An introduction to motives I: classical motives and motivic L-functions

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The exposition here follows the lecture delivered at the summer school, and hence, contains neither precision, breadth of comprehension, nor depth of insight. The goal rather is the curious one of providing a loose introduction to the excellent introductions that already exist, together with scattered parenthetical commentary. The inadequate nature of the exposition is certainly worst in the third section. As a remedy, the article of Schneider [40] is recommended as a good starting point for the complete novice, and that of Nekovar [37] might be consulted for more streamlined formalism. For the Bloch-Kato conjectures, the paper of Fontaine and Perrin-Riou [20] contains a very systematic treatment, while Kato [27] is certainly hard to surpass for inspiration. Kings [30], on the other hand, gives a nice summary of results (up to 2003).

1 Motivation

Given a variety X over $\mathbb{Q}$, it is hoped that a suitable analytic function

$$\zeta(X, s),$$

a *ζ-function* of X, encodes important arithmetic invariants of X. The terminology of course stems from the fundamental function

$$\zeta(\mathbb{Q}, s) = \sum_{n=1}^{\infty} n^{-s}$$

named by Riemann, which is interpreted in this general context as the zeta function of $\mathrm{Spec}(\mathbb{Q})$. A general zeta function should generalize Riemann's function in a manner similar to Dedekind's extension to number fields. Recall that the latter can be defined by replacing the sum over positive integers by a sum over ideals:

$$\zeta(F, s) = \sum_{I} N(I)^{-s}$$

where I runs over the non-zero ideals of the ring of integers $\mathcal{O}_F$ and $N(I) = |\mathcal{O}_F/I|$, and that $\zeta(F, s)$ has a simple pole at $s = 1$ (corresponding to the trivial motive factor of $\mathrm{Spec}(F)$, as it turns out) with

$$(s-1)\zeta(F, s)|_{s=1} = \frac{2^{r_1}(2\pi)^{r_2} h_F R_F}{w_F \sqrt{|D_F|}}$$

By the unique factorization of ideals, $\zeta(F, s)$ can also be written as an Euler product

$$\prod_{\mathcal{P}} (1 - N(\mathcal{P})^{-s})^{-1}$$

as $\mathcal{P}$ runs over the maximal ideals of $\mathcal{O}_F$, that is, the closed points of $\mathrm{Spec}(\mathcal{O}_F)$. Now, if a scheme $\mathcal{Y}$ is of finite type over $\mathbb{Z}$, then for any closed point $y \in \mathcal{Y}$, its residue field $k(y)$ is finite. Write $N(y) := |k(y)|$. We can then form an Euler product [41]

$$Z(\mathcal{Y}, s) := \prod_{y \in \mathcal{Y}_0} (1 - N(y)^{-s})^{-1},$$

where $(\cdot)_0$ denotes the set of closed points for any scheme $(\cdot)$. In the case when the map

$$\mathcal{Y} \to \mathrm{Spec}(\mathbb{Z})$$

factors through $\mathrm{Spec}(\mathbb{F}_p)$, $Z(\mathcal{Y}, s)$ reduces to Weil's zeta function for a variety over a finite field (with the substitution $p^{-s} \mapsto t$ if a formal variable has intervened as in [41], section 1.6).

When we are starting with $X/\mathbb{Q}$, which we assume throughout to be proper and smooth, a straightforward imitation of Dedekind's definition might involve taking an integral model $\mathcal{X}$ of X, which is a proper flat scheme of finite-type over $\mathbb{Z}$ with X as generic fiber, and defining

$$\zeta(X, s) \text{“} := \text{”} Z(\mathcal{X}, s) = \prod_{x \in \mathcal{X}_0} (1 - N(x)^{-s})^{-1}$$

The problem with this approach is that the function thus obtained will depend on the model, and there is no general method for choosing a canonical one. However, there will be some set S of primes such that there is a model $\mathcal{X}_S$ over $\mathrm{Spec}(\mathbb{Z}[1/S])$ which is furthermore *smooth*. Even though such a $\mathbb{Z}[1/S]$-model need be no more canonical, it does turn out that the incomplete zeta function

$$\zeta_S(X, s) := \prod_{x \in (\mathcal{X}_S)_0} (1 - N(x)^{-s})^{-1}$$

is independent of the model. (More on this point below.) So there are good elementary generalizations of incomplete zeta functions. We note in this connection that

$$Z(\mathcal{X}, s) = \prod_p Z(\mathcal{X}_p, s)$$

where

$$\mathcal{X}_p = \mathcal{X} \otimes \mathbb{F}_p$$

is the reduction of $\mathcal{X}$ modulo p, so that that

$$\zeta_S(X, s) = \prod_{p \notin S} Z(\mathcal{X}_p, s)$$

is the result of deleting a few Euler factors. Thus, the problem of defining a canonical zeta function becomes one of inserting canonical factors for the primes of bad reduction. It is not impossible that there is a theory of integrals models that isolates a class that is canonical enough to yield a good definition of $\zeta(X, s)$. But the current approach proceeds instead to break up partial zeta functions into natural factors

$$\zeta_S(X, s) = \prod L_S(M_i, s)^{\pm 1},$$

according to the way X is decomposed into constituent motives $\{M_i\}$ in a suitable category. (It is not much of an exaggeration to say that the decomposition of zeta functions is the main empirical phenomenon leading to the hypothesis of a category of motives.) The incomplete L-functions $L_S(M_i, s)$ of the M_i should then encode arithmetic invariants of the M_i, which, in turn, refine the arithmetic invariants of X. It is believed that good analytic properties must be established to access the invariants efficiently, including functional equations. This, in turn, requires us to complete the L-functions using

cohomological machinery in general. The completed L-functions then will lead to a completed zeta function.

A simple illustration is provided by the elementary example of an elliptic curve $E/\mathbb{Q}$ with affine equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

Let $\mathcal{E}_S$ be a smooth and proper $\mathbb{Z}[1/S]$ model. Then

$$\zeta_S(E, s) := Z(\mathcal{E}_S, s)$$

It is not very hard to check that

$$\zeta_S(E, s) = \zeta_S(\mathbb{Q}, s)\zeta_S(\mathbb{Q}, s-1)/L_S(H^1(E), s)$$

([43], V.2.4) illustrating the kind of decomposition alluded to above. Here

$$\zeta_S(\mathbb{Q}, s) = \sum_{\{(n,p)=1, \forall p \in S\}} n^{-s} = \prod_{p \notin S} (1 - p^{-s})^{-1}$$

is a standard incomplete zeta function and

$$L_S(H^1(E), s) = \prod_{p \notin S} L_p(H^1(E), s)$$

is the incomplete L-function of E with factors defined by

$$L_p(H^1(E), s) = \frac{1}{1 - a_p p^{-s} + p^{1-2s}}$$

Here $a_p = p + 1 - N_p$ and N_p is the number of points on E mod p. $L_S(H^1(E), s)$ turns out to be the partial L-function corresponding to the motivic factor $H^1(E)$ of E.

We can put in Euler factors for $p \in S$. It is obvious how to do it for $\zeta_S(\mathbb{Q}, s)$ and $\zeta_S(\mathbb{Q}, s-1)$ giving us the Riemann zeta function $\zeta(\mathbb{Q}, s)$ and its shift $\zeta(\mathbb{Q}, s-1)$ respectively. For the incomplete $L_S(H^1(E), s)$, we put in the factors according to a recipe determined by the reduction of E at p:

$$L_p(H^1(E), s) = \begin{cases} 1/(1 - p^{-s}) & \text{split multiplicative;} \\ 1/(1 + p^{-s}) & \text{non-split multiplicative;} \\ 1 & \text{additive.} \end{cases}$$

([44], II.10) and define

$$L(H^1(E), s) := \prod_p L_p(H^1(E), s)$$

Here we have used the breakdown of the incomplete zeta function into three factors as an aid in defining the full zeta function of E. However, this case is somewhat misleading in that there *is* a canonical model that could have been used instead, namely, the Weierstrass minimal model

$$\mathcal{E}$$

that appears in basic textbooks. In fact, one can check that

$$\zeta(E, s) = \zeta(\mathbb{Q}, s)\zeta(\mathbb{Q}, s-1)/L(H^1(E), s) = Z(\mathcal{E}, s)$$

as follows from the trace formula ([43], V.2) for the Frobenius map on elliptic curves for $p \notin S$, and a much easier counting argument for $p \in S$. So this would seem to be an instance where the naive extension of Dedekind's method works out. Nevertheless, we explain how the bad factors can be obtained without reference to the model, starting at this point to use the language of étale cohomology

[35]. In the sequel, we fix an algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$, closures $\bar{\mathbb{Q}}_p$ of $\mathbb{Q}_p$, and embeddings $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$. Therefore, we have embeddings of Galois groups

$$G_p := \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \hookrightarrow G := \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$$

The residue field of $\bar{\mathbb{Q}}_p$ is an algebraic closure $\bar{\mathbb{F}}_p$ of $\mathbb{F}_p$, and we have an exact sequence

$$0 \to I_p \to G_p \to \mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p) \to 0$$

defining the inertia subgroup I_p. Denote by Fr_p the generator of $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$ that takes x to $x^{1/p}$. Finally, $\bar{\mathcal{E}_p}$ denotes the base-change of $\mathcal{E}_p$ to $\bar{\mathbb{F}}_p$ and $\bar{E}$ the base-change of E to $\bar{\mathbb{Q}}$. We need the étale cohomology

$$H^1(\bar{E}, \mathbb{Q}_l)$$

for primes l, and

$$H^1(\bar{\mathcal{E}_p}, \mathbb{Q}_l)$$

for $l \neq p$. By the Lefschetz trace formula ([35], VI.12.3),

$$Z(\mathcal{E}_p, s) = \frac{\det([I - p^{-s}Fr_p] | H^1(\bar{\mathcal{E}_p}, \mathbb{Q}_l))}{\det([I - p^{-s}Fr_p] | H^0(\bar{\mathcal{E}_p}, \mathbb{Q}_l)) \det([I - p^{-s}Fr_p] | H^2(\bar{\mathcal{E}_p}, \mathbb{Q}_l))}$$

But for each $i = 0, 1, 2$,

$$H^i(\bar{\mathcal{E}_p}, \mathbb{Q}_l) \simeq H^i(\bar{E}, \mathbb{Q}_l)^{I_p}$$

the superscript referring to the subspace of elements fixed by the inertia action. (For H^0 and H^2, this is an easy exercise. The H^1 case is slightly harder. See [35], proof of theorem V.3.5. Although the discussion there is given for smooth surfaces fibered over 'geometric' curves, it is rather straightforward to adapt it to the present situation.) For $p \notin S$, any pair $X \hookrightarrow \mathcal{X}$ as above satisfies

$$H^i(\bar{\mathcal{X}_p}, \mathbb{Q}_l) \simeq H^i(\bar{X}, \mathbb{Q}_l)$$

where the I_p-action must be trivial, and provides the reason that the incomplete zeta function is independent of the model ([35], VI.4.1). In any case, it ends up that the bad factor could have been written

$$Z(\mathcal{E}_p, s) = \frac{\det([I - p^{-s}Fr_p] | H^1(\bar{E}, \mathbb{Q}_l)^{I_p})}{\det([I - p^{-s}Fr_p] | H^0(\bar{E}, \mathbb{Q}_l)^{I_p}) \det([I - p^{-s}Fr_p] | H^2(\bar{E}, \mathbb{Q}_l)^{I_p})}$$

in a way that refers only to E. It is this formula that generalizes to arbitrary motives.

Since we have thus far been entirely cavalier about convergence, we note in passing that Hasse's bound $|a_p| \leq 2\sqrt{p}$ ([43], V.II) implies that the Euler product converges for $Re(s) > 3/2$.

To control fine analytic properties, one establishes a relation to automorphic L-functions. For elliptic curves such a relation can be made explicit by computing the conductor

$$N_E := \prod_{p \in S} p^{f_p}$$

Here

$$f_p = ord_p(\Delta_E) + 1 - m_E$$

where Δ_E is the discriminant of E and m_E is the number of geometric components (that is, components over $\bar{\mathbb{F}}_p$) of the special fiber of the Neron model of E. Even though this formula for f_p again refers to the model, it can be defined purely in terms of the Galois action on $H^1(\bar{E}, \mathbb{Q}_l)$ ([44], IV.10).

The well-known and deep fact, established through the work of Wiles, Taylor-Wiles, and Breuil-Conrad-Diamond-Taylor ([50], [49], [10]), is that L has an analytic continuation to the complex plane. More precisely,

$$L(E, s) = L(f_E, s) = \frac{1}{(2\pi)^s \Gamma(s)} \int_0^\infty f_E(iy) y^{s-1} dy$$

$$= \frac{1}{(2\pi)^s \Gamma(s)} [\int_{1/\sqrt{N_E}}^{\infty} f_E(iy) y^{s-1} dy + w_E \int_{1/\sqrt{N_E}}^{\infty} f_E(iy) y^{1-s} dy]$$

for a normalized weight 2 new cusp form f_E of level N_E which is an eigenvector for the Hecke operators, determined by a q-expansion

$$f_E = q + a_2 q^2 + \cdots$$

where the a_p have to be the same as those for E and the general coefficient is determined by those with prime index. The number $w_E = \pm 1$ in this formula is an intrinsic invariant of the elliptic curve, the so-called *sign of the functional equation*, discussed further below. Because the weight 2 cusp forms of level N_E form a finite dimensional space, it is easy to see that f_E is completely determined by a finite computation of the a_p's, and that this integral formula can then be used to compute L-values.

The celebrated conjecture of Birch and Swinnerton-dyer (BSD) [5] says

$$\text{ord}_{s=1} L(E, s) = \text{rank} E(\mathbb{Q})$$

The equality is known if $\text{ord}_{s=1} L(E, s) \leq 1$ by the work of Gross-Zagier and Kolyvagin ([22], [31]). Recall that, according to Mordell's theorem,

$$E(\mathbb{Q}) = \mathbb{Z}^r \times E(\mathbb{Q})_{tor}$$

where the finite abelian group $E(\mathbb{Q})_{tor}$ is easily computed using the Nagell-Lutz theorem ([43], VIII.7). This conjecture promises to give an analytic approach to understanding the elusive rank r. However, it must be admitted that even though the L-function is computable, the utility of this equality for actually computing the rank of an elliptic curve is somewhat ambiguous. This is because the order of zero of an analytic function might not be possible to determine using a finite computation. We will discuss below how the vanishing itself can be computably determined using the refined version of this conjecture. On the other hand, an extremely useful viewpoint on the order of vanishing arises from the functional equation. That is to say, one inserts a *gamma factor*, determined by the Hodge theory of E and viewed as the contribution of the prime at infinity. With another correction factor contributed by the conductor, we arrive at a further completion:

$$\Lambda(E, s) := (2\pi)^s \Gamma(s) N_E^{s/2} L(E, s)$$

which then satisfies a functional equation

$$\Lambda(E, 2 - s) = w_E \Lambda(E, s).$$

In fact, the sign w_E can be expressed as a product of local terms

$$w_E = \prod_p w_{E,p}$$

each of which can be computed in a straightforward fashion ([6], section 6).

A significant corollary is that the *parity* of the order is determined by the sign of w_E, usually referred to as the *sign of the functional equation*. For example, if $w_E = -1$, then clearly

$$L(E, 1) = 0.$$

Suppose you can check $L'(E, 1) \neq 0$ using the equality with $L'(f, 1)$ (non-vanishing *can* be verified!). Then we conclude that $E(\mathbb{Q})$ has rank one. Thus, one can produce many examples where the refined analysis of the L-function, including the functional equation and computation, gives us the complete structure of $E(\mathbb{Q})$. (For deeper developments in this direction, see [34].)

The BSD conjecture continues as follows. If r is the order of vanishing, then

$$(s-1)^{-r} L(E, s)|_{s=1} = |Ш(E)| R_E \Omega \prod_p c_p / |E(\mathbb{Q})_{tor}|^2.$$

That is to say, the L-function purportedly encodes refined Diophantine invariants of E, which we proceed to describe briefly ([43], C.16).

The set $E(\mathbb{Q})_{tor}$, which occurred already above, is the (finite) torsion subgroup of $E(\mathbb{Q})$. The c_p refer to the Tamagawa numbers at primes p, consisting of the index

$$c_p = (E(\mathbb{Q}_p) : E^0(\mathbb{Q}_p))$$

where $E^0(\mathbb{Q}_p) \subset E(\mathbb{Q}_p)$ is the subset of points that reduce to the connected component of the identity in the Neron model of E ([43], VII.2). In particular, $c_p = 1$ for primes of good reduction. The difficult rational term is the order of $Ш(E)$, the Tate-Shafarevich group of E, conjectured to be finite. It is defined as the kernel

$$0 \to Ш(E) \to H^1(G, E(\bar{\mathbb{Q}})) \to \prod_p H^1(G_p, E(\bar{\mathbb{Q}}_p))$$

of the localization map on classifying spaces of torsors for E in the étale topology of $\mathrm{Spec}(\mathbb{Q})$ as p runs over all primes of $\mathbb{Q}$.

Then there are the transcendental terms: Ω, the real period (or twice that), defined as an integral

$$\Omega = \int_{E(\mathbb{R})} |\omega|$$

where $\omega = dx/(2y+a_1x+a_3)$ is an invariant differential obtained from a minimal Weierstrass equation. The period can be easily computed, but the inaccessible part is the regulator R_E. This is the co-volume of the lattice $E(\mathbb{Q})/E(\mathbb{Q})_{tor}$ inside the inner product space

$$([E(\mathbb{Q})/E(\mathbb{Q})_{tor}] \otimes \mathbb{R}, < \cdot, \cdot >)$$

where $< \cdot, \cdot >$ is defined by the Neron-Tate canonical height. Thus, if $\{P_1, P_2, \ldots, P_r\}$ is a basis for for $E(\mathbb{Q})/E(\mathbb{Q})(tor)$, then

$$R_E := |\det(< P_i, P_j >)|$$

Obviously, computation of R_E would require knowledge of the Mordell-Weil group. On the other hand, since the formula gives a computable bound for the denominator of $L(E,1)/\Omega$ when $R_E = 1$, assuming its validity allows us to verify the vanishing of $L(H^1(E), 1)$ after a finite computation. Furthermore, it is important to note that there is a rather efficient algorithm for obtaining the full Mordell-Weil group, once one is equipped with the conjecture in full, as described in [33]. Meanwhile, in relation to this general circle of conjectures, it still may be worth occasionally recalling that the classical method of descent, as described for example in [43], provides an algorithm for computing the Mordell-Weil group, which is guaranteed to terminate only assuming the finiteness of Ш, even with no prior knowledge at all of L-functions.

We refer the reader to [39] for an accessible report on the BSD conjecture, covering work up to 2002.

The known relations between L-functions and arithmetic are expected to generalize vastly. As indicated above, L-functions are defined using Galois actions on étale cohomology and completed using Hodge theory.

Before we summarize the relevant definitions, we recall the big picture represented by the following conjectures:

(1) Hasse-Weil: the completed L-function $L(M,s)$of a pure motive M has a meromorphic continuation to the complex plane and satifies a functional equation. This conjecture is supposed to be addressed by Langlands' program, which says 'Motivic L-functions are automorphic L-functions' ([32]).

(2) Conjectures about values:

(a) Deligne's conjecture gives the value, up to a rational number, of $L(M,m)$ whenever m belongs to the so-called 'critical strip,' using a general definition of periods coming from a comparison between rational De Rham and topological cohomologies ([13]);

(b) The Beilinson conjectures continues the discussion about $L(M,m)$ when m is an arbitrary integer. It relates orders of vanishing and principal values (up to a rational number) to the rank and co-volume of motivic cohomology using a generalized regulator map ([1], [2], [3], [4]).

(c) The Bloch-Kato conjectures determine these rational values in a precise way using Tamagawa numbers (or determinants) for Galois representations via p-adic Hodge theory ([9], [27], [20]).

2 Definitions

Let $X/\mathbb{Q}$ be a smooth projective variety as before. Associated to X is a well-known collection of cohomology groups, the *realizations* of the motive(s) of X [47].

$H^n_l(X) = H^n_{et}(\bar{X}, \mathbb{Q}_l)$ for each prime l, the $\mathbb{Q}_l$-coefficient étale cohomology of degree n. This carries a natural action of $G = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$.

$H^n_{DR}(X) := H^n(X, \Omega^{\cdot}_X)$, the algebraic De Rham cohomology equipped with a Hodge filtration given by

$$F^i H^n_{DR}(X) = H^n(X, \Omega^{\geq i}) \hookrightarrow H^n_{DR}(X)$$

for each i.

$H^n_B(X) := H^n(X(\mathbb{C}), \mathbb{Q})$, the $\mathbb{Q}$-coefficient singular cohomology of the complex manifold $X(\mathbb{C})$ equipped with an involution F_∞ induced by the anti-holomorphic complex conjugation acting on the complex points.

The completed L-function of the motive $H^n(X)$ requires us to use all these structures, but the definition of the motive itself is not needed.

These cohomology groups are bound together by an intricate system of canonical comparison isomorphisms. For example,

$$H^n_B(X) \otimes_{\mathbb{Q}} \mathbb{Q}_l \simeq H^n_l(X)$$

preserving the action of F_∞, the complex conjugation. And then,

$$H^n_B(X) \otimes_{\mathbb{Q}} \mathbb{C} \simeq H^n_{DR}(X) \otimes_{\mathbb{Q}} \mathbb{C}.$$

This isomorphism endows the pair $(H^n_B(X), H^n_{DR}(X) \otimes \mathbb{R})$ with a *rational Hodge structure* of weight n 'defined over $\mathbb{R}$.' That is, we have a direct sum decomposition

$$H^n_B(X) \otimes_{\mathbb{Q}} \mathbb{C} \simeq \oplus H^{p,q}(X)$$

where

$$H^{p,q} := F^p \cap \bar{F}^q$$

and

$$F_\infty(H^{p,q}) = H^{q,p}.$$

Here, $\bar{F}^q$ refers to the image of F^q under the action of complex conjugation c acting on the coefficients of $H^n_B(X) \otimes_{\mathbb{Q}} \mathbb{C}$ (that ends up giving the same image as F_∞). In fact,

$$(H^n_B(X) \otimes_{\mathbb{Q}} \mathbb{C})^{F_\infty \otimes c} = H^n_{DR} \otimes \mathbb{R}.$$

At non-archimedean places, there is an important analogue. For any prime p, we have

$$D_{DR}(H^n_p(X)) := (H^n_p(X) \otimes_{\mathbb{Q}_p} B_{DR})^{G_p} \simeq H^n_{DR}(X) \otimes_{\mathbb{Q}} \mathbb{Q}_p$$

where B_{DR} is Fontaine's ring of p-adic periods [19].

These structures taken together motivate the following observation. Regardless of its precise definition, a motive M should have associated to it a collection of objects as above that we call *a pure system of realizations* that make up a category $\mathcal{R}$. This is a collection

$$R(M) = \{\{M_l\}, M_{DR}, M_B\}$$

where each M_l is a continuous linear representation of G on a (finite-dimensional) $\mathbb{Q}_l$-vector space, M_{DR} is a filtered $\mathbb{Q}$-vector space, and M_B is a $\mathbb{Q}$-vector space with an involution F_∞. These vector spaces should all have the same dimension and be equipped with a system of comparison isomorphisms as above. The data must be subject to further subtle constraints having to do with local Galois representations.

That is to say recall the exact sequence:

$$0 \to I_p \to G_p \xrightarrow{v} \mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p) \to 0$$

where $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p) \simeq \hat{\mathbb{Z}}$ is topologically generated by the inverse Fr_p of the p-power map. For $l \neq p$, I_p has a tame l-quotient

$$t_l : I_p \to I_{p,l}$$

with the structure

$$I_{p,l} \simeq \hat{\mathbb{Z}}_l(1) \simeq \varprojlim \mu_{l^n}$$

as a module for $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$. Define

$$W_p := v^{-1}(Fr_p^{\mathbb{Z}}) \subset G_p,$$

the *Weil group* at p. It is convenient to analyze the data of M_l using an associated *Weil-Deligne* (W-D) representation [46]

$$WD_p(M_l)$$

for each p, consisting of a representation r of W_p such that $r|I_p$ has finite image and a nilpotent operator N_p acting on the representation.

These satisfy a compatibility condition

$$r(\phi_p) N_p r(\phi_p^{-1}) = p^{-1} N_p$$

for any lift $\phi_p \in W_p$ of Fr_p.

The construction of $WD_p(M_l)$ for $p \neq l$ uses the fact that the action of G_p when restricted to some finite index subgroup G_p' is semi-stable, i.e., its inertia subgroup I_p' acts unipotently. Hence, the action of I_p' can be expressed as

$$\sigma \mapsto \exp(t_l(\sigma) N_p)$$

for a nilpotent N_p. Then the representation r on W_p is given by

$$r(\phi_p^n \sigma) = \phi_p^n \sigma \exp(-t_l(\sigma) N_p)$$

for some choice of ϕ_p. In fact, since the data (ϕ_p, N_p) determine the Weil-Deligne representation, it is usual to identify the representation with such a pair.

For $p = l$, we use the fact that any De Rham representation is potentially semistable [11], and hence, gives us a filtered (ϕ_p, N_p) module via

$$M_p \mapsto (M_p \otimes_{\mathbb{Q}_p} B_{st})^{G_p'}$$

If $G_p' = G_p$ (that is, if the representation itself is semi-stable), then this gives us a Weil-Deligne representation in an obvious way by defining

$$r(g) = \phi_p^n$$

if g maps to $Fr_p^n \in \mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$. In [20], I.1, it is explained in detail how one extracts a Weil-Deligne representation from the data in the general case.

The viewpoint of the Weil-Deligne representation allows us to parametrize the information of the Galois representations in a form that does not use the topology of $\mathbb{Q}_l$. It provides, thereby, a suitable framework for comparing the representations as l varies, and makes natural the connection to complex automorphic forms [48]. Furthermore, one creates thereby a rather precise analogy with the theory of limit mixed Hodge structures [24].

Now define the *Frobenius semi-simplification* $WD_p(M_l)^{ss}$ of $WD_p(M_l)$ by replacing ϕ_p with its semi-simple part. With the terminology thus introduced, here are the constraints we impose on our pure system of realizations:

'Good reduction almost everywhere': We assume that there exists a finite set S of primes such that $WD_p(M_l)$ is unramified for all $p \notin S$, i.e., $N_p = 0$ and I_p acts trivially.

'Algebraicity and independence of l': There exists a Frobenius semi-simple W-D representation $WD_p(M)$ over $\bar{\mathbb{Q}}$ such that

$$WD_p(M) \otimes_{\bar{\mathbb{Q}}} \bar{\mathbb{Q}}_l \simeq WD_p^{ss}(M_l) \otimes_{\bar{\mathbb{Q}}} \bar{\mathbb{Q}}_l$$

Subject to these conditions, the collection $\{M_l\}$ is then referred to as a *strongly compatible* system of l-adic representations.

'Weil conjecture': There should exist an integer n, called the *weight* of M, such that the eigenvavlues of Fr_p acting on $WD_p(M)$ for $p \notin S$ have all Archimedean absolute values equal to $p^{n/2}$. Furthermore, the Hodge structure M_B should be pure of weight n.

'Purity of the monodromy filtration': If we denote by $Mn_\cdot$ the unique increasing filtration on $WD_p(M)$ such that [12]

-$Mn_{-k} = 0$, $Mn_k = WD_p(M)$ for sufficiently large k;

-$N(Mn_k) \subset Mn_{k-2}$;

-and N induces isomorphisms

$$N^k : Gr_k^{Mn}(WD_p(M)) \simeq Gr_{-k}^{Mn}(WD_p(M));$$

then, for each k, the associated graded piece

$$Gr_k^{Mn}(WD_p(M))$$

has all Frobenius eigenvalues of archimedean absolute value $p^{(n+k)/2}$.

It should be remarked that in general, we need to allow coefficients in E_λ for the representations where E is a number field and E_λ are completions. Such coefficient systems arise naturally when considering direct summands of $\mathbb{Q}_l$-representations or *motives with extra endomorphisms*, e.g., abelian varieties with CM. We will omit this generality in this summary. Another interesting view is that the bi-grading

$$M_B \otimes \mathbb{C} \simeq \oplus M^{p,q},$$

which is compatible with the complex conjugation of coefficients, corresponds to a representation of the group

$$Res_{\mathbb{R}}^{\mathbb{C}}(\mathbf{G}_m).$$

Together with the action of

$$F_\infty \circ C$$

it can be viewed as a representation of the real Weil group $W_{\mathbb{R}}$ ([46]) with points given by

$$W_{\mathbb{R}}(\mathbb{R}) = \mathbb{C}^* \cup \mathbb{C}^* j$$

where $j^2 = -1$ and $jzj^{-1} = \bar{z}$. Here, C is the Weil operator defined by

$$C|M^{pq} = i^{q-p}$$

It is conjectured that the realizations

$$H^n(X) = (\{H^n_l(X)\}, H^n_B(X), H^n_{DR}(X))$$

coming from a smooth projective variety X satisfy the algebraicity, independence of l, and purity conditions even for $p \in S$.

The category of pure motives should be comprised of objects in $\mathcal{R}$ of *geometric origin*, a notion without an entirely precise interpretation [14]. For example, we need to admit at every stage duals (homology) and tensor products of all objects considered. Objects that are not generated in an obvious way from those of the form

$$H^n(X)$$

arise via images (or kernels) under pull-backs and push-forwards in cohomology induced by maps of varieties, as well as $\mathbb{Q}$-linear combinations of geometric maps. We should also be able to compose pull-backs with push-forwards. Such compositions give rise to the idea of using *correspondences* modulo homological equivalence as morphisms [13]. Once morphisms are constructed in this manner, we naturally obtain new objects using the decomposition of

$$End(H^n(X)),$$

which is a semi-simple $\mathbb{Q}$-algebra subject to one of the standard conjectures saying that numerical equivalence and homological equivalence coincide [26].

One can consider also a category of mixed systems of realizations by requiring a weight filtration

$$\cdots \subset W_{n-1}M \subset W_nM \subset W_{n+1}M \subset$$

compatible with all the comparisons and such that each graded quotient

$$Gr^n_W(M)$$

is a pure system of realizations of weight n. Mixed motives should be those of geometric origin such as the cohomology of varieties that are not necessarily smooth or proper. But then, we need to include objects like (finite-dimensional quotients of)

$$\mathbb{Q}[[\pi_1]] = \varprojlim \mathbb{Q}[\pi_1]/I^n,$$

i.e., group algebras of fundamental groups completed with respect to the augmentation ideal, or the (co)-homology of (co-)simplicial varieties [25].

Given a pure system M of realizations we can define its L-function $L(M, s)$ as an Euler product

$$L(M, s) = \prod_p L_p(M, s)$$

with

$$L_p(M, s) = \frac{1}{\det[(1 - p^{-s}Fr_p)|(WD_p(M))^{I_p=1, N_p=0}]}.$$

If M is of weight n, then the product converges (and hence is non-zero) for $Re(s) > n/2 + 1$. For some conceptual motivation for this definition based on duality in the function field case, see [17]. (The point is that the inertia fixed part is the stalk of the intermediate extension of the étale sheaf corresponding to M_l.)

There is also a factor at ∞ depending upon the representation $M_B \otimes \mathbb{C}$ of $W_{\mathbb{R}}$. Define

$$\Gamma_{\mathbb{R}} := \pi^{-s/2}\Gamma(s/2)$$

$$\Gamma_{\mathbb{C}} := 2(2\pi)^{-s}\Gamma(s)$$

$$h^{pq} := \dim M^{pq}$$

$$h^{p,\pm} := \dim M^{pp,\pm(-1)^p}$$

where the signs in the superscript refer to the ± 1 eigenspaces of the F_∞-action. Then

$$L_\infty(M,s)$$

is defined as

$$\prod_{p<q} \Gamma_{\mathbb{C}}(s-p)^{h^{pq}}$$

for odd n, and

$$\prod_{p<q} \Gamma_{\mathbb{C}}(s-p)^{h^{pq}} \Gamma_{\mathbb{R}}(s-n/2)^{h^{n/2+}} \Gamma_{\mathbb{R}}(s-n/2+1)^{h^{n/2-}}$$

for n even ([42]).

It is conjectured that

$$\Lambda(M,s) = L_\infty(M,s)L(M,s)$$

has a meromorphic continuation to $\mathbb{C}$ and satisfies a functional equation

$$\Lambda(M,s) = \epsilon(M,s)\Lambda(M^*, 1-s)$$

where the epsilon factor has the form $\epsilon(M,s) = cf^s$, $c \in \mathbb{C}^*$ and f a positive integer. Note that the contribution of the conductor has also been incorporated into this factor. (For a precise discussion of the factor in the case $M = H^n(X)$, see [42].) As alluded to above, the general expectation is that this conjecture will be dealt with by the Langlands' program.

3 Conjectures on zeros, poles, and values

A list of overall references to this section should include the papers mentioned in the introduction as well as the original articles [7], [13], [1], [2], [3], [4], and [9]. Having mentioned thus the sources, we will then proceed to be somewhat sloppy with specific citation.

Here is some convenient notation:

$\mathbb{Q}$: trivial system of realizations.

$\mathbb{Q}(1) := H^2(\mathbf{P}^1)^*$

$\mathbb{Q}(i) = \mathbb{Q}(1)^{\otimes i}$ for $i \geq 0$

$\mathbb{Q}(i) = \mathrm{Hom}(\mathbb{Q}(-i), \mathbb{Q})$ for $i < 0$.

For a system M of realizations we define its *Tate twists* by tensor products with $\mathbb{Q}(i)$:

$$M(i) := M \otimes \mathbb{Q}(i)$$

Then for any smooth projective variety of dim d, we have ([35], VI.6)

$$H^{2d}(X) \simeq \mathbb{Q}(-d)$$

and a perfect pairing

$$H^i(X) \times H^{2d-i}(X) \to H^{2d}(X)$$

Repeated cup product with the cohomology class of a hyperplane gives us the hard Lefschetz theorem [15]

$$H^i(X) \simeq H^{2d-i}(X)(d-i).$$

The effect of the twisting on realizations is such that $M(n)_l$ is the tensor product of M_l with the n-th power of the $\mathbb{Q}_l$ cyclotomic character giving the action of G on the l-power roots of unity, and

$$F^i(M(n)_{DR}) = F^{n+i}M_{DR}$$

with a corresponding shift in Hodge numbers h^{pq}. Furthermore,

$$F_\infty|M(n)_B = (F_\infty|M_B) \otimes (-1)^n$$

and for the L-functions,

$$L(M(n), s) = L(M, s + n).$$

These facts all follow in an elementary way from the structure of $H^2(\mathbf{P}^1)$.

The conjectures deal with the behavior of $L(M, s)$ at an integral value m of s. To state them without spending time on categorical preliminaries, we will focus on the case where M is

$$H^n(X) = (\{H^n_l(X)\}, H^n_{DR}(X), H^n_B(X))$$

for a smooth projective variety X of dimension d. Assume that $H^n(X)$ is a pure system of realizations so that the analytic continuation and functional equation hold true. Since we have

$$H^n(X)^* \simeq H^{2d-n}(X)(d) \simeq H^n(X)(n)$$

the conjectured functional equation relates

$$L(H^n(X), s)$$

and

$$L(H^n(X)(n), 1 - s) = L(H^n(X), n + 1 - s),$$

while the center of reflection is

$$(n + 1)/2.$$

We will therefore confine interest mostly to

$$m \geq (n + 1)/2$$

or, equivalently,

$$n + 1 - m \leq (n + 1)/2,$$

and refer to $\mathrm{Re}(s) = (n + 1)/2$ as the critical line. (The reader should consult Nekovar's elegant article [37] for a careful discussion of how to relate the points addressed here to those on the right of the critical line.) In fact, it is conceptually convenient to parametrize by the letter m the integers $\geq (n+1)/2$, but to focus then on the values at the points $n+1-m$. In the discussion of orders, however, we will make explicit the case of $m = n/2$, and hence, the possible pole at $n + 1 - m = n/2 + 1$ (just to the right of the critical line), whose importance is evident from the classical example of Dedekind's zeta functions. As we will explain below, the general geometric importance of this pole is related to Tate's conjecture on the cohomology classes of algebraic cycles [47]. We remark also that the study of $L(H^n(X), s)$ near m corresponds to the study of $L(H^n(X)(m), s)$ near $s = 0$. Thus, when the conjectures are formulated in terms of values at zero in the literature, one encounters the assumption that the weight $n - 2m$ is negative.

We start then with the conjecture on orders. The simple case arising from an elliptic curve was reviewed already where $n = 1$ and $m = n+1-m$ is the reflection point $(n+1)/2 = 1$. The conjecture of Birch and Swinnerton-Dyer says

$$\mathrm{ord}_{s=1}L(H^1(E), s) = \mathrm{rank}E(\mathbb{Q})$$

Now, an element

$$x \in E(\mathbb{Q})$$

gives rise to an extension in the category $\mathcal{R}$ of realizations

$$\delta(x) \in Ext^1_{\mathcal{R}}(\mathbb{Q}, H^1(E)(1))$$

via Kummer theory. In the p-adic realization, for example, this is obtained in the inverse limit from the exact sequences

$$0 \to E[p^n] \to E(\bar{\mathbb{Q}}) \xrightarrow{n} E(\bar{\mathbb{Q}}) \to 0$$

of Galois modules as the image of $x \in H^0(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), E(\bar{\mathbb{Q}}))$ under the connecting homomorphisms. It is conjectured that when $\mathcal{R}$ is replaced by a suitable category of motives, this is the only way to construct such extensions. This notion conveys the basic flavor of conjectures on orders in the case of odd weight.

For an example in the even weight case, let $X = \mathrm{Spec}(F)$ for a number field F, assumed for simplicity to be Galois over $\mathbb{Q}$. Then

$$\zeta(F, s) = L(H^0(\mathrm{Spec}(F)), s)$$

which breaks up as into a product of Artin L-functions

$$\zeta(F, s) = \prod_{\rho} L(\rho, s)$$

as $\rho : \mathrm{Gal}(F/\mathbb{Q}) \to Aut(V)$ runs over finite-dimensional representations on $\mathbb{Q}$-vector spaces. This is the most basic example of a motivic decomposition. In fact, any finite-dimensional $\mathbb{Q}$-representation of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ defines an Artin motive M_ρ and it is a theorem that

$$\mathrm{ord}_{s=1} L(\rho, s) = -\mathrm{dim}\mathrm{Hom}_{AM}(\mathbb{Q}, M_\rho),$$

where the Hom occurs inside the category of Artin motives [13]. Note that $s = 1$ in this case is the integer point $n/2 + 1$ just to the right of the critical line mentioned above.

The general conjecture about orders says

$$\mathrm{ord}_{s=n+1-m} L(H^n(X), s)$$

$$= \mathrm{dim} Ext^1_{Mot_{\mathbb{Z}}}(\mathbb{Q}, H^n(X)(m)) - \mathrm{dim}\mathrm{Hom}_{Mot_{\mathbb{Z}}}(\mathbb{Q}, H^n(X)(m)).$$

The Hom and Ext should occur inside a conjectural category of mixed motives over $\mathbb{Z}$ with $\mathbb{Q}$-coefficients. For weight reasons, the Hom term should vanishes unless $n = 2m$, in which case the Ext term should vanish. (As would follow from the K-theoretic interpretation of the Ext group. See below.) That is, in the pure situation we are considering, only one term or the other is supposed to occur.

The point just to the right of the critical line is of interest in the even weight situation when $n = 2m$, where the formula predicts

$$\mathrm{ord}_{s=m+1} L(H^{2m}(X), s) = -\mathrm{dim}\mathrm{Hom}_{Mot_{\mathbb{Z}}}(\mathbb{Q}, H^{2m}(X)(m))$$

generalizing the pole of the Artin L-function ($m = 0$). As for an explicit connection to arithmetic geometry, it is expected that

$$\mathrm{Hom}_{Mot_{\mathbb{Z}}}(\mathbb{Q}, H^{2m}(X)(m)) \simeq [CH^m(X)/CH^m(X)^0] \otimes \mathbb{Q}.$$

Of course the isomorphism should arise via a cycle map

$$CH^m(X) \to H^{2m}(X)(m)$$

killing the cycles $CH^m(X)^0$ homologically equivalent to zero.

As we move to the left, we encounter the point $m = (n+1)/2$ for n odd ($n = 2m-1$), and the conjecture predicts the order of vanishing at the central critical point:

$$\mathrm{ord}_{s=m} L(H^{2m-1}(X), s) = \mathrm{dim} Ext^1_{Mot_{\mathbb{Z}}}(\mathbb{Q}, H^{2m-1}(X)(m)).$$

It is then expected that

$$Ext^1_{Mot_{\mathbb{Z}}}(\mathbb{Q}, H^{2m-1}(X)(m)) \simeq CH^m(X)^0 \otimes \mathbb{Q}$$

The map from cycles to extensions generalizes Kummer theory: given a representative Z for a class in $CH^m(X)^0$, we get an exact sequence

$$0 \to H^{2m-1}(X)(m) \to H^{2m-1}(X \setminus Z)(m)$$

$$\stackrel{\delta}{\to} H^{2m}_Z(X)(m) \to H^{2m}(X)(m).$$

There is a local cycle class

$$cl(Z) \in H^{2m}_Z(X)(m)$$

that maps to zero in $H^{2m}(X)(m)$, giving rise to the desired extension:

$$0 \to H^{2m-1}(X)(m) \to \delta^{-1}(cl(Z)) \to \mathbb{Q} \to 0$$

These two classical points, central critical, $n+1-m = m = (n+1)/2$ for n odd, and just right of it, $n+1-m = n/2+1$, for n even, are somewhat exceptional. As $n+1-m$ moves further left ($m > n/2+1$), one expects

$$Ext^1_{Mot_{\mathbb{Z}}}(\mathbb{Q}, H^n(X)(m)) = H^{n+1}_{M,\mathbb{Z}}(X, \mathbb{Q}(m))$$

with the last group, often referred to as *motivic cohomology*, defined using K-theory :

$$Im[(K_{2m-n-1}(\mathcal{X}))^{(m)} \to (K_{2m-n-1}(X))^{(m)}],$$

where $\mathcal{X}$ is a proper flat regular $\mathbb{Z}$-model for X (whose existence needs to be assumed) and the superscript for K-theory refers to the subspaces cut out by the Adams operators [21]. An alternative is to use Bloch's higher Chow groups [8]:

$$Im[CH^{n+1}(\mathcal{X}, 2m-n-1) \otimes \mathbb{Q} \to CH^{n+1}(X, 2m-n-1) \otimes \mathbb{Q}].$$

The latter interpretation, carrying with it the hope of representing motivic cohomology classes quite explicitly, is more popular lately [18]. In fact, when $m > n/2+1$, the conjectured functional equation implies

$$\mathrm{ord}_{s=n+1-m} L(H^n(X), s) = \mathrm{dim} Ext^1_{MHS^{\mathbb{R}}_{\mathbb{R}}}(\mathbb{R}, H^n_B(X)(m) \otimes \mathbb{R})$$

where the extension occurs inside the category of real mixed Hodge structures defined over $\mathbb{R}$. So the statement on the order of vanishing follows from the conjecture that the Hodge realization functor induces an isomorphism

$$Ext^1_{Mot_{\mathbb{Z}}}(\mathbb{Q}, H^n(X)(m)) \otimes \mathbb{R} \simeq Ext^1_{MHS^{\mathbb{R}}_{\mathbb{R}}}(\mathbb{R}, H^n_B(X)(m) \otimes \mathbb{R}).$$

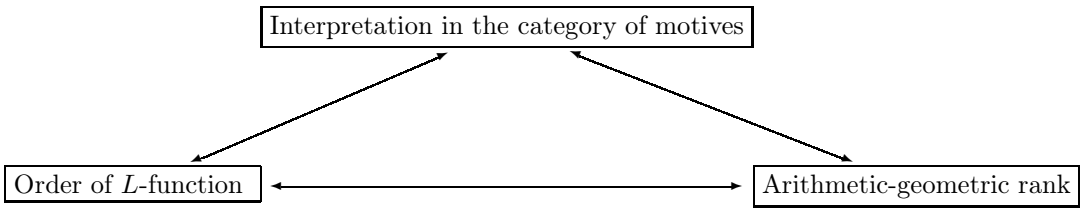
It has been emphasized by Deligne that the regulator map (discussed below) *is* this realization functor.

Our outline thus far should already make clear that the conceptual structure of the conjectures falls into two parts:

(1) Relation between L functions and Ext and Hom groups in category of motives;

(2) geometric interpretation of Ext and Hom groups.

That is to say, in addition to the difficult problem of drawing the lower edge of the following triangle



the further problem has been created of constructing a category that realizes the upper vertex.

There is a computation, convenient in practice, of the real Ext group via Deligne cohomology:

$$Ext^1_{MHS^{\mathbb{R}}_{\mathbb{R}}}(\mathbb{R}, H^n_B(X)(m) \otimes \mathbb{R}) \simeq H^{n+1}_D(X_{\mathbb{R}}, \mathbb{R}(m)).$$

Using the explicit nature of Deligne cohomology, one can construct regulator maps

$$H^{n+1}_{M,\mathbb{Z}}(X, \mathbb{Q}(m)) \to Ext^1_{MHS^{\mathbb{R}}_{\mathbb{R}}}(\mathbb{R}, H^n_B(X)(m) \otimes \mathbb{R})$$

that can be studied independently of a category of motives. For example, in essentially all known cases of the Beilinson conjectures (e.g. [2],[16]), one constructs subgroups

$$L \subset H^{n+1}_{M,\mathbb{Z}}(X, \mathbb{Q}(m)),$$

conjecturally of full rank, and studies their images in Deligne cohomology.

As we move on to the conjectures on values, we start with the central critical ones due to Bloch and Beilinson. Thus, $m = n + 1 - m = (n + 1)/2$ (and $n = 2m - 1$). We then have an isomorphism

$$F^m H^{2m-1}_{DR}(X) \otimes \mathbb{R} \simeq [H^{2m-1}_B(X)(m-1)]^{(-1)^{m-1}} \otimes \mathbb{R}$$

which gives rise to the *period isomorphism*

$$p = p(H^{2m-1}(X, m)) : \det(F^m H^{2m-1}_{DR}(X)) \otimes \mathbb{R} \simeq \det[[H^{2m-1}_B(X)(m-1)]^{(-1)^{m-1}}] \otimes \mathbb{R}$$

of real vector spaces of dimension one, where the det refers to top exterior powers. That is to say,

$$\det(F^m H^{2m-1}_{DR}(X))$$

and

$$\det[[H^{2m-1}_B(X)(m-1)]^{(-1)^{m-1}}]$$

are viewed as two $\mathbb{Q}$-lines sitting inside the same real line. An additional transcendental contribution comes from the *height pairing*, conjectured to be non-degenerate:

$$CH^m(X)^0 \times CH^{\dim(X)+1-m}(X)^0 \to \mathbb{R}$$

whose determinant gives us a regulator

$$r = r(H^{2m-1}(X)(m)) \in \mathbb{R}^*/\mathbb{Q}^*.$$

Recall that conjecturally

$$d_m := ord_{s=m} L(H^{2m-1}(X), s) = \dim CH^m(X)^0 \otimes \mathbb{Q}$$

For any motive M, denote by

$$L^*(M, m)$$

the leading coefficient of $L(M,s)$ at $s=m$. So

$$L^*(H^{2m-1}(X),m)=\lim_{s\to m}(s-m)^{-d_m}L(H^{2m-1}(X),s).$$

The general conjecture on central critical values then says

$$r\cdot p[\det(F^mH^{2m-1}_{DR}(X))]=L^*(H^{2m-1}(X),m)\det[[H^{2m-1}_B(X)(m-1)]^{(-1)^{m-1}}]$$

inside $\det[[H^{2m-1}_B(X)(m-1)]^{(-1)^{m-1}}]\otimes\mathbb{R}$.

Moving left to the values at $n+1-m\le n/2$ ($m\ge n/2+1$) we point out first that the period isomorphism is replaced by an exact sequence:

$$0\to F^mH^n_{DR}(X)\otimes\mathbb{R}\to[H^n_B(X)(m-1)]^{(-1)^{m-1}}\otimes\mathbb{R}$$

$$\to Ext^1_{MHS^{\mathbb{R}}_{\mathbb{R}}}(\mathbb{R},H^n_B(X)(m)\otimes\mathbb{R})\to 0$$

Thus, the transcendental part should incorporate a $\mathbb{Q}$-structure on

$$Ext^1_{MHS^{\mathbb{R}}_{\mathbb{R}}}(\mathbb{R},H^n_B(X)(m)\otimes\mathbb{R}).$$

We will skip for a moment the classically interesting point $n+1-m=n/2$ for n even and assume $n+1-m<n/2$. Therefore, $m>n/2+1$ lies in the region of convergence of the L-function. Then the $\mathbb{Q}$ structure is expected to come from the conjectured *regulator isomorphism*:

$$H^{n+1}_{M,\mathbb{Z}}(X,\mathbb{Q}(m)))\otimes\mathbb{R}\simeq Ext^1_{MHS^{\mathbb{R}}_{\mathbb{R}}}(\mathbb{R},H^n_B(X)(m)\otimes\mathbb{R})$$

We are led thus to an isomorphism

$$c=c(H^n(X)(m)):[\det(H^{n+1}_{M,\mathbb{Z}}(X,\mathbb{Q}(m)))]\otimes\mathbb{R}\simeq[\det F^mH^n_{DR}(X)]^{-1}\otimes\det([H^n_B(X)(m-1)]^{(-1)^{m-1}})\otimes\mathbb{R}$$

and Beilinson's conjecture says

$$c[\det(H^{n+1}_{M,\mathbb{Z}}(X,\mathbb{Q}(m)))]=L^*(H^n(X),n+1-m)[\det F^mH^n_{DR}(X)]^{-1}\otimes\det([H^n_B(X)(m-1)]^{(-1)^{m-1}}).$$

The cases where

$$Ext^1_{MHS^{\mathbb{R}}_{\mathbb{R}}}(\mathbb{R},H^n_B(X)(m)\otimes\mathbb{R})=0$$

are the *critical values* considered by Deligne [13], and Beilinson's conjecture in that case reduces to Deligne's conjecture, relating L-values to the discrepancy between $\mathbb{Q}$-structures coming from Betti and De Rham cohomology.

Finally, we return to the value at $n+1-m=n/2$ ($m=n/2+1$) for $n=2m-2$ even. Here, the conjecture is identical to the situation further left except the regulator involves maps from both motivic cohomology

$$H^{n+1}_{M,\mathbb{Z}}(X,\mathbb{Q}(m))$$

and

$$CH^{m-1}(X)$$

in a manner similar to how the central critical value incorporates periods and a height pairing. That is, we have a map

$$[CH^{m-1}(X)/CH^{m-1}(X)^0]\to[H^{2m-2}_B(X)(m-1)]^{(-1)^{m-1}}$$

coming from the cycle map that induces an injection

$$[CH^{m-1}(X)/CH^{m-1}(X)^0]\hookrightarrow Ext^1_{MHS^{\mathbb{R}}_{\mathbb{R}}}(\mathbb{R},H^n_B(X)(m)\otimes\mathbb{R})$$

via the quotient map

$$[H_B^n(X)(m-1)]^{(-1)^{m-1}} \otimes \mathbb{R} \to Ext^1_{MHS^{\mathbb{R}}_{\mathbb{R}}}(\mathbb{R}, H_B^n(X)(m) \otimes \mathbb{R}).$$

We remark then only that the conjectured isomorphism is

$$H^{n+1}_{M,\mathbb{Z}}(X,m) \otimes \mathbb{R} \oplus [CH^{m-1}(X)/CH^{m-1}(X)^0] \otimes \mathbb{R}$$

$$\simeq Ext^1_{MHS^{\mathbb{R}}_{\mathbb{R}}}(\mathbb{R}, H_B^n(X)(m) \otimes \mathbb{R}).$$

The point is that in order to give the correct order of zero (dimension of motivic cohomology) to the left of the critical strip, the poles to the right of the critical strip must be canceled out.

In the Bloch-Kato conjectures, isomorphisms are normalized more carefully, comparing certain integral structures one prime at a time. More precisely, the conjecture concerns the p-adic valuation of the 'rational part' of the L-function for each prime p. We give here just a flavor of the conjecture by discussing a range of cases that doesn't involve too many definitions. There is a theory of determinants of perfect complexes over a principal ring R that goes with the conjectures [20], where the useful facts are:

(1) If M is a finitely generated free module, then $\det(M)$ is the top exterior power.

(2) If K is the field of fractions of R, then for any finitely generated module M, the map $M \to M \otimes K$ induces

$$\det(M) \otimes_R K \simeq \det(M \otimes_R K)$$

canonically.

(3) If M is finitely generated and M' is its torsion-free quotient, then the natural isomorphism

$$\det(M) \otimes K \simeq \det(M') \otimes K$$

takes a generator of $\det(M)$ to $1/r$ times a generator of $\det(M')$, where r is a characteristic element for the torsion submodule of M.

We note in regard to these facts that the determinant module is always free of rank one, so it is only the morphisms that are the relevant data. Now assume that $m > \min\{n, \dim(X)\}$ so that $F^m H^n_{DR} = 0$ and

$$[H_B^n(X)(m-1)^{(-1)^{m-1}} \otimes \mathbb{R}] \simeq Ext^1_{MHS^{\mathbb{R}}_{\mathbb{R}}}(\mathbb{R}, H_B^n(X)(m) \otimes \mathbb{R}).$$

Therefore, we should have

$$H^{n+1}_{M,\mathbb{Z}}(X, \mathbb{Q}(m)) \otimes \mathbb{R} \simeq Ext^1_{MHS^{\mathbb{R}}_{\mathbb{R}}}(\mathbb{R}, H_B^n(X)(m) \otimes \mathbb{R})$$

$$\simeq [H^n(X(\mathbb{C}), \mathbb{Z}(m-1))^{(-1)^{m-1}} \otimes \mathbb{R}].$$

Let γ be a generator of

$$\det(H^n(X(\mathbb{C}), \mathbb{Z}(m-1))^{(-1)^{m-1}})$$

and let

$$\omega_\gamma \in \det(H^{n+1}_{M,\mathbb{Z}}(X, \mathbb{Q}(m)))$$

be an element mapping to

$$L^*(H^n(X)(m), n+1-m)\gamma.$$

As before, let S be a set of primes including those of bad reduction for X, the Archimedean prime, and a fixed prime p. We denote by G_S the Galois group of the maximal extension of $\mathbb{Q}$ unramified outside the primes in S. There is a Chern-class map

$$H^{n+1}_{M,\mathbb{Z}}(X, \mathbb{Q}(m)) \to H^1(G_S, H^n_p(X)(m))$$

that conjecturally induces an isomorphism

$$H^{n+1}_{M,\mathbb{Z}}(X,\mathbb{Q}(m))\otimes\mathbb{Q}_p \simeq H^1(G_S,H^n_p(X)(m))$$

That is to say, there are *two* maps

$$\begin{array}{ccc} H^{n+1}_{M,\mathbb{Z}}(X,\mathbb{Q}(m)) & \longrightarrow & Ext^1_{MHS^{\mathbb{R}}_{\mathbb{R}}}(\mathbb{R},H^n_B(X)(m)\otimes\mathbb{R}) \\ \downarrow & & \\ H^1(G_S,H^n_p(X)(m)) & & \end{array}$$

that are both supposed to induce isomorphisms upon changing coefficients, and with which the rational part will be controlled. Let

$$z_\gamma \in \det(H^1(G_S,H^n_p(X)(m)))$$

be the image of ω_γ so that we have the diagram

$$\begin{array}{ccc} \omega_\gamma & \longmapsto & L^*(H^n(X)(m),n+1-m)\gamma \\ \downarrow & & \\ z_g & & \end{array}$$

Then the conjecture is that

$$[\det(H^1(G_S,H^1(G,H^n_p(X)(m))):\mathbb{Z}_p z_\gamma] = |H^0(G_S,H^1_f(G,H^n_p(X)(m)))||H^2(G_S,H^1_f(G,H^n_p(X)(m)))|$$

It is easy to see that the H^0 term is finite, but the finiteness of the H^2, like that of *Sha* in the case of elliptic curves, must also be conjectured. In the uniform formalism, the desired equality is interpreted as the assertion that z_γ is a $\mathbb{Z}_p$-basis of the determinant of the perfect $\mathbb{Z}_p$-complex $R\Gamma(G_S,H^n_p(X)(m))$.

Extraction of the rational part is supposed to lead eventually to a *p-adic* L-function

$$\mathcal{L}^{(p)}(H^n(X))$$

that exercises control over Galois cohomology (i.e., Selmer groups) and Diophantine invariants. This p-adic theory appears so far to be the best strategy for applying the theory of L-functions to the elucidation of Diophantine structures ([38], [28], [45]).

4 Remark

We conclude with the warning that there is a conspicuous deficiency in theory of motives: This is that even in the best of possible worlds (ours), only *abelian* invariants are accessible, such as

$$CH^m(X).$$

These abelian invariants do not yield in general information about

$$X(\mathbb{Q})$$

and leave thereby untouched the most basic questions of Diophantine geometry. This is an artifact of the fact that the theory of motives as presently developed is implicitly modeled on the theory of abelian varieties and H_1. Attempts to redress this deficiency for certain varieties are contained in

Grothendieck's *anabelian program*

([23], [36]) that concerns itself with the theory of *pro-finite* π_1*'s*. The technology of motives ends up contributing here as well because the Diophantine aspect of this theory [29] assigns an interesting role to *motivic fundamental groups* [14], where *Ext* groups are replaced by

classifying spaces for non-abelian torsors

However, what is entirely missing as yet is an analogue of the L-function.

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