# Evaluations of Evidence Combination Rules in Terms of Statistical Sensitivity and Divergence

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***Abstract*—The theory of belief functions is one of the most important tools in information fusion and uncertainty reasoning. Dempster's rule of combination and its related modified versions are used to combine independent pieces of evidence. However, until now there is still no solid evaluation criteria and methods for these combination rules. In this paper, we look on the evidence combination as a procedure of estimation and then we propose a set of criteria to evaluate the sensitivity and divergence of different combination rules by using for reference the mean square error (MSE), the bias and the variance. Numerical examples and simulations are used to illustrate our proposed evaluation criteria. Related analyses are also provided.**

***Keywords*—belief functions; evidence combination; evaluation; sensitivity; divergence.**

## I. INTRODUCTION

The theory of belief functions, also called Dempster-Shafer evidence Theory(DST) [1], is one of the most important theories and methods in information fusion and uncertainty reasoning. It can distinguish 'unknown' and 'imprecision' and propose a way to fuse or combine different pieces of evidence by using the commutative and associative Dempster's rule of combination.

Dempster's rule of combination can bring counter-intuitive combination results in some cases [2], [3], so there have emerged several improved and modified alternative evidence combination rules, where counter-intuitive behaviors are imputed to the combination rule itself, especially the way to deal with the conflicting mass assignments. The representative works include Yager's rule [4], Florea's robust combination rule (RCR) [5], disjunctive rule [6], Dubois and Prade's rule [7], proportional conflict redistribution rule (PCR) [8], and mean rule [9], etc.

As aforementioned, several combination rules are available including Dempster's rule and its alternatives. Then, how to evaluate them? This is crucial for the practical use of the combination rules. The qualitative criterion is that the combination results should be intuitive and rational [10]. Up to now, there is still no solid performance evaluation approaches for combination rules, especially for establishing quantitative criteria. In this paper, we propose to interprept the evidence combination as a procedure of estimation [11]; therefore, a combination rule is regarded as an estimator. So, we define some statistical criteria on sensitivity and divergence for the different combination rules by using for reference the idea of Mean Square Error (MSE) and its decomposition in estimation. By adding small errors to the original pieces of evidence (i.e., the "input" of the "estimator"), we check the mean square error, the variance, and the bias of the combination result ("output" of the estimator) caused by adding some noise to describe the sensitivity and divergence of the given combination rule. Distance of evidence [12] is used in our work to define the variance, the bias and other related criteria. Simulation results are provided to illustrate our proposed evaluation approaches. Dempster's rule and major available alternative rules are evaluated and analyzed using the new evaluation approaches.

## II. BASICS OF DST

Dempster-Shafer evidence theory (DST) [1] has been developed by Shafer in 1976 based on previous works of Dempster. In evidence theory, the elements in frame of discernment (FOD) $\Theta$ are mutually exclusive and exhaustive. Define $m : 2^{\Theta} \to [0,1]$ as a basic belief assignment (BBA, also called mass function) satisfying:

$$\sum_{A \in 2^{\Theta}} m(A) = 1, \quad m(\emptyset) = 0 \tag{1}$$

If $m(A) > 0$, $A$ is called a focal element. In DST, two reliable independent bodies of evidence (BOEs) $m_1(\cdot)$ and $m_2(\cdot)$ are combined using Dempster's rule of combination as follows. $\forall A \in 2^{\Theta}$ :

$$m(A) = \begin{cases} 0, A = \emptyset \\ \frac{\sum_{A_i \cap B_j = A} m_1(A_i) m_2(B_j)}{1-K}, A \neq \emptyset \end{cases} \tag{2}$$

where

$$K = \sum\nolimits_{A_i \cap B_j = \emptyset} m_1(A_i) m_2(B_j) \tag{3}$$

represents the total conflicting or contradictory mass assignments. Obviously, from Eq. (2), it can be verified that Dempster's rule is both commutative and associative. For Dempster's rule of combination, the conflicting mass assignments are discarded through a classical normalization step.

As firstly pointed out by Zadeh [2], Dempster's rule has been criticized for its counter-intuitive behaviors[1]. DST's validity has also been argued [3]. There have emerged several alternatives of evidence combination rules aiming to suppress the counter-intuitive behaviors of classical Dempster's rule. See [8] for details.

To measure the dissimilarity between different BBAs, the distance of evidence can be used. Jousselme's distance [13] is one of the most commonly used distance of evidence, which is defined as

$$d_J(m_1, m_2) = \sqrt{\frac{1}{2} \cdot (m_1 - m_2)^T \mathbf{Jac}\ (m_1 - m_2)} \tag{4}$$

where the element $J_{ij} \triangleq \mathbf{Jac}(A_i, B_j)$ of Jaccard's weighting matrix $\mathbf{Jac}$ is defined as

$$\mathbf{Jac}(A_i, B_j) = \frac{|A_i \cap B_j|}{|A_i \cup B_j|} \tag{5}$$

There are also other types of distance of evidence [12], [14]. We choose to use Jousselme's distance of evidence in this paper, because it has been proved to be a strict distance metric [15].

## III. Some major alternative combination rules

In this section, some major combination rules in evidence theory other than Dempster's rule are briefly introduced. For all $A \in 2^\Theta$

*1) Yager's rule [4]:*

$$\begin{cases} m(\emptyset) = 0 \\ m_{Yager}(A) = \sum\limits_{B_i \cap C_j = A \neq \emptyset} m_1(B_i) m_2(C_j) \\ m(\Theta) = m_1(\Theta) m_2(\Theta) + \sum\limits_{B \cap C = \emptyset} m_1(B_i) m_2(C_j) \end{cases} \tag{6}$$

In Yager's rule, the conflict mass assignments are assigned to the total set of the FOD $\Theta$.

*2) Disjunctive rule [6]:*

$$\begin{cases} m(\emptyset) = 0 \\ m_{Dis}(A) = \sum\limits_{B_i \cup C_j = A} m_1(B_i) m_2(C_j) \end{cases} \tag{7}$$

This rule reflects the disjunctive consensus.

*3) Dubois & Prade's rule (D&P rule) [7]:*

$$\begin{cases} m(\emptyset) = 0 \\ \quad m_{DP}(A) = \sum\limits_{B_i \cap C_j = A \neq \emptyset} m_1(B_i) m_2(C_j) \\ \qquad + \sum\limits_{B_i \cap C_j = \emptyset, B_i \cup C_j = A} m_1(B_i) m_2(C_j) \end{cases} \tag{8}$$

This rule admits that the two sources are reliable when they are not in conflict, but only one of them is right when a conflict occurs.

[1] According to the viewpoint of proponents of Dempster's rule, the counter-intuitive behavior is imputed to the sensors, the data or the BOEs obtained from different sources, but not to Dempster's rule itself.

*4) Robust Combination Rule (RCR, or Florea's rule) [5]:*

$$m_{RCR}(A) = \alpha(K) m_{Dis}(A) + \beta(K) m_{Conj}(A) \tag{9}$$

where $m_{Dis}$ is the BBA obtained using the disjunctive rule, $m_{Conj}$ is the BBA obtained using the conjunctive rule, and $\alpha(K), \beta(K)$ are the weights, which should satisfy

$$\alpha(K) + (1 - K)\beta(K) = 1 \tag{10}$$

where $K$ is the conflict coefficient defined in Eq. (3). Robust combination rule can be considered as a weighted summation of the BBAs obtained using the disjunctive rule and the conjunctive rule, respectively.

*5) PCR5 [8]:* Proportional Conflict Redistribution rule 5 (PCR5) redistributes the partial conflicting mass to the elements involved. in the partial conflict, considering the canonical form of the partial conflict. PCR5 is the most mathematically exact redistribution of conflicting mass to non-empty sets following the logic of the conjunctive rule.

$m_{PCR5}(\emptyset) = 0$

and $\forall X \in 2^\Theta \setminus \{\emptyset\}$

$$m_{PCR5}(A) = \sum_{\substack{X_1, X_2 \in 2^\Theta \\ X_1 \cap X_2 = X}} m_1(X_1) m_2(X_2) + \sum_{\substack{X_2 \in 2^\Theta \\ X_2 \cap X = \emptyset}} [\frac{m_1(X)^2 m_2(X_2)}{m_1(X) + m_2(X_2)} + \frac{m_2(X)^2 m_1(X_2)}{m_2(X) + m_1(X_2)}] \tag{11}$$

In fact there exists another rule PCR6 that coincides with PCR5 when combining two sources, but differs from PCR5 when combining more than two sources altogether and PCR6 is considered more efficient than PCR5 because it is compatible with classical frequentist probability estimate [16].

*6) Mean rule [9]:*

$$m_{mean}(A) = \frac{1}{n} \sum\nolimits_{i=1}^{n} m_i(A) \tag{12}$$

By using this rule, we can find the average of the BBAs to be combined.

For the purpose of the practical use of different combination rules, the evaluation criteria are required. In the next section, the available evaluation criteria or properties of evidence combination rules are beriefly introduced.

## IV. Properties of combination rules as qualitative criteria

*1) Commutativity [17]:* The combination of two BBAs $m_1$ and $m_2$ using some rule $R$ does not depend on the order of the two BBA, i.e.,

$$R(m_1, m_2) = R(m_2, m_1) \tag{13}$$

All the combination rules aforementioned in Section III are commutative.

*2) Associativity [17]:* The combination result of multiple BBAs does not depend on the order of the BBAs to be combined. For example, when there are 3 BBAs,

$$R(R(m_1, m_2), m_3) = R(m_1, R(m_2, m_3)) \tag{14}$$

Dempster's rule and disjunctive rule are associative. The other rules introduced in Section III are not associative. The property of associativity is important to facilitate the implementation of the distributed information fusion system. But it should be noted that it is not necessarily efficient in term of quality of fusion result. Non-associative rules are able to provide better performances in general than associative rules [16].

*3) Neutral impact of the vacuous belief [17]:* The combination rule preserves the neutral impact of the vacuous BBA, i.e., when $m_2$ is $m(\Theta) = 1$,

$$R(m_1, m_2) = m_1 \tag{15}$$

All the rules aforementioned in Section III but the mean rule, satisfy this property.

These criteria are qualitative and they correspond to good (interesting) properties that a rule could satisfy. It should be noted that these "expected good" properties do not warrant that a real efficient fusion rule must absolutely satisfy them. Therefore, these properties are not enough to the evaluations of combination rules. In this paper, we propose some quantitative evaluation criteria for combination rules.

## V. Statistical Sensitivity and Divergence of Combination Rules

Here, we develop a group of criteria for combination rules in terms of sensitivity and divergence. The idea of Mean Square Error (MSE) and its decomposition are used as a basic framework for such a development.

### A. Mean Square Error and its decomposition

For an estimate $\hat{x}$ of the scalar estimand $x$, the MSE is defined as

$$\mathrm{MSE}(\hat{x}) = E[(\hat{x} - x)^2] \tag{16}$$

MSE can be decomposed as

$$\begin{aligned} \mathrm{MSE}(\hat{x}) &= E[(\hat{x} - E(\hat{x}))^2] + E[(E(\hat{x}) - x)^2] \\ &= \mathrm{Var}(\hat{x}) + (\mathrm{Bias}(\hat{x}, x))^2 \end{aligned} \tag{17}$$

The MSE is equal to the sum of the variance and the squared bias of the estimator or of the estimations. The variance can represent the divergence of the estimation results. The bias can represent the sensitivity of the estimator.

### B. Criteria for statistical sensitivity and divergence

If we consider the procedure of evidence combination with a given rule as an estimator (as illustrated in Fig. 1), then we can consider the combination results as the estimations.

So, we can use for reference the MSE and its decompositions to measure the error, the variance, and the bias of the combination results based on the given combination rule. Here we attempt to design some criteria related to the sensitivity and divergence of combination rules. We use the change of the combination results after adding small noise to the original BBA to reflect the sensitivity and divergence of a combination rule. If under a given small noise, a combination rule bring out smaller variance and smaller bias, then such a rule is less divergent and less sensitive, based on which, the sensitivity and divergence of combination rules can be evaluated. The definitions of MSE, variance and bias for combination rules, and the evaluation procedure are as follows.

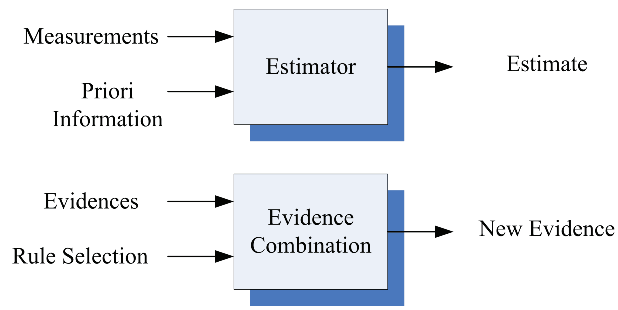


Fig. 1. Evidence combination and Estimation.

**Step 1:** Randomly generate a BBA $m$. Add random noise to $m$ for $N$ times, respectively. In each time, the noise is $\epsilon_i$ (small values), where $i = 1, ..., N$. The noise sequence is denoted by $\boldsymbol{\epsilon} = [\epsilon_1, \epsilon_2, ..., \epsilon_N]$. Here each $\epsilon_i$ is a small real number (negative or positive) close to zero. Then, we can obtain a sequence of noised BBAs as

$$\mathbf{m}' = [m_1, m_2, ..., m_N] \tag{18}$$

It should be noted that all the noised BBAs are normalized.

**Step 2:** Generate original combination results sequence with a combination rule $R$

$$\begin{aligned} \mathbf{m}_c &= [m_c^1, m_c^2, ..., m_c^N] \\ &= [R(m, m), R(m, m), ..., R(m, m)] \end{aligned} \tag{19}$$

The length of $\mathbf{m}_c$ is $N$.

**Step 3:** Generate combination results sequence by combining BBAs with noise and the original BBAs using the rule of $R$

$$\begin{aligned} \mathbf{m}_{cn} &= [m_{cn}^1, m_{cn}^2, ..., m_{cn}^N] \\ &= [R(m_1, m), R(m_2, m), ..., R(m_N, m)] \end{aligned} \tag{20}$$

**Step 4:** Calculate the MSE of $\mathbf{m}_{cn}$ as

$$\begin{aligned} \mathrm{MSE_{BBA}}(\mathbf{m}_{cn}) &= \frac{1}{N} \sum_{i=1}^{N} \left[ d_J(m_c^i, m_{cn}^i) \right]^2 \\ &= \frac{1}{N} \sum_{i=1}^{N} \left[ d_J(R(m, m), R(m, m_i)) \right]^2 \end{aligned} \tag{21}$$

where $d_J$ is Jousselme's distance defined in Eq. (3). $\mathrm{MSE_{BBA}}$ represents the error between the original combination results and the results obtained using BBAs with noise.

We can also calculate the relative MSE by removing the effect of the noise amplitude as follows

$$\mathrm{MSE}'_{\mathrm{BBA}}(\mathbf{m}_{cn}) = \frac{\mathrm{MSE_{BBA}}(\mathbf{m}_{cn})}{\|\boldsymbol{\epsilon}\|^2} \tag{22}$$

**Step 5:** Calculate the variance of $\mathbf{m}_{cn}$ as

$$\begin{aligned}\mathrm{Var}_{\mathrm{BBA}}(\mathbf{m}_{cn}) &= \tfrac{1}{N}\sum_{i=1}^{N}\left[d_J(m_{cn}^i,\bar{m}_{cn})\right]^2 \\ &= \tfrac{1}{N}\sum_{i=1}^{N}\left[d_J(R(m,m_i),\tfrac{1}{N}\sum_{j=1}^{N}R(m,m_j))\right]^2\end{aligned} \quad (23)$$

where $\bar{m}_{cn} = \frac{1}{N}\sum_{j=1}^{N} m_{cn}^j = \frac{1}{N}\sum_{j=1}^{N} R(m,m_j)$. $\mathrm{Var}_{\mathrm{BBA}}$ represents the fluctuations of the combination results obtained using BBAs with noise.

Then, calculate the relative variance by removing the effect of the noise amplitude as follows

$$\mathrm{Var}'_{\mathrm{BBA}}(\mathbf{m}_{cn}) = \frac{\mathrm{Var}_{\mathrm{BBA}}(\mathbf{m}_{cn})}{Var(\boldsymbol{\epsilon})} \quad (24)$$

Relative variance in fact represents the degree of amplification or reduction of the variances between and after combination.

**Step 6:** Calculate the bias of $\mathbf{m}_{cn}$ as

$$\begin{aligned}\mathrm{Bias}_{\mathrm{BBA}}(\mathbf{m}_{cn}) &= \sqrt{\tfrac{1}{N}\sum_{i=1}^{N}\left[d_J(m_c^i,\bar{m}_{cn})\right]^2} \\ &= \sqrt{\tfrac{1}{N}\sum_{i=1}^{N}\left[d_J(R(m,m),\tfrac{1}{N}\sum_{j=1}^{N}R(m,m_j))\right]^2} \\ &= d_J(R(m,m),\tfrac{1}{N}\sum_{j=1}^{N}R(m,m_j))\end{aligned} \quad (25)$$

$\mathrm{Bias}_{\mathrm{BBA}}$ represents the difference between the expectation of the combination results obtained using BBAs with noise and the original combination results.

Then, calculate the relative bias by removing the effect of the noise amplitude as follows

$$\mathrm{Bias}'_{\mathrm{BBA}}(\mathbf{m}_{cn}) = \frac{\mathrm{Bias}_{\mathrm{BBA}}(\mathbf{m}_{cn})}{\|\boldsymbol{\epsilon}\|} \quad (26)$$

Regenerate randomly a new original BBA $m$ for $M$ times. In each time, re-do Step 1 to Step 6. Based on the $M$ groups of results, calculate the averaged $\mathrm{MSE}'_{\mathrm{BBA}}$, the averaged $\mathrm{Var}'_{\mathrm{BBA}}$, and the averaged $\mathrm{Bias}'_{\mathrm{BBA}}$. These three indices are called the statistical MSE, the statistical variance, and the statistical bias of the combination rule $R$. We jointly use these indices (quantitative criteria) to describe the statistical sensitivity and divergence of a given combination rule $R$.

Relative MSE is a comprehensive index. Larger relative MSE intuitively means larger sensitivity. However, relative MSE is insufficient to evaluate a combination rule. So we should further use its decomposition (including the relative variance and the relative bias) for a deeper analysis.

High relative bias values represent high sensitivity. it represents high degree of departure from the origin. It can reflect a given combination rule's capability of sensitive response to the changes in input evidences. It represents the "agility" of a combination rule. Moderate relative bias values are preferred, which means the balance or trade-off between the robustness and the sensitivity.

Relative variance in fact represents the degree of amplification or reduction of the variances between and after combination. In the evaluation procedure, for all the combination rule, the variance of the noise are the same (using the same noise sequence for different rules). So, high relative variance values also represent high divergence among all the combination results using a given combination rule when adding noise. Small relative variance values are preferred, which represent the high cohesion of a given combination rule.

In this work, we propose a statistical evaluation approach for evidence combination rules based on Monte-Carlo simulation. To implement the statistical evaluation of a combination rule according to the method introduced here, two problems should be resolved at first. One is the way of adding noise and the other is the way of random generation of BBA.

### C. Method I for adding noise

Method I for adding noise is designed to evaluate the effect of the slight value change of the mass of the existing focal element. Suppose that $m$ is a BBA defined on FOD $\Theta$. First, we find the primary focal element (the focal element having the highest mass assignment) [2], i.e., the focal element $A_i$ satisfying

$$i = \arg\max_{j, A_j \subseteq \Theta} m(A_j) \quad (27)$$

Second, add the noise $\epsilon$ to the mass assignment of the primary focal element.

$$m'(A_i) = m(A_i)\cdot(1+\epsilon) \quad (28)$$

Then, for the mass assignments of other focal elements in original BBA,

$$m'(A_j) = m(A_j) - \frac{m(A_j)}{1-m(A_i)}\cdot\epsilon\cdot m(A_i), \forall j \neq i \quad (29)$$

$m'$ is the generated BBA with noise. It is easy to verify that

$$\sum\nolimits_{B\subseteq\Theta} m'(B) = 1 \quad (30)$$

It can be seen that the change of mass assignment for the primary focal element is the most significant when compared with those of other focal elements. The change of the mass assignment for primary focal element is redistributed to all the other focal elements according to the ratio among their corresponding mass assignments. BBAs are generated according to Algorithm 1 below [12].

For method I for adding noise, some restrictions should be adopted for the values of original BBA and the noise added to make sure that the noised BBA $m'$ satisfies the defintion of BBA. The restriction are as shown in Eq. (31).

$$0 \leq (1+\epsilon)\cdot\max_{A}(m(A)) \leq 1, \forall A \in 2^{\Theta} \quad (31)$$

[2] For example, when $\Theta = \{\theta_1,\theta_2\}$ and $m(\{\theta_1\}) = 0.8, m(\{\theta_2\}) = 0.1, m(\Theta) = 0.1$, the primary focal element is $\{\theta_1\}$. When $\Theta = \{\theta_1,\theta_2\}$ and $m(\{\theta_1\}) = 0.45, m(\{\theta_2\}) = 0.45, m(\Theta) = 0.1$, the primary focal elements are $\{\theta_1\}$ and $\{\theta_2\}$.

**Algorithm 1**. Random generation of BBA

**Input**: $\Theta$: Frame of discernment;
$N_{max}$: Maximum number of focal elements
**Output**: **Output**: $m$: BBA
Generate $\mathcal{P}(\Theta)$, which is the power set of $\Theta$;
Generate a random permutation of $\mathcal{P}(\Theta) \rightarrow \mathcal{R}(\Theta)$;
Generate an integer between 1 and $N_{max} \rightarrow l$;
**FOReach** First $k$ elements of $\mathcal{R}(\Theta)$ do
Generate a value within $[0, 1] \rightarrow m_i$, $i = 1, ..., l$;
**END**
Normalize the vector $\mathbf{m} = [m_1, ..., m_l] \rightarrow \mathbf{m}'$;
$m(A_i) = m'_i$;

where $m$ is the original BBA.

### D. Method II for adding noise

Method II for adding noise is designed to evaluate the effect of creating new focal elements. Suppose that $m$ is a BBA with a special structure defined on FOD $\Theta$. The focal elements are some singletons $\{\theta_i\}$ and the total set $\Theta$. First, find out a pair of singletons $\{\theta_i\}$ and $\{\theta_j\}$.

Second, create a new focal element $\{\theta_i, \theta_j\}$ with the mass value of $\epsilon$, i.e., $m'(\{\theta_i, \theta_j\}) = \epsilon$.

Then, the mass values for focal elements $\{\theta_i\}$ and $\{\theta_j\}$ are regenerated as

$$\begin{cases} m'(\{\theta_i\}) = m(\{\theta_i\}) - \epsilon \cdot \frac{m(\{\theta_i\})}{m(\{\theta_i\})+m(\{\theta_j\})} \\ m'(\{\theta_j\}) = m(\{\theta_j\}) - \epsilon \cdot \frac{m(\{\theta_j\})}{m(\{\theta_i\})+m(\{\theta_j\})} \end{cases} \quad (32)$$

Obviously, one has $\sum_{B\subseteq\Theta} m'(B) = 1$.

The BBAs with special structure (with only some singletons and the total set focal elements) are generated according to Algorithm 2 below:

**Algorithm 2**. Random generation of BBA

**Input**: $\Theta$: Frame of discernment;
$n$: Cardinality of $\Theta$;
$N_{max}$: Maximum number of focal elements
**Output**: $m$: BBA
Generate $\mathcal{P}(\Theta)$, which is the power set of $\Theta$;
Generate a random permutation of $\mathcal{P}(\Theta) \rightarrow \mathcal{R}(\Theta)$;
**FOR** $i = 1 : N_{max} - 1$
Generate an integers $j$ between 1 and $n$;
Generate a focal element $F_i : \{\theta_j\}$;
**END**
Generate a focal element $F_{N_{max}} : \Theta$.
**FOR** $i = 1 : N_{max}$
Generate a value within $[0, 1] \rightarrow m_i$;
**END**
Normalize the vector $m = [m_1, ..., m_{N_{max}}] \rightarrow m'$;
$m(F_i) = m'_i$;

For method II for adding noise, some restrictions should be adopted for the values of original BBA and the noise added to make sure that the noised BBA $m'$ satisfies the defintion of BBA. According to Eq. (32), the restriction can be obtained as shown in Eq. (33). For all the available singleton focal element $\{\theta_i\}$ in original BBA,

$$0 \leq m(\{\theta_i\}) \cdot (1 - \frac{\epsilon}{\sum_{j, m(\{\theta_j\})>0} m(\{\theta_j\})}) \leq 1, \forall A \in 2^\Theta \quad (33)$$

where $m$ is the original BBA.

### E. A simple illustrative example

Here an illustrative example of single cycle calculation of the evaluation indices is provided by using Method I for adding noise. By referring to this illustrative example, evaluations by using Method II are easy to implement.

A BBA $m$ defined on the FOD $\Theta = \{\theta_1, \theta_2, \theta_3\}$ is $m(\{\theta_1\}) = 0.6, m(\{\theta_2\}) = 0.3, m(\{\theta_1, \theta_2, \theta_3\}) = 0.1$.

Suppose that the noise sequence is $\boldsymbol{\epsilon} = [-0.1, -0.05, -0.02, 0.02, 0.05, 0.1]$.

It can be seen that the restrictions in Eq. (31) are not violated.

According to the Step 1, we generate the sequence six noised BBA $\mathbf{m}' = [m_1, m_2, m_3, m_4, m_5, m_6]$ as follows:

$$\begin{aligned} &m_1(\{\theta_1\}) = 0.5, m_1(\{\theta_2\}) = 0.375, m_1(\{\theta_1, \theta_2, \theta_3\}) = 0.125; \\ &m_2(\{\theta_1\}) = 0.55, m_2(\{\theta_2\}) = 0.3375, m_2(\{\theta_1, \theta_2, \theta_3\}) = 0.1125; \\ &m_3(\{\theta_1\}) = 0.58, m_3(\{\theta_2\}) = 0.315, m_3(\{\theta_1, \theta_2, \theta_3\}) = 0.105; \\ &m_4(\{\theta_1\}) = 0.62, m_4(\{\theta_2\}) = 0.285, m_4(\{\theta_1, \theta_2, \theta_3\}) = 0.095; \\ &m_5(\{\theta_1\}) = 0.65, m_5(\{\theta_2\}) = 0.2625, m_5(\{\theta_1, \theta_2, \theta_3\}) = 0.0875; \\ &m_6(\{\theta_1\}) = 0.7, m_6(\{\theta_2\}) = 0.225, m_6(\{\theta_1, \theta_2, \theta_3\}) = 0.075. \end{aligned}$$

Here we use Dempster's rule of combination. Then, according to the Step 2, the original combination sequence $\mathbf{m}_c = [m_c^1, m_c^2, ..., m_c^6]$ is as

$$\begin{aligned} &\forall i = 1, ..., 6. \\ &m_c^i(\{\theta_1\}) = 0.75, m_c^i(\{\theta_2\}) = 0.2344, m_c^i(\{\theta_1, \theta_2\, \theta_3\}) = 0.0156. \end{aligned}$$

Then according to the Step 3, the sequence of combination results by combining BBAs with noise and the original BBAs $\mathbf{m}_{cn} = [m_{cn}^1, m_{cn}^2, ..., m_{cn}^6]$ is as

$$\begin{aligned} &m_{cn}^1(\{\theta_1\}) = 0.6627, m_{cn}^1(\{\theta_2\}) = 0.3162, \\ &m_{cn}^1(\{\theta_1, \theta_2, \theta_3\}) = 0.0211; \\ &m_{cn}^2(\{\theta_1\}) = 0.7076, m_{cn}^2(\{\theta_2\}) = 0.2741, \\ &m_{cn}^2(\{\theta_1, \theta_2, \theta_3\}) = 0.0183; \\ &m_{cn}^3(\{\theta_1\}) = 0.7334, m_{cn}^3(\{\theta_2\}) = 0.2500, \\ &m_{cn}^3(\{\theta_1, \theta_2, \theta_3\}) = 0.0167; \\ &m_{cn}^4(\{\theta_1\}) = 0.7662, m_{cn}^4(\{\theta_2\}) = 0.2192, \\ &m_{cn}^4(\{\theta_1, \theta_2, \theta_3\}) = 0.0146; \\ &m_{cn}^5(\{\theta_1\}) = 0.7895, m_{cn}^5(\{\theta_2\}) = 0.1973, \\ &m_{cn}^5(\{\theta_1, \theta_2, \theta_3\}) = 0.0132; \\ &m_{cn}^6(\{\theta_1\}) = 0.8257, m_{cn}^6(\{\theta_2\}) = 0.1634, \\ &m_{cn}^6(\{\theta_1, \theta_2, \theta_3\}) = 0.0109. \end{aligned}$$

For the noise sequence,

$$\|\boldsymbol{\epsilon}\|^2 = 0.0258$$

$$Var(\boldsymbol{\epsilon}) = 0.0043$$

According to the Step 4, the value of MSE is

$$\text{MSE}_{\text{BBA}}(\mathbf{m}_{cn}) = 0.00270$$

$$\text{MSE}'_{\text{BBA}}(\mathbf{m}_{cn}) = 0.00270/0.0258 = 0.104752$$

According to the Step 5, the value of variance is

$$\mathrm{Var}_{\mathrm{BBA}}(\mathbf{m}_{cn}) = 0.002697$$

$$\mathrm{Var}'_{\mathrm{BBA}}(\mathbf{m}_{cn}) = 0.002697/0.0043 = 0.6272$$

In the final, according to the Step 5, the value of bias is

$$\mathrm{Bias}_{\mathrm{BBA}}(\mathbf{m}_{cn}) = 0.002406$$

$$\mathrm{Bias}'_{\mathrm{BBA}}(\mathbf{m}_{cn}) = 0.002406/\sqrt{0.0258} = 0.014978$$

The above is the illustration of one-cycle procedure. One can use other combination rules to do these steps. Randomly generate BBAs and repeat all the steps, then we can obtain the final statistical evaluation results.

## VI. SIMULATIONS

### A. Simulation I: using Method I for adding noise

In our simulations, the cardinality of the FOD is 3. In random generation of BBAs, the number of focal elements has been set to 5. The length of the noise sequence is 50 (the noise value starts from -0.1, with an increasing step of 0.004, up to 0.1. Of course, the zero value for noise is not considered because it corresponds to noiseless case.). In each simulation cycle, seven combination rules including Dempster's rules and other alternatives aforementioned in Section III are used, respectively. We have repeated the Monte Carlo simulation with 100 runs. In random generation of original BBAs, the restrictions in Eq. (31) are not violated. The statistical results are listed in Tables I-III. The ranks of the relative MSE, relative variance and relative bias are obtained based on the descending order.

It should be noted that when using RCR in our simulation, the weights are generate as follows.

$$\begin{cases} \alpha(K) = \frac{K}{1-K+K^2} \\ \beta(K) = \frac{1-K}{1-K+K^2} \end{cases} \tag{34}$$

TABLE I. COMPARISONS IN TERMS OF MSE

| Combination Rules | $\mathrm{MSE}'_{\mathrm{BBA}}$ | Rank |
|---|---|---|
| Dempster's rule | 0.0010758 | 1 |
| Yager's rule | 0.0005743 | 5 |
| Disjunctive rule | 0.0004298 | 7 |
| D&P rule | 0.0006247 | 4 |
| RCR | 0.0007152 | 3 |
| PCR5 | 0.0010505 | 2 |
| Mean rule | 0.0005711 | 6 |

TABLE II. COMPARISONS IN TERMS OF VARIANCE

| Combination Rules | $\mathrm{Var}'_{\mathrm{BBA}}$ | Rank |
|---|---|---|
| Dempster's rule | 1.7260 | 1 |
| Yager's rule | 0.9568 | 6 |
| Disjunctive rule | 0.7469 | 7 |
| D&P rule | 1.0226 | 4 |
| RCR | 1.2788 | 3 |
| PCR5 | 1.6801 | 2 |
| Mean rule | 1 | 5 |

TABLE III. COMPARISONS IN TERMS OF BIAS

| Combination Rules | $\mathrm{Bias}'_{\mathrm{BBA}}$ | Rank |
|---|---|---|
| Dempster's rule | $0.81132*10^{-7}$ | 1 |
| Yager's rule | $0.59228*10^{-7}$ | 2 |
| Disjunctive rule | $0.41949*10^{-7}$ | 4 |
| D&P rule | $0.49075*10^{-7}$ | 3 |
| RCR | $0.39592*10^{-7}$ | 5 |
| PCR5 | $0.38066*10^{-7}$ | 6 |
| Mean rule | 0 | 7 |

As we can see in Tables I - III, Dempster's rule are the most sensitive to the mass change according to the criterion of the relative bias, and it also has highest degree of divergence according to the criterion of relative variance. Mean rule is the most insensitive to the mass change according to the criteria of relative bias, and it is always a rule with smaller divergence according to the criterion of the relative variance. Yager's rule is always more sensitive to the mass change and is always not so divergent.PCR5 rule is not so sensitive to the mass change according to the criterion of Bias (rank 6), and it is not so divergent according to the criterion of the relative variance. The Robust combination rule (RCR), Dubois & Prade's rule (D&P rule) are always moderate to the mass change in terms of sensitivity and in terms of divergence. So, PCR5 and RCR are more moderate rules; thus, they are relatively good choices for practical use.

### B. Simulation II: using Method II for adding noise

In our simulations, the cardinality of the FOD is 3. In generation of BBAs, the total set $\Theta$ is used as a focal element and the number of singleton focal elements has been set to 2. The length of the noise sequence is 50 (the noise value starts at 0.002 with an increasing step of 0.002, up to 0.1.) In each simulation cycle, seven combination rules including Dempster's rules and other alternatives aforementioned in Section III are used, respectively. We have repeated the Monte Carlo simulation with 100 runs. In random generation of original BBAs, the restrictions in Eq. (33) are not violated. The statistical results are listed in Tables IV-VI. The ranks of the relative MSE, relative variance and relative bias are obtained based on the descending order.

The derivation of weights of RCR has been done in the same manner as for the Simulation I.

TABLE IV. COMPARISONS IN TERMS OF MSE

| Combination Rules | $\mathrm{MSE}'_{\mathrm{BBA}}$ | Rank |
|---|---|---|
| Dempster's rule | 0.0050 | 3 |
| Yager's rule | 0.0038 | 5 |
| Disjunctive rule | 0.0033 | 6 |
| D&P rule | 0.0014 | 7 |
| RCR | 0.0078 | 1 |
| PCR5 | 0.0043 | 4 |
| Mean rule | 0.0056 | 2 |

TABLE V. COMPARISONS IN TERMS OF VARIANCE

| Combination Rules | $\mathrm{Var}'_{\mathrm{BBA}}$ | Rank |
|---|---|---|
| Dempster's rule | 0.9218 | 3 |
| Yager's rule | 0.7019 | 5 |
| Disjunctive rule | 0.5434 | 6 |
| D&P rule | 0.2714 | 7 |
| RCR | 1.3528 | 1 |
| PCR5 | 0.8014 | 4 |
| Mean rule | 1 | 2 |

As we can see in Tables IV - VI, RCR is the most sensitive to the change of focal elements according to the criterion of the relative bias, and it also has highest degree

TABLE VI. COMPARISONS IN TERMS OF BIAS

| Combination Rules | $\text{Bias}'_{\text{BBA}}$ | Rank |
|---|---|---|
| Dempster's rule | 0.0016 | 3 |
| Yager's rule | 0.0012 | 5 |
| Disjunctive rule | 0.0011 | 6 |
| D&P rule | 0.0004 | 7 |
| RCR | 0.0025 | 1 |
| PCR5 | 0.0013 | 4 |
| Mean rule | 0.0018 | 2 |

of divergence according to the criterion of relative variance. Dubois & Prade's rule (D&P rule) is the most insensitive rule according to the criteria of relative bias, and it is always a rule with smaller divergence according to the criterion of the relative variance. Yager's rule is always insensitive and is always not so divergent. Mean rule is sensitive to the change of focal element according to the criterion of Bias (rank 2), and it is divergent according to the criterion of the relative variance. Dempster's rule is not so sensitive to the change of focal element. The PCR5 and Yager's rules are always moderate to the change of focal elements in terms of sensitivity and in terms of divergence.

According to simulations results, we see that the different methods of adding noises impact differently the results of the comparative evaluations. However, we have shown that no matter the method adopted (by keeping the original core of the BBA, or modifying it slightly), PCR5 provides quite robust results for combining two BBA's and thus offers practical interests from this standpoint.

## VII. CONCLUSION

In this paper we have proposed a group of statistical criteria for evaluating the sensitivity of different combination rules with respect to the noise perturbations. The design is based on the classical measures of performance like MSE, variance, and bias encountered in the estimation theory. We don't rank the rules according to their a priori "good expected" properties. Moderate relative bias values are preferred, which means the balance or trade-off between the robustness and the sensitivity. Small relative variance values are preferred, which represent the high cohesion of a given combination rule. Seven widely used evidence combination rules were evaluated using the new proposed evaluation criteria. PCR5 is a moderate rule which is good for the practical use for combining two BBAs. For combining more than two BBAs, we expect that PCR6 will be a good choice, but we need to make more investigations in future to evaluate precisely its performances.

In this work, we have added some noises to BBAs mainly by modifying the mass assignments of the primary focal element and by creating new focal elements. In our future work, we will try to use other methods to add noise to BBAs, e.g., eliminating some of original focal elements. In our Monte-Carlo simulations, there is no pre-settings of mass assignments for the BBAs. In this paper, in each cycle we only generate one BBA, based on which, we generate a sequence of BBAs by adding small noise. The BBAs to be combined are the original BBAs and the BBAs with small noise. In our future work, we will try to generate two BBA sequences and add noise to them, respectively, where we can use some special BBAs in the evaluation procedure, e.g., BBAs to be combined are high conflicting. Then we can do more specific performance evaluations on the combination rules. In this paper, we did only focus on the property of sensitivity and divergence. The evaluation criteria of other aspects of evidence combination are also required for evaluating and designing new combination rules, which will be investigated in future research works and forthcoming publications.


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