ON CARMICHAËL'S CONJECTURE

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Introduction.

Carmichaël's conjecture is the following: "the equation $\varphi(x) = n$ cannot have a unique solution, $(\forall)n \in \mathbb{N}$, where φ is the Euler's function". R. K. Guy presented in [1] some results on this conjecture; Carmichaël himself proved that, if n_0 does not verify his conjecture, then $n_0 > 10^{37}$; V. L. Klee [2] improved to $n_0 > 10^{400}$, and P. Masai & A. Valette increased to $n_0 > 10^{10000}$. C. Pomerance [4] wrote on this subject too.

In this article we prove that the equation $\varphi(x) = n$ admits a finite number of solutions, we find the general form of these solutions, also we prove that, if x_0 is the unique solution of this equation (for a $n \in \mathbb{N}$), then x_0 is a multiple of $2^2 \cdot 3^2 \cdot 7^2 \cdot 43^2$ (and $x_0 > 10^{10000}$ from [3]).

In the last paragraph we extend the result to: x_0 is a multiple of a product of a very large number of primes.

§1. Let x_0 be a solution of the equation $\varphi(x) = n$. We consider n fixed. We'll try to construct another solution $y_0 \neq x_0$.

The first method:

We decompose $x_0 = a \cdot b$ with a, b integers such that $(a, b) = 1$.

we look for an $a' \neq a$ such that $\varphi(a') = \varphi(a)$ and $(a', b) = 1$; it results that $y_0 = a' \cdot b$.

The second method:

Let's consider $x_0 = q_1^{\beta_1} \ldots q_r^{\beta_r}$, where all $\beta_i \in \mathbb{N}^*$, and $q_1, \ldots, q_r$ are distinct primes two by two; we look for an integer q such that $(q, x_0) = 1$ and $\varphi(q)$ divides $x_0 / (q_1, \ldots, q_r)$; then $y_0 = x_0 q / \varphi(q)$.

We immediately see that we can consider q as prime.

The author conjectures that for any integer $x_0 \geq 2$ it is possible to find, by means of one of these methods, a $y_0 \neq x_0$ such that $\varphi(y_0) = \varphi(x_0)$.

Lemma 1. The equation $\varphi(x) = n$ admits a finite number of solutions, $(\forall)n \in \mathbb{N}$.

Proof. The cases $n = 0, 1$ are trivial.

Let's consider n to be fixed, $n \geq 2$. Let $p_1 < p_2 < ... < p_s \leq n+1$ be the sequence of prime numbers. If x_0 is a solution of our equation (1) then x_0 has the form $x_0 = p_1^{\alpha_1} ... p_s^{\alpha_s}$, with all $\alpha_i \in \mathbb{N}$. Each α_i is limited, because:

$(\forall) i \in \{1, 2, ..., s\}$, $(\exists) a_i \in \mathbb{N}$: $p_i^{\alpha_i} \geq n$.

Whence $0 \leq \alpha_i \leq a_i + 1$, for all i. Thus, we find a wide limitation for the number of solutions: $\prod_{i=1}^{s} (a_i + 2)$

Lemma 2. Any solution of this equation has the form (1) and (2):

$$x_0 = n \cdot \left(\frac{p_1}{p_1 - 1} \right)^{\varepsilon_1} .... \left(\frac{p_s}{p_s - 1} \right)^{\varepsilon_s} \in \mathbb{Z} \ ,$$

where, for $1 \leq i \leq s$, we have $\varepsilon_i = 0$ if $\alpha_i = 0$, or $\varepsilon_i = 1$ if $\alpha_i \neq 0$.

Of course**,** $n = \varphi(x_0) = x_0 \left(\frac{p_1}{p_1 - 1} \right)^{\varepsilon_1} .... \left(\frac{p_s}{p_s - 1} \right)^{\varepsilon_s}$,

whence it results the second form of x_0.

From (2) we find another limitation for the number of the solutions: $2^s - 1$ because each ε_i has only two values, and at least one is not equal to zero.

§2. We suppose that x_0 is the unique solution of this equation.

Lemma 3. x_0 is a multiple of $2^2 \cdot 3^2 \cdot 7^2 \cdot 43^2$.

Proof. We apply our second method.

Because $\varphi(0) = \varphi(3)$ and $\varphi(1) = \varphi(2)$ we take $x_0 \geq 4$.

If $2 \nmid x_0$ then there is $y_0 = 2x_0 \neq x_0$ such that $\varphi(y_0) = \varphi(x_0)$, hence $2 \mid x_0$; if $4 \nmid x_0$, then we can take $y_0 = x_0 / 2$.

If $3 \nmid x_0$ then $y_0 = 3x_0 / 2$, hence $3 \mid x_0$; if $9 \nmid x_0$ then $y_0 = 2x_0 / 3$, hence $9 \mid x_0$; whence $4 \cdot 9 \mid x_0$.

If $7 \nmid x_0$ then $y_0 = 7x_0 / 6$, hence $7 \mid x_0$; if $49 \nmid x_0$ then $y_0 = 6x_0 / 7$ hence $49 \mid x_0$; whence $4 \cdot 9 \cdot 49 \mid x_0$.

If $43 \nmid x_0$ then $y_0 = 43x_0 / 42$, hence $43 \mid x_0$; if $43^2 \nmid x_0$ then $y_0 = 42x_0 / 43$, hence $43^2 \mid x_0$; whence $2^2 \cdot 3^2 \cdot 7^2 \cdot 43^2 \mid x_0$.

Thus $x_0 = 2^{\gamma_1} \cdot 3^{\gamma_2} \cdot 7^{\gamma_3} \cdot 43^{\gamma_4} \cdot t$, with all $\gamma_i \geq 2$ and (t, $2 \cdot 3 \cdot 7 \cdot 43$) = 1 and $x_0 > 10^{10000}$ because $n_0 > 10^{10000}$.

§3. Let's consider $\gamma_1 \geq 3$. If $5 \nmid x_0$ then $5x_0 / 4 = y_0$, hence $5 \mid x_0$; if $25 \nmid x_0$ then $y_0 = 4x_0 / 5$, whence $25 \mid x_0$.

We construct the recurrent set M of prime numbers:

a) the elements $2, 3, 5 \in M$;
b) if the distinct odd elements $e_1, ..., e_n \in M$ and $b_m = 1 + 2^m \cdot e_1, ..., e_n$ is prime, with $m = 1$ or $m = 2$, then $b_m \in M$;

c) any element belonging to M is obtained by the utilization (a finite number of times) of the rules a) or b) only.

The author conjectures that M is infinite, which solves this case, because it results that there is an infinite number of primes which divide x_0. This is absurd.

For example 2, 3, 5, 7, 11, 13, 23, 29, 31, 43, 47, 53, 61, … belong to M.

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The method from §3 could be continued as a tree (for $\gamma_2 \geq 3$ afterwards $\gamma_3 \geq 3$, etc.) but its ramifications are very complicated…

§4. **A Property for a Counter-Example to Carmichael Conjecture**.

Carmichaël has conjectured that:
$(\forall)\ n \in \mathbb{N}$, $(\exists)\ m \in \mathbb{N}$, with $m \neq n$, for which $\varphi(n) = \varphi(m)$, where φ is Euler's totient function.

There are many papers on this subject, but the author cites the papers which have influenced him, especially Klee's papers.

Let n be a counterexample to Carmichaël's conjecture.

Grosswald has proved that n_0 is a multiple of 32, Donnelly has pushed the result to a multiple of 2^{14}, and Klee to a multiple of $2^{42} \cdot 3^{47}$, Smarandache has shown that n is a multiple of $2^2 \cdot 3^2 \cdot 7^2 \cdot 43^2$. Masai & Valette have bounded
n > 10^{10000}.

In this paragraph we will extend these results to: n is a multiple of a product of a very large number of primes.

We construct a recurrent set M such that:

a) the elements $2, 3 \in M$;

b) if the distinct elements $2, 3, q_1, ..., q_r \in M$ and $p = 1 + 2^a \cdot 3^b \cdot q_1 \cdots q_r$ is a prime, where $a \in \{0, 1, 2, ..., 41\}$ and $b \in \{0, 1, 2, ..., 46\}$, then $p \in M$; $r \geq 0$;

c) any element belonging to M is obtained only by the utilization (a finite number of times) of the rules a) or b).

Of course, all elements from M are primes.

Let n be a multiple of $2^{42} \cdot 3^{47}$;

if $5 \nmid n$ then there exists $m = 5n/4 \neq n$ such that $\varphi(n) = \varphi(m)$; hence $5 \mid n$; whence $5 \in M$;

if $5^2 \nmid n$ then there exists $m = 4n/5 \neq n$ with our property; hence $5^2 \mid n$;

analogously, if $7 \nmid n$ we can take $m = 7n/6 \neq n$, hence $7 \mid n$; if $7^2 \nmid n$ we can take $m = 6n/7 \neq n$; whence $7 \in M$ and $7^2 \mid n$; etc.

The method continues until it isn't possible to add any other prime to M, by its construction.

For example, from the 168 primes smaller than 1000, only 17 of them do not belong to M (namely: 101, 151, 197, 251, 401, 491, 503, 601, 607, 677, 701, 727, 751, 809, 883, 907, 983); all other 151 primes belong to M.

Note $M = \{2, 3, p_1, p_2, ..., p_s, ...\}$, then n is a multiple of $2^{42} \cdot 3^{47} \cdot p_1^2 \cdot p_2^2 \cdots p_s^2 \cdots$
From our example, it results that M contains at least 151 elements, hence $s \geq 149$.

If M is infinite then there is no counterexample n, whence Carmichaël's conjecture is solved.

(The author conjectures M is infinite.)

Using a computer it is possible to find a very large number of primes, which divide n, using the construction method of M, and trying to find a new prime p if $p-1$ is a product of primes only from M.

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[First part published in "Gamma", XXV, Year VIII, No. 3, June 1986, pp. 4-5; and second part in "Gamma", XXIV, Year VIII, No. 2, February 1986, pp. 13-14.]