Written Exam
Modern Control Systems
(Et 3-015)

On Wednesday the 3rd of November 1999 from 14:00 PM to 17:00 PM

Read the following VERY carefully.

- The exam is composed of 2 Parts: theory and exercises. The £rst part is WITHOUT any books or notes and it will last **40 minutes**. At the end of the 40 minutes, the exam will be collected and after 10 minutes the second part of **130 minutes** will start. You are allowed to use anything you want for the second part:
  - Part I, 40 minutes (end at 14:40) , no books, hand in at the end.
  - Break, 10 minutes, (end at 14:50) collect books, sheets and the material you are allowed to use for Part II.
  - Part II, 130 minutes (end at 17:00), with books, sheets.....

  If you are ready with Part I earlier than the time given, you can start working on Part II, but **WITHOUT BOOK** and on a **NEW SHEET OF PAPER**.
- NEVER talk with your neighbor.
- If the answers are not easily readable, the corresponding answer will be given 0 points. Therefore WRITE CLEARLY.
- Read every question well before answering.
- Write ALL your reasoning steps on paper.
- Write your name and student number clearly readable on EACH piece of paper.
- Good luck !

Technische Universiteit Delft
Faculteit der Informatie Technologie en Systemen
Vakgroep Regeltechniek
Mekelweg 4
2628 CD Delft

Part I: theory

(Closed Book)

Question 1

(weight:3)

Given: For a transfer function $F(s)$, due to the Cauchy's theorem of complex analysis, we have that:

$$\frac{1}{2\pi j} \oint \frac{F'(s)}{F(s)}\, ds = Z - P$$

where the integral is along a closed curve encircling Z zeros and P poles of $F(s)$.

Asked: Prove the Nyquist criterium formula

$$N_{enc} = Z - P$$

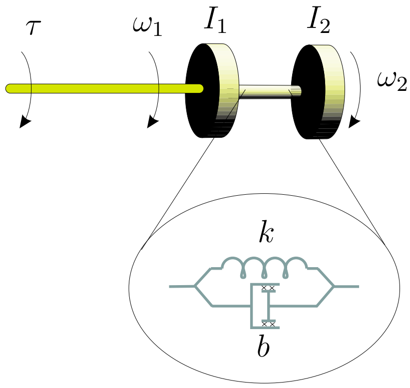
where N_{enc} is the number of clockwise encirclements around the origin of $F(s)$ for values of s along a closed curve enclosing Z zeros of $F(s)$ and P poles of $F(s)$.

Part II: exercises

(Open Book)

Question 2 (weight:2)

Given: The following system with input torque τ, representing a typical transmission.



The system is composed of two turning wheels of rotational inertia I_1 and I_2 and an axis connecting the two wheels which is characterised to be elastic and at the same time dissipating energy as schematically shown in the £gure. The axis used to apply the torque τ can be considered perfectly rigid and of neglectable inertia. The energy stored in the elastic axis k is equal to

$$E(\Delta\theta) = \arctan^2 \Delta\theta$$

and the damping effect can be modeled by the linear relation

$$\tau = b\Delta\omega$$

Asked:

1. Draw a bond-graph of the system considering τ as the input torque.
2. Annotate the bond-graph and calculate from it the differential equation of the form

$$\dot{x} = f(x, \tau)$$

describing the dynamics of the system. I recall that

$$\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$$

Question 3

(weight:3)

Given: The following system in state space form is given:

$$\dot{x} = \begin{pmatrix} 3 & b \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \quad (1)$$

$$y = \begin{pmatrix} 2 & 0 \end{pmatrix} x \quad (2)$$

Asked:

1. Calculate for what values of b is the system controllable.
2. Calculate for what values of b is the system observable.
3. Sketch the amplitude frequency response of the system for $b = 1$.
4. Calculate the feedback gains K such that the poles of the closed loop system for $b = 1$ have a natural frequency of $\omega_n = 3$ rad/s and a damping of $\zeta = 0.9$.

Question 4 (weight:3)

Given: The transfer function of a system $G(s)$ is:

$$G(s) = \frac{K(\tau s + 1)}{(2s+1)(4s+1)(s+1)}$$

with $\tau = 8$ sec. The system is put in a feedback loop as follows:

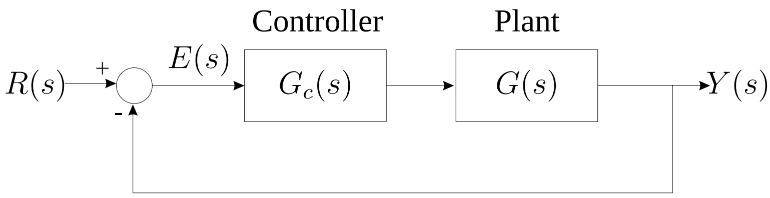


Figure 1: The feedback system

Asked:

1. Sketch the Nyquist plot of $G(s)$. For what values of K is the closed loop system stable?
2. Write for this case the 6^{th} order **polynomial** equation which should be solved to calculate the cross-over frequency when $G_c(s) = 1$ and $K = 10$: **you do not need to solve the equation !**
3. Suppose that the solutions to the polynomial equation of the previous point are:

   $$\begin{pmatrix} 0+3.2629j & 0-3.2629j & 3.0577 & -3.0577 & 0+0.1247j & 0-0.1247j \end{pmatrix}.$$

   What is the phase margin of the system ?
4. Thermal effects create drift in the time constant τ and it is useful to analyse the consequence of such a variation. Draw the root locus of the feedback system for $G_c(s) = 1$ and $K = 10$ for variations of the time constant τ from 0 to ∞. To make this part it is useful to know that the solutions to the equation $8s^3 + 14s^2 + 7s + 11 = 0$ are

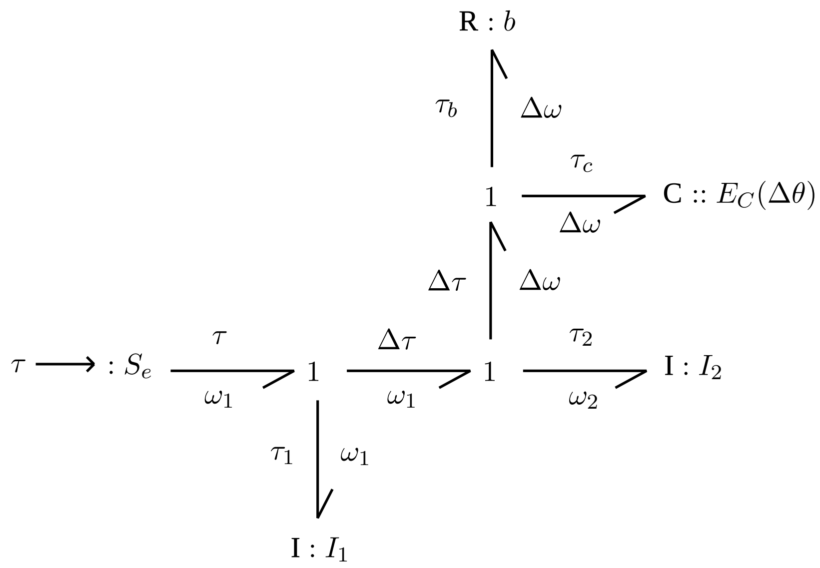
   $$\begin{pmatrix} -1.70883 & -0.0205858 + 0.896784j & -0.0205858 - 0.896784j \end{pmatrix}$$

5. To get a good response, it is asked to have a *phase margin* of about $60°$. Calculate a £rst approximation of a *phase lead compensator* $G_c(s)$ with 0 dB *DC gain* which gives an improvement of the *phase margin* of about $30°$.

Answers exam of November 3, 1999

Question 2

1. Bond-graph



2. Nonlinear differential equations:

$$\begin{pmatrix} \dot{p_1} \\ \dot{p_2} \\ \dot{\Delta\theta} \end{pmatrix} = \begin{pmatrix} -\frac{b}{I_1} \cdot p_1 + \frac{b}{I_2} \cdot p_2 - 2 \cdot \frac{arctan \Delta\theta}{1+\Delta\theta^2} + \tau \\ \frac{b}{I_1} \cdot p_1 - \frac{b}{I_2} \cdot p_2 + 2 \cdot \frac{arctan \Delta\theta}{1+\Delta\theta^2} \\ \frac{1}{I_1} \cdot p_1 - \frac{1}{I_2} \cdot p_2 \end{pmatrix}$$

Question 3

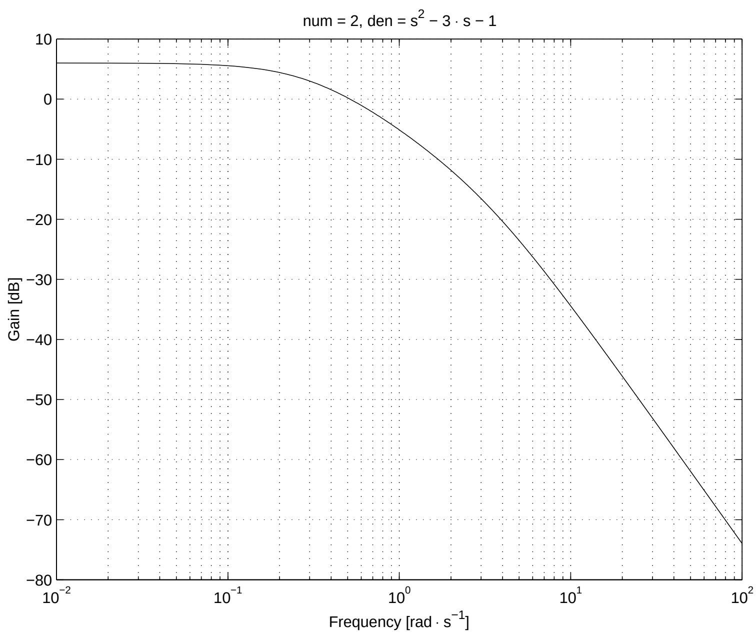
$$\underline{\dot{x}} = \begin{pmatrix} 3 & b \\ 1 & 0 \end{pmatrix} \cdot \underline{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \underline{u}$$

$$y = \begin{pmatrix} 2 & 0 \end{pmatrix} \cdot \underline{x}$$

1. $b \neq 0$
2. $b \neq 0$
3. Transfer function:

$$G(s) = H \cdot (sI - F)^{-1} \cdot G = \frac{2}{s^2 - 3 \cdot s - 1}$$

Poles: $p_1 = \frac{3}{2} + \frac{1}{2} \cdot \sqrt{13}$, $p_2 = \frac{3}{2} - \frac{1}{2} \cdot \sqrt{13}$



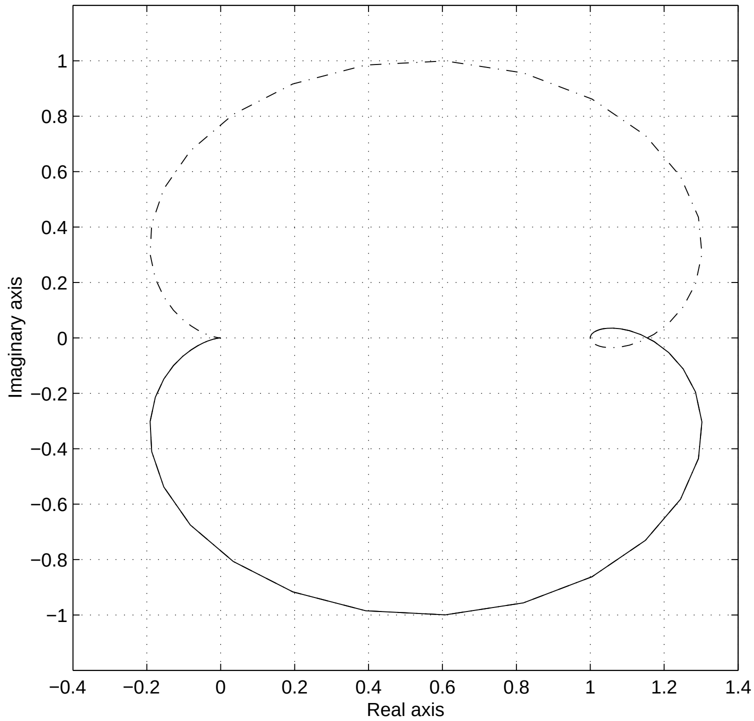
4. $K = \begin{pmatrix} 35.2 & 8.4 \end{pmatrix}$

Question 4

$$G(s) = \frac{K \cdot (\tau \cdot s + 1)}{(2 \cdot s + 1) \cdot (4 \cdot s + 1) \cdot (s + 1)}$$

$\tau = 8$ sec.

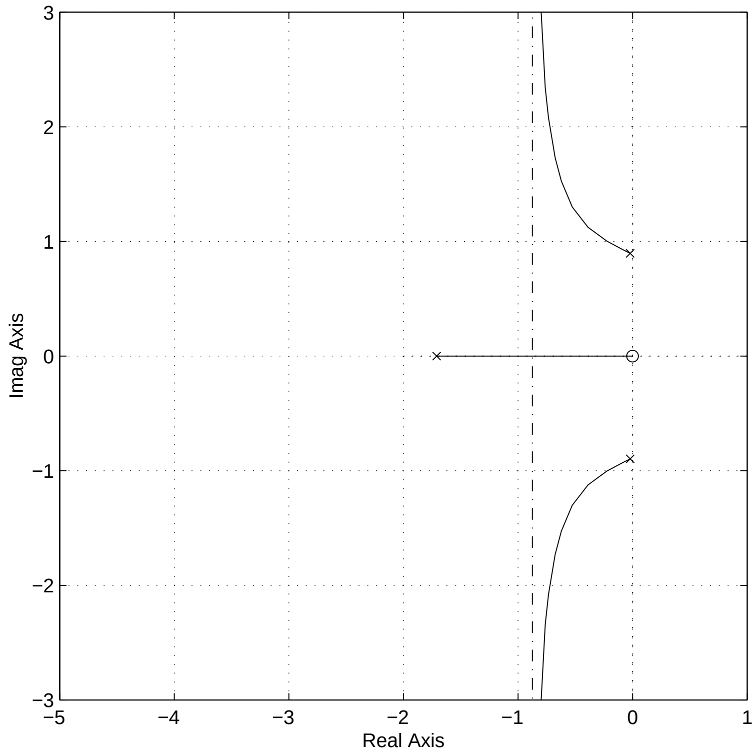
1. K is stable for: $0 < K < \infty$



2. $64 \cdot {\omega_c}^6 + 84 \cdot {\omega_c}^4 - 6379 \cdot {\omega_c}^2 - 99 = 0$
3. Cross-over frequency: $\omega_c = 3.0577\ rad \cdot s^{-1}$

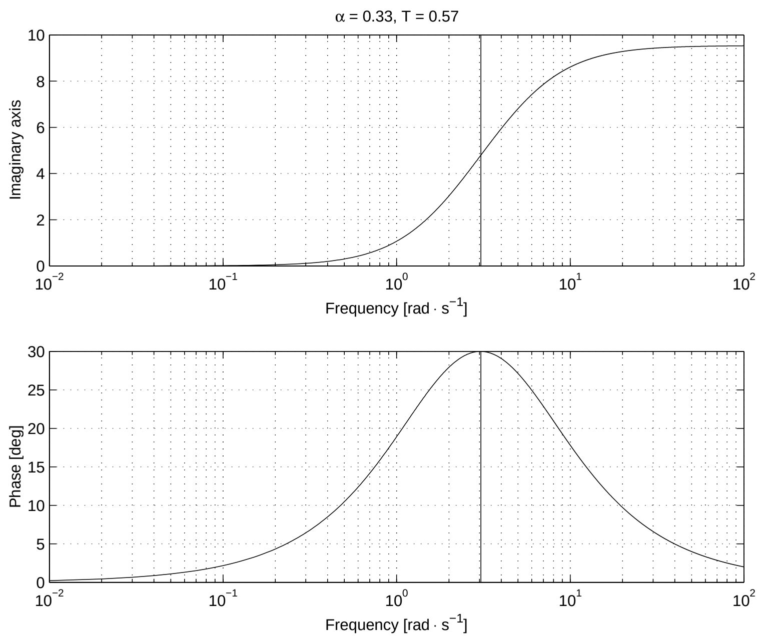
   $GM = \angle G(i \cdot \omega_c) + 180° = 29.7°$

4. Root-locus for $\tau : 0 \rightarrow \infty$



5. Lead-compensator:

$$D_{lead}(s) = \frac{T \cdot s + 1}{\alpha \cdot T \cdot s + 1} = \frac{0.57 \cdot s + 1}{0.19 \cdot s + 1}$$



Written Exam
Modern Control Systems
(Et 3-015)

On Tuesday the 1st of February 2000 from 9:00 AM to 12:00 PM

Read the following VERY carefully.

- The exam is composed of 2 Parts: theory and exercises. The £rst part is WITHOUT any books or notes and it will last **40 minutes**. At the end of the 40 minutes, the exam will be collected and after 10 minutes the second part of **130 minutes** will start. You are allowed to use anything you want for the second part:
  - Part I, 40 minutes (end at 9:40) , no books, hand in at the end.
  - Break, 10 minutes, (end at 9:50) collect books, sheets and the material you are allowed to use for Part II.
  - Part II, 130 minutes (end at 12:00), with books, sheets.....

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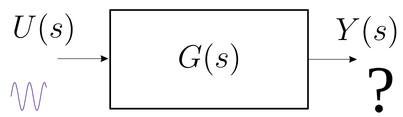
Part I: theory

(Closed Book)

Question 1

(weight:3)

Given: Consider a linear, time invariant, strictly stable, system



with transfer function $G(s)$. Suppose to supply as an input a sinusoidal input equal to:

$$u(t) = M \sin \omega t$$

of magnitude $M > 0$ and frequency ω rad/sec.

Asked: Prove that, waiting long enough, the output of the system for the given sinusoidal input will be

$$y(t) \simeq M\,|G(j\omega)| \sin(\omega t + \angle \underline{G(j\omega)}\,)$$

I recall that:

$$\mathcal{L}\left\{\sin(\omega t)\right\} = \frac{\omega}{s^2+\omega^2} \quad , \quad \mathcal{L}\left\{e^{\alpha t}\right\} = \frac{1}{s-\alpha} \quad , \quad G(s^*) = G(s)^*$$

and the Euler formula

$$\sin(\alpha) = \frac{e^{j\alpha} - e^{-j\alpha}}{2j}.$$

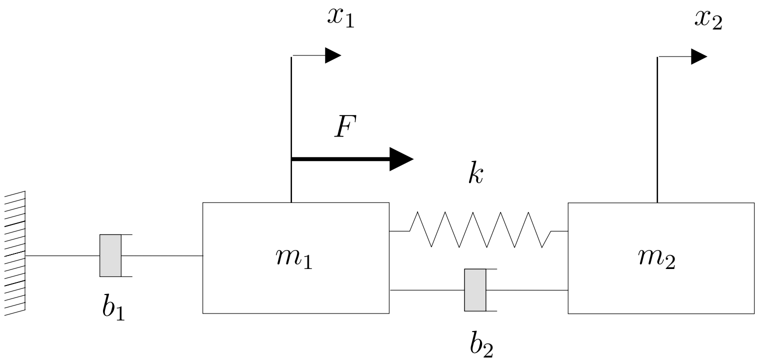
Part II: exercises

(Open Book)

Question 2

(weight:3)

Given: The input of the system illustrated below is the force F applied to the mass m_1.



The system is composed of two masses m_1 and m_2, damper b_1 connects mass m_1 to the wall, while damper b_2 and spring k connect the two masses. The energy stored in the spring k is equal to

$$E(\Delta x) = \frac{1}{2} \cdot k_1 \cdot \Delta x^2 + \frac{1}{4} \cdot k_2 \cdot \Delta x^4$$

where $\Delta x = x_1 - x_2$. The damping effects can be modeled by the linear relation

$$F_b = b \cdot v$$

where for each damper the the accompanying damping coef£cient b and velocity v needs to be substituted.

Asked:

1. Draw a bond-graph of the system considering F as the input force.
2. Annotate the bond-graph and calculate from it the state space differential equation of the form

$$\dot{\underline{x}} = f(\underline{x}, F)$$

describing the dynamics of the system.

Question 3 (weight:1)

Given: The following system in state space form is given:

$$\dot{\underline{x}} = \begin{pmatrix} b & 0 \\ 2 & 1 \end{pmatrix} \underline{x} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u \qquad (1)$$
$$y = \begin{pmatrix} 0 & 1 \end{pmatrix} \underline{x}$$

Asked:

1. Calculate for what values of b is the system controllable.
2. Calculate for what values of b is the system observable.
3. Sketch the **amplitude** frequency response of the system for $b = 1$.

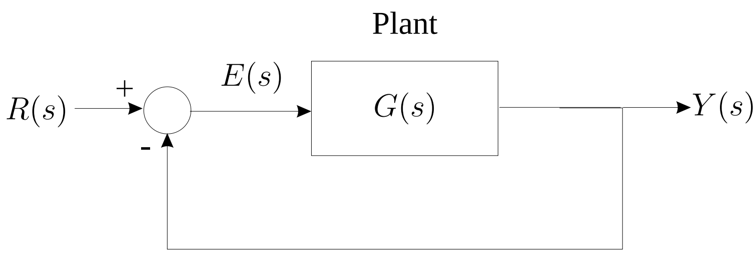
Question 4

(weight:3)

Given: The transfer function of a system $G(s)$ is:

$$G(s) = \frac{K \cdot (\tau s + 1)}{(2s+1) \cdot (s+1)}$$

The system is put in a feedback loop as follows:



Asked: Thermal effects create drift in the time constant τ and it is useful to analyse the consequence of such a variation. Draw the root locus for the poles of the transfer function $\frac{Y(s)}{R(s)}$ for $K = 8.125$ and for variations of the time constant τ from 0 to ∞.

Question 5

(weight:2)

Given: The transfer function of a system $G(s)$ is:

$$G(s) = \frac{K \cdot (0.5s + 1)}{(s+1) \cdot (3s+1)}$$

Asked:

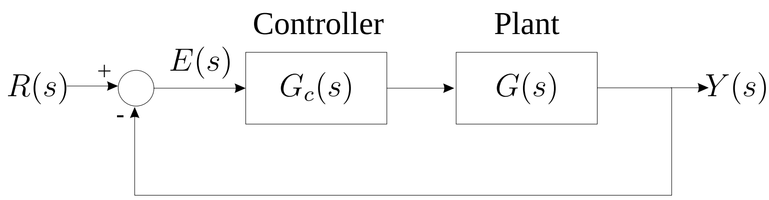
1. Sketch the Nyquist plot of $G(s)$ for $K = 2$ (**Be precise about the shape or the Nyquist plot around the origin !**)
2. Give the value of the gain margin of $G(s)$ for $K = 2$.
3. Show graphically in your sketch the phase margin of $G(s)$ for $K = 2$.

Question 6 (weight:2)

Given: The transfer function of a system $G(s)$ is:

$$G(s) = \frac{K \cdot (0.1s + 1)}{(s+1) \cdot (2s+1) \cdot (3s+1)}$$

with $K = 5$. The system is put in a feedback loop as follows:



Asked:

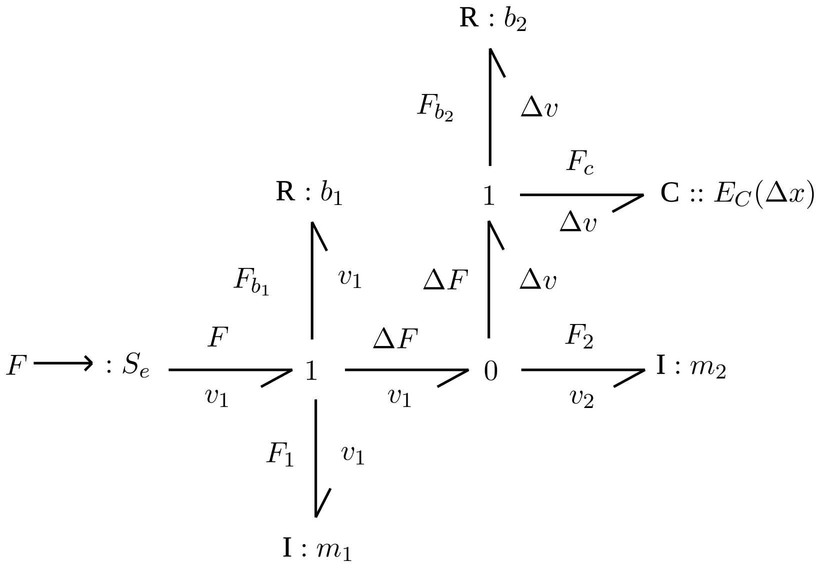
1. To get a good response, it is asked to have a *phase margin* of about $60°$. Calculate a £rst approximation of a *phase lead compensator* $G_c(s)$ with 0 dB *DC Gain* which gives an improvement of the *phase margin* of about $30°$. The cross-over frequency of the closed-loop system with $G_c(s) = 1$ is equal to $\omega_c = 0.71$ rad/sec.

2. What is the system type of the closed-loop system? The *steady state error* should be approximately $e_{ss} = 1/20$. This can be obtained by implementing a *phase lag compensator*. The controller $G_c(s)$ consists of the *phase lag compensator* in series with the computed *phase lead compensator*. Design the *phase lag compensator* and make sure that it does not generate an extra gain at frequencies higher than 0.25 rad/sec.

Answers exam of February 1, 2000

Question 2

1. Bond-graph



2. Nonlinear differential equations:

$$\begin{pmatrix} \dot{p_1} \\ \dot{p_2} \\ \dot{\Delta x} \end{pmatrix} = \begin{pmatrix} -(\frac{b_1}{m_1} + \frac{b_2}{m_1}) \cdot p_1 + \frac{b_2}{m_2} \cdot p_2 - (k_1 + k_2 \cdot \Delta x^2) \cdot \Delta x + F \\ \frac{b_2}{m_1} \cdot p_1 - \frac{b_2}{m_2} \cdot p_2 + (k_1 + k_2 \cdot \Delta x^2) \cdot \Delta x \\ \frac{1}{m_1} \cdot p_1 - \frac{1}{m_2} \cdot p_2 \end{pmatrix}$$

Question 3

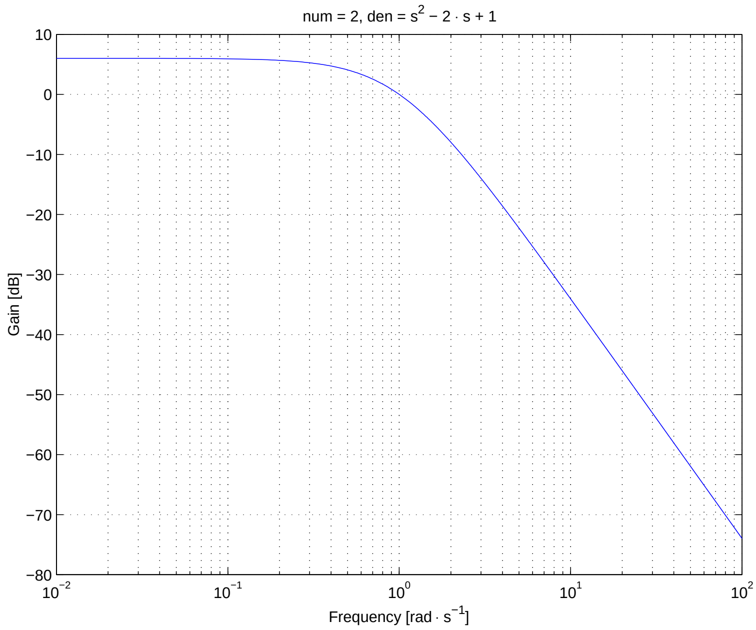
$$\underline{\dot{x}} = \begin{pmatrix} b & 0 \\ 2 & 1 \end{pmatrix} \cdot \underline{x} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \underline{u}$$

$$y = \begin{pmatrix} 0 & 1 \end{pmatrix} \cdot \underline{x}$$

1. All values of b
2. All values of b
3. Transfer function:

$$G(s) = H \cdot (sI - F)^{-1} \cdot G = \frac{2}{s^2 - 2 \cdot s + 1} = \frac{2}{(s-1)^2}$$

Poles: $p_{1,2} = 1$ (Instable system)

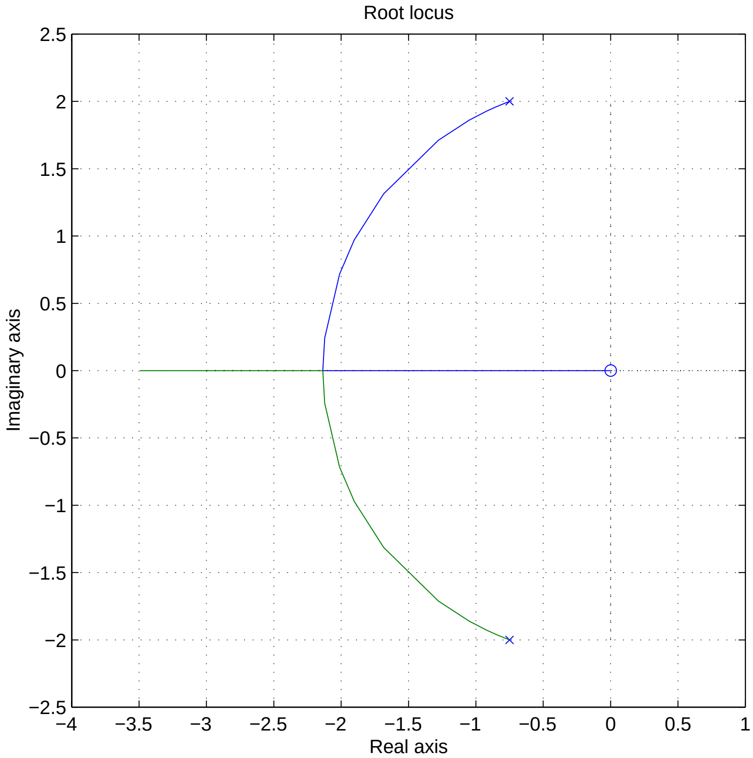


Question 4

$$F(s) = -\frac{1}{\tau} \to F(s) = \frac{K \cdot s}{2s^2 + 3s + (K+1)}$$

$K = 8.125.$

Zero: $z_1 = 0$, Poles: $p_{1,2} = -0.75 \pm 2 \cdot i$

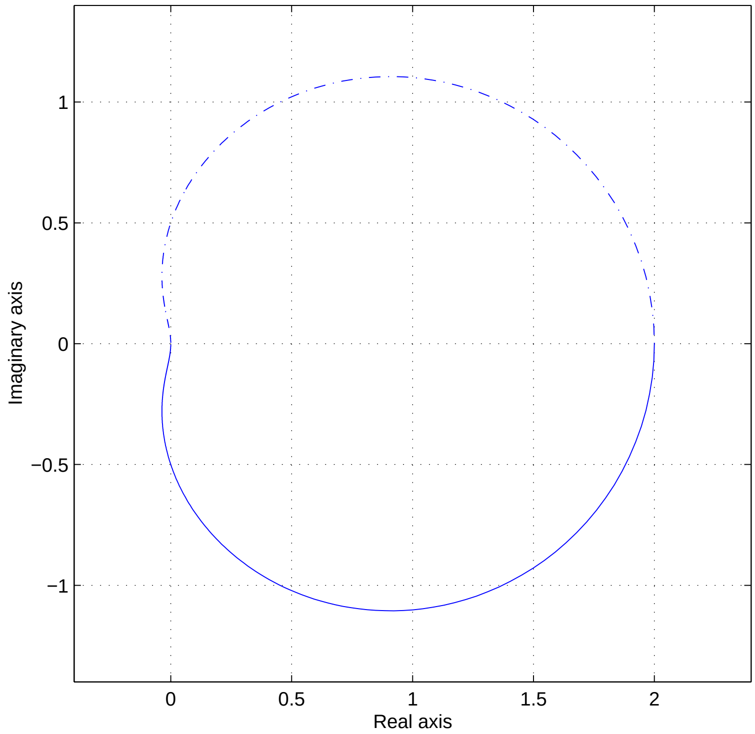


Question 5

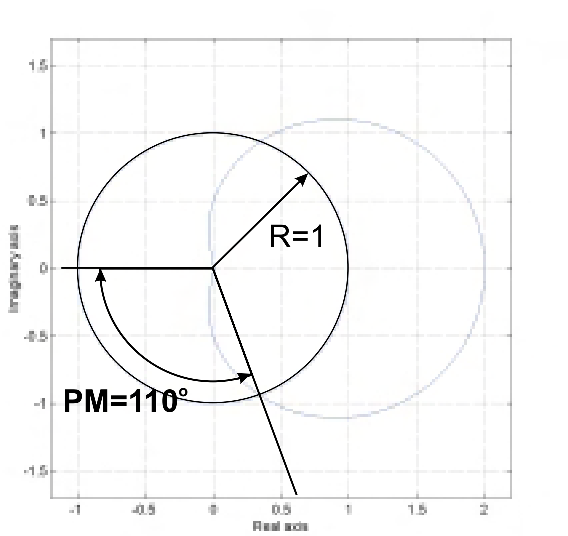
$$G(s) = \frac{K \cdot (0.5s + 1)}{(s + 1) \cdot (3s + 1)}$$

$K = 2.$

1. Nyquist plot.



2. $GM = \infty$
3. $PM = 110°$

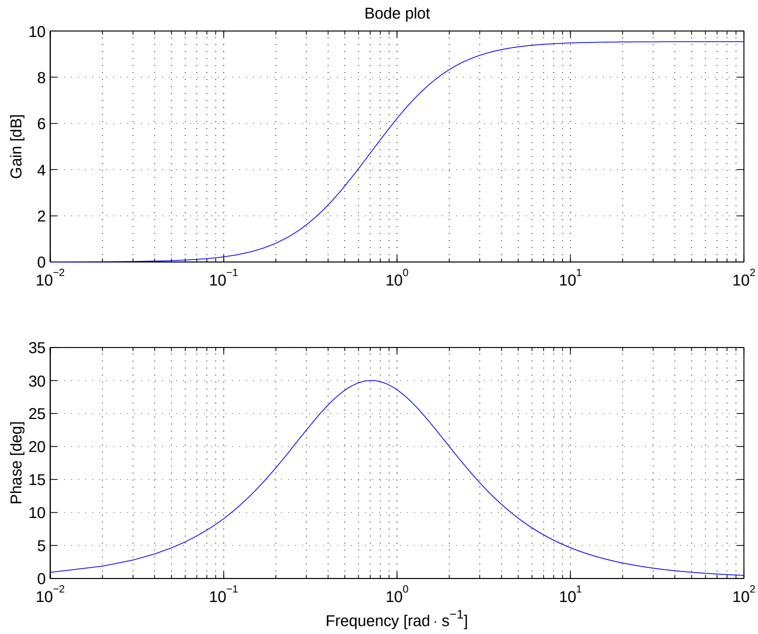


Question 6

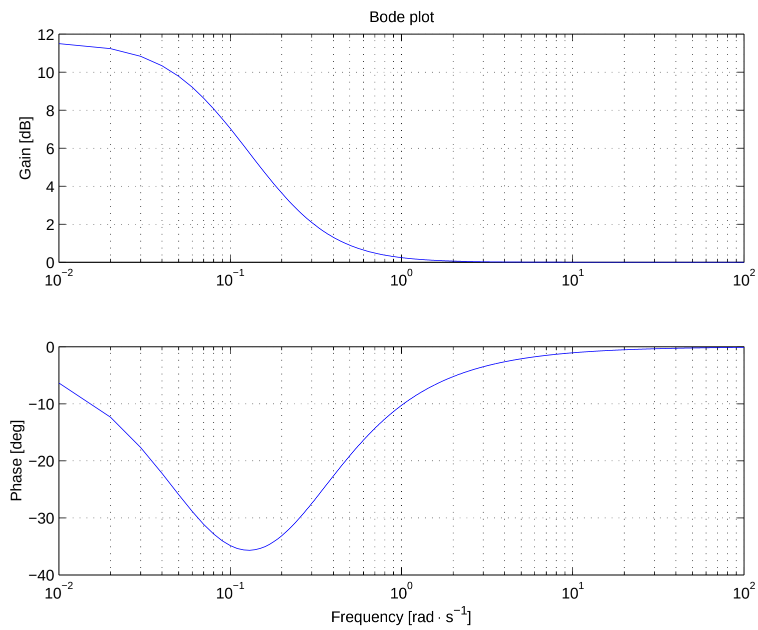
$$G(s) = \frac{K \cdot (0.1s + 1)}{(s+1) \cdot (2s+1) \cdot (3s+1)}$$

$$K = 5.$$

1. $G_{lead} = \frac{2.44s+1}{0.81s+1}$



2. System type: 0
   $G_{lag} = 3.8 \cdot \frac{4s+1}{15.2s+1}$



Written Exam
Modern Control Systems
(Et 3-015)

On Wednesday the 1st of November 2000 from 2:00 PM to 5:00 PM

Read the following VERY carefully.

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  - Break, 10 minutes, (end at 2:50 PM) collect books, sheets and the material you are allowed to use for Part II.
  - Part II, 130 minutes (end at 5:00 PM), with books, sheets.....

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- Write ALL your reasoning steps on paper.
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- Good luck !

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Vakgroep Regeltechniek
Mekelweg 4
2628 CD Delft

Part I: theory

(Closed Book)

Question 1

(weight:1)

Define the vector margin of the loop transfer function $L(s)$, and prove that it is equal to the inverse of the peaking value of the module of the sensitivity function $S(j\omega)$. Write all your reasoning and be as precise as possible.

Question 2

(weight:4)

Given: A system in state space form:

$$\dot{x} = Fx + Gu$$
$$y = Hx + Ju$$

Asked: Show using the theorem of Carley-Hamilton that we can express the input-output dynamics in the following form:

$$\sum_{i=0}^{n} a_i \frac{d^i}{dt^i} y(t) = \sum_{j=0}^{m} b_j \frac{d^j}{dt^j} u(t)$$

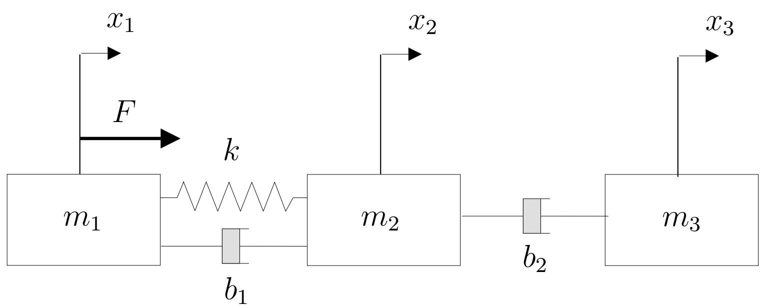
Part II: exercises

(Open Book)

Question 3

(weight:3)

Given: The input of the system illustrated below is the force F applied to the mass m_1.



The system is composed of three masses m_1, m_2 and m_2, two dampers b_1, b_2 and a spring k. The energy stored in the spring k is equal to

$$E(\Delta x) = \frac{1}{2} \cdot k_1 \cdot \Delta x^2 + \frac{1}{4} \cdot k_2 \cdot \sin^4(\Delta x)$$

where $\Delta x = x_1 - x_2$. The damping effects can be modeled by the linear relation

$$F_b = b \cdot v$$

where for each damper the the accompanying damping coefficient b and velocity v needs to be substituted.

Asked:

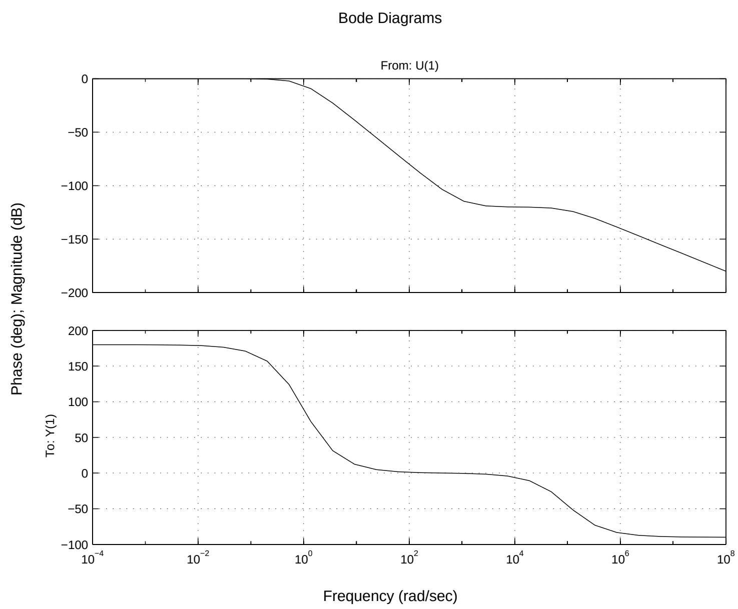
1. Draw a bond-graph of the system considering F as the input force.
2. Annotate the bond-graph and calculate from it the state space differential equation of the form

   $$\dot{\underline{x}} = f(\underline{x}, F)$$

   describing the dynamics of the system.

Question 4 (weight:2)

Given: The following Bode plot of magnitude and phase of a system $G(s)$



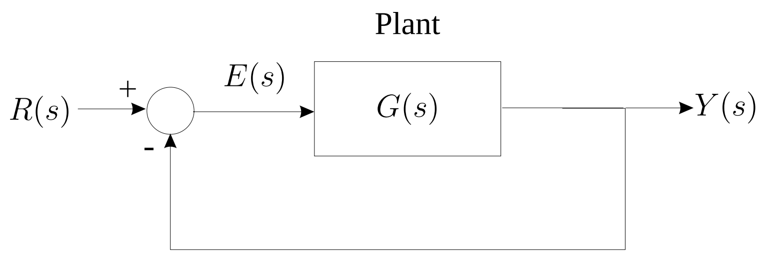
Asked: Give an expression of $G(s)$. Observe carefully the phase plot !

Question 5 (weight:3)

Given: The transfer function of a system $G(s)$ is:

$$G(s) = \frac{K \cdot (s+1)}{(2s+1) \cdot (\tau s+1)}$$

The system is put in a feedback loop as follows:



Asked: Thermal effects create drift in the time constant τ and it is useful to analyse the consequence of such a variation. Draw the root locus for the poles of the transfer function $\frac{Y(s)}{R(s)}$ for $K = 1$ and for variations of the time constant τ from 0 to ∞. Compute all the details like break-points and departure/arrival angles if they are applicable for this specific case.

Question 6 (weight:3)

Given: The transfer function of a system $G(s)$ is:

$$G(s) = \frac{K \cdot (s+0.1)}{(s+3)(s+4)(s^2+10s+26)}$$

Asked:

1. Sketch the Nyquist plot of $G(s)$ for $K = 1$ **(Be precise about the shape of the Nyquist plot for small ω !)**
2. Given that the solutions to the following equation:

$$281.8\,x - 106.3\,x^3 + x^5 = 0$$

   are $x = -10.2, +10.2, 0, 1.7, -1.7$, calculate analytically the value of the gain margin in dB for $K = 1$.

SOLUTION OF NOVEMBER 2000 EXAM

IMPORTANT NOTE:
Be critical at all time. Although this document has been prepared with great care, it is always possible that there are some (minor) errors left. If you do find one, please inform us.

Question 1

Define the vector margin of the loop transfer function L(s), and prove that it is equal to the inverse of the peaking value of the module of the sensitivity function $S(j\omega)$. Write all your reasoning and be as precise as possible.

Answer

Loop transfer function: $L(j\omega) = K \cdot G(j\omega)$

Sensitivity function: $S(j\omega) = \dfrac{1}{1 + K \cdot G(j\omega)} = \dfrac{1}{1 + L(j\omega)}$

The vector margin $\alpha_{\min}$ is the minimum distance between the Nyquist plot of $L(j\omega)$ and the point -1

$$
\begin{aligned}
\rightarrow \alpha_{\min} &= \min_{\omega} |L(j\omega) - (-1)| \\
&= \min_{\omega} |L(j\omega) + 1| \\
&= \min_{\omega} \left| \frac{L(j\omega) + 1}{1} \right| \\
&= \min_{\omega} \frac{1}{|S(j\omega)|} \\
&= \frac{1}{\max_{\omega} |S(j\omega)|}
\end{aligned}
$$

See also page 424, Fig. 6.79 of the textbook!

Question 2

A system in state space form:

$$\begin{aligned}\dot{x} &= Fx + Gu \\ y &= Hx + Ju\end{aligned}$$

Show using the theorem of Carley-Hamilton that we can express the input-output dynamics in the following form:

$$\sum_{i=0}^{n} a_i \frac{d^i}{dt^i} y(t) = \sum_{j=0}^{m} b_j \frac{d^j}{dt^j} u(t) \tag{1}$$

Answer

Characteristic polynomial:

$$\det(\lambda I - F) = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0 \tag{2}$$

Cayley-Hamilton:

$$F^n + a_{n-1}F^{n-1} + \ldots + a_1 F + a_0 I = 0 \tag{3}$$

where the parameters $a_{n-1}, \ldots, a_0$ result from the characteristic equation.

The left hand side of Equation **??** can be written as

$$\begin{aligned}\sum_{i=0}^{n} a_i \frac{d^i}{dt^i} y(t) &= \\ a_n \frac{d^n}{dt^n} y(t) + a_{n-1} \frac{d^{n-1}}{dt^{n-1}} y(t) + \ldots + a_1 \frac{d}{dt} y(t) + a_0 y(t) &= \\ a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y^{(1)} + a_0 y\end{aligned}$$

where $y, y^{(1)}, \ldots, y^{(n-1)}, y^{(n)}$ can be obtained from the state space description:

$$\begin{aligned}&y = Hx + Ju \\ &y^{(1)} = Hx^{(1)} + Ju^{(1)} = HFx + HGu + J^{(1)} \\ &\vdots \\ &y^{(n-1)} = HFx^{(n-1)} + HFx^{(n-2)}Gu + \ldots + HGu^{(n-2)} + Ju^{(n-1)}\end{aligned}$$

$$y^{(n)} = HFx^{(n)} + HFx^{(n-1)}Gu + \ldots + HGu^{(n-1)} + Ju^{(n)}$$

Setting $a_n, \ldots, a_0$ equal to the coefficients of the characteristic polynomial, Equation (??), and substitution of $y, y^{(1)}, \ldots, y^{(n-1)}, y^{(n)}$ results in

$$y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y^{(1)} + a_0 y =$$

$$\begin{aligned}
HF^{(n)}x + HF^{(n-1)}Gu + \ldots + HGu^{(n-1)} + Ju^{(n)} + \\
a_{n-1} \cdot \left(HF^{(n-1)}x + HF^{(n-2)}Gu + \ldots HGu^{(n-2)} + Ju^{(n-1)}\right) + \\
\vdots \\
a_1 \cdot \left(HFx + HGu + Ju^{(1)}\right) + \\
a_0 Ju^{(1)} =
\end{aligned}$$

$$\begin{aligned}
H\left(F^{(n)} + a_{n-1}F^{(n-1)} + \ldots + a_1 F^{(1)} + a_0 I\right)x + \\
\left(HF^{(n-1)}G + a_{n-1}HF^{(n-2)}G + \ldots + a_1 HG + a_0 J\right)u + \\
\left(HF^{(n-2)}G + a_{n-1}HF^{(n-3)}G + \ldots + a_1 J\right)u^{(1)} + \\
\vdots \\
Ju^{(n)}
\end{aligned}$$

Substitution of Equation (??) results in:

$$y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y^{(1)} + a_0 y =$$

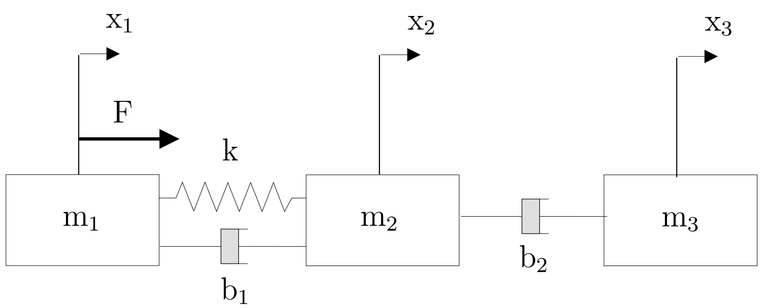
$$\begin{aligned}
\left(HF^{(n-1)}G + a_{n-1}HF^{(n-2)}G + \ldots + a_1 HG + a_0 J\right)u + \\
\left(HF^{(n-2)}G + a_{n-1}HF^{(n-3)}G + \ldots + a_1 J\right)u^{(1)} + \\
\vdots \\
Ju^{(n)} =
\end{aligned}$$

$$\begin{aligned}
\left(HF^{(n-1)}G + a_{n-1}HF^{(n-2)}G + \ldots + a_1 HG + a_0 J\right)u(t) + \\
\left(HF^{(n-2)}G + a_{n-1}HF^{(n-3)}G + \ldots + a_1 J\right)\frac{d}{dt}u(t) + \\
\vdots \\
J\frac{d^n}{dt^n}u(t) =
\end{aligned}$$

$$\sum_{j=0}^{m} b_j \frac{d^j}{dt^j}u(t),\ m \le n$$

Question 3

The input of the system illustrated below is the force F applied to the mass m_1.



The system is composed of three masses m_1, m_2 and m_3, two dampers b_1, b_2 and a spring k. The energy stored in the spring k is equal to

$$E(\Delta x) = \frac{1}{2} \cdot k_1 \cdot \Delta x^2 + \frac{1}{4} \cdot k_2 \cdot \sin^4(\Delta x)$$

where $\Delta x = x_1 - x_2$. The damping effects can be modeled by the linear relation

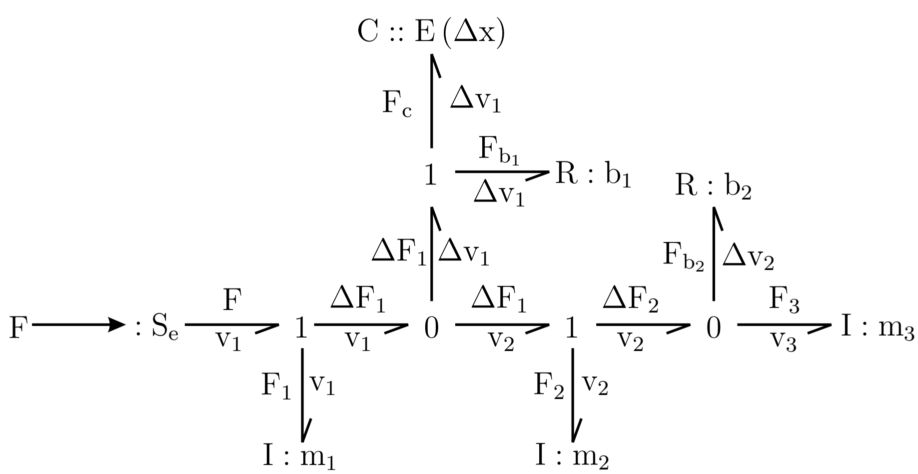
$$F_b = b \cdot v$$

where for each damper the the accompanying damping coefficient b and velocity v needs to be substituted.

1. Draw a bond-graph of the system considering F as the input force.

   Answer

   The Bond Graph of the system is as follows:



2. Annotate the bond-graph and calculate from it the state space differential equation of the form

$$\dot{\underline{x}} = f(\underline{x}, F)$$

describing the dynamics of the system.

Answer

Inertia 1
$$F_1 = \dot{p}_1$$
$$v_1 = \frac{p_1}{m_1}$$

Inertia 2
$$F_2 = \dot{p}_2$$
$$v_2 = \frac{p_2}{m_2}$$

Inertia 3
$$F_3 = \dot{p}_3$$
$$v_3 = \frac{p_3}{m_3}$$

Resistor 1 $F_{b_1} = b_1 \cdot \Delta v_1$

Resistor 2 $F_{b_2} = b_2 \cdot \Delta v_2$

Spring
$$\Delta\dot{x} = \Delta v_1$$
$$F_C = k_1 \cdot \Delta x + k_2 \cdot \sin^3(\Delta x) \cdot \cos(\Delta x)$$

1-junctions
$$F = \Delta F_1 + F_1$$
$$\Delta F_1 = F_C + F_{b_1}$$
$$\Delta F_1 = F_2 + \Delta F_2$$

0-junctions
$$v_1 = v_2 + \Delta v_1$$
$$v_2 = v_3 + \Delta v_2$$

Solving the equations

$$\begin{aligned}
F_1 &= F - \Delta F_1 \\
&= F - F_C - F_{b_1} \\
\dot{p}_1 &= F - k_1 \cdot \Delta x - k_2 \cdot \sin^3(\Delta x) \cdot \cos(\Delta x) - b_1 \cdot \Delta v_1
\end{aligned}$$

$$\begin{aligned}
F_2 &= \Delta F_1 - \Delta F_2 \\
&= F_C + F_{b_1} - F_{b_2} \\
\dot{p}_2 &= k_1 \cdot \Delta x + k_2 \cdot \sin^3(\Delta x) \cdot \cos(\Delta x) + b_1 \cdot \Delta v_1 - b_2 \cdot \Delta v_2
\end{aligned}$$

$$\begin{aligned}
F_3 &= \Delta F_2 \\
&= F_{b_2} \\
\dot{p}_3 &= b_2 \cdot \Delta v_2
\end{aligned}$$

$$\Delta\dot{x} = \Delta v_1$$

$$\Delta v_1 = v_1 - v_2 = \frac{p_1}{m_1} - \frac{p_2}{m_2}$$

$$\Delta v_2 = v_2 - v_3 = \frac{p_2}{m_2} - \frac{p_3}{m_3}$$

<u>Express dynamic equations as a function of the state variables</u>

$$\dot{p}_1 = F - k_1 \cdot \Delta x - k_2 \cdot \sin^3(\Delta x) \cdot \cos(\Delta x) - b_1 \cdot (\frac{p_1}{m_1} - \frac{p_2}{m_2})$$

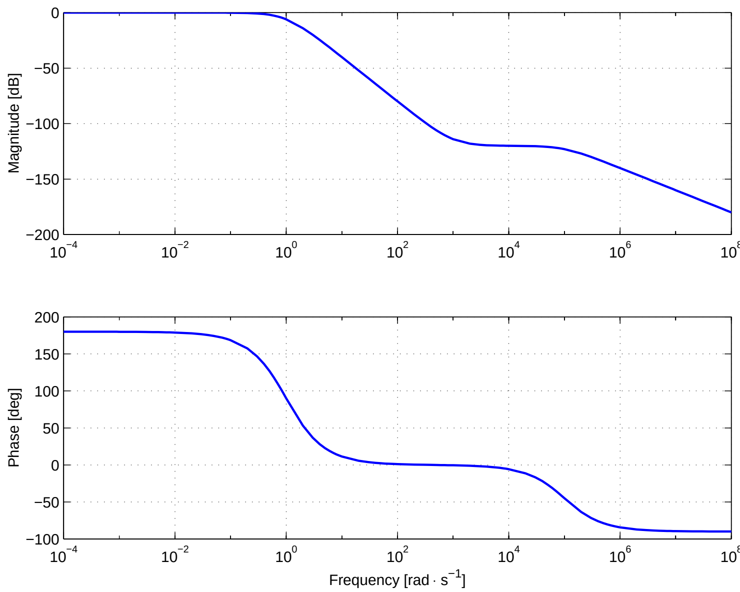
$$\dot{p}_2 = k_1 \cdot \Delta x + k_2 \cdot \sin^3(\Delta x) \cdot \cos(\Delta x) + b_1 \cdot (\frac{p_1}{m_1} - \frac{p_2}{m_2}) - b_2 \cdot (\frac{p_2}{m_2} - \frac{p_3}{m_3})$$

$$\dot{p}_3 = b_2 \cdot (\frac{p_2}{m_2} - \frac{p_3}{m_3})$$

$$\Delta\dot{x} = \frac{p_1}{m_1} - \frac{p_2}{m_2}$$

Question 4

The following Bode plot of magnitude and phase of a system G(s)



Give an expression of G(s). Observe carefully the phase plot !

Answer

In general a system transfer function can be subdivided in a number of subsystem transfer functions with a relative simple structure. In order to be able to find the system transfer function G(s), we need to identify the subsystem transfer functions $G_i(s)$. First we need to identify the area where each subsystem transfer function is valid, and then we determine the subsystem transfer function itself. Transitions from one subsystem transfer function to another are marked by so-called break points, that are evident in both the magnitude and phase plot of the system transfer function G(s).

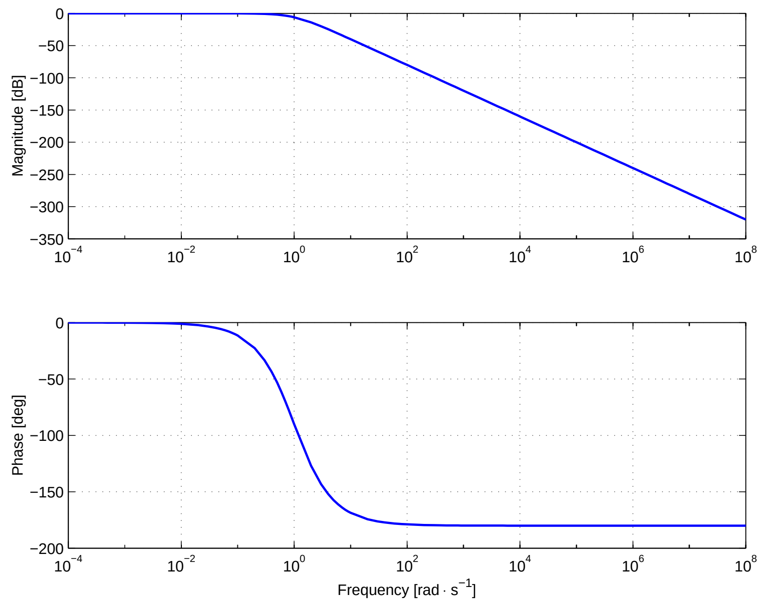
Break points magnitude:	$\omega = 1$	slope: 0 dB/dec $\rightarrow -40$ dB/dec
	$\omega = 10^3$	slope: $-$ 40 dB/dec $\rightarrow$ 0 dB/dec
	$\omega = 10^5$	slope: 0 dB/dec $\rightarrow -20$ dB/dec
Break points phase:	$\omega = 1$	phase: $180^\circ \rightarrow 0^\circ$
	$\omega = 10^5$	phase: $0^\circ \rightarrow -90^\circ$

Two things stand out immediately. First of all there are three break points evident in the magnitude plot and only two in the phase plot, and secondly, the phase plot starts at 180°. We have to keep this in mind.

Let's consider the **magnitude** plot first. Around the break point $\omega = 1$ the slope of the Bode plot decreases from 0 dB/dec to -40 dB/dec. Before $\omega = 1$ the Bode plot behaves like a constant gain, and after $\omega = 1$ the Bode plot behaves like a double integrator. This effect is obtained by the following subsystem transfer function

$$G_1(s) = \frac{1}{(s+1)^2}$$

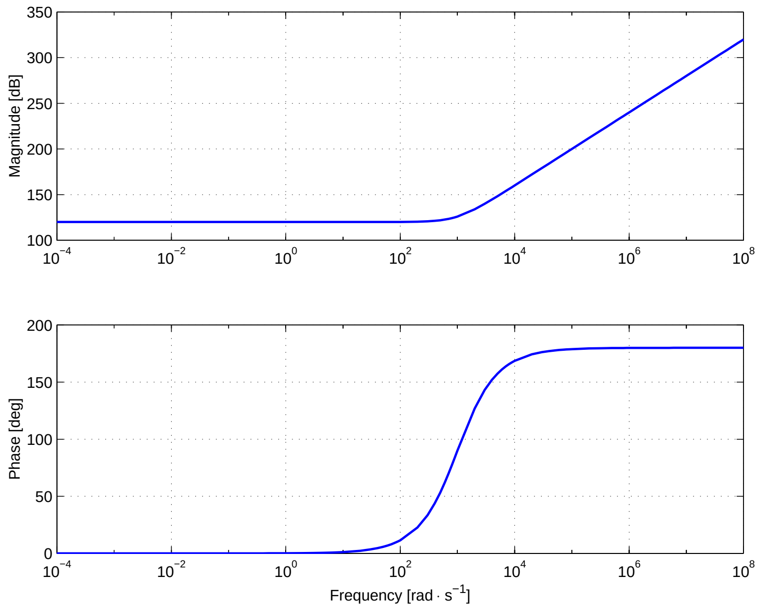
The Bode plot of $G_1(s)$ is illustrated below.



Around $\omega = 10^3$ the slope increases from -40 dB/dec to 0 dB/dec. This implies that around this frequency the slope of -40 dB/dec is compensated for by a slope of 40 dB/dec. Adding up these slopes results in a slope of 0 dB/dec. The second element of the Bode plot behaves like a double differentiator from $\omega = 10^3$ on. This effect is obtained by the following subsystem transfer function

$$G_2(s) = \frac{(s+10^3)^2}{1}$$

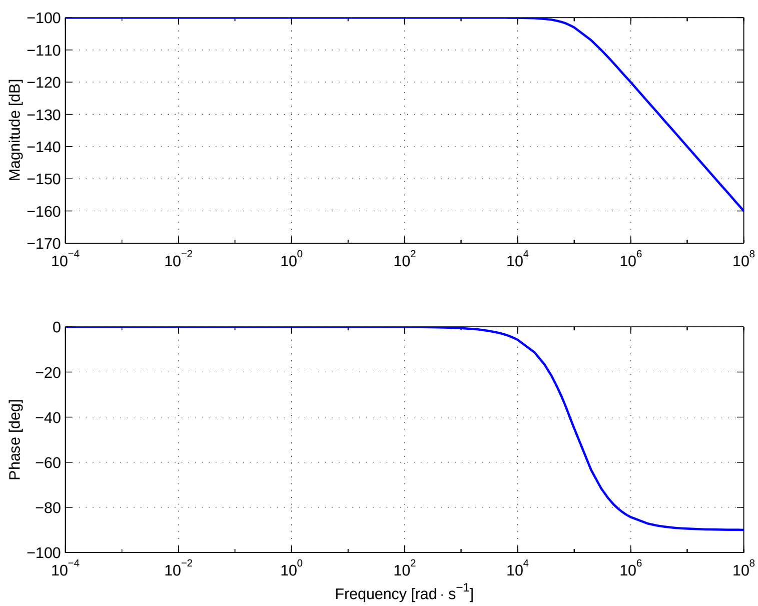
The Bode plot of $G_2(s)$ is illustrated below.



Around $\omega = 10^5$ the slope decreases again to -20 dB/dec. The third element of the Bode plot behaves therefore as a single integrator after $\omega = 10^5$. This effect is obtained by the following subsystem transfer function

$$G_3(s) = \frac{1}{(s + 10^5)}$$

The Bode plot of $G_3(s)$ is illustrated below.



Together these three elements form the following transfer function

$$G(s) = G_1(s)G_2(s)G_3(s) = \frac{(s + 10^3)^2}{(s + 1)^2(s + 10^5)} \rightarrow G(i\omega) = \frac{(i\omega + 10^3)^2}{(i\omega + 1)^2(i\omega + 10^5)}$$

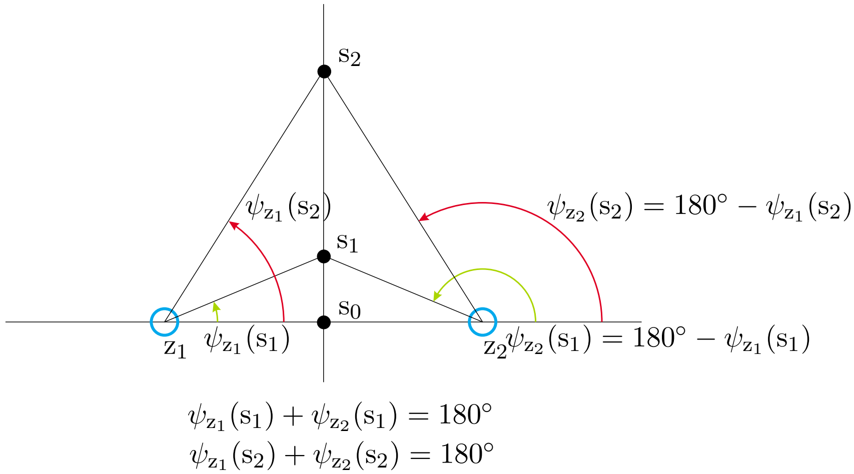
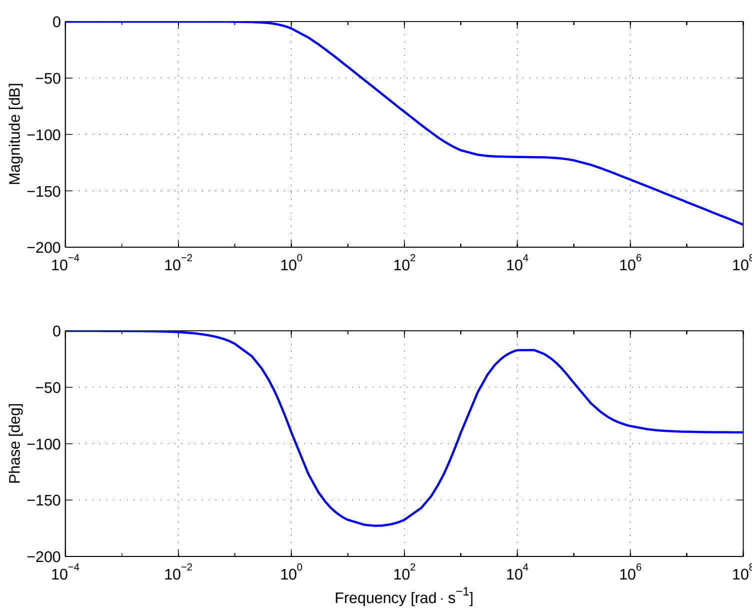


Figure 1: Illustration of the joint contribution to the argument of a zero in the right half plane and a zero in the left half plane, on equal distance from the origin

For $\omega = 0$ the gain is $|G(0)| = 10$. Since the gain is equal to 1 according to the Bode plot, the transfer function $G(s)$ has to be multiplied by $K = 0.1$.

$$\mathrm{G(s)} = \frac{0.1 \cdot (\mathrm{s} + 10^3)^2}{(\mathrm{s}+1)^2 \cdot (\mathrm{s} + 10^5)}$$

The Bode plot of G(s) is illustrated below.

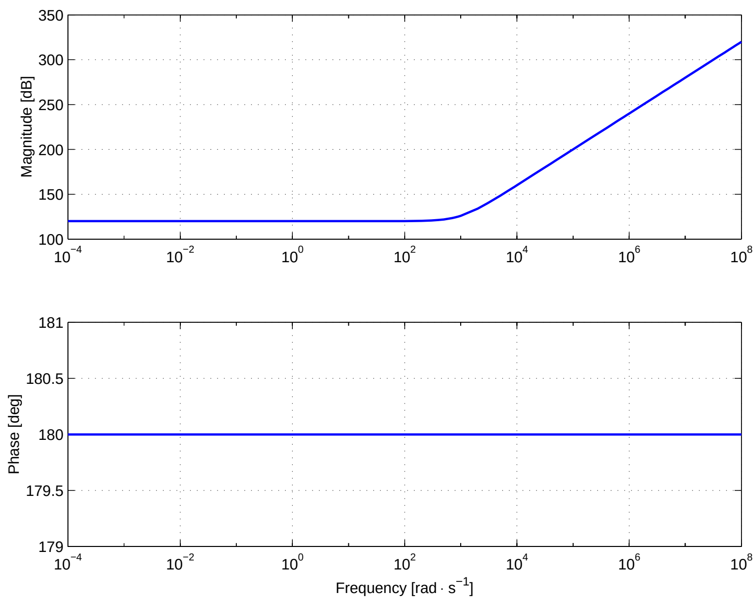


From the resulting Bode plot we see that indeed the magnitude plot is correct, but the phase plot is not. The phase starts at 180° for $\omega = 0$, as mentioned before. This indicates an odd absolute number of poles minus zeros in the right half plane ($|\mathrm{m} - \mathrm{n}|$). Every pole or zero on the real axis of the left half plane, contributes 0° to the phase of G(j0), while every pole or zero on the right half plane contributes ±180° to the phase of G(j0). Studying the phase plot again, we see that around the frequency $\omega = 10^3$ the phase is not changing, while the magnitude is. This can only indicate that the two zeros of $G_2(s)$ are on the real axis, one in the left half plane and one in the right half plane, on equal distance from the origin.

This is illustrated in Figure **??**. For $s_0 = j0$ we have $\angle\psi_{z_1}(s_0) = 0°$ and $\angle\psi_{z_2}(s_0) = 180°$. Continuing along the positive imaginary axis (increasing ω), it is shown that the increase of $\angle\psi_{z_1}(s)$ is equal to the decrease of $\angle\psi_{z_2}(s)$, and therefore there joint contribution to the phase remains $\angle\psi_{z_1}(j\omega) + \angle\psi_{z_2}(j\omega) = 180°$. The joint magnitude is changing with ω, this is why at $\omega = 10^3$ there is a break point evident in the magnitude plot, but not in the phase plot. The subsystem transfer function $G_2(s)$ should therefore be as follows:

$$G_2(s) = \frac{(s + 10^3)(s - 10^3)}{1}$$

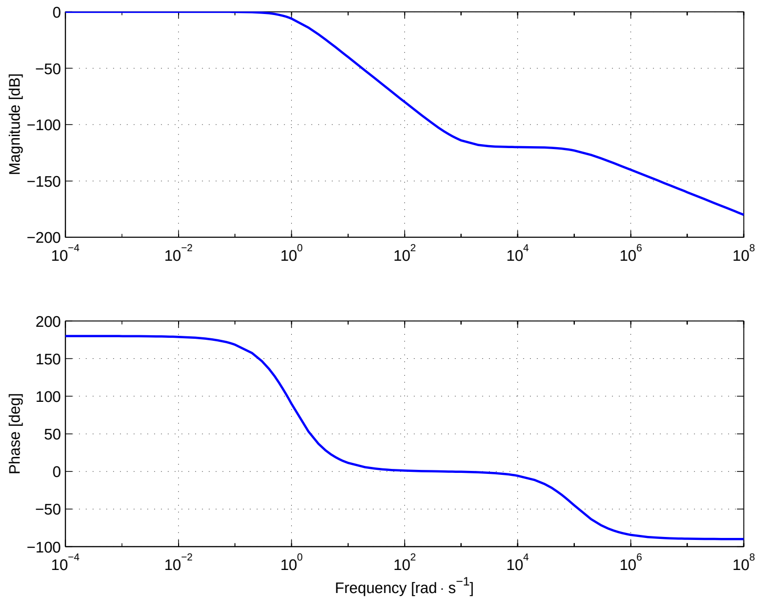
The Bode plot of $G_2(s)$ is illustrated below. Note that indeed the phase remains constant, while the magnitude changes with ω.



The transfer $G(s)$ is now

$$G(s) = \frac{0.1 \cdot (s + 10^3)(s - 10^3)}{(s + 1)^2 \cdot (s + 10^5)}$$

The Bode plot of $G(s)$ is illustrated below.



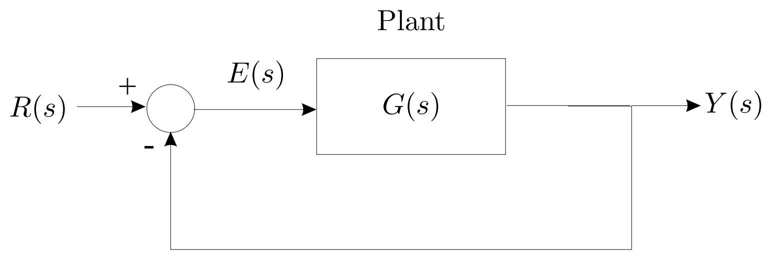
Magnitude [dB]
0
−50
−100
−150
−200
Phase [deg]
200
150
100
50
0
−50
−100
10^{-4}
10^{-2}
10^{0}
10^{2}
10^{4}
10^{6}
10^{8}
Frequency [rad · s^{-1}]

Question 5

The transfer function of a system $G(s)$ is:

$$G(s) = \frac{K \cdot (s+1)}{(2s+1) \cdot (\tau s + 1)}$$

The system is put in a feedback loop as follows:



Thermal effects create drift in the time constant τ and it is useful to analyse the consequence of such a variation. Draw the root locus for the poles of the transfer function $\frac{Y(s)}{R(s)}$ for $K = 1$ and for variations of the time constant τ from 0 to ∞. Compute all the details like break-points and departure/arrival angles if they are applicable for this specific case.

Answer

The transfer function from input to output of the closed-loop plant is

$$\mathrm{G(s)} = \frac{\mathrm{K(s+1)}}{(2\mathrm{s}+1)(\tau \mathrm{s}+1)}$$

The characteristic equation is therefore

$$1 + \frac{\mathrm{K(s+1)}}{(2\mathrm{s}+1)(\tau \mathrm{s}+1)} = 0$$

For $K = 1$ this results in

$$1 + \frac{(\mathrm{s}+1)}{(2\mathrm{s}+1)(\tau \mathrm{s}+1)} = 0$$

Since we are interested in the root-locus for variations in τ, this needs to be rewritten to the form

$$\mathrm{F(s)} = -\frac{1}{\tau}$$

which is obtained as follows:

$$
\begin{aligned}
1 + \frac{(s+1)}{(2s+1)(\tau s+1)} &= 0 \rightarrow \\
(2s+1)(\tau s+1) + (s+1) &= 0 \rightarrow \\
\tau s(2s+1) + (3s+2) &= 0 \rightarrow \\
(3s+2) &= -\tau s(2s+1) \rightarrow \\
\frac{(3s+2)}{s(2s+1)} &= -\tau \rightarrow \\
\frac{s(2s+1)}{(3s+2)} &= -\frac{1}{\tau}
\end{aligned}
$$

and therefore:

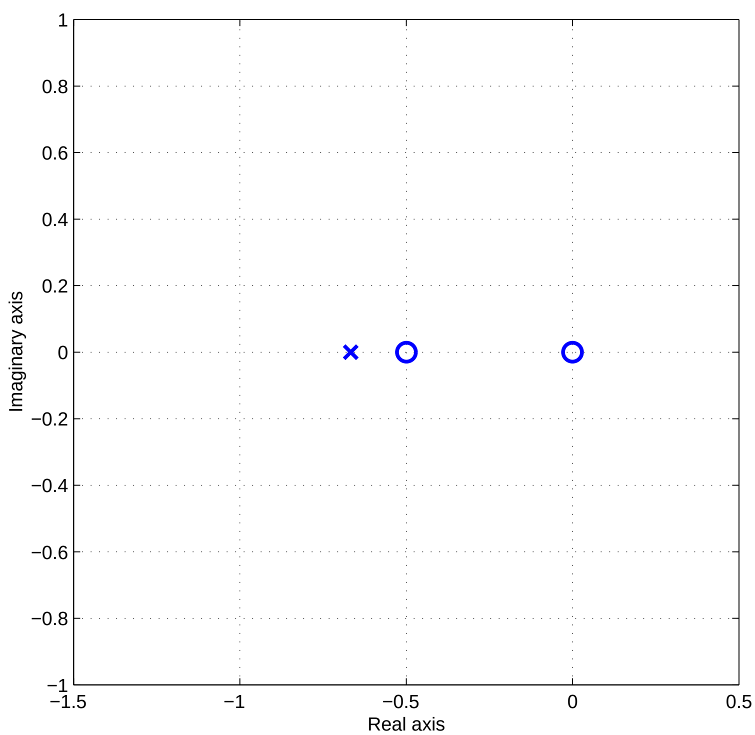
$$F(s) = \frac{s(2s+1)}{(3s+2)}$$

1. *Draw the axes of the s-plane to a suitable scale and enter an $\times$ on this plane for each pole of $F(s)$ and a $\circ$ for each zero of $F(s)$.*

   The poles and zeros of $F(s)$ are

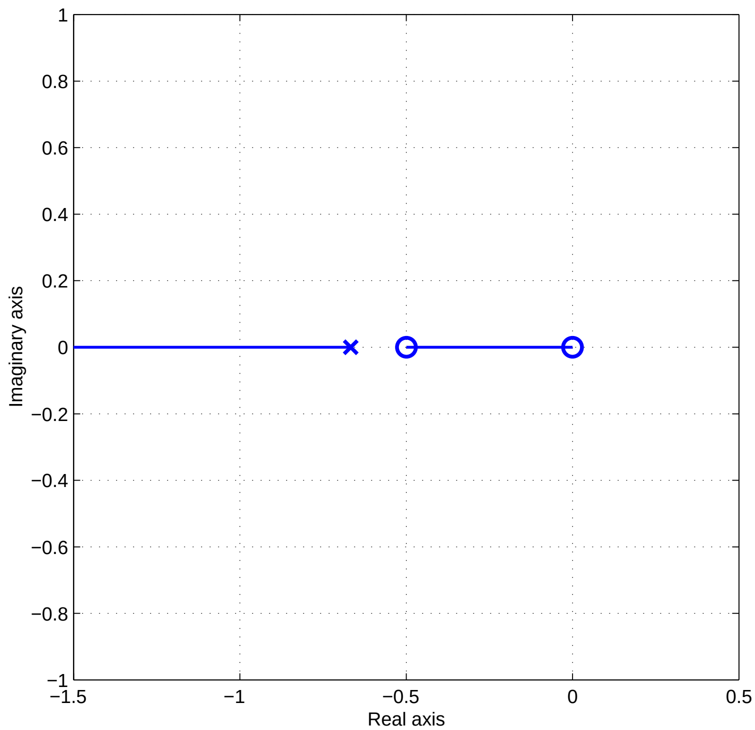
   poles: $p_1 = -\frac{2}{3}$

   zeros: $z_1 = 0$, $z_2 = -0.5$



2. *Find the real axis portions of the locus.*

The real axis portion of the root locus is located between z_1 and z_2 and left of the p_1. Since there is one pole and there are two zeros, one pole will come from infinity.



3. *Draw the asymptotes for large values of τ.*

$$|n-m| = \text{number of asymptotes}$$

$$\alpha = \frac{\sum p_i - \sum z_i}{|n-m|}$$

$$\phi_l = \frac{180° + 360° \cdot (l-1)}{|n-m|}, \; l = 1, 2, ..., |n-m|$$

The number of asymptotes is equal to the absolute difference in the number of poles and the number of zeros. Since there is one pole and there are two zeros, the number of asymptotes is equal to one. For $\tau > \infty$ the pole p_1 moves towards one zero, while another pole comes from infinity to the other zero.

To compute the origin α of the asympote we use the following expression

$$\alpha = \frac{\sum_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{Re(p_i)} - \sum_{\mathrm{j=1}}^{\mathrm{m}} \mathrm{Re(z_j)}}{|\mathrm{n-m}|} = \frac{-\frac{2}{3} + 0.5}{|1-2|} = \frac{-1}{1} = -1$$

where n is the number poles and m is the number of zeros of $F(s)$.

The departure angles of the asymptote is computed as follows (take absolute values for l and $n-m$)

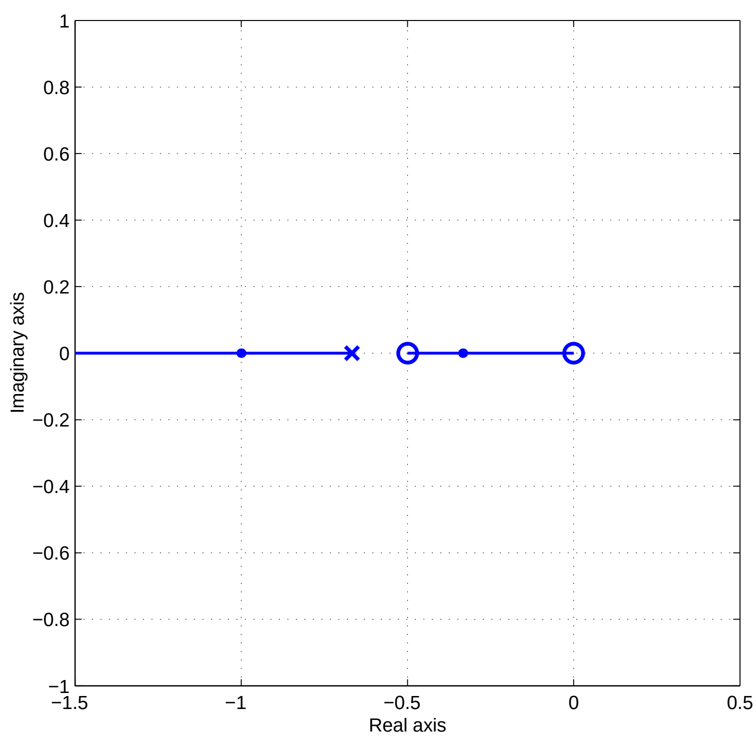
$$\phi_1 = \frac{180° + 360° \cdot 0}{1} = \frac{180°}{1} = 180°$$

4. *Compute locus departure angles from the poles and arrival angles at the zeros*

$$q\phi_{dep} = \sum \psi_i - \sum \phi_i - 180° - 360° \cdot l$$
$$q\psi_{arr} = \sum \phi_i - \sum \psi_i + 180° + 360° \cdot l$$

where q is the order of the pole or zero and l takes on q integer values so that the angles are between $\pm 180°$.

Since all the poles and zeros are located on the real axis, there is no need to compute the departure and arrival angles.



5. *Estimate (or compute) the points where the locus crosses the imaginary axis*

   This is not the case for $\tau : 0^+ > \infty$.

6. *Estimate locations of multiple roots, especially on the real axis, and determine the arrival and departure angles at these locations.*

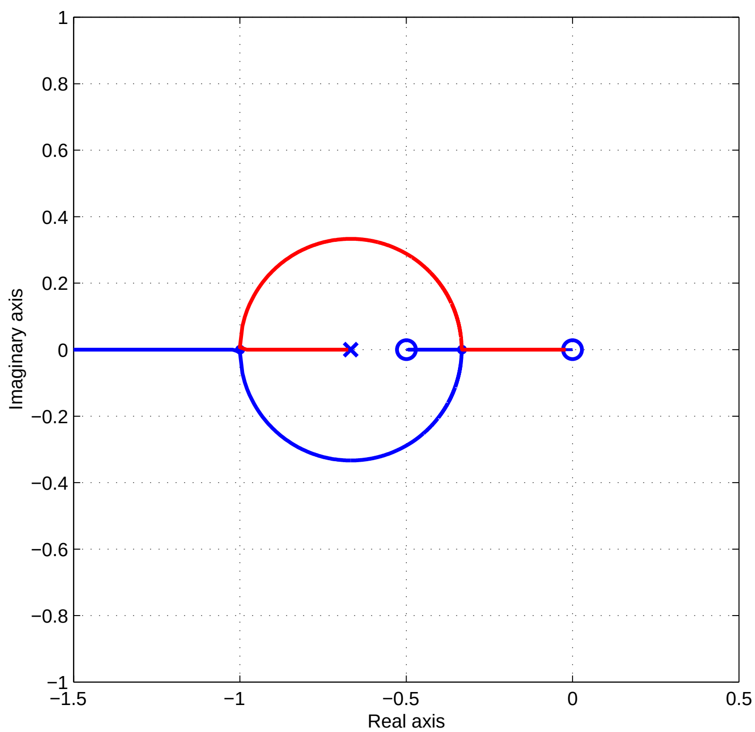
$$\frac{d}{ds}\left(-\frac{1}{F(s)}\right)_{s=s_0} = 0$$

$$\frac{d}{ds}\left(-\frac{3s+2}{s(2s+1)}\right)_{s=s_0} = 0$$

$$\left(\frac{6s^2+8s+2}{s^2(2s+1)^2}\right)_{s=s_0} = 0$$

$$p_{1,2} = \frac{-8 \pm \sqrt{64-4.6.2}}{12} = \frac{-8 \pm 4}{12} = -\frac{1}{3}, -1$$

7. *Complete the sketch*



Question 6

The transfer function of a system G(s) is:

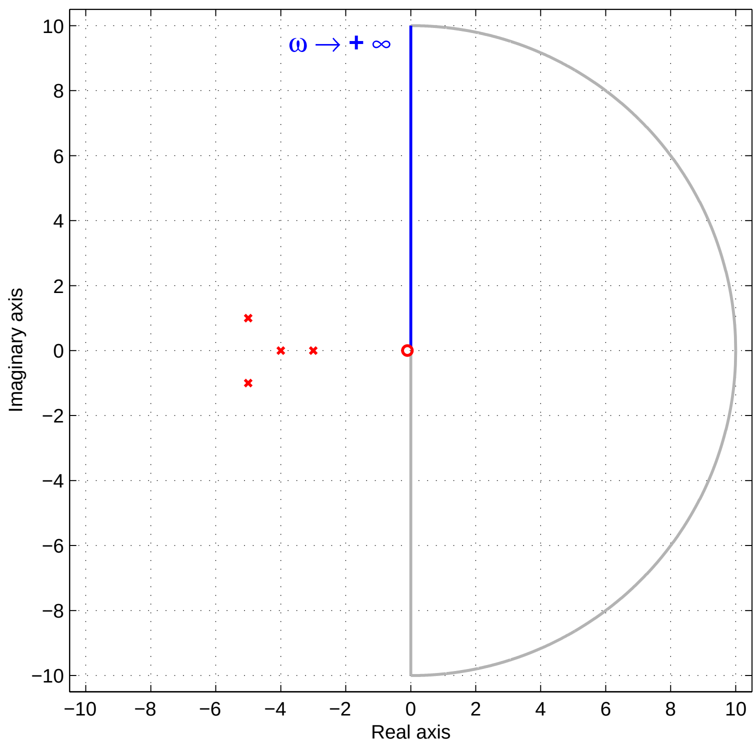
$$G(s) = \frac{K \cdot (s + 0.1)}{(s + 3)(s + 4)(s^2 + 10s + 26)}$$

1. Sketch the Nyquist plot of G(s) for K = 1 (**Be precise about the shape of the Nyquist plot for small ω !**)

Answer

To be able to sketch the Nyquist plot, we first need to compute the zeros and poles of the transfer function G(s):

zeros: $z_1 = -0.1$
poles: $p_1 = -3$, $p_2 = -4$, $p_{3,4} = -5 \pm i$



The Nyquist contour is illustrated in Figure **??**, as well as the poles and zeros. The Nyquist contour encircles the entire right half plane (see fig. 6.16 of the textbook). Every point on the Nyquist contour has a gain and argument when substituted in the transfer function G(s) and therefore represents a point in the complex plane.

The approach to sketch a Nyquist plot is as follows

(a) Sketch the part of the Nyquist plot that is related to that part of the Nyquist contour along the positive imaginary axis

(b) Mirror the result in the real axis, which represents the part of the part of the Nyquist plot that is related to the Nyquist contour along the negative imaginary axis.

(c) If necessary, close the Nyquist plot. This is the part of the Nyquist plot that is related to the half circle with radius 0^+ and/or ∞.

The positive imaginary axis of the Nyquist contour

The positive imagninary axis of the Nyquist contour is represented by substitution of $s = j\omega$ in $G(s)$ for $\omega : 0^+ \to \infty$. Substitution of $s = j\omega$ results in and K = 1

$$\mathrm{G}(\mathrm{j}\omega) = \frac{(\mathrm{j}\omega + 0.1)}{(\mathrm{j}\omega + 3)(\mathrm{j}\omega + 4)(-\omega^2 + 10\mathrm{j}\omega + 26)}$$

Substitution of $\omega = 0^+$ in $\mathrm{G}(\mathrm{i} \cdot \omega)$ results in $|G(i \cdot \omega)| = \frac{0.1}{3 \cdot 4 \cdot 26} = \frac{1}{3120} \approx 3.2 \cdot 10^{-3}$. The argument for $\omega = 0^+$ is $\angle G(i \cdot \omega) = 0°$. This can also be determined by using Figure **??**. For $\omega = 0^+$ the argument of $\mathrm{G}(\mathrm{i} \cdot \omega)$ due to the zero is $0°$, while the argument of $\mathrm{G}(\mathrm{i} \cdot \omega)$ due to the two real poles is $-0°$ and the joint contribution of the two complex poles is $-0°$. The total argument is therefore $\angle \mathrm{G}(\mathrm{i} \cdot 0^+) = 0°$.

Substitution of $\omega = \infty$ in $G(j \cdot \omega)$ results in $|G(j \cdot \omega)| = 0$. The argument for $\omega = \infty$ is $\angle G(j \cdot \omega) = -270°$. This can also be determined by using Figure **??**. For $\omega = \infty$ the argument of $G(i \cdot \omega)$ due to the zero is $90°$, while the argument of $G(i \cdot \omega)$ due to each of the poles is $-90°$. The total argument for $\omega = \infty$ is therefore $\angle \mathrm{G}(\mathrm{i} \cdot \infty) = 90° - 4 \cdot 90° = -270°$.

Now we can sketch the part of the Nyquist plot that is related to the positive imaginary axis of the Nyquist contour. The result is illustrated in the figure below. Note that the the magnitude and phase of the Nyquist plot first increases. This is due to the position of the poles and zeros in the complex plane. The zero is by far the closest to the origin and therefore has the biggest influence on the magnitude and phase at low frequencies. Because the contribution of a zero to the magnitude and phase is positive, the magnitude and phase of the Nyquist plot increase at low frequencies. At higher frequencies the four poles become more and more dominant and the phase decreases again until the phase is $-270°$ for $\omega = \infty$ and the magnitude decreases to zero.

The negative imaginary axis of the Nyquist contour

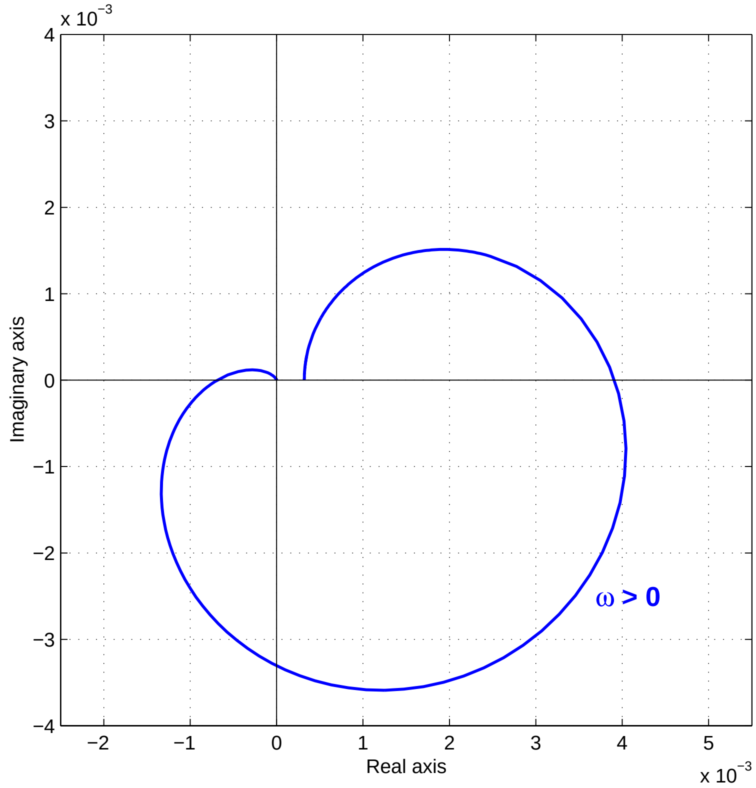
This part of the Nyquist plot is obtained by mirroring the part of the Nyquist plot that is related to the positive imaginary axis of the Nyquist contour in the real axis. The result is illustrated in the figure below.

Closing the Nyquist plot

If necessary the Nyquist plot needs to be closed (representing those parts of the Nyquist plot related to the parts of the Nyquist contour with the half circle with radius 0^+ and/or ∞. In this case the Nyquist plot is already closed. The final result is therefore the Nyquist plot illustrated in the figure below.

2. Given that the solutions to the following equation:

$$281.8\,\mathrm{x} - 106.3\,\mathrm{x}^3 + \mathrm{x}^5 = 0$$



are x = −10.2, +10.2, 0, 1.7, −1.7, calculate analytically the value of the gain margin in dB for K = 1.

Answer

The gain margin is expressed as (see Figure 6.32, pag. 375 of the textbook)

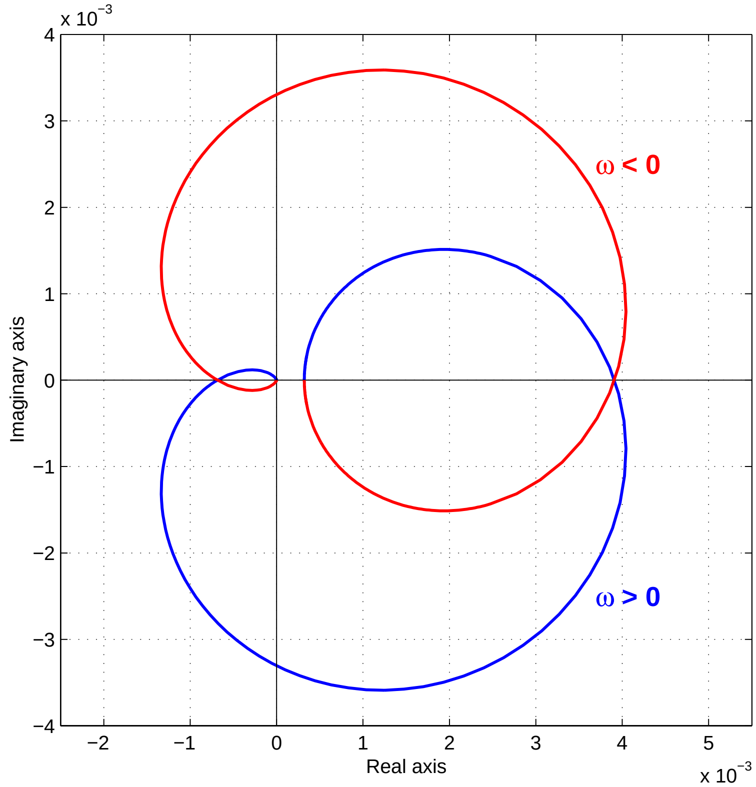
$$\mathrm{GM} = \frac{1}{|\mathrm{K}\cdot\mathrm{G}(\mathrm{j}\omega)|_{\angle \mathrm{K}\cdot\mathrm{G}(\mathrm{j}\omega)=180^\circ}}$$

To compute the gain margin, we first need to compute the gain of $K \cdot G(s)$ when the phase is equal to $\angle K \cdot G(s) = 180°$. The expression for $|K \cdot G(s)|$ is obtained as follows

$$\mathrm{G}(\mathrm{s}) = \frac{\mathrm{K}(\mathrm{s}+0.1)}{(\mathrm{s}+3)(\mathrm{s}+4)(\mathrm{s}^2+10\mathrm{s}+26)}$$

$$\mathrm{G}(\mathrm{j}\omega) = \frac{\mathrm{K}(\mathrm{j}\omega+0.1)}{(\mathrm{j}\omega+3)(\mathrm{j}\omega+4)(-\omega^2+10\mathrm{j}\omega+26)}$$

$$\mathrm{G}(\mathrm{j}\omega) = \frac{\mathrm{K}(\mathrm{j}\omega+0.1)}{(\mathrm{j}\omega+3)(\mathrm{j}\omega+4)(-\omega^2+10\mathrm{j}\omega+26)},\ \mathrm{K}=1$$



$$G(j\omega) = \frac{(j\omega + 0.1)}{(-\omega^2 + 7j\omega + 12)(-\omega^2 + 10j\omega + 26)}$$

$$G(j\omega) = \frac{(j\omega + 0.1)}{\omega^4 - 17j\omega^3 - 108\omega^2 + 302j\omega + 312}$$

$$G(j\omega) = \frac{j\omega + 0.1}{\omega^4 - 17j\omega^3 - 108\omega^2 + 302j\omega + 312} \cdot \frac{\omega^4 + 17j\omega^3 - 108\omega^2 - 302i\omega + 312}{\omega^4 + 17j\omega^3 - 108\omega^2 - 302i\omega + 312}$$

$$G(j\omega) = \frac{j\omega^5 - 16.9\omega^4 - 106.3j\omega^3 + 291.2\omega^2 + 281.8i\omega + 31.2}{\omega^8 + 73\omega^6 + 2020\omega^4 + 23812\omega^2 + 97344}$$

$$G(j\omega) = \frac{(-16.9\omega^4 + 291.2\omega^2 + 31.2) + j(\omega^5 - 106.3\omega^3 + 281.8\omega)}{\omega^8 + 73\omega^6 + 2020\omega^4 + 23812\omega^2 + 97344} \tag{4}$$

Phase equal to 180° implies that the imaginary part of the gain should be equal to zero:

$$\omega^5 - 106.3\omega^3 + 281.8\omega = 0$$

The solutions of this equation are given: $\omega = -10.2, -1.7, 0, 1.7, 10.2$. One of these five answers is the one we are looking for. The negative answer we can skip straight away, since a frequency is positive. The answer $\omega = 0$ denotes the starting point, see sketch, and is also not the answer we are looking for. Substituting $\omega = 1.7$ and $\omega = 10.2$ in Equation **??**, we get:

$$\begin{aligned} \omega = 1.7 \rightarrow \mathrm{G}(\mathrm{j}\omega) &= 0.0040 \\ \omega = 10.2 \rightarrow \mathrm{G}(\mathrm{j}\omega) &\approx -0.000682 \end{aligned}$$

Since the phase of $\mathrm{G}(\mathrm{j}\omega)$ should be equal to $\angle \mathrm{G}(\mathrm{j}\omega) = 180°$, the real part should be negative. The answer we are looking for is therefore $\omega = 10.2$.

The gain margin is equal to

$$\mathrm{GM} = \frac{1}{|\mathrm{K} \cdot \mathrm{G}(\mathrm{j}\omega)|_{\angle \mathrm{K} \cdot \mathrm{G}(\mathrm{j}\omega) = 180°}} = \frac{1}{0.000682} \approx 1.466 \cdot 10^3$$

Written Exam
Modern Control Systems
(Et 3-101 AND Et 3-015)

On Tuesday the October 30, 2001 from 9:00 AM to 12:00 PM

Read the following VERY carefully.

- The exam is composed of 2 Parts: theory and exercises. The first part is WITHOUT any books or notes and it will last **40 minutes**. At the end of the 40 minutes, the exam will be collected and after 10 minutes the second part of **130 minutes** will start. You are allowed to use anything you want for the second part:
  - Part I, 40 minutes (end at 9:40 AM) , no books, hand in at the end.
  - Break, 10 minutes, (end at 9:50 AM) collect books, sheets and the material you are allowed to use for Part II.
  - Part II, 130 minutes (end at 12:00 PM), with books, sheets.....

  If you are ready with Part I earlier than the time given, you can start working on Part II, but **WITHOUT BOOK** and on a **NEW SHEET OF PAPER**.
- NEVER talk with your neighbor.
- If the answers are not easily readable, the corresponding answer will be given 0 points. Therefore WRITE CLEARLY.
- For Part II, fill in the answers on the answer sheet (but also hand in your calculations).
- Read every question well before answering.
- Write ALL your reasoning steps on paper.
- Write your name and student number clearly readable on EACH piece of paper.
- Good luck !

Delft University of Technology
Faculty of Information Technology and Systems
Control Systems Engineering
Mekelweg 4
2628 CD Delft

Part I: theory

(Closed Book)

Question 1

(weight:2)

Let the stable nominal loop transfer function be given by $L_0(j\omega)$, and the loop transfer function of the perturbed system by $L(j\omega)$. Assume the nominal stable loop transfer function leads to a stable closed loop.
Show that a sufficient condition for robust stability of the perturbed system is given by:

$$|L_0(j\omega) - L(j\omega)| < |1 + L_0(j\omega)| \ , \quad \forall \omega \in \mathbb{R}$$

Show that this is equivalent to the condition:

$$|T_0(j\omega)| < \frac{1}{|\Delta_{rel}(L_0(j\omega))|} \ , \quad \forall \omega \in \mathbb{R}$$

where $\Delta_{rel}(L_0(j\omega))$ is the relative perturbation of the loop transfer function L_0.
Write all your reasoning and be as precise as possible.

Question 2

(weight:2)

Given: The following Laplace relation

$$\frac{1}{(s+a)^m} \overset{\mathcal{L}^{-1}}{\rightarrow} \frac{1}{(m-1)!} t^{m-1} e^{-at}$$

Asked: Give an expression of the time responce of the system with transfer function

$$G(s) = \frac{1}{(s+1)^2}$$

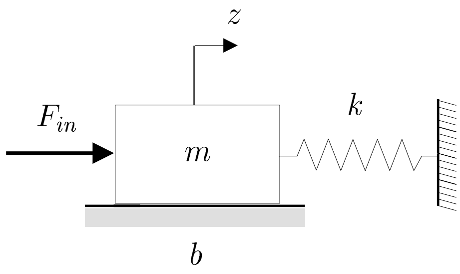
to a step input from an initially-at-rest condition at time $t = 0$.

Part II: exercises

(Open Book)

Question 3 (weight:3)

Given: The mass-spring system is illustrated below



where F_{in} is the input force, the force of friction b is described as $F_b = b_1 v + b_2 v^2$, and the energy function of the spring k is $E(z) = \frac{1}{2}k_1 z^2 + \frac{1}{4}k_2 z^4$. The non-linear state-space description is given:

$$\begin{pmatrix} \dot{p} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} F_{in} - \frac{b_1}{m}p - \frac{b_2}{m^2}p^2 - k_1 z - k_2 z^3 \\ \frac{1}{m}p \end{pmatrix}$$

$$y = p$$

Asked:

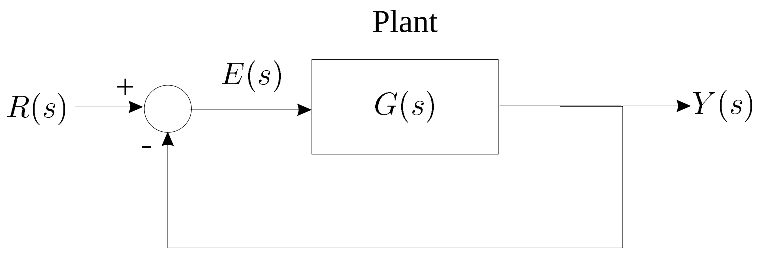
- Linearize the system around the equilibrium $\underline{x}_0 = (p_0, z_0)$ and $u_0 = F_{in_0}$.
- Give the state and input of the system $(\underline{x}_0, F_{in_0})$ when it is in equilibrium at the position $z_0 = 1$.
- For what values of $m, b_1, b_2, k_1,$ and k_2 is the system controllable at the equilibrium point.
- For what values of $m, b_1, b_2, k_1,$ and k_2 is the system observable at the equilibrium point.

Question 4 (weight:4)

Given: The transfer function of a system $G(s)$ is:

$$G(s) = \frac{K \cdot (s^2 + 4s + 20)}{s^3 + 2s^2 - 3s}$$

The system is put in a feedback loop as follows:



Asked: Draw the root locus for the poles of the transfer function $\frac{Y(s)}{R(s)}$ for K from 0 to ∞. Compute all the details like break-points, departure/arrival angles, intersections with the imaginary axis, etc. if they are applicable for this specific case.

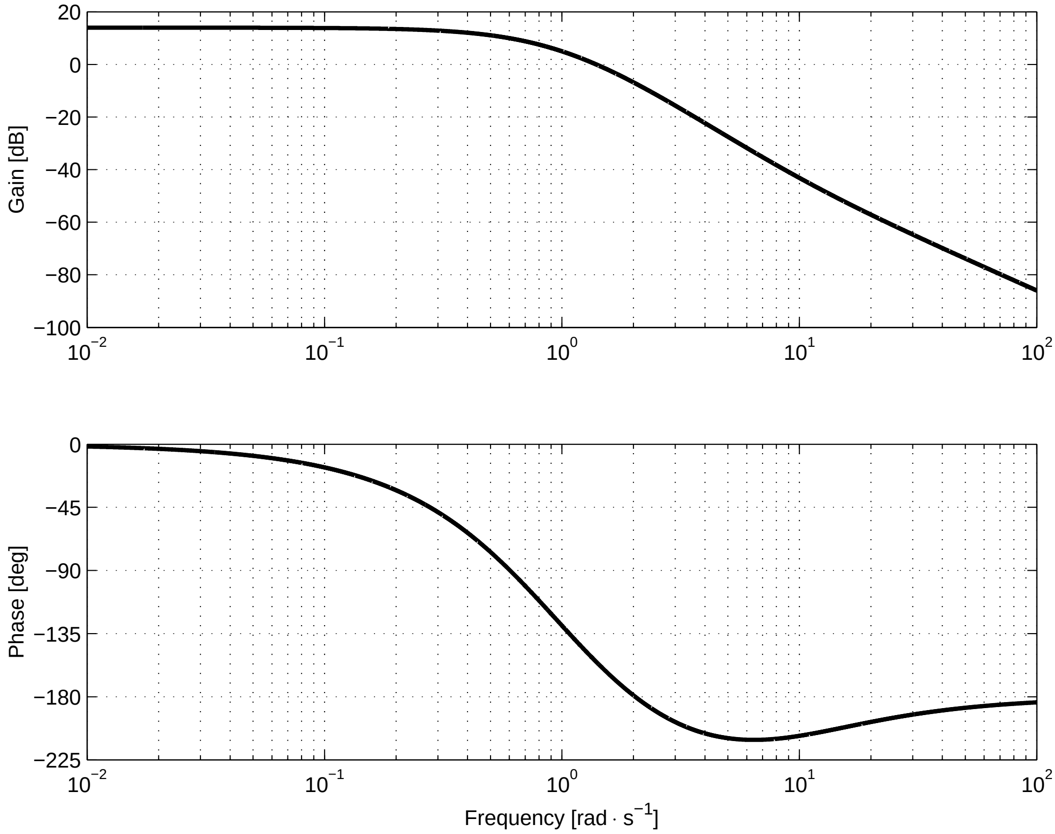
To compute the multiple roots, the following equation need to be solved:

$$\left.\frac{d}{ds}\left(-\frac{1}{F(s)}\right)\right|_{s=s_0} = 0$$

Give the equation. The answers to this equation in this particular case are $0.51, -1.88, -3.31 \pm 7.20i$. One of these answers is valid and should be used to sketch the root locus.

Question 5 (weight:3)

Given:



Asked:

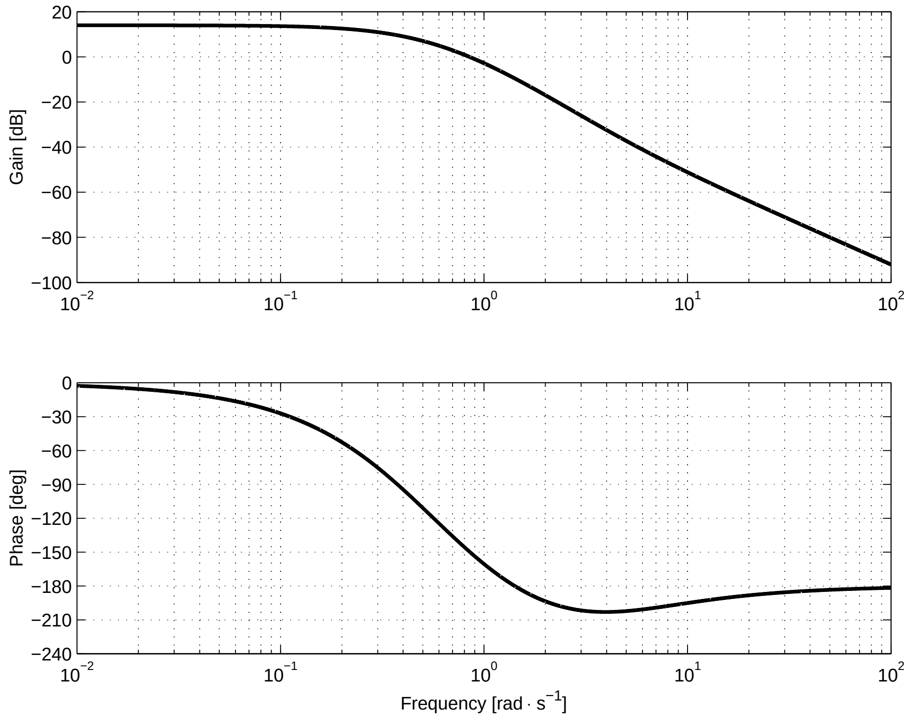
- Sketch the Nyquist plot based on the given Bode diagram.
- Estimate the gain margin, and phase margin and show how you obtain them from both the Bode diagram as well as the Nyquist plot.
- Using Nyquist's stability criterion, for what positive values of K will the closed-loop system be stable (the given Bode diagram is for $K = 1$).

Question 6 (weight:3)

Given: The transfer function of a system $G(s)$ is:

$$G(s) = \frac{K \cdot (0.2s + 1)}{(s+1) \cdot (2s+1)^2}$$

with $K = 5$. The system is put in a feedback loop as in Question 4. The Bode diagram of $G(s)$ is given:



Asked:

- To get a good response, it is asked to have a *phase margin* of about $60°$. Calculate a first approximation of a *phase lead compensator* $G_c(s) = D_{lead}(s) = \frac{T_1 s+1}{\alpha_1 T_1 s+1}$. Make the assumption that the gain is not manipulated by the lead compensator (which is not true). The loop transfer function becomes $L(s) = G_c(s)G(s)$.

- What is the system type of the closed-loop system? The *steady state error* should be approximately $e_{ss} = 1/20$. This can be obtained by implementing a *phase lag compensator* $D_{lag}(s) = \alpha_2 \cdot \frac{T_2 s+1}{\alpha_2 T_2 s+1}$. The controller consists then of the *phase lag compensator* in series with the computed *phase lead compensator,* $G_c(s) = D_{lag}(s)D_{lead}(s)$. Design the *phase lag compensator* and choose the corner frequency ω_2 to be a tenth of the cross-over frequency.

Answer Sheet (Part II)
Modern Control Systems
(Et 3-015 AND Et 3-101)
October 30, 2001

Name: Marcel Oosterom
Student number: -

Problem 2

$y(t) = 1 - e^{-t} - t \cdot e^{-t}$

Problem 3

$F = \begin{pmatrix} -\frac{b_1}{m} - \frac{2b_2}{m^2}p_0 & -k_1 - 3k_2 z_0^2 \\ \frac{1}{m} & 0 \end{pmatrix}$	$G = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$H = \begin{pmatrix} 1 & 0 \end{pmatrix}$	$J = 0$

$p_0 = 0$	$z_0 = 1$	$F_{in_0} = k_1 + k_2$

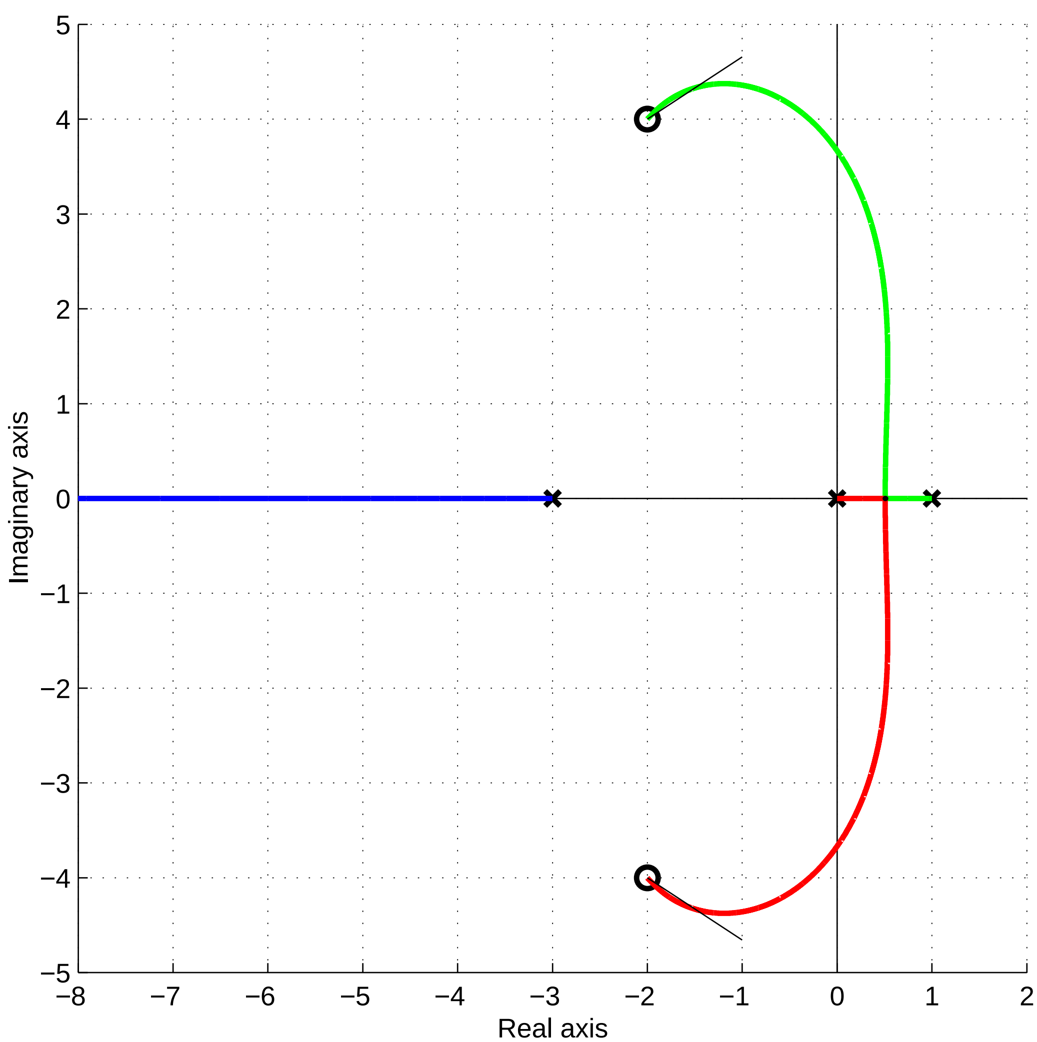
$\mathcal{C} = \begin{pmatrix} 1 & -\frac{b_1}{m} \\ 0 & \frac{1}{m} \end{pmatrix}$	Conditions for controllability: $m \neq \infty$

$\mathcal{O} = \begin{pmatrix} 1 & 0 \\ -\frac{b_1}{m} & -k_1 - 3k_2 \end{pmatrix}$	Conditions for observability: $k_1 \neq -3k_2$

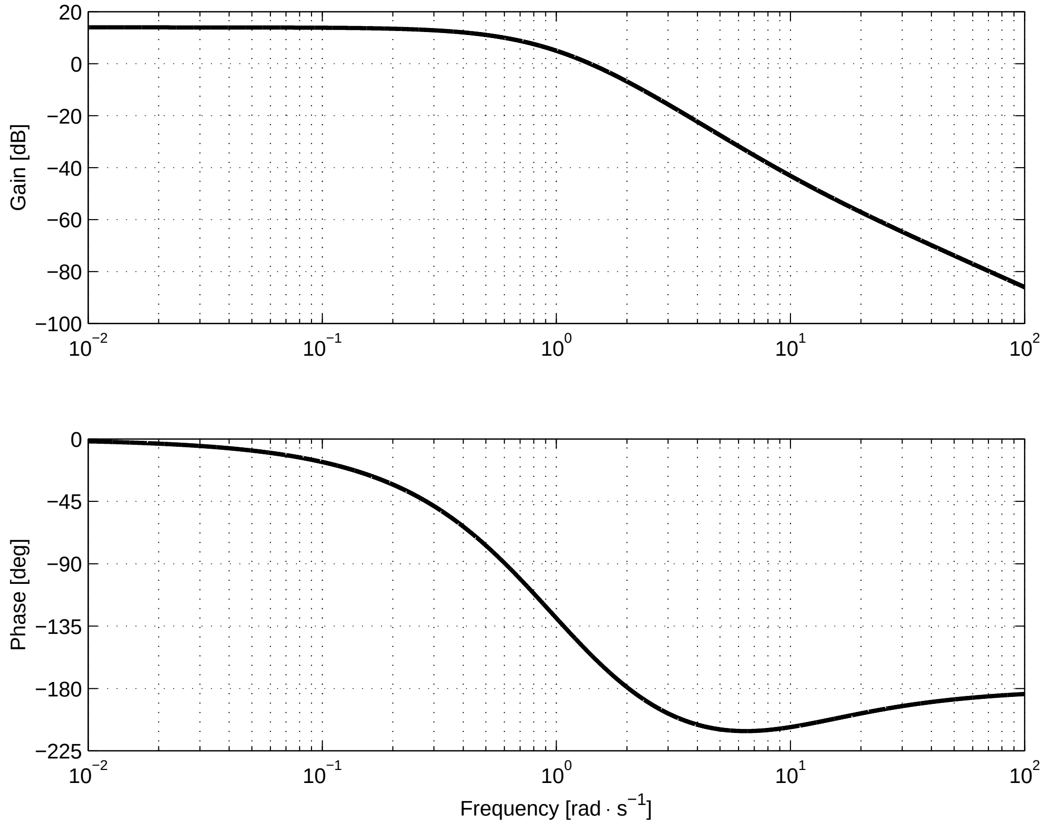
Problem 4

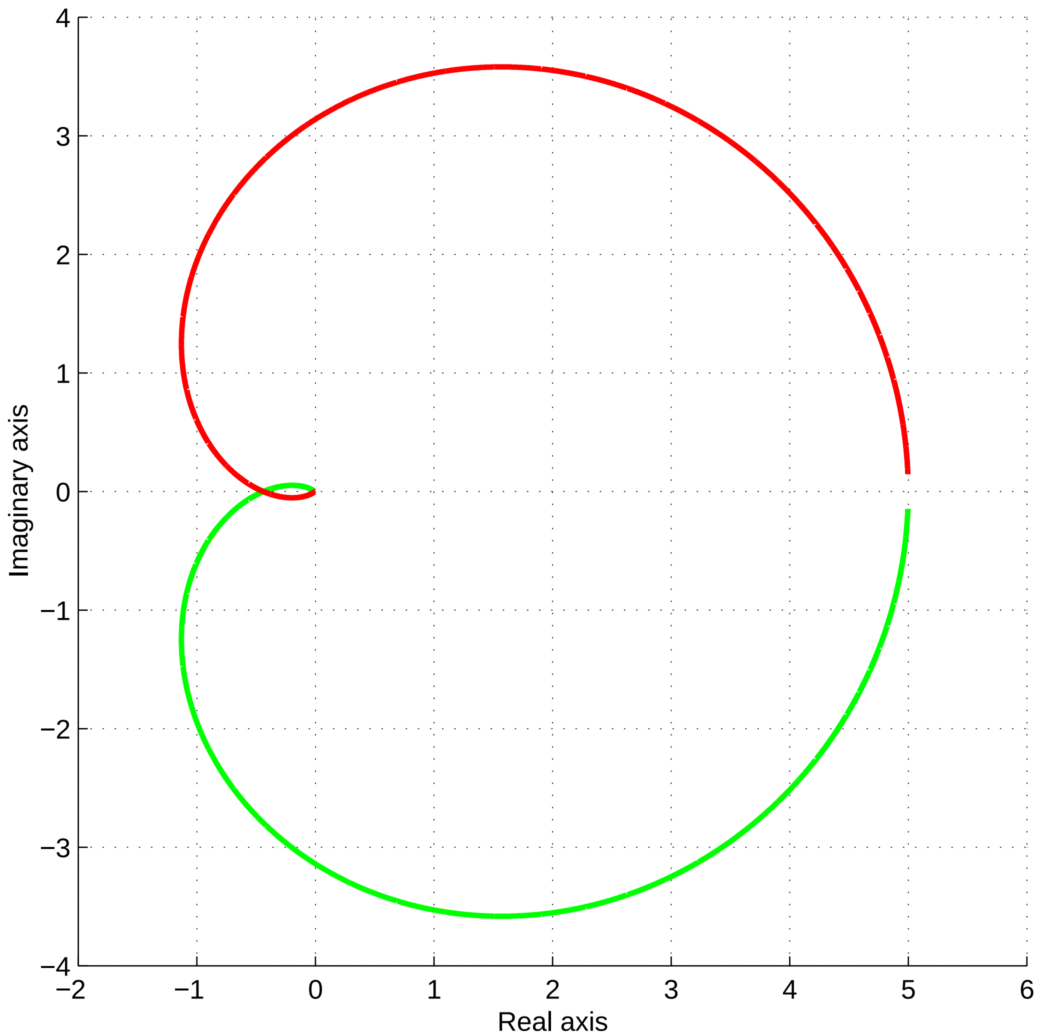
$F(s) = \dfrac{s^2 + 4s + 20}{s^3 + 2s^2 - 3s}$
poles: $-3, 0, 1$ zeros: $-2 \pm 4j$
real axis portion: $\langle -\infty, -3]$ and $[0, 1]$

asymptote(s) angle(s): $-180°$ origin: $+2$
arrival angles: $\psi_1 = 49°$ and $\psi_2 = -49°$
characteristic equation: $s^3 + s^2 \cdot (2+K) + s \cdot (-3+4K) + 20K = 0$
crossing with imaginary axis: $K \approx 4.12$ $p_{2,3} \approx \pm j \cdot 3.67$
equation to compute multiple-roots: $s^4 + 8s^3 + 71s^2 + 80s - 60 = 0$
position of multiple-roots: 0.51



Problem 5





$PM \approx 30^\circ$	$GM \approx 7.2dB \approx 2.29$
Conditions for stability: $0 < K < GM \rightarrow 0 < K < 2.29$	

Problem 6

lead compensator:	$\alpha_1 = \frac{1}{3}$	$T_1 = \frac{1}{0.9}\sqrt{3}$		
lag compensator:	system type: 0	corner frequency $\omega_2 = 0.09$	$\alpha_2 = 3.8$	$T_2 = \frac{1}{0.09}$

Written Exam
Control Systems 1 (Et 3-101)
/ Modern Control Systems (Et 3-015)

On Friday the 23rd of August 2002 from 14:00 to 17:00

Read the following VERY carefully.

- The exam is composed of 2 Parts: theory and exercises. The first part is WITHOUT any books or notes and it will last **45 minutes**. At the end of the 45 minutes, the exam will be collected and after 5 minutes the second part of **130 minutes** will start. You are allowed to use anything you want for the second part:
  - Part I, 45 minutes (end at 14:45) , no books, hand in at the end.
  - Break, 5 minutes (end at 14:50) collect books, sheets and the material you are allowed to use for Part II.
  - Part II, 130 minutes (end at 17:00), with books, sheets.....

  If you are ready with Part I earlier than the time given, you can start working on Part II, but **WITHOUT BOOK** and on a **NEW SHEET OF PAPER**.
- NEVER talk with your neighbor.
- If the answers are not easily readable, the corresponding answer will be given 0 points. Therefore WRITE CLEARLY.
- Read every question well before answering.
- Write ALL your reasoning steps on paper.
- Write your name and student number clearly readable on EACH piece of paper.
- Good luck !

Technische Universiteit Delft
Faculteit der Informatie Technologie en Systemen
Vakgroep Regeltechniek
Mekelweg 4
2628 CD Delft

Part I: theory

(Closed Book)

Question 1

(weight:2)

Given: The transfer function of a system $G(s)$ is:

$$G(s) = \frac{(s+2)}{(s+1)\cdot(3s+1)}$$

Asked:

1. Sketch the Nyquist plot of $G(s)$ **(Be precise about the shape of the Nyquist plot around the origin !)**
2. Give the value of the gain margin of $G(s)$.
3. Show graphically in your sketch the phase margin of $G(s)$.

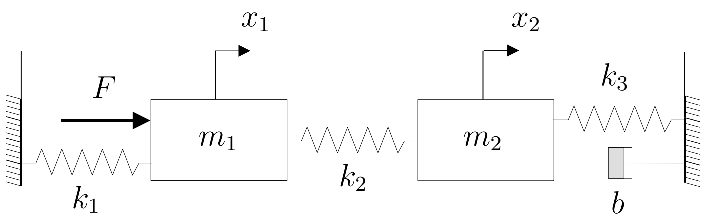
Part II: exercises

(Open Book)

Question 2

(weight:2)

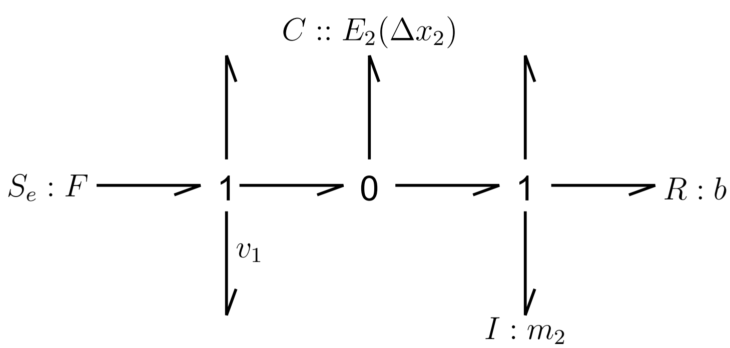
Given: The mass-spring system is illustrated below



where F is the input force, the force of friction b is described as $F_b = b_1 v + b_2 v^3$, and the energy function of the spring k_2 is $E_2(\Delta x_2) = \frac{1}{2}k_{21}\Delta x_2^2 + \frac{1}{4}k_{22}\Delta x_2^4$. The springs k_1 and k_3 are linear.

Asked:

1. Draw a bond-graph of the system considering F as the input force (**Note**: The walls are modelled as a flow source of zero velocity). The structure of the bond-graph can be simplified to the structure illustrated below.



2. Annotate the bond-graph and calculate from it the state space differential equation of the form

$$\dot{\underline{x}} = f(\underline{x}, F)$$

describing the dynamics of the system.

3. Linearize the system around the equilibrium $\underline{x}_0$ and $u_0 = F_0$.

Question 3 (weight:2)

Given: The following system in state space form is given:

$$\dot{x} = \begin{pmatrix} -1 & 0 \\ -3 & -3/2 \end{pmatrix} x + \begin{pmatrix} 2 \\ 2b \end{pmatrix} u \tag{1}$$

$$y = \begin{pmatrix} 1 & 2c \end{pmatrix} x \tag{2}$$

Asked:

1. Calculate for what values of b and c the system is controllable.
2. Calculate for what values of b and c the system is observable.
3. Sketch the amplitude frequency response of the system system for $b = 0$ and $c = 3/4$.
4. Calculate the state feedback such that the poles of the system for $b = 0$ and $c = 3/4$ has a natural frequency $\omega_n = 3$ rad/s and a damping $\zeta = 0.5$.
5. Calculate the estimator for the system with $b = 0$ and $c = 3/4$. The poles of the dynamics of the estimation error should be chosen in $\lambda_1 = -3$,$\lambda_2 = -4$.

Question 4 (weight:2)

Given: The loop-gain transfer function of a system $L(s)$ is

$$L(s) = \frac{2s + 10}{s^2}$$

Asked:

1. Sketch the Bode plot of the sensitivity function $S(j\omega)$.
2. Calculate the peaking value of the magnitude of the sensitivity function $S(j\omega)$.
3. Calculate the vector margin of the loop transfer function $L(s)$.

Question 5

(weight:2)

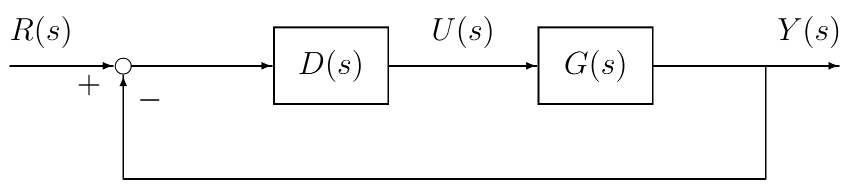
Given: The transfer function of a system $G(s)$ is:

$$G(s) = \frac{1}{(2s+1)(s+1)}$$

The system is put in a feedback loop with PI-controller

$$D(s) = 2(1 + \frac{1}{T_I s})$$

as follows:



Asked:

1. To analyse the variation of integrator parameter T_I, draw the root locus for the poles of the transfer function $\frac{Y(s)}{R(s)}$ for variations of $K = 1/T_I$ from 0 to ∞. Compute all details like break-points and departure/arrival angles if they are applicable for this specific case.
2. Determine for what values of T_I the closed loop is stable.

Solutions
Control Systems I

(Et 3-015 AND Et 3-101)
August 23, 2002

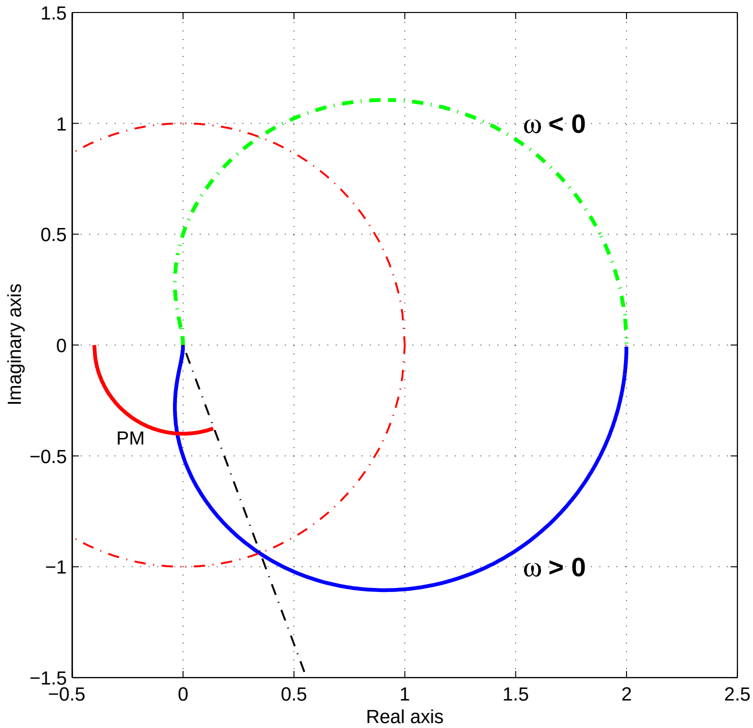
If you have questions concerning the provided solutions of this exam or if you think there is a mistake in the provided solutions, please contact me: Marcel Oosterom,
m.oosterom@its.tudelft.nl, phone: (27) 83371, room: HB 05.050.

Question 1

The transfer function of a system $G(s)$ is:

$$G(s) = \frac{(s+2)}{(s+1)\cdot(3s+1)}$$

1. Sketch the Nyquist plot of $G(s)$ **(Be precise about the shape of the Nyquist plot around the origin !)**



2. Give the value of the gain margin of $G(s)$.

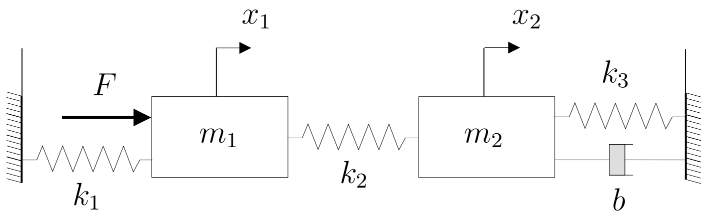
   $GM = \infty$

3. Show graphically in your sketch the phase margin of $G(s)$.

   See Figure above (PM).

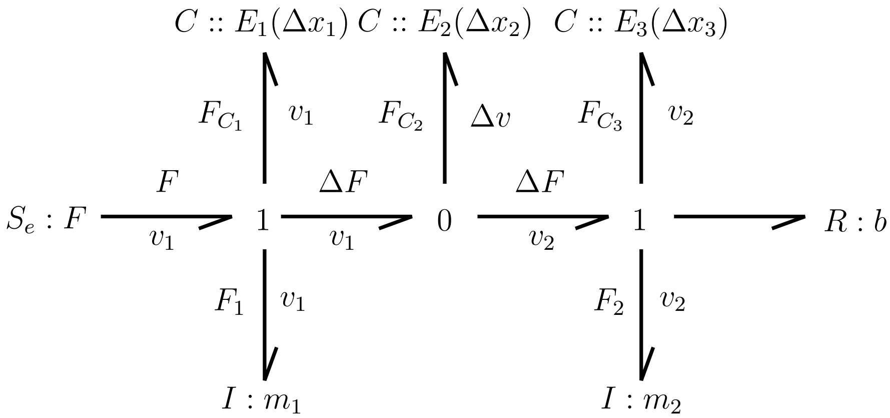
Question 2

The mass-spring system is illustrated below



where F is the input force, the force of friction b is described as $F_b = b_1 v + b_2 v^3$, and the energy function of the spring k_2 is $E_2(\Delta x_2) = \frac{1}{2}k_{21}\Delta x_2^2 + \frac{1}{4}k_{22}\Delta x_2^4$. The springs k_1 and k_3 are linear.

1. Draw a bond-graph of the system considering F as the input force (**Note**: The walls are modelled as a flow source of zero velocity). The structure of the bond-graph can be simplified to the structure illustrated below.



2. Annotate the bond-graph and calculate from it the state space differential equation of the form

$$\dot{\underline{x}} = f(\underline{x}, F)$$

describing the dynamics of the system.

$$\begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \\ \dot{\Delta x}_1 \\ \dot{\Delta x}_2 \\ \dot{\Delta x}_3 \end{bmatrix} = \begin{bmatrix} F - k_1\Delta x_1 - k_{12}\Delta x_2 - k_{22}\Delta x_2^3 \\ k_{12}\Delta x_2 + k_{22}\Delta x_2^3 - k_3\Delta x_3 - \frac{b_1}{m_2}p_2 - \frac{b_2}{m_2^3}p_2^3 \\ \frac{p_1}{m_1} \\ \frac{p_1}{m_1} - \frac{p_2}{m_2} \\ \frac{p_2}{m_2} \end{bmatrix}$$

3. Linearize the system around the equilibrium $\underline{x}_0$ and $u_0 = F_0$.

$$F = \begin{bmatrix} 0 & 0 & k_1 & -k_{12} - 3k_{22}\Delta x_{2_0}^2 & 0 \\ 0 & -\frac{b_1}{m_2} - 3\frac{b_2}{m_2^3}p_{2_0}^2 & 0 & k_{12} + 3k_{22}\Delta x_{2_0}^2 & -k_3 \\ \frac{b_1}{m_1} & 0 & 0 & 0 & 0 \\ \frac{b_1}{m_1} & -\frac{b_2}{m_2} & 0 & 0 & 0 \\ 0 & \frac{b_2}{m_2} & 0 & 0 & 0 \end{bmatrix}, \; G = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Question 3

The following system in state space form is given:

$$\dot{x} = \begin{pmatrix} -1 & 0 \\ -3 & -3/2 \end{pmatrix} x + \begin{pmatrix} 2 \\ 2b \end{pmatrix} u \quad (1)$$

$$y = \begin{pmatrix} 1 & 2c \end{pmatrix} x \quad (2)$$

1. Calculate for what values of b and c the system is controllable.

$$\mathcal{C} = \begin{bmatrix} G & FG \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 2b & -6-3b \end{bmatrix}$$

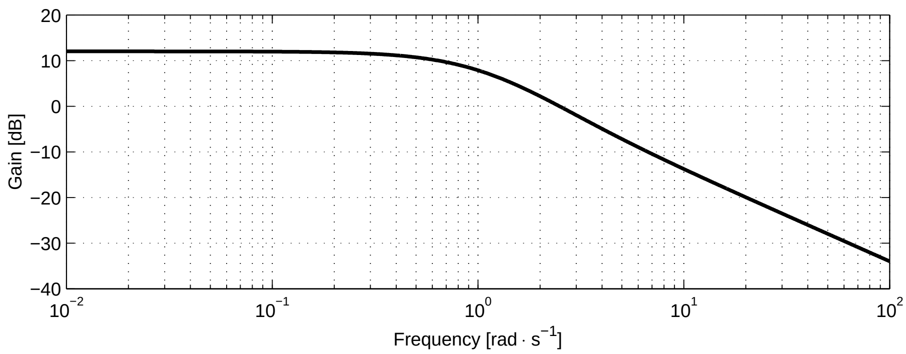
   Controllability: $b \neq -6$

2. Calculate for what values of b and c the system is observable.

$$\mathcal{O} = \begin{bmatrix} H \\ HF \end{bmatrix} = \begin{bmatrix} 1 & 2c \\ -1-6c & -3c \end{bmatrix}$$

   Observability: $c \neq 0 \vee c \neq \frac{1}{12}$

3. Sketch the amplitude frequency response of the system system for $b = 0$ and $c = 3/4$.



4. Calculate the state feedback such that the poles of the system for $b = 0$ and $c = 3/4$ has a natural frequency $\omega_n = 3$ rad/s and a damping $\zeta = 0.5$.

$$K = \begin{bmatrix} 0.25 & -1.125 \end{bmatrix}$$

5. Calculate the estimator for the system with $b = 0$ and $c = 3/4$. The poles of the dynamics of the estimation error should be chosen in $\lambda_1 = -3$,$\lambda_2 = -4$.

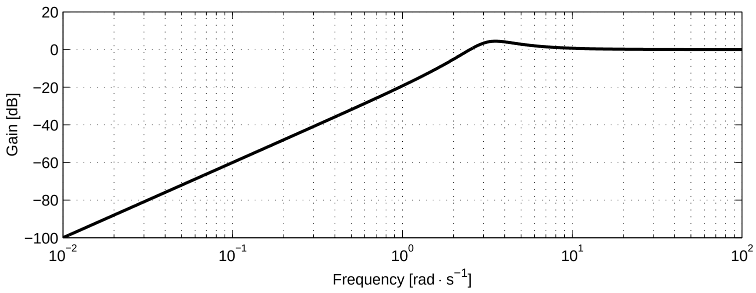
$$L = \begin{bmatrix} -1.5 \\ 4 \end{bmatrix}$$

Question 4

The loop-gain transfer function of a system $L(s)$ is

$$L(s) = \frac{2s + 10}{s^2}$$

1. Sketch the Bode plot of the sensitivity function $S(j\omega)$.



2. Calculate the peaking value of the magnitude of the sensitivity function $S(j\omega)$.

$$\max |S(j\omega)| = 1.66$$

3. Calculate the vector margin of the loop transfer function $L(s)$.

$$\alpha_{min} = \frac{1}{\max_\omega |S(j\omega)|} = 0.60$$

Question 5

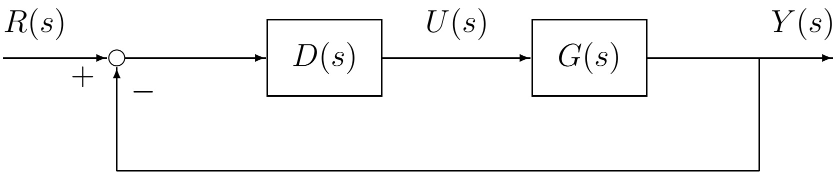
The transfer function of a system $G(s)$ is:

$$G(s) = \frac{1}{(2s+1)(s+1)}$$

The system is put in a feedback loop with PI-controller

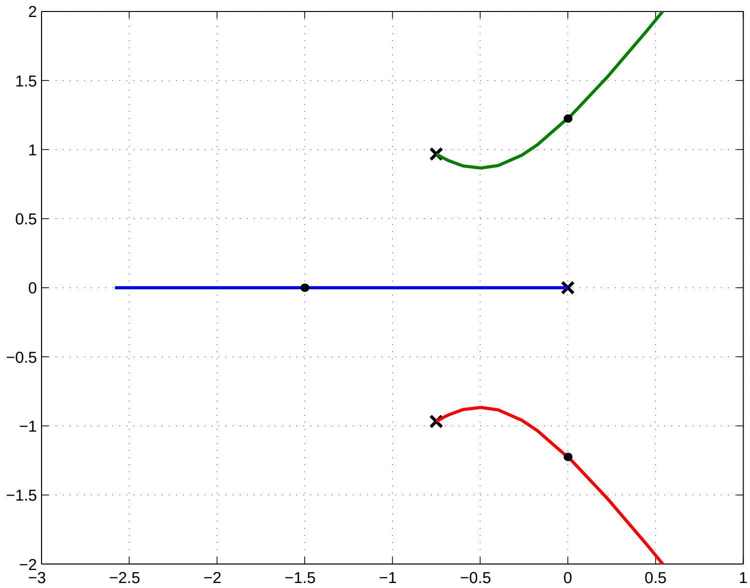
$$D(s) = 2(1 + \frac{1}{T_I s})$$

as follows:



1. To analyse the variation of integrator parameter T_I, draw the root locus for the poles of the transfer function $\frac{Y(s)}{R(s)}$ for variations of $K = 1/T_I$ from 0 to ∞. Compute all details like break-points and departure/arrival angles if they are applicable for this specific case.

$$F(s) = -\frac{1}{K'} \qquad F(s) = \frac{1}{s(s^2+1.5s+1.5)} \text{ and } K' = 2K$$



2. Determine for what values of T_I the closed loop is stable.

$$0 < K' < \frac{4}{9} \rightarrow 0 < K < \frac{2}{9} \rightarrow \frac{9}{2} < T_I < \infty$$

Written Exam
Control Systems 1 (Et 3-101)

On Tuesday the 29th of October 2002 from 9:00 to 12:00

Read the following VERY carefully.

- The exam consists of 6 pages with 5 exercises.
- You are only allowed to use the BOOK and a CALCULATOR.
- NEVER talk with your neighbor.
- If the answers are not easily readable, the corresponding answer will be given 0 points. Therefore WRITE CLEARLY.
- Read every question well before answering.
- Write ALL your reasoning steps on paper.
- Write your name and student number clearly readable on EACH piece of paper.
- Good luck !

Technische Universiteit Delft
Faculteit der Informatie Technologie en Systemen
Vakgroep Regeltechniek
Mekelweg 4
2628 CD Delft

Question 1

(weight:1.5)

Given: The differential equation for the *van der Pol* oscillator is given by

$$\ddot{y}(t) + (1 - y^2(t))\,\dot{y}(t) + y(t) = u(t)$$

Asked:

1. Calculate the differential equation of the form

$$\dot{x} = f(x, \tau)$$

   describing the dynamics of the system with $x = \begin{bmatrix} \dot{y} & y \end{bmatrix}^T$.

2. Linearize the system around an equilibrium x_0 and u_0.

3. For what values of u_0 do we obtain a stable linearized system ?

Question 2 (weight:2)

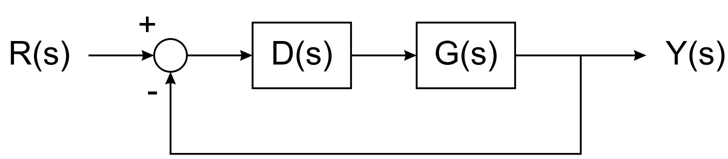
Given: The transfer function of a system $G(s)$ is:

$$G(s) = \frac{(s+1)(s+3)}{(s^2+2s+17)(s+4)}$$

The system is put in a feedback loop with the proportional gain

$$D(s) = K_P$$

as follows:



Asked:

1. To analyse the variation of the proportional gain K_P, draw the root locus for the poles of the transfer function $\frac{Y(s)}{R(s)}$ for variations of K_P from 0 to ∞. Compute all details like asymptotes, departure/arrival angles, etc. if they are applicable for this specific case.

   The four solutions of the equation that needs to be solved to determine the location of the multiple roots are -2.20, 3.39, $-4.60 \pm 2.30 \cdot j$. Give this equation.

2. For $K_P \geq K_1$ it holds that $\zeta_i \geq 0.8$, where ζ_i denotes the damping coefficient of the ith pole. Give the equation that needs to be solved to determine K_1.

Question 3 (weight:2.5)

Given: The transfer function of a system $G(s)$ is:

$$G(s) = \frac{K(s+0.2)}{(s+0.5)(s+1)^2(s+2)}$$

The system is put in a feedback loop as in Question 2.

Asked:

1. Sketch the Nyquist plot $L(s) = G(s)D(s)$ for $K = 10$ and $D(s) = 1$. **Be precise about the shape of the Nyquist plot for low frequencies!**.
2. Estimate the gain margin, and phase margin and show how you obtain them from the Nyquist plot.
3. To get a good response, it is asked to have a *phase margin* of about $60°$. Calculate a first approximation of a *phase lead compensator*

   $$D(s) = D_{lead}(s) = \frac{T_1 s + 1}{\alpha_1 T_1 s + 1}$$

   taking into account that the current phase margin is equal to $33°$ at a frequency of $1.6 \text{ rad} \cdot \text{sec}^{-1}$. Make the assumption that the gain is not manipulated by the lead compensator (which is not true).
4. What is the system type of the closed-loop system? Design a *phase lag compensator*

   $$D_{lag}(s) = \alpha_2 \cdot \frac{T_2 s + 1}{\alpha_2 T_2 s + 1}$$

   with a corner frequency ω_2 to be a tenth of the cross-over frequency such that the *steady state error* is equal to $e_{ss} = 1/20$. The controller consists then of the *phase lag compensator* in series with the computed *phase lead compensator*, $D(s) = D_{lag}(s)D_{lead}(s)$.

Question 4

(weight:2)

Given: The following system in state space form is given:

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ -3 & -4 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} x$$

Asked:

1. Calculate the state feedback such that the poles of the system have a natural frequency $\omega_n = 2$ rad/s and a damping $\zeta = 0.5$.
2. Calculate the estimator for the system. The poles of the dynamics of the estimation error should be chosen in $\lambda_1 = -2$,$\lambda_2 = -4$.

Question 5 (weight:2)

Given: The system $G(s)$ has a transfer function:

$$G(s) = \frac{s+\tau}{(s+2)}$$

where the parameter τ is not exactly known, but bounded by

$$1 \leq \tau \leq 2$$

This system can be described as a nominal system

$$G_0(s) = \frac{(s+1)}{(s+2)}$$

with multiplicative uncertainty $l(s)$.

Asked:

1. Describe the unstructured multiplicative uncertainty $l(s)$ as a function of the uncertain τ.
2. Show that $|l(j\omega)| \leq 1$ for all frequencies.
3. Prove that the controller $D(s) = \dfrac{1}{s+1}$ robustly stabilizes the system.

Solutions
Control Systems I
Et 3-101
October 29, 2002

If you have questions concerning the provided solutions of this exam or if you think there is a mistake in the provided solutions, please contact me: Marcel Oosterom, m.oosterom@its.tudelft.nl, phone: (27) 83371, room: HB 05.050.

Question 1

The differential equation for the *van der Pol* oscillator is given by

$$\ddot{y}(t) + (1 - y^2(t))\,\dot{y}(t) + y(t) = u(t)$$

1. Calculate the differential equation of the form

$$\dot{x} = f(x, \tau)$$

describing the dynamics of the system with $x = \begin{bmatrix} \dot{y} & y \end{bmatrix}^T$.
Nonlinear differential equations:

$$\begin{bmatrix} \ddot{y} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -(1 - x_2^2(t))\,x_1(t) - x_2(t) + u(t) \\ x_1(t) \end{bmatrix}$$

2. Linearize the system around an equilibrium x_0 and u_0.
Equilibrium state:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = 0$$

it immediately follows that $x_1(t) = 0$ and $x_2(t) = u_0$.

Linear equations:

$$\frac{\partial f_1}{\partial x_1}|_{(x_0,u_0)} = -(1 - x_2^2(t))|_{x_0} = -(1 - u_0^2)$$

$$\frac{\partial f_1}{\partial x_2}|_{(x_0,u_0)} = (2\,x_2(t)x_1(t) - 1)|_{x_0} = -1$$

$$\frac{\partial f_2}{\partial x_1}|_{(x_0,u_0)} = 1 \qquad , \qquad \frac{\partial f_2}{\partial x_2}|_{(x_0,u_0)} = 0$$

$$\frac{\partial f_1}{\partial u}|_{(x_0,u_0)} = 1 \qquad , \qquad \frac{\partial f_2}{\partial u}|_{(x_0,u_0)} = 0$$

$$\delta\dot{x} = \begin{bmatrix} u_0^2 - 1 & -1 \\ 1 & 0 \end{bmatrix}\delta x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \delta u$$

3. For what values of u_0 do we obtain a stable linearized system ?
   Eigenvalues of the matrix F can be found by solving

$$\det \begin{bmatrix} \lambda - u_0^2 + 1 & 1 \\ -1 & \lambda \end{bmatrix} = (\lambda - u_0^2 + 1)\lambda + 1 = \lambda^2 + (1 - u_0^2)\lambda + 1$$

   eigenvalues:

$$\lambda_{1,2} = (u_0^2 - 1)/2 \pm 0.5\sqrt{(u_0^2 - 1)^2 - 4}$$

   Therefore if $u_0^2 < 1$, the linearized model is stable.

Question 2

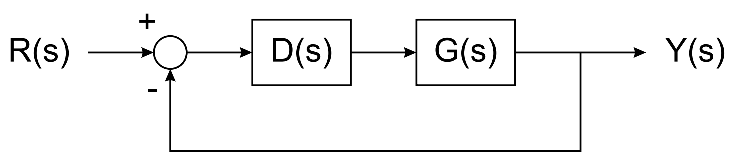
The transfer function of a system $G(s)$ is:

$$G(s) = \frac{(s+1)(s+3)}{(s^2+2s+17)(s+4)}$$

The system is put in a feedback loop with the proportional gain

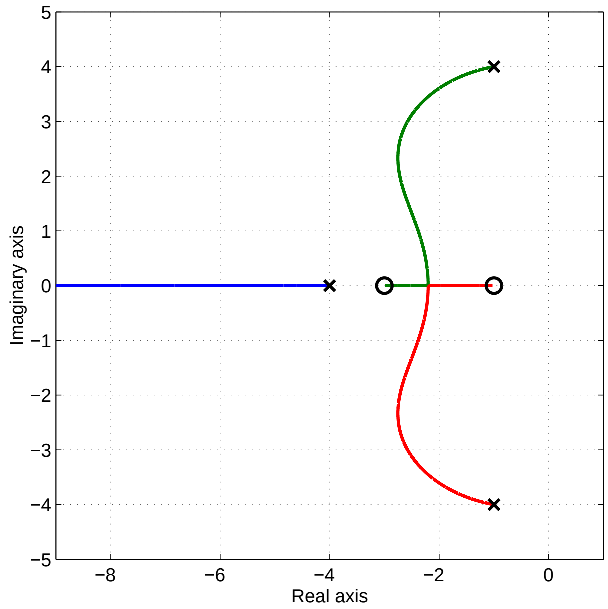
$$D(s) = K_P$$

as follows:



1. To analyse the variation of the proportional gain K_P, draw the root locus for the poles of the transfer function $\frac{Y(s)}{R(s)}$ for variations of K_P from 0 to ∞. Compute all details like asymptotes, departure/arrival angles, etc. if they are applicable for this specific case.

   The four solutions of the equation that needs to be solved to determine the location of the multiple roots are -2.20, 3.39, $-4.60 \pm 2.30 \cdot j$. Give this equation.



$$\frac{\partial}{\partial s}\left(-\frac{1}{F(s)}\right) = 0 \rightarrow -s^4 - 8s^3 - 8s^2 + 100s + 197 = 0$$

2. For $K_P \geq K_1$ it holds that $\zeta_i \geq 0.8$, where ζ_i denotes the damping coefficient of the ith pole. Give the equation that needs to be solved to determine K_1.

$$(s^2 + 2s + 17) \cdot (s + 4) + K_1(s^2 + 4s + 3) = (s^2 + 2 \cdot 0.8 \cdot \omega_n \cdot s + \omega_n^2) \cdot (s - p_3)$$

The left-hand side of this equation represents the characteristic polynomial as a function of K_1, while the right-hand side represents the desired polynomial. The desired polynomial is convolution of a complex-conjugate pair of poles with a relative damping of 0.8 and an unknown frequency and a third real pole p_3, from which the location is not yet known either.

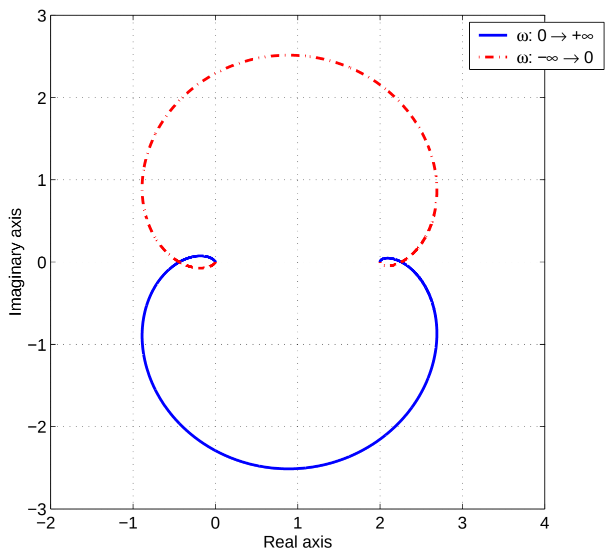
Question 3

The transfer function of a system $G(s)$ is:

$$G(s) = \frac{K(s + 0.2)}{(s + 0.5)(s + 1)^2(s + 2)}$$

The system is put in a feedback loop as in Question 2.

1. Sketch the Nyquist plot $L(s) = G(s)D(s)$ for $K = 10$ and $D(s) = 1$. **Be precise about the shape of the Nyquist plot for low frequencies!**.



2. Estimate the gain margin, and phase margin and show how you obtain them from the Nyquist plot.

$GM \approx 2.29$ and $PM \approx 33°$

3. To get a good response, it is asked to have a *phase margin* of about $60°$. Calculate a first approximation of a *phase lead compensator*

$$D(s) = D_{lead}(s) = \frac{T_1 s + 1}{\alpha_1 T_1 s + 1}$$

taking into account that cross-over frequency is $1.6 \text{ rad} \cdot \text{sec}^{-1}$. Make the assumption that the gain is not manipulated by the lead compensator (which is not true).

$\alpha_1 = 0.38$ and $T_1 = 1.02$

4. What is the system type of the closed-loop system? Design a *phase lag compensator*

$$D_{lag}(s) = \alpha_2 \cdot \frac{T_2 s + 1}{\alpha_2 T_2 s + 1}$$

with a corner frequency ω_2 to be a tenth of the cross-over frequency such that the *steady state error* is equal to $e_{ss} = 1/20$. The controller consists then of the *phase lag compensator* in series with the computed *phase lead compensator*,
$D(s) = D_{lag}(s) D_{lead}(s)$.

The closed-loop system with lead compensator is of type 0.

$\alpha_2 = 9.5$ and $T_2 = 6.25$

Question 4

The following system in state space form is given:

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ -3 & -4 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} x$$

1. Calculate the state feedback such that the poles of the system have a natural frequency $\omega_n = 2$ rad/s and a damping $\zeta = 0.5$.
From what explained at p.122 of the book, we should have as characteristic polynomial:

$$\alpha_c(s) = s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + 2s + 4$$

To place the poles we can use Ackermann's formula

$$K = \begin{bmatrix} 0 & 1 \end{bmatrix} \mathcal{C}^{-1} \alpha_c(F)$$

We compute:

$$\mathcal{C} = \begin{bmatrix} G & FG \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & -3 \end{bmatrix} \Rightarrow \mathcal{C}^{-1} = \frac{1}{3} \begin{bmatrix} 3 & -1 \\ 0 & -1 \end{bmatrix}$$

and

$$\alpha_c(F) = F^2 + 3F + 9I = \begin{bmatrix} -1 & 0 \\ -3 & -4 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ -3 & -4 \end{bmatrix} + 2 \begin{bmatrix} -1 & 0 \\ -3 & -4 \end{bmatrix} + 4 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 9 & 12 \end{bmatrix}$$

And therefore

$$K = \begin{bmatrix} 0 & 1 \end{bmatrix} \frac{1}{3} \begin{bmatrix} 3 & -1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 9 & 12 \end{bmatrix} = \begin{bmatrix} -3 & -4 \end{bmatrix}$$

Check:

$$(\det[sI - F + GK]) = \det\left(sI - \begin{bmatrix} 2 & 4 \\ -3 & -4 \end{bmatrix}\right) = s^2 + 2s + 4$$

2. Calculate the estimator for the system. The poles of the dynamics of the estimation error should be chosen in $\lambda_1 = -2$,$\lambda_2 = -4$.
   Due to the asked error dynamics, the estimator characteristic polynomial should be:

   $$\alpha_e(s) = (s+2)(s+4) = s^2 + 6s + 8$$

   The observer gains can be simply directly calculated with Ackermann's estimator formula.

   $$L = \alpha_e(F)\mathcal{O}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

   We compute:

   $$\mathcal{O} = \begin{bmatrix} H \\ HF \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} \Rightarrow \mathcal{O}^{-1} = \frac{1}{3} \begin{bmatrix} -4 & -1 \\ 3 & 0 \end{bmatrix}$$

   and

   $$\alpha_e(F) = F^2 + 6F + 8I = \begin{bmatrix} -1 & 0 \\ -3 & -4 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ -3 & -4 \end{bmatrix} + 6 \begin{bmatrix} -1 & 0 \\ -3 & -4 \end{bmatrix} + 8 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ -3 & 0 \end{bmatrix}$$

   And therefore

   $$L = \begin{bmatrix} 3 & 0 \\ -3 & 0 \end{bmatrix} \frac{1}{3} \begin{bmatrix} -4 & -1 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

   Check:

   $$(\det[sI - F + LH]) = \det\left(sI - \begin{bmatrix} -1 & 1 \\ -3 & -5 \end{bmatrix}\right) = s^2 + 6s + 8$$

Question 5

The system $G(s)$ has a transfer function:

$$G(s) = \frac{s + \tau}{(s+2)}$$

where the parameter τ is not exactly known, but bounded by

$$1 \le \tau \le 2$$

This system can be described as a nominal system

$$G_0(s) = \frac{(s+1)}{(s+2)}$$

with multiplicative uncertainty $l(s)$.

1. Describe the unstructured multiplicative uncertainty $l(s)$ as a function of the uncertain τ.

$$G(s) = \frac{s+\tau}{(s+2)} \qquad G_0(s) = \frac{(s+1)}{(s+2)}$$

From $G(s) = G_0(s)(1 + l(s))$ it follows

$$l(s) = G(s)/G_0(s) - 1 = \frac{s+\tau}{(s+2)}\frac{(s+2)}{(s+1)} - 1 = \frac{\tau - 1}{(s+1)}$$

2. Show that $|l(j\omega)| \leq 1$ for all frequencies.
Find maximum

$$\max_{\omega} |l(j\omega)| = \max_{\omega} \frac{\tau - 1}{\sqrt{1+\omega^2}} = \frac{\tau - 1}{\sqrt{1+0^2}} = \tau - 1$$

For all $1 \leq \tau \leq 2$ we find

$$|l(j\omega)| \leq 1$$

3. Prove that the controller $D(s) = \dfrac{1}{s+1}$ robustly stabilizes the system.
To prove that the controller $D(s)$ robustly stabilizes the system, we have to prove that

$$|T^{-1}(j\omega)| > |l(j\omega)| \quad \text{for all } \omega$$

Compute $T(s)$ for $G_0(s) = \frac{(s+1)}{(s+2)}$ and $D(s) = \frac{1}{s+1}$:

$$T(s) = \frac{G_0 D}{1 + G_0 D} = \frac{\frac{(s+1)}{(s+2)}\frac{1}{s+1}}{1 + \frac{(s+1)}{(s+2)}\frac{1}{s+1}} = \frac{1}{s+3}$$

Find minimum

$$\min_{\omega} |T^{-1}(j\omega)| = \min_{\omega} |j\omega + 3| = \min_{\omega} \sqrt{\omega^2 + 9} = \sqrt{0+9} = 3$$

Therefore for all ω we find

$$|T^{-1}(j\omega)| > |l(j\omega)|$$

which proves robust stability.

Control Systems (SC3020et)

(http://www.dcsc.tudelft.nl/sc3020et/index.html)

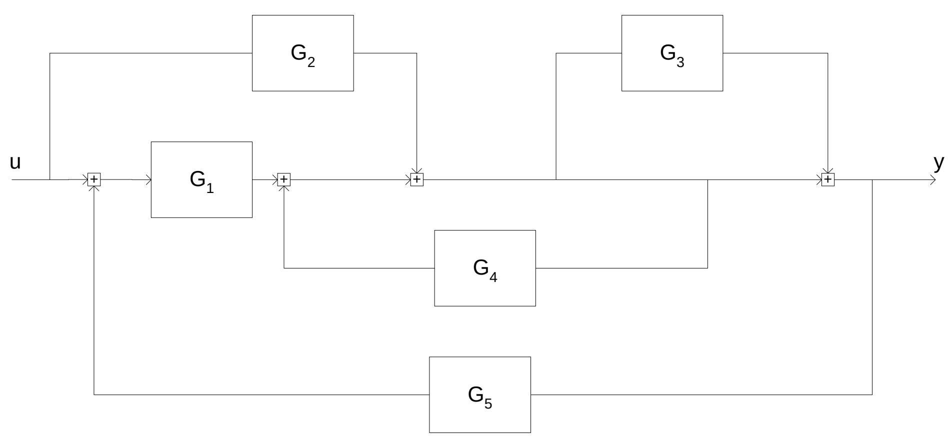
Instruction stencil no. 1

September 11, 2003

Name:	Student Number:

Problem 1: Transfer function of Block Diagram

Compute the transfer function from u to y for the following interconnection of systems $G_i, \quad i = 1 \ldots 5$.



Problem 2: Stability problem (3.44)

Modify the Routh criterion so that it applies to the case where all the poles are to be to left of -α when $\alpha > 0$. Apply the modified test to the polynomial

$$s^3 + (6 + K)s^2 + (5 + 6K)s + 5K = 0 \tag{1}$$

finding those values of K for which all poles have a real part less than -1.

Problem 3: Non-minimum Phase System (3.35)

Consider the two non-minimum phase systems

$$G_1(s) = -\frac{2(s-1)}{(s+1)(s+2)} \tag{2}$$

$$G_2(s) = \frac{3(s-1)(s-2)}{(s+1)(s+2)(s+3)} \tag{3}$$

a) Sketch the unit step responses for $G_1(s)$ and $G_2(s)$, paying close attention to the transient part of the response.
b) Explain the difference in the behavior of the two responses as it relates to the zero locations.
c) Consider a stable, strictly proper system (i.e. m zeros and n poles, where $m < n$). Let $y(t)$ denote the step response of the system. The step response is said to have undershoot if it initially starts off in the wrong direction (also called inverse response). Prove that a stable, strictly proper system has an undershoot if and only if its transfer function has an odd number of real RHP zeros.

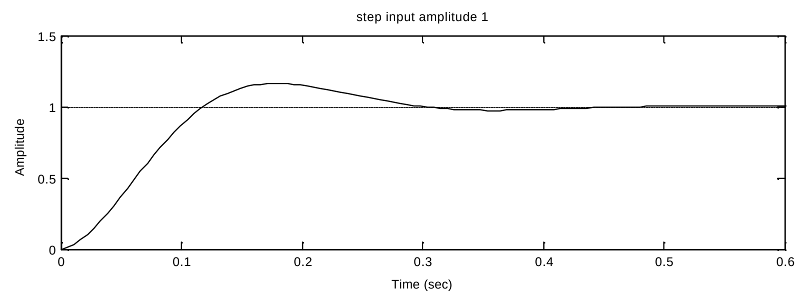
Problem 4: Performance specifications on 2nd order system (4.30)

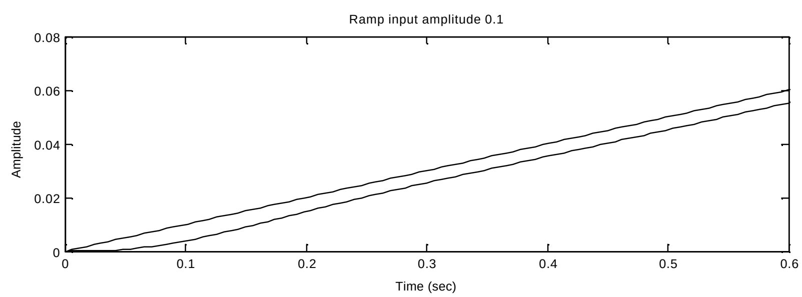
A position control system has the closed loop transfer function (meter/meter) given by

$$\frac{Y(s)}{R(s)} = \frac{b_0 s + b_1}{s^2 + a_1 s + a_2} \tag{4}$$

1) What is the system type?
2) Choose the parameters (a_1, a_2, b_0, b_1) so that the following specifications are satisfied simultaneously:

- The rise time $t_r < 0.1$ sec.
- The overshoot $M_p < 20\%$
- The settling time $t_s < 0.5$ sec.
- Steady state error to a step reference is zero
- Steady state error to a ramp input of 0.1 m/sec is not more than 1 mm.


step input amplitude 1
1.5
1
0.5
0
Amplitude
0
0.1
0.2
0.3
0.4
0.5
0.6
Time (sec)


Ramp input amplitude 0.1
0.08
0.06
0.04
0.02
0
Amplitude
0
0.1
0.2
0.3
0.4
0.5
0.6
Time (sec)

Control Systems (SC3020et)

(http://www.dcsc.tudelft.nl/sc3020et/index.html)

Instruction stencil no. 1: answers[1]

September 11, 2003

Problem 1: Transfer function of Block Diagram

When cutting the interconnection as shown in Figure , both signals y, z can be explicitly written in terms of w and u.

$$z = (G_2 + G_1)u + (G_4 + (1 + G_3)G_5G_1)w$$

$$y = (1 + G_3)w$$

Since $w = z$, we get from the first equation

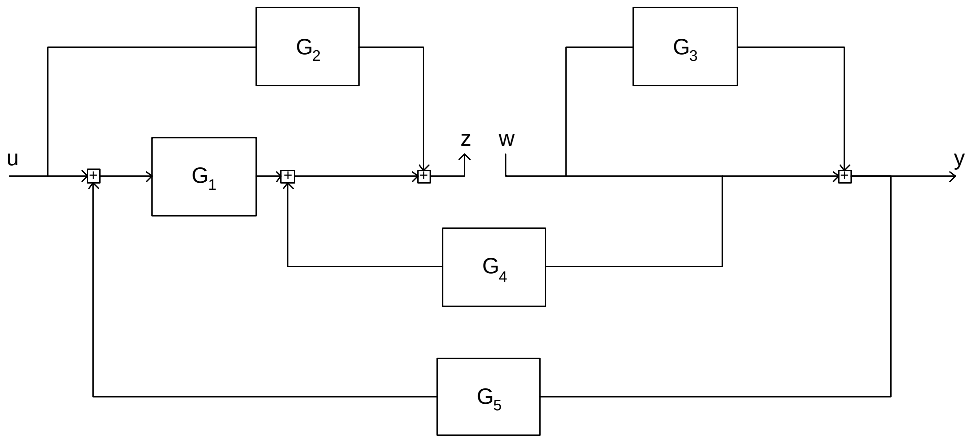
$$(I - (G_4 + (1 + G_3)G_5G_1))z = (G_2 + G_1)u$$

$$z = (I - (G_4 + (1 + G_3)G_5G_1))^{-1}(G_2 + G_1)u$$

Therefore we get

$$y = (1 + G_3)(I - (G_4 + (1 + G_3)G_5G_1))^{-1}(G_2 + G_1)u$$

[1] For remarks and questions please contact Sjoerd Dietz, 278-1660



Problem 2: Stability problem (3.44)

The standard Routh criterion for a polynomial proves whether all roots are in the open LHP. The trick is to introduce a new $z = s - \alpha$ which defines a new polynomial $\hat{p}(z)$. Then, all roots of $p(s)$ are left to the vertical axes at $-\alpha$ if and only if the $\hat{p}(z)$ has all roots in the open LHP

$$p(s) = s^3 + (6+K)s^2 + (5+6K)s + 5K \tag{1}$$

$$\hat{p}(z) = (z+\alpha)^3 + (6+K)(z+\alpha)^2 + (5+6K)(z+\alpha) + 5K \tag{2}$$

The Routh criterion for $alpha = 1$ becomes

$$z^3 + 9z^2 + 20z + 12 + Kz^2 + 8Kz + 12K = 0 \tag{3}$$

or

$$z^3 + (9+K)z^2 + (20+8K)z + 12(1+K) = 0 \tag{4}$$

The corresponding Routh array is

Row 3	1	20+8K
Row 2	9+K	12(1+K)
Row 1	b1	0
Row 0	c1	

in which

$$b1 = \frac{a_1 a_2 - a_3}{a_1} = 8(K+7)(K+3)/(9+K) \tag{5}$$

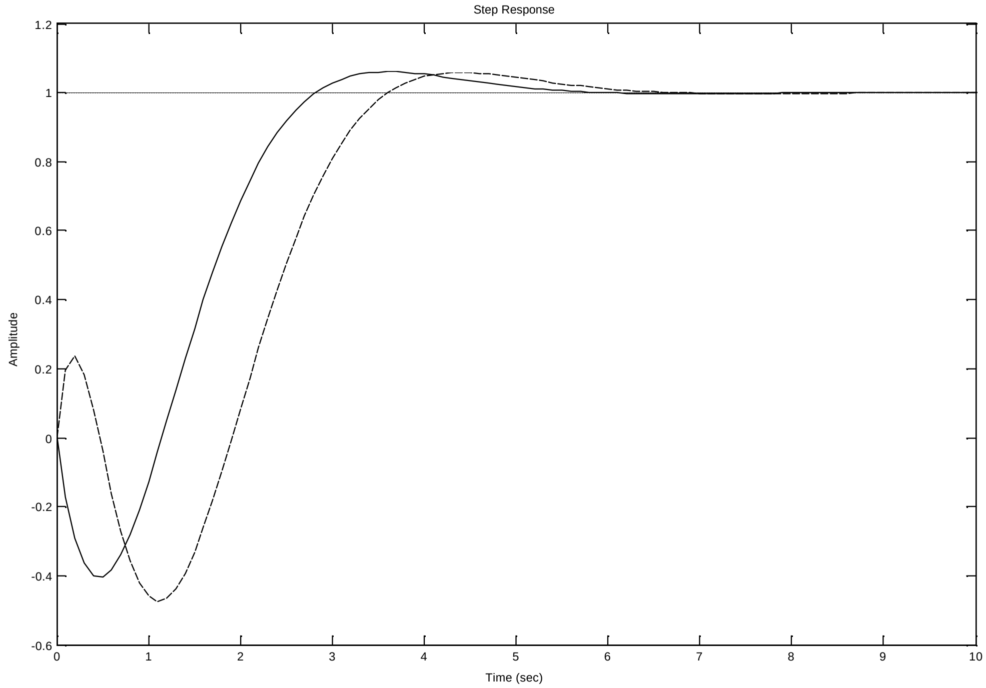
$$c1 = \frac{b_1 a_3 - a_1 b_2}{b_1} = 12(1+K) \tag{6}$$

The most restricting is condition $c_1 > 0$. The stability of the polynomial therefore becomes

$$K \in \{(-1, \infty)\} \tag{7}$$

.

Figure 1: Step-response of G_1 (full) and G_2 (dashed)



Problem 3: Non-minimum Phase System (3.35)

Consider the two non-minimum phase systems

$$G_1(s) = -\frac{2(s-1)}{(s+1)(s+2)} \tag{8}$$

$$G_2(s) = \frac{3(s-1)(s-2)}{(s+1)(s+2)(s+3)} \tag{9}$$

0.1 Part a

In Figure 1 the response is shown. In this assignment, one is intended to use the implications of the Laplace transform, as discussed in Appendix A of the book. Consider the general form

$$G(s) = \frac{\prod_{i=1}^{m} (s+b_i)}{\prod_{j=1}^{n} (s+a_j)} \tag{10}$$

in which we require $m < n$ since we know $G(s)$ to be strictly proper. An inverse response occurs when initial response is in the opposite direction to the 'eventual' behavior (here we have a steady state value). Let $y(t)$ be the response due to a unit step input, and $Y(s)$ be its Laplace transform. Then we know

$$Y(s) = G(s)u(s) = G(s)\frac{1}{s} \tag{11}$$

The steady state gain K_{ss} of G is easily found

$$K_{ss} = \lim_{s \to 0} G(s) = \frac{\prod_{i=1}^{m} b_i}{\prod_{j=1}^{n} a_j} \tag{12}$$

Since G is stable, $a_i > 0, i = 1 \ldots n$ so that the steady state gain of (10) is negative only for odd number of RHP zeros $(b_i < 0)$ and positive for even number of RHP zeros.

Now let us look at the initial response. First we express the time derivative $\frac{d}{dt}y(t)$ in terms of the signal $Y(s)$. Equation (A.11) from the book reads

$$\mathcal{L}(\frac{d}{dt}y(t)) = -y(0^-) + sY(s) \tag{13}$$

Assuming the system output is initially zero, we get that

$$\mathcal{L}(\frac{d}{dt}y(t)) = sG(s)u(s) = G(s) \tag{14}$$

2

Finally using equation (A.22), the initial response of signal f(t) with $\mathcal{L}(f(t)) = F(s)$ is

$$f(0^+) = \lim_{s \to \infty} sF(s) \tag{15}$$

so that we get

$$\frac{d}{dt}y(0^+) = \lim_{s \to \infty} sG(s) \tag{16}$$

. For the transfer function of the form (10), this value always equals 1.
Summarizing, an inverse response for a strictly proper and stable $G(s)$ occurs when it has an odd number of poles.

[2]Note that $G(s)$ should be read as the Laplace transform of a *signal*, rather than the transfer function of a linear system

Problem 4: Performance specifications on 2nd order system (4.30)

The position control system has the closed loop transfer function (meter/meter) given by

$$\frac{G(s) = Y(s)}{R(s)} = \frac{b_0 s + b_1}{s^2 + a_1 s + a_2} = \frac{b_0 s + b_1}{s^2 + 2\zeta\omega_n s + {\omega_n}^2} \tag{17}$$

0.2 Part a

- Rise time.
  As can be found in the slides, the rise time for a second order system is approximately $\frac{1.8}{\omega_n}$. Hence, $\omega_n > 18$, i.e. $a_2 > 324$

- Overshoot in percentage $M = 100e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}$. For $M = 20, \zeta \approx 0.45$ so we end up with ζ¿0.45. This corresponds to the requirement $\frac{a_1}{2\sqrt{a_2}} > 0.45$

- Settling time, $t_s < 0.5$ s. This is directly related to the negative real part σ of the roots of $R(s)$. Assume that we require 1% settling time.

  $$t_{sn\%} = \frac{1}{\sigma} ln \frac{100}{1} \approx \frac{1}{\sigma} 4.6 \tag{18}$$

  The desired settling time of 0.5 seconds corresponds to $\sigma = -9.2$.

- Steady state error to a step reference is zero. This means that the system steady state gain is one. This immediately requires $b_1 = a_2$

- For this specification it is useful to introduce a new system

  $$H(s) = G(s)\frac{1}{s} \tag{19}$$

  The system output is the integrated value of the original system. It is evident that the step response of $H(s)$ equals the response due to a ramp input to $G(s)$ (using the same magnitude of course). The requirement is that a unit step input results in 0.01 m steady state error. The steady state gain of $H(s)$ equals $\frac{b_0}{a_2}$. Hence, $\frac{b_0}{a_2} < 0.01$

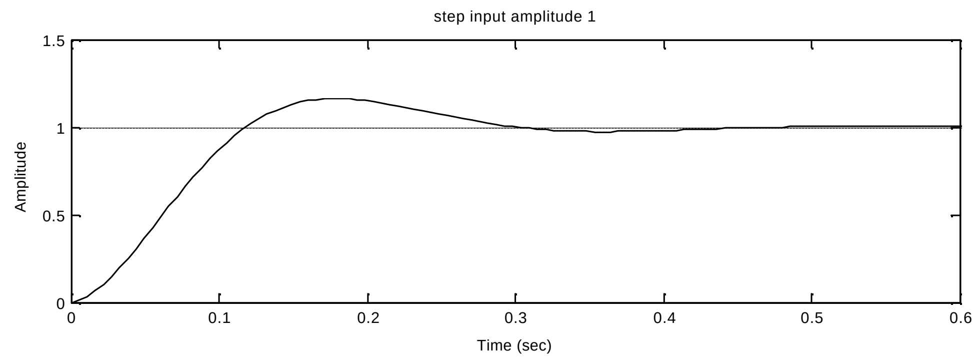
Summarizing all specifications (Sp 1-5)

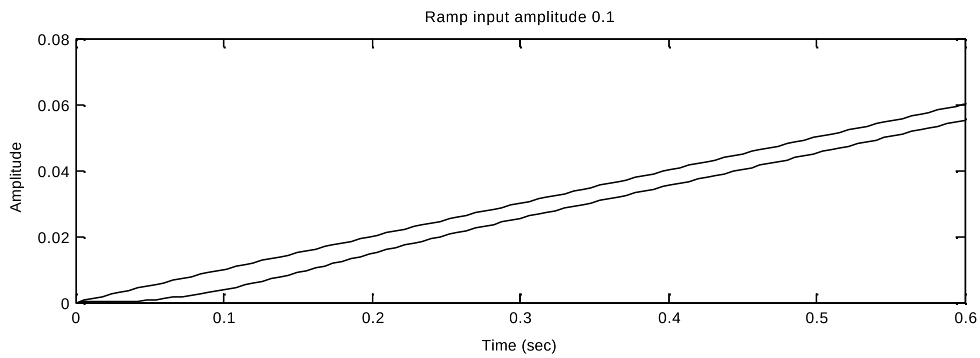
1. $a_2 > 324$
2. $\frac{a_1}{2\sqrt{a_2}} > 0.45$
3. real part of the poles < -9.2
4. $b_1 = a_2$
5. $\frac{b_0}{a_2} < 0.01$

Note that we can explicitly compute the poles of $G(s)$

$$p_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2} \tag{20}$$

Let us assume that $a_1^2 - 4a_2$ is negative (Note that for equality in specification (2), $\sqrt{(0.9^2 a_2) - 4a_2} \approx \sqrt{-3.2a_2}$ which justifies the assumption). By setting $a_1 = 20$, we get the real part of the poles at -10, which satisfies (Sp 3). Set $a_2 = 400$, then (Sp 1-2) are also satisfied. Finally, take $b_1 = a_2$ and choose $b_0 = 1$ such that (Sp 5) is satisfied.


step input amplitude 1
1.5
1
0.5
0
Amplitude
0
0.1
0.2
0.3
0.4
0.5
0.6
Time (sec)


Ramp input amplitude 0.1
0.08
0.06
0.04
0.02
0
Amplitude
0
0.1
0.2
0.3
0.4
0.5
0.6
Time (sec)

Control Systems (SC3020et): Instruction stencil no. 2

due October 16^{th}, 2003

Name:	Student Number:

0.1 Partial-fraction expansion

Give the partial-fraction expansion of the following transfer function

$$G(s) = \frac{(s+3)}{(s+1)^2 \cdot (s+2)} \tag{1}$$

0.2 Sketching of the Nyquist plot

The following system is given:

$$G(s) = \frac{s+1}{(s-1)^2} \tag{2}$$

Sketch the Nyquist plot of $G(s)$ and use the plot to determine for which values of K the closed-loop system is stable.

0.3 Root Locus

Sketch the root locus for the characteristic equation of the system represented by

$$L(s) = \frac{s+1}{s(s+1)(s+2)} \tag{3}$$

Compute all the characteristics that are involved, i.e. centroids, crossing imaginary axis, angle of departure and angle of arrival, location of breakaway points. Determine also the value of the root locus gain for which the complex conjugate poles have a damping ratio of 0.5.

0.4 Design Lead Compensator

A numerically controlled machine has a transfer function given by

$$G(s) = \frac{1}{s(s+1)} \tag{4}$$

Performance specifications of the system in the unity feedback configuration are satisfied if the closed loop poles are located at $s = -1 \pm j\sqrt{3}$.

1. Show that this specification cannot be achieved by choosing proportional control alone.
2. Design a lead compensator $D(s) = K\frac{s+z}{s+p}$ that will meet the specification

Control Systems (SC3020et)

(http://www.dcsc.tudelft.nl/sc3020et/index.html)

Instruction stencil no. 2: answers[1]

Problem 1: Partial-fraction expansion (see textbook p. 98)

The transfer function

$$G(s) = \frac{(s+3)}{(s+1)^2 \cdot (s+2)}$$

can be rewritten to

$$G(s) = \frac{C_1}{s+2} + \frac{C_2}{s+1} + \frac{C_3}{(s+1)^2}$$

The constants C_1, C_2, and C_3 are computed as follows

$$C_1 = (s+2) \cdot G(s)|_{s=-2} = \frac{s+3}{(s+1)^2}|_{s=-2} = \frac{1}{(-1)^2} = 1$$

$$C_2 = \frac{d}{ds}(s+1)^2 \cdot G(s)|_{s=-1} = \frac{d}{ds}\frac{s+3}{s+2}|_{s=-1} = \frac{-1}{(s+2)^2}|_{s=-1} = \frac{-1}{1^2} = 1$$

$$C_3 = (s+1)^2 \cdot G(s)|_{s=-1} = \frac{s+3}{s+2}|_{s=-1} = \frac{2}{1} = 2$$

Substitution of C_1, C_2, and C_3 results in

$$G(s) = \frac{1}{(s+2)} - \frac{1}{(s+1)} + \frac{2}{(s+1)^2}$$

Problem 2: Sketching of the Nyquist plot

The problem is to sketch the Nyquist plot of the transfer function

$$G(s) = \frac{s+1}{(s-1)^2}$$

The location of the poles and zeros are illustrated in the figure below.

The Nyquist contour encircles the entire right half plane (see fig. 6.16 of the textbook). Every point on the Nyquist contour has a gain and argument when substituted in the transfer function $G(s)$ and therefore represents a point in the complex plane.

[1]For remarks and questions please contact Sjoerd Dietz, 278-1660

The approach to sketch a Nyquist plot is as follows

1. Sketch the part of the Nyquist plot that is related to that part of the Nyquist contour along the positive imaginary axis
2. Mirror the result in the real axis, which represents the part of the part of the Nyquist plot that is related to the Nyquist contour along the negative imaginary axis.

<u>The positive imaginary axis of the Nyquist contour</u>

The positive imagninary axis of the Nyquist contour is represented by substitution of $s = i \cdot \omega$ in $G(s)$ for $\omega : 0^+ \to \infty$. Substitution of $s = i \cdot \omega$ results in

$$G(i \cdot \omega) = \frac{i \cdot \omega + 1}{(i \cdot \omega - 1)^2} = \frac{i \cdot \omega + 1}{-\omega^2 - 2 \cdot i \cdot \omega + 1}$$

Substitution of $\omega = 0^+$ in $G(i \cdot \omega)$ results in $|K \cdot G(i \cdot \omega)| = 1$. The argument for $\omega = 0^+$ is $\angle G(i \cdot \omega) = 0°$. This can also be determined by using the illustration above. For $\omega = 0^+$ the argument of $G(i \cdot \omega)$ due to the zero is $0°$, while the argument of $G(i \cdot \omega)$ due to each of the poles is $-180°$ (note: a zero has a positive contribution to the argument, while a pole has a negative contribution to the argument). The total argument for $\omega = 0^+$ is therefore

$$\angle G(i \cdot \omega) = 0° - 2 \cdot 180° = 0°$$

Substitution of $\omega = \infty$ in $G(i \cdot \omega)$ results in $|K \cdot G(i \cdot \omega)| = 0$. The argument for $\omega = \infty$ is $\angle G(i \cdot \omega) = -90°$. This can also be determined by using the illustration above. For $\omega = \infty$ the argument of $G(i \cdot \omega)$ due to the zero is $90°$, while the argument of $G(i \cdot \omega)$ due to each of the poles is $-90°$. The total argument for $\omega = \infty$ is therefore

$$\angle G(i \cdot \omega) = 90° - 2 \cdot 90° = -90°$$

For $\omega : 0^+ \to \infty$ the argument due to the zero increases from $0°$ to $90°$. The argument due to each of the poles decreases from $-180°$ to $-90°$. The total argument therefore increases from $0°$ to $270°$ for $\omega : 0^+ \to \infty$.

The Nyquist plot for $\omega : 0^+ \to \infty$ intersects both the imaginary axis and the real axis. Although it is not strictly necessary to compute the location of these intersection points to sketch the Nyquist pot, in this case it is important with respect to the next question (in particular the intersection point of the real axis).

The intersection point of the Nyquist plot with the real axis is determined by solving the following equation

$$\frac{i \cdot \omega + 1}{-\omega^2 - 2 \cdot i \cdot \omega + 1} = a$$

Besides the solution $\omega = 0$ and $a = 1$ this results in

$$\omega = \sqrt{3} \approx 1.73$$
$$a = -0.5$$

The intersection of the Nyquist plot and the real axis is therefore located in the point -0.5.

The intersection point of the Nyquist plot with the imaginary axis is determined by solving the following equation

$$\frac{i \cdot \omega + 1}{-\omega^2 - 2 \cdot i \cdot \omega + 1} = b \cdot i$$

This results in

$$b = \sqrt{\tfrac{3}{4}} \approx 0.866$$
$$\omega = \frac{1}{2 \cdot \sqrt{\frac{3}{4}}} \approx 0.577$$

The intersection with the imaginary axis is therefore located in the point $\sqrt{\frac{3}{4}} \cdot i \approx 0.866 \cdot i$.

Now we can sketch the part of the Nyquist plot that is related to the positive imaginary axis of the Nyquist contour. The result is illustrated in Figure 1.

The negative imaginary axis of the Nyquist contour

This part of the Nyquist plot is obtained by mirroring the part of the Nyquist plot that is related to the positive imaginary axis of the Nyquist contour in the real axis. The result is illustrated by the dotted line in Figure 1.

The Nyquist stability criterion The Nyquist criterion is $Z = N + P$ where N is the number of clockwise encirclements of the point -1, P the number of instable poles of the open-loop system and Z the number of zeros of the open-loop system in the right half plane (number of instable poles of the closed-loop system). For stability, Z should be equal to zero. Substitution of $Z = 0$ results in

$$N = -P$$

The transfer function $K \cdot G(s)$ has two unstable poles, and therefore $P = 2$. Substitution of $P = 2$ results in $N = -2$ for stability. The closed-loop system is stable if the Nyquist plot results in two counter clockwise encirclements of the point -1 (or the Nyquist plot of $G(s)$ results in two counter clockwise encirclements of the point $-\frac{1}{K}$). For stability we therefore demand

$$-\frac{1}{2} < -\frac{1}{K} < 0$$

The closed-loop system is therefore stable for

$$K > 2$$

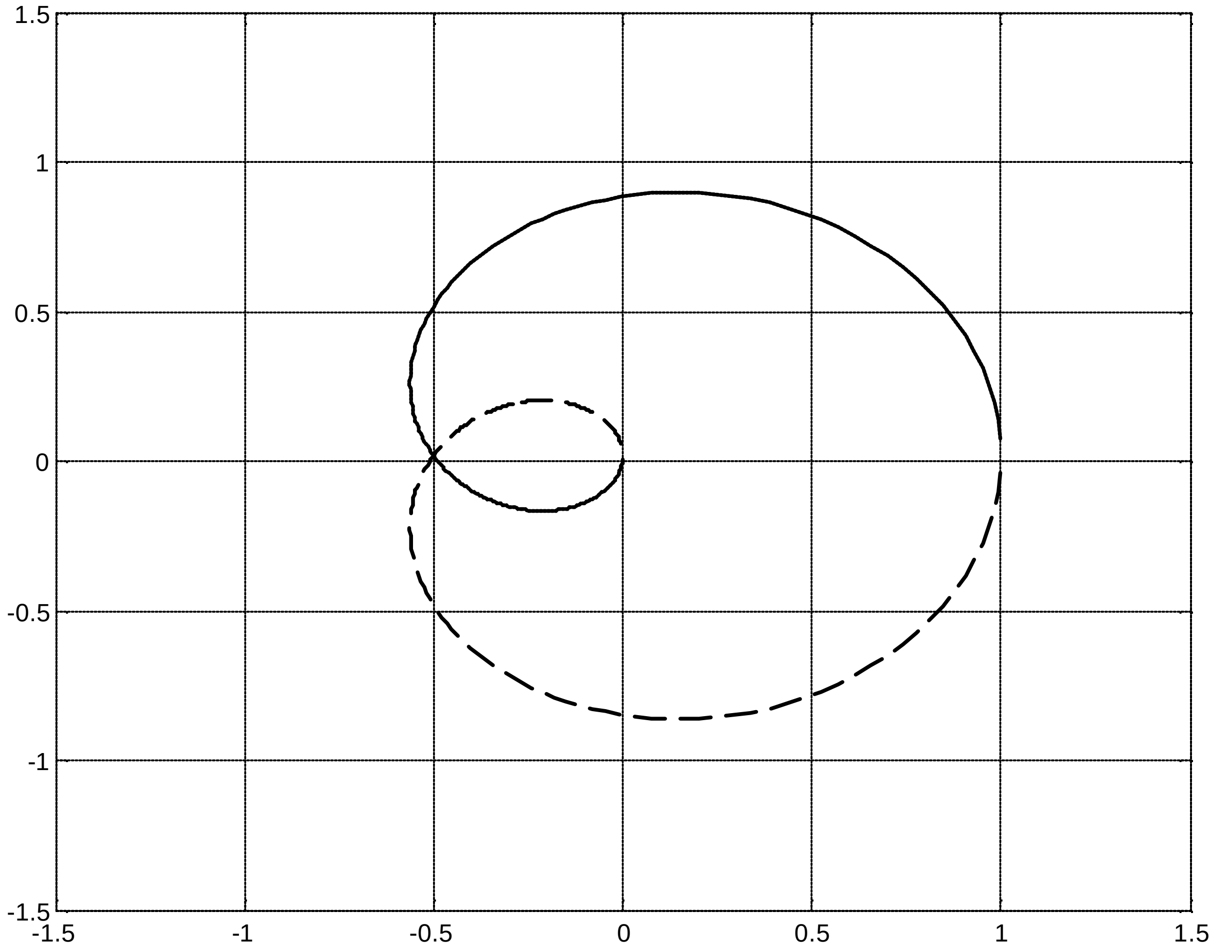


Figure 1: Nyquist plot. $\omega = 0^+ \ldots \infty$ (full) and $\omega = -\infty \ldots 0^+$ (dotted)

Problem 3: Root Locus

The system is represented by

$$L(s) = \frac{s+1}{s(s+1)(s+2)} \tag{1}$$

One observes that it has a common factor $(s+1)$. It is allowed to simply cancel this term. The function $L(s)$ is identical to the function

$$P(s) = \frac{1}{s(s+2)} \tag{2}$$

The reason that such a transfer function appears in the literature is the following. For a given state-space description of a system

$$\dot{x} = Ax + Buy = Cx + Du \tag{3}$$

the corresponding transfer function is

$$G(s) = C(sI - A)^{-1}B + D \tag{4}$$

Vice versa, each transfer function $G(s)$ can be represented by a state-space representation (infinitely many). When for given system matrices (A, B, C, D) the pair (A, B) is uncontrollable then there is an eigenvalue of A that will not turn up as a pole in the transfer function $G(s)$. Namely, computing (4) will lead to a form in which sometimes a commone factor in the numerator and denominator is left. This is then called pole-zero cancellation. In fact, $G(s)$ only captures the input/output behaviour of the system. Note that the same holds when the pair (C, A) is unobservable.

For computing the root locus, we can either consider $L(s)$ or we can use $P(s)$. We look for the roots of

$$1 + KL(s) = 0 \tag{5}$$

$$s(s+1)(s+2) + (s+1)K = 0 \tag{6}$$

$$(s+1)(s(s+2) + K) = 0 \tag{7}$$

This equation shows that we have this equation satisfied for $s = -1$ independent of K.

Since $P(s)$ has a pole excess of 2, we have 2 asymptotes. Using equation (5.21) from the book

$$\phi_l = \frac{180° + 360°(l-1)}{2} \qquad l = 1, 2 \tag{8}$$

which yields $\phi_l = 90°, \phi_l = 270°$.

To compute the break-away point, we apply (5.37) from the book. With $a(s) = s(s+2)$ and $b(s) = K$ we get for this condition to hold

$$K(2s+2) = 0 \tag{9}$$

This corresponds to $s = -1$. The value of K can be found by substituting $s = -1$ into (7) which gives $K = 1$.

The resulting root locus is given in Figure (2). Note that Matlab plots a pole and zero on top of each other at $s = -1$ when using the command `rlocus.m`. This is shown in Figure 3.

A final remark about pole-zero cancellations. There has been written a lot about zeros and poles and there are many different definitions. Since it always has to do with uncontrollable and /or unobservable dynamics, it is strongly recommended to work with state-space respresentations. Controllability and observability are well-defined properties of the matrix pairs (A, B) and (C, A). To our opinion, using a form $L(s)$ as given in this excercise is mathmatically confusing.

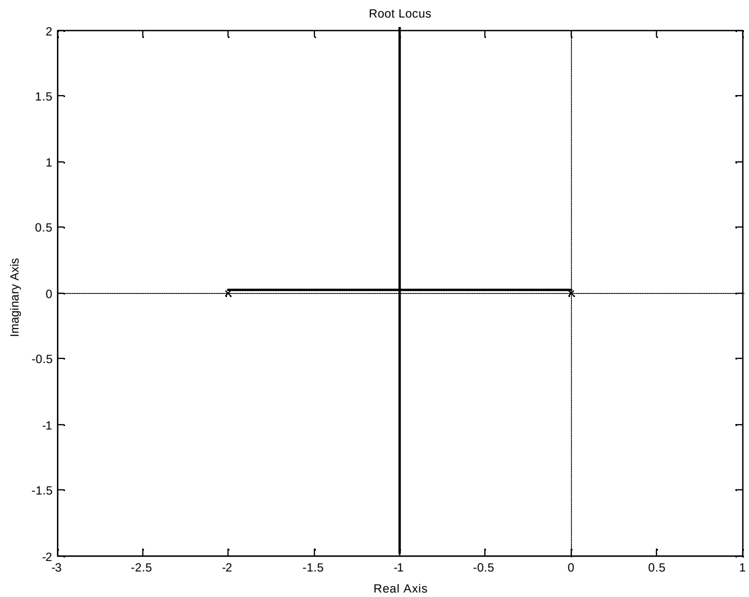


Figure 2: Root locus (horizontal line should of course be *on* the real axis) $P(s)$

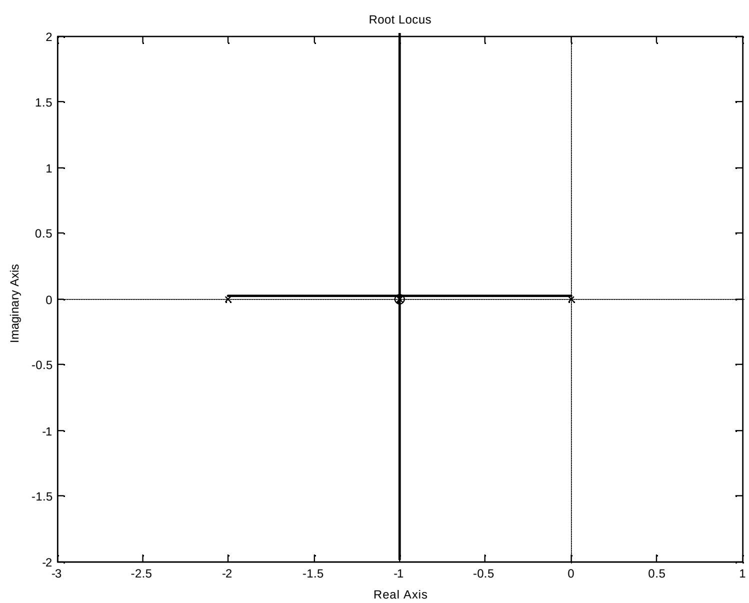


Figure 3: Root locus $L(s)$. (horizontal line should of course be *on* the real axis). As shown, Matlab also plots the fixed root $s = -1$ and also plots a zero there to express the fact that as $K \to \infty$ the closed loop poles go to the zeros of $L(s)$.

Problem 4: Design Lead Compensator

$$G(s) = \frac{1}{s(s+1)} \tag{10}$$

Performance specifications of the system in the unity feedback configuration are satisfied if the closed loop poles are located at $s = -1 \pm j\sqrt{3}$.

Part 1.
The poles of the closed loop system are the zeros of the characteristic equation

$$1 + G(s)K = 0 \tag{11}$$

Substituting $G(s)$ yields $s^2 + s + K = 0$. The corresponding roots become

$$s_{1,2} = -\frac{-1 \pm \sqrt{1-4K}}{2} \tag{12}$$

From this we immediately see that closed loop poles at $s = -1 \pm \sqrt{3}$ are impossible using a constant feedback gain K.

Part 2.
When using a lead compensator $D(s) = K\frac{s+z}{s+p}$ the characteristic equation yields $(s^2+s)(s+p) + K(s+z) = 0$. This equation has to be satisfied for $s = -1 + j\sqrt{3}$. It is clear that there infinitely many solutions (K, p, z) that render both the real and imaginary part zero. One has to solve 2 equations in 3 unknowns (K, p, z)

Alternatively, we can reason as follows. The open-loop system G has the value $G(-1 + j\sqrt{3}) = -1/4 + \frac{sqrt3}{12}$ with a corresponding angle is $150°$. In order to satisfy $G(s)D(s) = -1$, we have to add a phase lead $30°$. Let us call

$$D_0(s) = \frac{s+z}{s+p} \tag{13}$$

then we have

$$\angle \frac{-1 + \sqrt{3} + z}{-1 + \sqrt{3} + p} = 30° \tag{14}$$

Choosing $z = 1$ (this will cancel out the pole $s = -1$ of $G(s)$) yields $p = 2$.
In order to find the gain K we substitute D_0 into the characteristic equation, i.e.

$$G(s)D_0(s) = -\frac{1}{K} \tag{15}$$

Since the choice of $z = 1$ causes a pole-zero cancellation we get

$$\frac{1}{s(s+2)}|_{s=-1+j\sqrt{3}} = -1/4 \tag{16}$$

The lead filter that does the job therefore is

$$D(s) = 4\frac{s+1}{s+2} \tag{17}$$