N. Syassen, D. Bauer, E. Hansis, S. Dürr, and G. Rempe, 2006, "Preparation of a quantum state with one molecule at each site of an optical lattice," Nature Phys. 2, 692–695.
- von Stecher, J., J. P. D'Incao, and C. H. Greene, 2009, "Signatures of universal four-body phenomena and their relation to the Efimov effect," Nature Phys. 5, 417–421.
- Vuletić, V., A. J. Kerman, C. Chin, and S. Chu, 1999, "Observation of low-field Feshbach resonances in collsions of cesium atoms." Phys. Rev. Lett. 82, 1406.
- Wang, Y., and B. D. Esry, 2009, "Efimov trimer formation via ultracold four-body recombination," Phys. Rev. Lett. 102, 133201.
- Weber, C., G. Barontini, J. Catani, G. Thalhammer, M. Inguscio, and F. Minardi, 2008, "Association of ultracold double-species bosonic molecules," Phys. Rev. A 78, 061601(R).
- Weber, J. M., E. Leber, M.-W. Ruf, and H. Hotop, 1999, "Nuclear-excited Feshbach resonances in electron attachment to molecular clusters," Phys. Rev. Lett. 82, 515–519.
- Weber, T., J. Herbig, M. Mark, H.-C. Nägerl, and R. Grimm, 2003a, "Bose-Einstein condensation of cesium," Science **299**, 232–235.
- Weber, T., J. Herbig, M. Mark, H.-C. Nägerl, and R. Grimm, 2003b, "Three-body recombination at large scattering lengths in an ultracold atomic gas," Phys. Rev. Lett. 91, 123201.
- Weck, P. F., and N. Balakrishnan, 2005, "Quantum dynamics of the Li+HF \rightarrow H+LiF reaction at ultralow temperatures," J. Chem. Phys. **122**, 154309.
- Werner, J., A. Griesmaier, S. Hensler, J. Stuhler, T. Pfau, A. Simoni, and E. Tiesinga, 2005, "Observation of Feshbach resonances in an ultracold gas of ⁵²Cr," Phys. Rev. Lett. 94, 183201.
- Widera, A., O. Mandel, M. Greiner, S. Kreim, T. W. Hänsch, and I. Bloch, 2004, "Entanglement interferometry for precision measurement of atomic scattering properties," Phys. Rev. Lett. 92, 160406.
- Wille, E., F. M. Spiegelhalder, G. Kerner, D. Naik, A. Trenkwalder, G. Hendl, F. Schreck, R. Grimm, T. G.

- Tiecke, J. T. M. Walraven, S. J. J. M. F. Kokkelmans, E. Tiesinga, and P. S. Julienne, 2008, "Exploring an ultracold Fermi-Fermi mixture: Interspecies Feshbach resonances and scattering properties of ⁶Li and ⁴⁰K," Phys. Rev. Lett. **100**, 053201.
- Winkler, K., F. Lang, G. Thalhammer, P. v. d. Straten, R. Grimm, and J. Hecker Denschlag, 2007, "Coherent optical transfer of Feshbach molecules to a lower vibrational state," Phys. Rev. Lett. 98, 043201.
- Winkler, K., G. Thalhammer, F. Lang, R. Grimm, J. Hecker Denschlag, A. J. Daley, A. Kantian, H. P. Büchler, and P. Zoller, 2006, "Repulsively bound atom pairs in an optical lattice," Nature 441, 853–856.
- Wynar, R., R. S. Freeland, D. J. Han, C. Ryu, and D. J. Heinzen, 2000, "Molecules in a Bose-Einstein condensate," Science 287, 1016–1019.
- Xu, K., T. Mukaiyama, J. Abo-Shaeer, J. Chin, D. Miller, and W. Ketterle, 2003, "Formation of quantum-degenerate sodium molecules," Phys. Rev. Lett. 91, 210402.
- Yan, Z.-C., J. F. Babb, A. Dalgarno, and G. W. F. Drake, 1996, "Variational calculations of dispersion coefficients for interactions among H, He, and Li atoms," Phys. Rev. A 54, 2824–2833.
- Yi, S., and L. You, 2002, "Probing dipolar effects with condensate shape oscillation," Phys. Rev. A 66, 013607.
- Yurovsky, V. A., M. Olshanii, and D. S. Weiss, 2008, "Collisions, correlations, and integrability in atom wave guides," Adv. At. Mol. Opt. Phys. 55, 61–138.
- Zaccanti, M., C. D'Errico, F. Ferlaino, G. Roati, M. Inguscio, and G. Modugno, 2006, "Control of the interaction in a Fermi-Bose mixture," Phys. Rev. A 41, 041605(R).
- Zaccanti, M., G. Modugno, C. D'Errico, M. Fattori, G. Roati, and M. Inguscio, 2008, talk at DAMOP-2008, 27-31 May 2008, State College, Pennsylvania, USA.
- Zahzam, N., T. Vogt, M. Mudrich, D. Comparat, and P. Pillet, 2006, "Atom-molecule collisions in an optically trapped gas," Phys. Rev. Lett. 96, 023202.
- Zelevinsky, T., M. M. Boyd, A. D. Ludlow, T. Ido, J. Ye, R. Ciurylo, P. Naidon, and P. S. Julienne, 2006, "Narrow

- line photoassociation in an optical lattice," Phys. Rev. Lett. **96**, 203201.
- Zhang, J., E. G. M. van Kempen, T. Bourdel, L. Khaykovich,
 J. Cubizolles, F. Chevy, M. Teichmann, L. Tarruell, S. J.
 J. M. F. Kokkelmans, and C. Salomon, 2005, in Atomic Physics 19, edited by L. Marcassa, V. Bagnato, and
 K. Helmerson, volume 770 of AIP conference proceedings,
 228–237, ISBN 0-7354-0255-8, ISSN 0094-243X, 19th International Conference on Atomic Physics, Rio de Janeiro,
 Brazil, July 25-30, 2004.
- Zhang, J., E. G. M. van Kempen, T. Bourdel, L. Khaykovich, J. Cubizolles, F. Chevy, M. Teichmann, L. Taruell, S. J. J. M. F. Kokkelmans, and C. Salomon, 2004, "P-wave Feshbach resonances of ultracold ⁶Li," Phys. Rev. A 70, 030702(R).
- Zirbel, J. J., K.-K. Ni, S. Ospelkaus, J. P. D'Incao, C. E. Wieman, J. Ye, and D. S. Jin, 2008a, "Collisional stability of fermionic Feshbach molecules," Phys. Rev. Lett. 100, 143201.
- Zirbel, J. J., K.-K. Ni, S. Ospelkaus, T. L. Nicholson, M. L. Olsen, C. E. Wieman, J. Ye, D. S. Jin, and P. S. Julienne, 2008b, "Heteronuclear molecules in an optical dipole trap," Phys. Rev. A 78, 013416.
- Zwierlein, M. W., J. R. Abo-Shaeer, A. Schirotzek, C. H. Schunck, and W. Ketterle, 2005, "Vortices and superfluidity in a strongly interacting Fermi gas," Nature 435, 1047–1051.
- Zwierlein, M. W., A. Schirotzek, C. H. Schunck, and W. Ketterle, 2006, "Fermionic superfluidity with imbalanced spin populations," Science 311, 492–496.
- Zwierlein, M. W., C. A. Stan, C. H. Schunck, S. M. F. Raupach, S. Gupta, Z. Hadzibabic, and W. Ketterle, 2003, "Observation of Bose-Einstein condensation of molecules," Phys. Rev. Lett. 91, 250401.
- Zwierlein, M. W., C. A. Stan, C. H. Schunck, S. M. F. Raupach, A. J. Kerman, and W. Ketterle, 2004, "Condensation of pairs of fermionic atoms near a Feshbach resonance," Phys. Rev. Lett. 92, 120403.