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Generalized Baer-Invariant of a Pair of Groups and Marginal Extension

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Abstract

In this paper, we give connection between the order of the generalized Baer-invariant of a pair of finite groups and its factor groups, when v is considered to be the specific variety. Moreover, we give a necessary and sufficient condition in which the generalized Baer-invariant of a pair of groups can be embedded into the generalized Baer-invariant of pair of its factor groups.

Keywords: Varieties of groups; Generalized Baer-invariant; Marginal extension

Introduction

We assume that the reader is familiar with the notions of the verbal subgroup V(G), and the marginal subgroup V*(G) , associated with a variety of groups $\mathcal{V}$ and a group G; see [11] for more information on varieties of groups. Let $\mathcal{V}$ and $\mathcal{W}$ be two varieties of groups defined by the sets of laws V and W, respectively. Let N be a normal subgroup of a group G, then we define $\left[NV^{*}G\right]$ to be the subgroup of G generated by the elements of the following set:

$$\left\{v(g_1, g_2, \ldots, g_i n, \ldots, g_r)v(g_1, g_2, \ldots, g_r)^{-1} \;\middle|\; 1 \le i \le r, v \in V, g_1, \ldots, g_r \in G, n \in N\right\}.$$

It is easily checked that $\left[NV^{*}G\right]$ is the least normal subgroup T of G such that N/T is contained in V*(G/T); see [2]. In 1976, Leedham-Green and McKay [5] introduced the following generalized version of the Baer-invariant of a group whit respect to two varieties $\mathcal{V}$ and $\mathcal{W}$. Let G be an arbitrary group in $\mathcal{W}$ with a free presentation $1 \to R \to F \to G \to 1$, in which F is a free group. Clearly, 1=W(G) = W(F)R/R and hence W(F) $\subseteq$ R, therefore,

$$1 \to R / W(F) \to F / W(F) \to G \to 1,$$

is a $\mathcal{W}$-free presentation of the group G, then

$$\mathcal{WV}M(G) = \frac{R / W(F) \cap V(F / W(F))}{\left[R / W(F)V^{*}(F / W(F))\right]} = \frac{W(F)(R \cap V(F))}{W(F)\left[RV^{*}F\right]}$$

is generalized Baer-invariant of the group G in $\mathcal{W}$ with respect to the variety $\mathcal{V}$ (see [6]). Now if N is a normal subgroup of the group G for a suitable normal subgroup S of the free group F, we have N $\cong$ S/R. Then we can define the generalized Baer-invariant of the pair of

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groups with respect to two varieties $\mathcal{V}$ and $\mathcal{W}$ as follows:

$$\mathcal{WV}M(G,N)=\frac{R/W(F)\cap\left[S/W(F)V^*(F/W(F))\right]}{\left[R/W(F)V^*(F/W(F))\right]}$$

$$=\frac{W(F)(R\cap\left[SV^*F\right])}{W(F)\left[RV^*F\right]}.$$

One may check that $\mathcal{WV}M(G, N)$ is always abelian and independent of the free presentation of G. In particular, if $\mathcal{W}$ is the variety of all group, then $\mathcal{WV}M(G,N)=\mathcal{V}M(G,N)$ which is Baer-invariant of the pair of groups (G,N), see [9]. Also, if $\mathcal{V}$ is the variety of abelian group and N be a complement in G, then the Baer-invariant of the pair (G,N) will be

$$\mathcal{V}M(G,N)=\frac{R\cap[S,F]}{[R,F]}=M(G,N),$$

which is the Schur multiplier of a pair of groups; see [7].

The following lemma gives the basic properties of the verbal and marginal subgroups of a group G with respect to the variety $\mathcal{V}$ which is useful in our investigation, so you may see [2].

Lemma 0.1. Let $\mathcal{V}$ be a variety of groups defined by a set of laws V and N be a normal subgroup of a given group G. Then

(i) $G\in\mathcal{V}\Leftrightarrow V(G)=1\Leftrightarrow V^*(G)=G$;

(ii) $V(G/N)=V(G)N/N$ and $V^*(G/N)\supseteq V^*(G)N/N$;

(iii) $N\subseteq V^*(G)\Leftrightarrow\left[NV^*G\right]=1$;

(iv) $V(N)\subseteq\left[NV^*G\right]\subseteq N\cap V(G)$. In particular, $V(G)=\left[GV^*G\right]$;

(v) $V\left(V^*(G)\right)=1$ and $V^*(G/V(G))=G/V(G)$.

Variety $\mathcal{V}$ is called a Schur-Baer variety if for any group G in which the marginal factor group $G/V^*(G)$ is finite, then the verbal subgroup V(G) is also finite. In 2002, Moghaddam et al. [8] proved that for finite group G, $\mathcal{V}M(G)$ is finite with respect to a Schur-Baer variety $\mathcal{V}$. In the following lemma we prove similar result for the $\mathcal{WV}M(G,N)$ and $\mathcal{WV}M(G)$.

Lemma 0.2. Let $\mathcal{V}$ be a Schur-Baer variety and G be a finite group in $\mathcal{W}$ with a normal subgroup N. Then there exists a group H with a normal subgroup K such that

$$\left|\left[NV^*G\right]\right|\left|\mathcal{WV}M(G,N)\right|=\left|\left[KV^*H\right]\right|<\infty.$$

In particular, $|V(G)||\mathcal{WV}M(G)|=|V(H)|<\infty.$

Proof. Let $G=F/R$ be a free presentation for the group G and S be a normal subgroup of the free group F such that $N\cong S/R$. Lemma 0.1 implies that

$$\frac{R}{W(F)\left[RV^*F\right]}\subseteq V^*\left(\frac{F}{W(F)\left[RV^*F\right]}\right).$$

Let $H=F/W(F)\left[RV^*F\right]$ and $K=S/W(F)\left[RV^*F\right]$, then $\left|\frac{H}{V^*(H)}\right|<|G|<\infty$ and $\left|\left[KV^*H\right]\right|\le|V(H)|<\infty$. But

$$\left|\left[KV^*H\right]\right|=\left|\frac{W(F)\left[SV^*F\right]}{W(F)\left[RV^*F\right]}\right|$$

$$=\left|\frac{W(F)\left[SV^*F\right]}{W(F)(R\cap\left[SV^*F\right])}\right|\left|\frac{W(F)(R\cap\left[SV^*F\right])}{W(F)\left[RV^*F\right]}\right|.$$

Also,

$$\left[NV^*G\right]=\frac{\left[SV^*F\right]R}{R}=\frac{W(F)\left[SV^*F\right]R}{R}$$

$$\cong\frac{W(F)\left[SV^*F\right]}{W(F)(R\cap\left[RV^*F\right])}.$$

Thus the result holds.

It is interesting to know the connection between the generalized Baer-invariant of a pair of finite groups (G,N) and its factor groups. Jones [3] gave some inequalities for the Schur multiplier of a finite groups G and its factor group. Moghaddam et al. [10] generalized these inequalities to two varieties of groups. In the next section, we give generalized version of these inequalities for the generalized Baer-invariant of a pair of groups and its factor groups (Theorem 1.2). Finally, a necessary and sufficient condition will be given in which the Baer-invariant of a pair of group may be embedded into the generalized Baer-invariant of a pair of its factor groups (Theorem 2.4).

Results

1. Some Exact Sequences

In the following lemma we present some exact sequences for the Baer-invariant of a pair of groups and its factor groups.

Lemma 1.1. Let G be a group with a free presentation $1\to R\to F\to G\to 1$ and S,T be normal subgroups of the free group F such that $T\subseteq S$, $S/R\cong N$ and $T/R\cong K$. then the following sequences are exact:

(i) $$1\to \frac{R\cap[TV^*F]}{W(F)[RV^*F]\cap[TV^*F]}\to \mathcal{WV}M(G,N)$$

$$\to \mathcal{WV}M(G/K,N/K)\to \frac{K\cap[NV^*G]}{[KV^*G]}\to 1;$$

(ii) $$1\to \mathcal{WV}M(G,K)\to \mathcal{WV}M(G,N)$$

$$\to \mathcal{WV}M(G/K,N/K)$$

$$\to \frac{K}{[KV^*G]}\to \frac{N}{[NV^*G]}\to \frac{N}{[NV^*G]K}\to 1;$$

(iii) Moreover, if K is contained in $V^*(G)$, then the following sequence is exact:

$$1\to \frac{R\cap[SV^*F]}{W(F)[TV^*F]\cap[SV^*F]}\to \mathcal{WV}M(G/K,N/K)$$

$$\to K\to \frac{N}{[NV^*G]}\to \frac{N}{[NV^*G]K}\to 1.$$

Proof. By considering the definition which has been mentioned before, we have:

$$\mathcal{WV}M(G,K)=\frac{W(F)(R\cap[TV^*F])}{W(F)[RV^*F]},$$

$$\mathcal{WV}M(G,N)=\frac{W(F)(R\cap[SV^*F])}{W(F)[RV^*F]}$$

$$\mathcal{WV}M(G/K,N/K)=\frac{W(F)(T\cap[SV^*F])}{W(F)[TV^*F]},$$

$$\frac{K\cap[NV^*G]}{[NV^*G]}=\frac{(T\cap[SV^*F])R}{[TV^*F]R}$$

Now one can easily check that the sequences (i) and (ii) are exact.

(iii) Using the assumption and Lemma 0.1, we have $W(F)[TV^*F]\subseteq R$. Therefore, one can easily check that the following sequence is exact:

$$1\to \frac{R\cap[SV^*F]}{W(F)[TV^*F]\cap[SV^*F]}\to \frac{W(F)(T\cap[SV^*F])}{W(F)[TV^*F]}$$

$$\to T/R\to \frac{S}{[SV^*F]R}\to \frac{S}{[SV^*F]T}\to 1.$$

The extension e: $1\to A\to G\to H\to 1$ is said to be the $\mathcal{V}$-marginal extension of the group A by H with respect to the variety $\mathcal{V}$, if $A\subseteq V^*(G)$. Moreover, if we take $\mathcal{W}$ and $\mathcal{V}$ to be the varieties of all groups and abelian groups, respectively, then from (i) we conclude the following exact sequence, which is [7]

$$1\to \frac{R\cap[T,F]}{[R,F]}\to M(G,N)$$

$$\to M(G/K\ ,\ N/K)\to \frac{K\cap[N,G]}{[K,G]}\to 1.$$

By assuming K to be the central subgroup of G and considering the epimorphism

$$\frac{T}{R}\otimes\frac{F}{RF'}\to\frac{[T,F]}{[R,F]}$$

$$xR\otimes yF'\mapsto [x,y][R,F],$$

one obtains the following exact sequences which are generalizations of those considered by Ganea [1] and Stallings' [13] when N=G.

$$K\otimes G\to M(G,N)\to M(G/K,N/K)$$

$$\to\frac{K\cap[N,G]}{[K,G]}\to 1,$$

$$1\to M(G,K)\to M(G,N)\to M(G/K,N/K)$$

$$\to K\to\frac{N}{[N,G]}\to\frac{N}{[N,G]K}\to 1.$$

Let (G,N) be a pair of finite groups and $\mathcal{V}$ be a Schur-Baer variety then by Lemma 0.2, we have $\mathcal{WV}M(G,N)$ and $\mathcal{WV}M(G)$ as finite groups. Therefore, throughout the rest of this section we always assume that $\mathcal{V}$ is a variety of groups which enjoys Schur-Baer property.

Now using Lemma 1.1, we are able to prove the following theorem of this section which is a generalization of Theorem 2.1 of [3].

Theorem 1.2. Let G be a finite group with a free presentation $1 \to R \to F \to G \to 1$.

Let S and T be normal subgroups of the free group F such that $T \subseteq S$, $S/R \cong N$ and $T/R \cong K$, then

(i) $\left|K \cap \left[NV^*G\right]\right| \left|\mathcal{WV}M(G,N)\right|$

$$= \left|\mathcal{WV}M(G/K,\ N/K)\right| \left|\frac{W(F)\left[TV^*F\right]}{W(F)\left[RV^*F\right]}\right|,$$

(ii) $d\left(\mathcal{WV}M(G,N)\right)$

$$\le d(\mathcal{WV}M(G/K,N/K)) + d\left(\frac{W(F)\left[TV^*F\right]}{W(F)\left[RV^*F\right]}\right),$$

(iii) $e(\mathcal{WV}M(G,N))$

$$\le e(\mathcal{WV}M(G/K,N/K)) + e\left(\frac{W(F)\left[TV^*F\right]}{W(F)\left[RV^*F\right]}\right).$$

Proof. By Lemma 1.1(i), we have

$$\left|\mathcal{WV}M(G,N)\right| = |L| \left|\frac{R \cap \left[TV^*F\right]}{W(F)\left[RV^*F\right] \cap \left[TV^*F\right]}\right|$$

and

$$\frac{\mathcal{WV}M(G/K,N/K)}{L} \cong \frac{K \cap \left[NV^*G\right]}{\left[KV^*G\right]},$$

where

$$L = \operatorname{Im}\left(\mathcal{WV}M(G,N) \to \mathcal{WV}M(G/K,N/K)\right),$$

as in Lemma 1.1 (i). So it is easily seen that

$$\left|K \cap \left[NV^*G\right]\right| \left|\mathcal{WV}M(G,N)\right|$$

$$= \left|K \cap \left[NV^*G\right]\right| |L| \left|\frac{R \cap \left[TV^*F\right]}{W(F)\left[RV^*F\right] \cap \left[TV^*F\right]}\right|$$

$$= \left|\left[KV^*G\right]\right| \left|\mathcal{WV}M(G/K,N/K)\right| \left|\frac{R \cap \left[TV^*F\right]}{W(F)\left[RV^*F\right] \cap \left[TV^*F\right]}\right|.$$

But

$$\left[KV^*G\right] \cong \frac{\left[TV^*F\right]}{R \cap \left[TV^*F\right]}$$

and

$$\frac{\left[TV^*F\right]}{W(F)\left[RV^*F\right] \cap \left[TV^*F\right]} \Big/ \frac{R \cap \left[TV^*F\right]}{W(F)\left[RV^*F\right] \cap \left[TV^*F\right]} \cong \frac{\left[TV^*F\right]}{R \cap \left[TV^*F\right]}.$$

Hence, we get

$$\left|K \cap \left[NV^*G\right]\right| \left|\mathcal{WV}M(G,N)\right|$$

$$= \left|\mathcal{WV}M(G/K,N/K)\right| \left|\frac{\left[TV^*F\right]}{W(F)\left[RV^*F\right] \cap \left[TV^*F\right]}\right|$$

$$= \left|\mathcal{WV}M(G/K,N/K)\right| \left|\frac{W(F)\left[TV^*F\right]}{W(F)\left[RV^*F\right]}\right|,$$

which implies (i) Similarly, we can prove (ii) and (iii).

By Lemma 1.1 and Theorem 1.2, we have the following corollaries.

Corollary 1.3. Let G be a finite group with two normal subgroups K and N such that $K \subseteq N$. Then the following conditions are equivalent:

(i) sequence $1 \to \mathcal{WV}M(G/K,N/K) \to \frac{K}{\left[KV^*G\right]} \to \frac{N}{\left[NV^*G\right]} \to \frac{N}{\left[KV^*G\right]K} \to 1$ is exact;

(ii) $\mathcal{WV}M(G,K) = \mathcal{WV}M(G,N)$;

(iii) $\mathcal{WV}M(G/K,N/K) \cong \frac{K \cap \left[NV^*G\right]}{\left[KV^*G\right]}$.

Proof. By the definition of the generalized Baer-invariant of the pair of groups and Lemma 1.1(i), we have the following exact sequence:

$$1 \to \mathcal{WV}M(G,K) \to \mathcal{WV}M(G,N) \to \mathcal{WV}M(G/K,N/K) \to \frac{K \cap \left[NV^*G\right]}{\left[KV^*G\right]} \to 1.$$

It is easily check that (ii) and (iii) are equivalent. Also, by Lemma 1.1(ii), (i) and (ii) are equivalent.

Corollary 1.4. Let G be a finite group with a free presentation $1 \to R \to F \to G \to 1$. Let S and T be normal subgroups of the free group F such that $T \subseteq S$, $S/R \cong N$ and $T/R \cong K$. If K is contained in $V^*(G)$, then

$$\left|\left[KV^*G\right]\right|\left|\mathcal{WV}M(G/K,N/K)\right| = \left|K \cap \left[NV^*G\right]\right|\left|\frac{W(F)\left[SV^*F\right]}{W(F)\left[TV^*F\right]}\right|.$$

2. Subgroup $(WV^*)^*(G)$

In this section, a necessary and sufficient condition will be given in which the generalized Baer-invariant of a pair of groups may be embedded into the generalized Baer-invariant of a pair of its factor groups with respect to two varieties of groups.

Let $\mathcal{V}$ and $\mathcal{W}$ be two varieties of groups defined by sets of laws V and W, respectively. Let E be an arbitrary group and G a group in $\mathcal{W}$. Let $\psi : E \to G$ be an epimorphism such that $Kre\psi \subseteq V^*(E)$. We denoted by $(WV^*)^*(G)$ the intersection of all subgroups of the from $\psi(V^*(E))$. Clearly, $(WV^*)^*(G)$ is a characteristic subgroup of G and contained in $V^*(G)$. In particular, if $\mathcal{W}$ is the variety of all groups and $\mathcal{V}$ is the variety of abelian groups then this subgroup is denoted by $Z^*(G)$ as in $[4]$. The following lemma whose proof is straightforward plays an essential role in proving the main theorem of this section.

Lemma 2.1. Let G be a group in the variety $\mathcal{W}$ with a free presentation $1 \to R \to F \overset{\pi}{\to} G \to 1$ and $1 \to A \to B \to G \to 1$ be a $\mathcal{V}$-marginal extension of A by G. Then there exists a homomorphism $\beta : \dfrac{F}{W(F)\left[RV^*F\right]} \to B$ such that the following diagram is commutative:

$$\begin{array}{ccccccc} 1 \to & \dfrac{R}{W(F)\left[RV^*F\right]} & \to & \dfrac{F}{W(F)\left[RV^*F\right]} & \overset{\bar{\pi}}{\to} & G \to 1 \\ & \beta_1 \downarrow & & \beta \downarrow & & 1_G \downarrow \\ 1 \to & A & \to & B & \to & G \to 1, \end{array}$$

where β_1 is the restriction of β and $\bar{\pi}$ is the induced homomorphism of π.

We keep the notations of the above lemma in the rest of this section. We also denote the factor group $F/\ W(F)\left[RV^*F\right]$ by $\bar{F}$.

Lemma 2.2. Let $\mathcal{V}$ and $\mathcal{W}$ be two varieties of groups and G be a group in the variety $\mathcal{W}$. For any free presentation $1 \to R \to F \overset{\pi}{\to} G \to 1$, we have $(WV^*)^*(G) = \bar{\pi}(V^*(\bar{F}))$.

Proof. Let $1 \to A \to E \overset{\varphi}{\to} G \to 1$ be a $\mathcal{V}$-marginal extension. By Lemma 2.1, there exists a homomorphism $\beta : \bar{F} \to E$ such that the corresponding diagram with the above $\mathcal{V}$-marginal extension in Lemma 2.1 is commutative. It is easy to check that $E = A\beta(\bar{F})$. Assume that $\bar{f} \in V^*(\bar{F})$ and $v = v(x_1, x_2, \ldots, x_n) \in V$. If $e_1, e_2, \ldots, e_n \in E$, then for each $i (1 \le i \le n)$, there exist elements $a_1, a_2, \ldots, a_n$ in A and $\bar{f_1}, \bar{f_2}, \ldots, \bar{f_n}$ in $\bar{F}$ such that $e_i = a_i\beta(\bar{f_i})$. So

$$\begin{aligned} & v\left(e_1, \ldots, e_i\beta\left(\bar{f}\right), \ldots, e_n\right) \\ & = v(a_1\beta\left(\bar{f_1}\right), \ldots, a_i\beta\left(\bar{f_i}\bar{f}\right), \ldots, a_n\beta(\bar{f_n})) \\ & = v\left(\beta\left(\bar{f_1}\right), \ldots, \beta\left(\bar{f_i}\bar{f}\right), \ldots, \beta\left(\bar{f_n}\right)\right) \\ & = \beta\left(v\left(\bar{f_1}, \ldots, \bar{f_n}\right)\right) = v\left(\beta\left(\bar{f_1}\right), \ldots, \beta\left(\bar{f_n}\right)\right) \\ & = v(e_1, \ldots, e_n). \end{aligned}$$

Therefore, $\beta(V^*(\bar{F})) \subseteq V^*(E)$. Now, one can deduce that $\bar{\pi}\left(V^*\left(\bar{F}\right)\right) = \varphi(\beta(V^*\left(\bar{F}\right))) \subseteq \varphi(V^*\left(E\right))$. Hence $\left(WV^*\right)^*\left(G\right) = \bar{\pi}\left(V^*\left(\bar{F}\right)\right)$.

Now, we note to the property of the natural map $\mathcal{WV}M\left(G,N\right) \to \mathcal{WV}M(G/K,N/K)$ as in Lemma 1.1. The following theorem generalizes Theorem 5.1 of [12].

Theorem 2.3. Let $\mathcal{V}$ and $\mathcal{W}$ be two varieties of groups and G be a finite group in the variety $\mathcal{W}$. Let N and K be normal subgroups of G such that $K \subseteq N$ and $K \subseteq V^*\left(G\right)$. If

$$\mathcal{WV}M\left(G,N\right) \cong \mathcal{WV}M(G/K,N/K)/(K \cap \left[NV^*G\right]),$$

then the natural map $\mathcal{WV}M\left(G,N\right) \to \mathcal{WV}M(G/K,N/K)$ is monomorphism.

Now we are able to prove the following theorem of this section which generalizes Theorem 3.2 of [10].

Theorem 2.4. Let $\mathcal{V}$ and $\mathcal{W}$ be two varieties of groups and G be a group in the variety $\mathcal{W}$. Let N and K be normal subgroups of G such that $K \subseteq N \cap V^*(G)$. Then $K \subseteq N \cap (WV^*)^*(G)$ if and only if the natural map $\mathcal{WV}M(G,N) \to \mathcal{WV}M(G/K,N/K)$ is monomorphism.

Proof. Let F/R $\cong$ G be a free presentation of G, and $K \cong T/R$ for a suitable normal subgroup T of F. By construction, the kernel of the natural map $\mathcal{WV}M(G,N) \to \mathcal{WV}M(G/K,N/K)$ is equal to $W(F)[TV^*F]/W(F)[RV^*F]$. Therefore, we only need to verify that $W(F)[TV^*F] = W(F)[RV^*F]$ if and only if $K \subseteq N \cap (WV^*)^*(G)$. Set $\bar{R} = R/W(F)[RV^*F]$ and $\bar{T} = T/W(F)[RV^*F]$. Then $W(F)[TV^*F] = W(F)[RV^*F]$ if and only if $\bar{T} \subseteq V^*(\bar{F})$. Also, by Lemma 2.2, $(WV^*)^*(G) = \bar{\pi}(V^*(\bar{F}))$. Consequently, we obtain that $\bar{\pi}(\bar{T}) \subseteq (WV^*)^*(G)$ if and only if $\bar{T} \subseteq V^*(\bar{F})$. Now the result follows since $\bar{\pi}(\bar{T}) = K$.

The following two corollaries follow from Theorem 2.4 and Lemma 1.1(i).

Corollary 2.5. let G be a finite group in the variety $\mathcal{W}$ with two normal subgroups K and N. Then $K \subseteq N \cap (WV^*)^*(G)$ if and only if

$$\left|K \cap [NV^*G]\right| \left|\mathcal{WV}M(G,N)\right| = \left|\mathcal{WV}M(G/K,N/K)\right|.$$

Corollary 2.6. Let G be a finite group in the variety $\mathcal{W}$ with a normal subgroup N such that $V^*(G) \subseteq N$. Then $(WV^*)^*(G)$ is trivial if and only if the natural map $\mathcal{WV}M(G,N) \to \mathcal{WV}M(G/x,N/x)$ has a non-trivial kernel for all non-zero elements x in $V^*(G)$.

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