Complex Analysis

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1 Complex numbers

Let us start with a heuristic description of the passage from the set $\mathbb{Q}$ of all rational numbers to the set $\mathbb{R}$ of all real numbers:

Rational numbers can be thought of as points on an infinite (horizontal) line once one has specified the positions of 0 and 1 (with 1 lying to the right hand side of the point 0). But the set $\mathbb{Q}$ of all such rational points does not fill the entire line. Real numbers then are (or correspond to) the points on that line, they form a set $\mathbb{R} \supset \mathbb{Q}$.

Indeed it turns out that the arithmetic operations on $\mathbb{Q}$, i.e. addition and multiplication, can be extended continuously from $\mathbb{Q}$ to $\mathbb{R} \supset \mathbb{Q}$ such that the following rules are satisfied:

1. The commutative laws: $a + b = b + a, ab = ba$
2. The associative laws: $a + (b + c) = (a + b) + c, a(bc) = (ab)c$
3. The distributive law: $a(b + c) = ab + ac$
4. $a + 0 = a, a \cdot 1 = a$
5. The equation $a + x = 0$ is solvable.
6. The equation $ax = 1$ is solvable for $a \neq 0$.

Remark 1.1.

1. Any set K with two distinguished elements $0, 1 \in K$ and two binary operations $+$ and $\cdot$ satisfying the above six conditions is called a *field*.
2. The solution of the equation $a + x = 0$ resp. $ax = 1$ in a field K is unique, we write it in the form $x = -a$ resp. $x = a^{-1}$. Fractions then are defined as:
$$\frac{a}{b} := ab^{-1} = b^{-1}a.$$
3. In a field K there are no zero divisors:
$$ab = 0 \Longleftrightarrow a = 0 \vee b = 0,$$
since if $ab = 0$ and $a \neq 0$, we have:
$$0 = a^{-1} \cdot (ab) = (a^{-1}a)b = b.$$

So it seems natural to ask whether the above extension process can be continued:

Is there a field $K \supsetneqq \mathbb{R}$ *the binary operations of which extend those on* $\mathbb{R}$*?*

If so, K becomes an $\mathbb{R}$-vector space: The scalar multiplication

$$\mathbb{R} \times K \longrightarrow K$$

is the restriction of the field multiplication

$$K \times K \longrightarrow K.$$

If even $n = \dim K < \infty$ we can choose a basis $e_1 := 1, e_2, ..., e_n$, such that every element $z \in K$ has the form

$$z = a + \sum_{\mu=2}^{n} a_\mu e_\mu$$

with unique real numbers $a, a_2, ..., a_n$. Then the multiplication is determined by the values

$$e_\mu e_\nu \in K, \quad 2 \leq \mu, \nu \leq n.$$

For $n = 2$ there is only the square $(e_2)^2 \in K$ to be considered. And by a good choice of the second base vector $e_2 \in K \setminus \mathbb{R}$ we can make it equal $-1 \in \mathbb{R}$.

Proposition 1.2. *Let* $K \supsetneqq \mathbb{R}$ *be a field. If* $\dim K = 2$ *holds for the dimension of the real vector space* K*, there is an element* $i \in K$ *such that*

$$i^2 = -1$$

and

$$K = \mathbb{R} + \mathbb{R}i.$$

Before we prove Prop.1.2, we remark, that the statement is true whenever $\dim K < \infty$, see Th.8.5.2.

Proof. Since $\dim K = 2$, we have only to hunt for an element $i \in K$ with $i^2 = -1$. Indeed, it is sufficient to find an element $j \in K \setminus \mathbb{R}$ with $j^2 \in \mathbb{R}$. In that case we have $j^2 < 0$: The case $j^2 = c \geq 0$ is impossible, since then

$$(j - \sqrt{c})(j + \sqrt{c}) = j^2 - c = 0,$$

but the field K does not admit zero divisors. Finally we rescale j to

$$i := \frac{j}{\sqrt{|c|}},$$

where $c = j^2$. To find j we start with any element $e = e_2 \in K \setminus \mathbb{R}$ and write

$$e^2 = a + be.$$

We write $j = x + ye$ and compute its square

$$(x + ye)^2 = x^2 + 2xye + (a + be)y^2 = (x^2 + ay^2) + (2x + by)y \cdot e$$

and see that $j = b - 2e$ will do. $\square$

Of course we have still to make sure that such a field really exists: The **complex plane** $\mathbb{C}$ is the real vector space

$$\mathbb{C} := \mathbb{R}^2,$$

endowed with the following multiplication

$$(a, b) \cdot (c, d) = (ac - bd, ad + bc).$$

Then the map

$$\mathbb{R} \longrightarrow \mathbb{C}, a \mapsto (a, 0)$$

preserves both addition and multiplication, and the second base vector $i := (0, 1)$ satisfies

$$i^2 = (-1, 0).$$

Identifying $a \in \mathbb{R}$ with $(a, 0) \in \mathbb{C}$ we arrive at

$$a + bi = (a, b)$$

and thus

$$\mathbb{C} = \{a + bi; a, b \in \mathbb{R}\}.$$

Both field operations have geometric interpretations: The addition is the addition of vectors in the plane, while the multiplication is best understood in polar coordinates. For $\varphi \in \mathbb{R}$ we define

$$\mathbf{e}(\varphi) := \cos(\varphi) + i \sin(\varphi),$$

any vector $x + iy \neq 0$ then can be written

$$x + iy = r\mathbf{e}(\varphi),$$

where $r = \sqrt{x^2 + y^2}$ is the length of the vector $x+iy$ and the angle φ is only determined up to an integer multiple of 2π, i.e.

$$x + iy = r\mathbf{e}(\varphi) = r\mathbf{e}(\varphi + 2\pi k), \forall k \in \mathbb{Z}.$$

This ambiguity in the choice of the angle φ, as intuitive as it is, is very important and should never be forgotten about: Indeed it is behind many phenomena in complex analysis!

Let us now consider the product of two vectors in polar coordinates: We obtain

$$r\mathbf{e}(\varphi) \cdot s\mathbf{e}(\psi) = (rs)\mathbf{e}(\varphi + \psi)$$

as a consequence of the addition theorems for trigonometric functions. Thus the length of the product of two vectors is the product of the lengths and the angles add. With that formula in mind we easily check the above six conditions - note in particular that

$$(r\mathbf{e}(\varphi))^{-1} = r^{-1}\mathbf{e}(-\varphi).$$

Let us add some useful remarks on the arithmetics of complex numbers: First of all

$$(-i)^2 = (-1)^2 i^2 = i^2 = -1,$$

so from the point of view of arithmetics, $-i \in \mathbb{C}$ should be as good as $i \in \mathbb{C}$. What do we mean by this? Let us consider the map

$$\mathbb{C} \longrightarrow \mathbb{C}, z = x + iy \mapsto \overline{z} := x + (-i)y = x - iy,$$

the reflection with respect to the "real axis" $\mathbb{R} \subset \mathbb{C}$, also called *complex conjugation.* It preserves both addition and multiplication

$$\overline{z + w} = \overline{z} + \overline{w}, \overline{zw} = \overline{z} \cdot \overline{w},$$

its fixed point set is the real line:

$$\overline{z} = z \Longleftrightarrow z \in \mathbb{R}.$$

The *real* and *imaginary part* of a complex number $z = x + iy$ are defined as

$$\mathrm{Re}(z) = x = \frac{z+\overline{z}}{2},\ \mathrm{Im}(z) = y = \frac{z-\overline{z}}{2i},$$

its *absolute value* $|z| \in \mathbb{R}_{\geq 0}$ is nothing but its euclidean length

$$|z| := \sqrt{x^2+y^2} = \sqrt{z\overline{z}}.$$

In particular, for $z = x + iy \neq 0$ we have

$$\frac{1}{z} = \frac{\overline{z}}{z\overline{z}} = \frac{\overline{z}}{|z|^2} = \frac{x}{|z|^2} - i\frac{y}{|z|^2}.$$

2 Elementary functions

Complex analysis deals with functions $f : G \longrightarrow \mathbb{C}$ defined on certain subsets $G \subset \mathbb{C}$. We shall discuss in this section some basic examples.

Let us start with $\mathbb{R}$-**linear maps** $f : \mathbb{C} \longrightarrow \mathbb{C}$, i.e. endomorphisms of the real vector space $\mathbb{C}$. Such a map can uniquely be written

$$f(z) = f(x+iy) = ax + by$$

with complex coefficients $a, b \in \mathbb{C}$. In fact $a = f(1), b = f(i)$.

If we don't want to split up into real and imaginary part, we can as well rewrite

$$f(z) = cz + d\overline{z}$$

with unique complex coefficients $c, d \in \mathbb{C}$ using the identities

$$x = \frac{z+\overline{z}}{2}, y = \frac{z-\overline{z}}{2i},$$

such that

$$c = \frac{1}{2}(a - ib),\ d = \frac{1}{2}(a + ib).$$

Note that an endomorphism $f : \mathbb{C} \longrightarrow \mathbb{C}$ is even $\mathbb{C}$-linear, i.e.

$$f(\lambda z) = \lambda f(z),\ \forall\ \lambda \in \mathbb{C},$$

iff $d = 0$. The condition is obviously sufficient; to see the necessity take $\lambda = i$ and compare the coefficients of the expressions for the right and the left hand side.

Starting with the constant and $\mathbb{R}$-linear maps and allowing all combinations of them using sums and products we obtain the **real polynomial maps** $f : \mathbb{C} \longrightarrow \mathbb{C}$; they are given as sums

$$f(z) = f(x+iy) = \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} a_{\mu\nu} x^\mu y^\nu$$

with unique complex coefficients $a_{\mu\nu} \in \mathbb{C}$. The function f coincides with its Taylor expansion at every point $z_0 = x_0 + iy_0 \in \mathbb{C}$, i.e.

$$f(z) = f(x+iy) = \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} c_{\mu\nu} (x-x_0)^\mu (y-y_0)^\nu$$

with

$$c_{\mu\nu} = \frac{1}{\mu!\nu!} \frac{\partial^{\mu+\nu} f}{\partial x^\mu \partial y^\nu}(z_0).$$

As a consequence $f \equiv 0$ near z_0 (by that we mean that f vanishes identically on some open disc

$$D_r(z_0) := \{z \in \mathbb{C}; |z - z_0| < r\}$$

of positive radius r) implies that $c_{\mu\nu} = 0$ for all μ, ν. So we obtain the following:

Theorem 2.1 (**Weak Identity Theorem for $\mathbb{R}$-polynomials.**). *Let $f, g : \mathbb{C} \longrightarrow \mathbb{C}$ be real polynomial maps, and $D := D_r(z_0)$ the disc with center z_0 and radius $r > 0$. Then*

$$f|_D = g|_D \Longrightarrow f = g.$$

Proof. Apply the above reasoning to the difference $f - g$. □

Finally, a real polynomial map may be written as a $(z, \overline{z})$-polynomial

$$f(z) = \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} b_{\mu\nu} z^\mu \overline{z}^\nu.$$

To see the uniqueness of the coefficients $b_{\mu\nu} \in \mathbb{C}$ we may use an argument analogous to the above one with the "complex partial derivatives" $\frac{\partial}{\partial z}, \frac{\partial}{\partial \overline{z}}$ to be introduced later on.

The **complex polynomial maps** $f : \mathbb{C} \longrightarrow \mathbb{C}$ are the real polynomials without $\overline{z}$-terms, i.e.

$$f(z) = \sum_{\mu=0}^{m} a_\mu z^\mu.$$

Though we can not draw the graph

$$\Gamma_f := \{(z, f(z)), z \in \mathbb{C}\} \subset \mathbb{C}^2$$

of such a function, we want to study its "geometry": Here the following concept is quite useful:

Definition 2.2. Let $f : G \longrightarrow \mathbb{C}$ be a complex valued function. The **fiber** $f^{-1}(w) \subset G$ over a point $w \in \mathbb{C}$ is defined as the set

$$f^{-1}(w) := \{z \in G; f(z) = w\},$$

so it consists of all points $z \in G$ lying "above" $w \in \mathbb{C}$ - where one thinks of the map f as a sort of projection.

Let us consider the power map

$$f(z) = z^n.$$

It is surjective: Its zero fiber consist of one point

$$f^{-1}(0) = \{0\},$$

while for $r > 0$ the fibre

$$f^{-1}(r\mathbf{e}(\varphi)) = \left\{ \sqrt[n]{r} \cdot \mathbf{e}\left(\frac{\varphi}{n} + k\frac{2\pi}{n}\right); k = 0, ..., n-1 \right\}$$

contains n different points. In particular $f^{-1}(1) \subset \mathbb{C}$ is the set of n-th roots of unity, which span a regular n-gon.

A **rational function** $f : G \longrightarrow \mathbb{C}$ is a function

$$f(z) = \frac{p(z)}{q(z)}$$

with complex polynomials $p, q : \mathbb{C} \longrightarrow \mathbb{C}$ and $q(z) \neq 0$ for $z \in G$. So the maximal choice of the domain of definition for a rational function is $G := \mathbb{C} \setminus N(q)$ with the finite set $N(q) := \{z \in \mathbb{C}; q(z) = 0\}$ of zeros of q.

The **complex exponential map**

$$\exp : \mathbb{C} \longrightarrow \mathbb{C}$$

extends the real exponential function to a function on the entire plane $\mathbb{C}$; it is defined by

$$\exp(z) = \exp(x + iy) := e^x \mathbf{e}(y) = e^x(\cos(y) + i\sin(y)).$$

It satisfies the functional equation

$$\exp(z_1 + z_2) = \exp(z_1)\exp(z_2)$$

and attains all nonzero values:

$$\exp(\mathbb{C}) = \mathbb{C}^* := \mathbb{C} \setminus \{0\}.$$

Namely, given $w = s\mathbf{e}(\psi) \in \mathbb{C}^*$ the point $z = \ln(s) + i\psi$ satisfies $\exp(z) = w$. Furthermore

$$\exp^{-1}(1) = 2\pi i\mathbb{Z}.$$

Now the functional equation implies that the w-fiber looks as follows

$$\exp^{-1}(w) = z + \exp^{-1}(1) = z + 2\pi i\mathbb{Z},$$

since $w = \exp(z_1) = \exp(z)$ implies $\exp(z_1 - z) = 1$.

The **complex trigonometric functions**

$$\sin(z) := \frac{1}{2i}(\exp(iz) - \exp(-iz))$$

and

$$\cos(z) := \frac{1}{2}(\exp(iz) + \exp(-iz))$$

extend the real trigonometric functions from the real line $\mathbb{R}$ to the complex plane $\mathbb{C}$. They are 2π-periodic functions

$$\sin(z + 2\pi) = \sin(z),\ \cos(z + 2\pi) = \cos(z)$$

and satisfy

$$\sin^2(z) + \cos^2(z) = 1.$$

But that does not imply that the complex trigonometric functions are bounded!

3 Convergence

Here we shall recall briefly some basic facts about convergence, continuity and open sets: Nothing of that is specific for $\mathbb{C}$ but rather holds in any $\mathbb{R}^n$.

Definition 3.1. A sequence (z_ν) of complex numbers is said to converge to $z_0 \in \mathbb{C}$, iff $\lim_{\nu\to\infty} |z_\nu - z_0| = 0$.

Remark 3.2.

1. $\lim_{\nu\to\infty} z_\nu = z_0$ iff $\lim_{\nu\to\infty} x_\nu = x_0$ and $\lim_{\nu\to\infty} y_\nu = y_0$. That follows from the estimates

$$|x_\nu - x_0|, |y_\nu - y_0| \leq |z_\nu - z_0| \leq |x_\nu - x_0| + |y_\nu - y_0|.$$

2. The sum and the product of two convergent sequences is convergent, the limit being the sum resp. product of the two limits.

3. If $\lim_{\nu\to\infty} z_\nu = z_0 \neq 0$, then $\lim_{\nu\to\infty} \frac{1}{z_\nu} = \frac{1}{z_0}$.

Definition 3.3. Let $D := D_r(z_0)$ be a disc and $D^* := D \setminus \{z_0\}$ the punctured disc. Assume $f : D^* \longrightarrow \mathbb{C}$ is a function. We write

$$\lim_{z\to z_0} f(z) = a,$$

if for any sequence $(z_\nu) \subset D^*$ converging to z_0 we have

$$\lim_{\nu\to\infty} f(z_\nu) = a.$$

The functions we are interested in are defined on open subsets of the complex plane:

Definition 3.4. A set $G \subset \mathbb{C}$ is called open, if, given any point $z_0 \in G$ there is some $r = r(z_0) > 0$, such that $D_r(z_0) \subset G$. By an open neighbourhood of $z_0 \in \mathbb{C}$ we mean any open set $U \ni z_0$.

Remark 3.5.

1. Unions and finite intersections of open sets are open.

2. A sequence z_ν converges to z_0 iff outside any open neighbourhood U of z_0 there are only finitely many z_ν.

Definition 3.6. Let $G \subset \mathbb{C}$ be open. A function $f : G \longrightarrow \mathbb{C}$ is called continuous at $z_0 \in G$ if

$$\lim_{z \to z_0} f(z) = f(z_0).$$

It is called continuous if it is continuous at every point of G.

Remark 3.7.

1. The function $f : G \longrightarrow \mathbb{C}$ is continuous at $z_0 \in G$ if for every open set $V \ni f(z_0)$ the inverse image

$$f^{-1}(V) := \{z \in G; f(z) \in V\}$$

contains some disk $D_r(z_0)$.

2. The function $f : G \longrightarrow \mathbb{C}$ is continuous if for every open set V the inverse image $f^{-1}(V) \subset G$ is open as well.

In order to construct interesting functions $f : G \longrightarrow \mathbb{C}$ one often represents them as limits: $f(z) = \lim_{n\to\infty} f_n(z)$ with functions $f_n : G \longrightarrow \mathbb{C}$. But if we want to show that certain properties of the functions f_n hold as well for the limit f, we need that the convergence is *uniform*.

Example 3.8. Let

$$g(z) := \begin{cases} 3|z|^2 - 2|z|^3 & , \quad \text{if } |z| \le 1 \\ 1 & , \quad \text{if } |z| \ge 1 \end{cases} .$$

Though the functions $f_n := g^n : \mathbb{C} \longrightarrow \mathbb{C}$ are continuous, indeed even differentiable, their pointwise limit, the function $f : \mathbb{C} \longrightarrow \mathbb{C}$ with

$$f(z) := \begin{cases} 0 & , \quad \text{if } |z| < 1 \\ 1 & , \quad \text{if } |z| \ge 1 \end{cases}$$

is not!

Definition 3.9. A sequence of functions $f_n : G \longrightarrow \mathbb{C}$ is said to *converge uniformly* to a function $f : G \longrightarrow \mathbb{C}$, if for any $\varepsilon > 0$ there is a natural number n_0, such that for $n \ge n_0$ the estimate $|f(z) - f_n(z)| < \varepsilon$ holds **simultaneously for all** $z \in G$.

The uniform limit of a sequence of continuous functions is again continuous:

Proposition 3.10. *If the sequence of continuous functions $f_n : G \longrightarrow \mathbb{C}$ converges uniformly to $f : G \longrightarrow \mathbb{C}$, then the function f is continuous as well.*

Proof. We have to show that $z_\nu \to z_0 \in G$ implies $f(z_\nu) \to f(z_0)$. Take some $\varepsilon > 0$. Choose $n \in \mathbb{N}$, such that the estimate $|f(z) - f_n(z)| < \varepsilon/3$ holds for all $z \in G$. Since f_n is continuous at z_0, we have $f_n(z_\nu) \to f_n(z_0)$. In particular we can find a $\nu_0 \in \mathbb{N}$, such that $|f_n(z_0) - f_n(z_\nu)| < \varepsilon/3$ holds for $\nu \geq \nu_0$. Now

$$|f(z_0) - f(z_\nu)| = |(f(z_0) - f_n(z_0)) + (f_n(z_0) - f_n(z_\nu)) + (f_n(z_\nu) - f(z_\nu))|$$

$$\leq |f(z_0) - f_n(z_0)| + |f_n(z_0) - f_n(z_\nu)| + |f_n(z_\nu) - f(z_\nu)| \leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

□

Example 3.11. Let $g_\nu : G \longrightarrow \mathbb{C}$ be continuous functions, assume that

$$|g_\nu(z)| \leq R_\nu, \ \forall \ z \in G$$

with reals $R_\nu > 0$, such that $\sum_{\nu=0}^{\infty} R_\nu < \infty$. Then the function $f : G \longrightarrow \mathbb{C}$ with

$$f(z) := \sum_{\nu=0}^{\infty} g_\nu(z)$$

is continuous, since the sequence of partial sums

$$f_n(z) = \sum_{\nu=0}^{n} g_\nu(z)$$

converges uniformly to f. Namely, given $\varepsilon > 0$, choose $n_0 \in \mathbb{N}$ such that $\sum_{\nu=n_0}^{\infty} R_\nu < \varepsilon$. Then for $n \geq n_0$ we have

$$|f(z) - f_n(z)| = \left| \sum_{\nu=n+1}^{\infty} g_\nu(z) \right| \leq \sum_{\nu=n+1}^{\infty} |g_\nu(z)| \leq \sum_{\nu=n+1}^{\infty} R_\nu < \varepsilon.$$

Finally we need the notion of a *connected* open set:

Definition 3.12. An open set $G \subset \mathbb{C}$ is called (path-)connected if for any two points $a, b \in G$ there is a path (=continuous map) $\gamma : I := [0,1] \longrightarrow G$ with $\gamma(0) = a, \gamma(1) = b$. A connected open set is also called a *domain*.

The following criterion is often quite useful:

Proposition 3.13. *An open set $G \subset \mathbb{C}$ is a domain iff it can not be written*

$$G = U \cup V$$

as the disjoint union of two non-empty open sets U and V .

Proof. The condition is necessary: We show that a point $a \in U$ can not be connected to a point $b \in V$ by a path $\gamma : I := [0,1] \longrightarrow G$ with $\gamma(0) = a, \gamma(1) = b$. Otherwise denote $t_0 \in I$ the least upper bound (supremum) of the set $\gamma^{-1}(U)$. Necessarily $\gamma(t_0) \in V$ - otherwise $\gamma(t_0) \in U$, and by continuity there would be some small interval $]t_0 - \varepsilon, t_0 + \varepsilon[$ such that even $\gamma(]t_0 - \varepsilon, t_0 + \varepsilon[) \subset U$, a contradiction to the fact that t_0 is an upper bound for the set $\gamma^{-1}(U)$. But if $\gamma(t_0) \in V$, then, once again by continuity, $\gamma(]t_0 - \varepsilon, t_0 + \varepsilon[) \subset V$ for some $\varepsilon > 0$, so $t_0 - \varepsilon$ is an upper bound for $\gamma^{-1}(U)$ as well and t_0 is not the least upper bound!

Now assume our criterion is satisfied. Take a point $a \in G$ and define

$$U := \{z \in G; \text{ there is a path in } G \text{ connecting } a \text{ with } z\}.$$

Furthermore let $V := G \setminus U$. We show that both U and V are open and conclude $G = U$. Since "being connectable by a path in G" is a transitive relation - in fact an equivalence relation - on G, it follows that G is path-connected.

The set U is open: Take $z \in U$ and choose a disc $D_r(z) \subset G$. Obviously all points in $D_r(z)$ can be connected by a path in $D_r(z) \subset G$ to its center z, hence also to $a \in G$. So $D_r(z) \subset U$.

The complement V is open as well: Given $z \in V$ choose again $D_r(z) \subset G$. If some point of $D_r(z)$ could be connected by a path in G with a, then z as well - with other words, $D_r(z) \subset V$.

□

4 Logarithm functions

The real exponential function $\mathbb{R} \longrightarrow \mathbb{R}_{>0}$ is bijective and has an inverse function $\ln : \mathbb{R}_{>0} \longrightarrow \mathbb{R}$, the natural logarithm. The complex exponential map is not injective due to the ambiguity of the angle in the polar coordinate description of a complex number. Even worse, there is no unique maximal domain of definition for the logarithm.

We start defining in general "logarithm functions" or "branches of the logarithm":

Definition 4.1. Let $G \subset \mathbb{C}$ be a domain. A continuous function $g : G \longrightarrow \mathbb{C}$ is called a *branch of the logarithm* on G if $\exp(g(z)) = z, \ \forall z \in G$.

Though the terminology: "**the** logarithm" seems to indicate it there is in general no preferred choice of a branch of the logarithm.

Lemma 4.2. *If $g, h : G \longrightarrow \mathbb{C}$ are branches of the logarithm on the domain G, then there is an integer $\ell \in \mathbb{Z}$ such that $h = g + 2\pi i\ell$.*

Proof. We have $\exp(g - h) \equiv 1$, and thus $g(z) - h(z) = 2\pi i\ell(z)$ with an integer $\ell(z) \in \mathbb{Z}$.. Since g and h are continuous, $\ell : G \longrightarrow \mathbb{Z} \subset \mathbb{C}$ is as well. But $\mathbb{Z} \subset \mathbb{C}$ is discrete (i.e. consists of isolated points only), so by the intermediate value theorem the continuous function $t \mapsto \ell(\gamma(t)) \in \mathbb{Z}$ is constant along any path $\gamma : I \longrightarrow G$. Since G is connected it follows that $\ell : G \longrightarrow \mathbb{Z}$ is constant. $\square$

A necessary condition for the existence of a branch of the logarithm on G is obviously that $G \subset \mathbb{C}^*$, the exponential function having no zeros. Unfortunately that condition is not sufficient. But let us look first at a positive example:

Example 4.3. 1. Let

$$G := \mathbb{C} \setminus \mathbb{R}_{\leq 0}$$

be the plane with the left real half line, i.e. the non-positive reals, removed. The *principal branch of the logarithm*

$$\mathrm{Log} : G \longrightarrow \mathbb{C}$$

is the function

$$\mathrm{Log}(z) := \ln|z| + i\mathrm{Arg}(z),$$

where the argument function $\mathrm{Arg} : G \longrightarrow \mathbb{R}$ is defined as follows

$$\mathrm{Arg}(z) := \begin{cases} -\arccos(\frac{x}{|z|}) & , \ \text{if } z = x + iy, y < 0 \\ \arctan(\frac{y}{x}) & , \ \text{if } z = x + iy, x > 0 \\ \arccos(\frac{x}{|z|}) & , \ \text{if } z = x + iy, y < 0 \end{cases}$$

with the inverse trigonometric functions $\arccos :]-1,1[\longrightarrow]0,\pi[$ and $\arctan : \mathbb{R} \longrightarrow]-\frac{\pi}{2}, \frac{\pi}{2}[$.

2. There is no branch of the logarithm $g : \mathbb{C}^* \longrightarrow \mathbb{C}$ on the punctured plane $\mathbb{C}^*$: Otherwise $g|_G$ would be a branch of the logarithm on G and thus $g|_G = \mathrm{Log} + 2\pi i\ell$. We may even assume $\ell = 0$ - replace g with $g - 2\pi i\ell$. But

$$g(-r) = \lim_{z\to -r, \mathrm{Im}(z)>0} \mathrm{Log}(z) = \ln(r) + \pi i$$

as well as

$$g(-r) = \lim_{z\to -r, \mathrm{Im}(z)<0} \mathrm{Log}(z) = \ln(r) - \pi i.$$

Intuitively, there is a branch of the logarithm $g : G \longrightarrow \mathbb{C}$ if and only if there is no loop in G around the origin. (A loop in G is a path $\gamma : I = [0,1] \longrightarrow G$ with $\gamma(1) = \gamma(0)$.)

3. On the plane with the half line $\mathbb{R}_{\geq 0}e^{i\vartheta}$ removed, i.e. on

$$G := \mathbb{C} \setminus \mathbb{R}_{\geq 0}e^{i\vartheta}$$

the function

$$g(z) = \mathrm{Log}(-e^{-i\vartheta}z) + i(\pi + \vartheta).$$

is a branch of the logarithm.

In order to determine a branch $g : G \longrightarrow \mathbb{C}$ of the logarithm on a domain G - if there is one - it is sufficient to fix the value $g(z_0)$ for some point $z_0 \in G$. Or, in special cases one can give an interval $]\vartheta, \vartheta + 2\pi i[$ such that $\mathrm{Im} g(z) \in]\vartheta, \vartheta + 2\pi i[$ for all $z \in G$. But let us mention here that there is a branch of the logarithm on

$$G := \mathbb{C}^* \setminus \{e^{t+it}; t \in \mathbb{R}\},$$

the punctured complex plane with a spiral removed, and its imaginary part attains all real values! Try to give an explicit definition!

5 Complex Differentiability

The notion of differentiability of a function $f : G \longrightarrow \mathbb{C}$ in a point $z \in G$ is literally the same as in real analysis of one variable. The difference is that differentiability of a complex valued function is quite a strong condition and, in contrast to the real situation, has striking consequences.

Definition 5.1. The function $f : G \longrightarrow \mathbb{C}$ is called differentiable (or $\mathbb{C}$-differentiable) in $z \in G$, if

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists. In that case we denote the limit $f'(z)$ and call it the (complex) derivative of f at z. The function f is called *holomorphic* at z, iff it is differentiable at every point of an open disc $D_r(z) \subset G$ around z. It is called holomorphic or differentiable, iff it is differentiable at every point in G. We denote

$$\mathcal{O}(G) := \{f : G \longrightarrow \mathbb{C}; f \text{ holomorphic}\}$$

the set of all holomorphic functions on G.

Example 5.2.

1. A constant function $f \equiv c$ is differentiable with derivative $f' \equiv 0$.
2. The function $f(z) = z$ is differentiable with derivative $f' \equiv 1$.
3. The complex conjugation $f(z) = \overline{z}$ is nowhere differentiable. (Note that functions of that type also exist in real analysis, but they are not that easy to define!). Indeed for $h = re^{i\varphi}$ we have

   $$\frac{\overline{z+h} - \overline{z}}{h} = \frac{\overline{h}}{h} = e^{-2i\varphi},$$

   i.e. along different rays through z there are different limits! So there is no limit in the whole!
4. The derivative preserves sums and satisfies the Leibniz rule: Assume that $f, g : G \longrightarrow \mathbb{C}$ are differentiable at $z \in G$. Then so are the sum $f + g$ and the product fg, and we have:

   $$(f+g)'(z) = f'(z) + g'(z), \ (fg)'(z) = f'(z)g(z) + f(z)g'(z).$$

In particular we obtain by induction on $n \in \mathbb{N}_{>0}$ that the power function $f(z) := z^n$ is differentiable with derivative $f'(z) = nz^{n-1}$. Complex polynomials are differentiable on $\mathbb{C}$.

5. An $\mathbb{R}$-linear map $f(z) = cz + d\overline{z}$ is differentiable at $0 \in \mathbb{C}$ iff it is differentiable everywhere iff $d = 0$ iff f is even $\mathbb{C}$-linear.

6. The set $\mathcal{O}(G)$ is closed with respect to both sums and products of functions.

7. If $f, g : G \longrightarrow \mathbb{C}$ are differentiable at $z \in G$ and $g(z) \neq 0$, so is $\frac{f}{g} : G \setminus N(g) \longrightarrow \mathbb{C}$ (where $N(g) \subset G$ denotes the set of zeros of g), and
$$\left(\frac{f}{g}\right)'(z) = \frac{f'(z)g(z) - f(z)g'(z)}{g(z)^2}.$$

8. The chain rule holds: Let $f : G \longrightarrow G' \subset \mathbb{C}$ and $g : G' \longrightarrow \mathbb{C}$ be functions. If f is differentiable at $z \in G$ and g at $w = f(z)$, then the composition $g \circ f : G \longrightarrow \mathbb{C}$ is differentiable at $z \in G$ and
$$(g \circ f)'(z) = g'(w)f'(z) = g'(f(z))f'(z).$$

Let us view at the above situation from the point of view of real analysis in the two variables x, y, taking $\mathbb{C}$ as the real plane. The functions differentiable in the sense of real analysis we shall call here $\mathbb{R}$-differentiable. We recall:

Definition 5.3. A function $f : G \longrightarrow \mathbb{C}$ is $\mathbb{R}$-differentiable at a point $z_0 \in G$, if there is an $\mathbb{R}$-linear map $L : \mathbb{C} \longrightarrow \mathbb{C}$, such that
$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0) - L(h)}{|h|} = 0,$$
or equivalently, h being a complex number (and not only a vector in $\mathbb{R}^n$),
$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0) - L(h)}{h} = 0.$$

The linear map L is uniquely determined by the above condition: It is called the **differential** of f at z_0 and also denoted
$$Df(z_0) := L.$$

Writing $h = s + it$ the differential $L : \mathbb{C} \longrightarrow \mathbb{C}$ of an $\mathbb{R}$-differentiable map $f : G \longrightarrow \mathbb{C}$ at some point $z_0 \in G$ takes the form

$$L(s + it) = as + bt,$$

where the coefficients $a, b \in \mathbb{C}$ are the partial derivatives

$$a = \frac{\partial f}{\partial x}(z_0),\ b = \frac{\partial f}{\partial y}(z_0).$$

On the other hand we may write

$$L(h) = ch + d\overline{h}$$

with $c = \frac{1}{2}(a - ib), d = \frac{1}{2}(a + ib)$. We are thus led to the following definition of "complex" partial derivatives:

Definition 5.4. For a function $f : G \longrightarrow \mathbb{C}$, which is $\mathbb{R}$-differentiable at $z_0 \in G$ with differential $L(h) = ch + d\overline{h}$ we define

$$\frac{\partial f}{\partial z}(z_0) := c,\ \frac{\partial f}{\partial \overline{z}}(z_0) := d.$$

Indeed

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$$

and

$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

The relation between real and complex differentiability is as follows:

Proposition 5.5. *A function $f : G \longrightarrow \mathbb{C}$ is $\mathbb{C}$-differentiable at $z_0 \in G$, iff it is $\mathbb{R}$-differentiable at z_0 and*

$$\frac{\partial f}{\partial \overline{z}}(z_0) = 0.$$

Proof. If f is $\mathbb{C}$-differentiable at z_0, then $L(h) := f'(z_0)h$ is the $\mathbb{R}$-linear map $L : \mathbb{C} \longrightarrow \mathbb{C}$ satisfying the above condition. On the other hand, given a function, which is $\mathbb{R}$-differentiable at $z_0 \in G$ with differential $L(h) = ch + d\overline{h}$, we have

$$\frac{f(z_0+h) - f(z_0)}{h} = \frac{L(h)}{h} + \varepsilon(h),$$

where $\lim_{h \to 0} \varepsilon(h) = 0$. So a function, which is $\mathbb{R}$-differentiable at $z_0 \in G$, is there $\mathbb{C}$-differentiable iff its differential $L : \mathbb{C} \longrightarrow \mathbb{C}$ at z_0 is $\mathbb{C}$-differentiable at $0 \in \mathbb{C}$, or equivalently: $L(h) = ch$ for some $c \in \mathbb{C}$, see 5.2.5. $\square$

In terms of ordinary partial derivatives we obtain:

Corollary 5.6. *For a function $f : G \longrightarrow \mathbb{C}$ and a point $z_0 \in G$ the following statements are equivalent:*

1. *f is $\mathbb{C}$-differentiable at z_0.*
2. *f is $\mathbb{R}$-differentiable at z_0 and there*
$$\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}$$
holds. With other words: The differential quotient of f at z_0 taken along the line $z_0 + \mathbb{R}$ coincides with that taken along $z_0 + i\mathbb{R}$.
3. *$f = u + iv$ is $\mathbb{R}$-differentiable at z_0 and there the* **Cauchy-Riemann equations**
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\ , \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$
hold.

Remark 5.7. A function $f : G \longrightarrow \mathbb{C}$ is $\mathbb{C}$-differentiable at $z_0 \in G$ iff

$$\frac{\partial f}{\partial \overline{z}}(z_0) = 0.$$

But be careful: We do not have

$$\frac{\partial f}{\partial z}(z_0) = \lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h},$$

since the right hand side in general does not exist, but if it exists, then the equality holds and furthermore $\frac{\partial f}{\partial \overline{z}}(z_0) = 0$.

The calculus for the first order differential operators $\frac{\partial}{\partial z}, \frac{\partial}{\partial \overline{z}}$ is analogous to that of partial derivatives: They satisfy

$$\frac{\partial z}{\partial z} = 1, \frac{\partial z}{\partial \overline{z}} = 0, \frac{\partial \overline{z}}{\partial z} = 0, \frac{\partial \overline{z}}{\partial \overline{z}} = 1,$$

are linear, satisfy the Leibniz rule and even the chain rule as for usual partial derivatives.

Example 5.8. 1. A real polynomial function

$$f(z) = \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} a_{\mu\nu} z^{\mu} \overline{z}^{\nu},$$

either is

(a) a complex polynomial and thus holomorphic everywhere

(b) or nowhere holomorphic.

First of all we have

$$a_{\mu\nu} = \frac{1}{\mu!\nu!} \frac{\partial^{\mu+\nu} f}{\partial z^{\mu} \partial \overline{z}^{\nu}}(0).$$

Thus the coefficients $a_{\mu\nu}$ are uniquely determined by f.

Now the $\overline{z}$-derivative

$$\frac{\partial f}{\partial \overline{z}} = \sum_{\mu=0}^{m} \sum_{\nu=1}^{n} \nu a_{\mu\nu} z^{\mu} \overline{z}^{\nu-1},$$

being a real polynomial as well, is either $\equiv 0$ - that means that all its coefficients vanish resp. that f was a complex polynomial - or does not vanish identically on any open subset, with other words, f is nowhere holomorphic.

2. The exponential function $\exp : \mathbb{C} \longrightarrow \mathbb{C}$ is holomorphic and $\exp' = \exp$. Indeed

$$\frac{\partial \exp}{\partial x}(z) = e^{x}\mathbf{e}(y), \frac{\partial \exp}{\partial y}(z) = e^{x} i \mathbf{e}(y).$$

3. A branch of the logarithm is holomorphic - to see that it is sufficient to look at branches on $\mathbb{C} \setminus \mathbb{R}_{\geq 0} e^{i\vartheta}$ resp. at the principal branch of the logarithm Log. Now the chain rule applied to $z = \exp(g(z))$ gives

$$1 = \exp'(g(z))g'(z) = \exp(g(z))g'(z) = zg'(z)$$

i.e. $g'(z) = \frac{1}{z}$.

6 Path integrals

The values of a holomorphic function $f \in \mathcal{O}(G)$ on the interior D of a closed disc $\overline{D} \subset G$ can be computed from its restriction $f|_{\partial D}$. This is the essential difference between differentiable functions on open parts of the real line and holomorphic functions on domains of the complex plane. If something similar would hold in the real situation (with closed intervals replacing closed discs), every differentiable function would be an affine linear function $f(x) = ax + b$!

In order to compute $f(z)$ for $z \in D$ from $f|_{\partial D}$, we need path integrals: This section is devoted to a careful discussion of that subject. Indeed we do not only treat it from the point of view of complex analysis, but explain the general background from real analysis in two variables as well.

First let us fix some notation:

Definition 6.1. For a domain $G \subset \mathbb{C}$ we denote

$$C^k(G)$$

the set of all complex valued functions, admitting continuous partial derivatives up to order k. In particular

$$C(G) := C^0(G)$$

is the set of all complex valued continuous functions on G.

The objects to be integrated along "paths" are not functions, but "differential forms". We start with a preparatory remark:

Remark 6.2. We denote

$$\mathcal{L}(\mathbb{C}) := \{L : \mathbb{C} \longrightarrow \mathbb{C}; L\ \mathbb{R}\text{-linear}\}$$

the complex vector space of all $\mathbb{R}$-linear maps from $\mathbb{C}$ to itself (with the scalar multiplication $(\lambda L)(h) := \lambda L(h)$). Recall that it admits two natural bases

$$\mathcal{L}(\mathbb{C}) = \mathbb{C} \cdot \mathrm{Re} \oplus \mathbb{C} \cdot \mathrm{Im},$$

where $\mathrm{Re}, \mathrm{Im} : \mathbb{C} \longrightarrow \mathbb{C}$ satisfy $\mathrm{Re}(x + iy) = x, \mathrm{Im}(x + iy) = y$, and

$$\mathcal{L}(\mathbb{C}) = \mathbb{C} \cdot \mathrm{id}_{\mathbb{C}} \oplus \mathbb{C} \cdot \overline{\mathrm{id}}_{\mathbb{C}},$$

where $\overline{\mathrm{id}}_{\mathbb{C}}$ is complex conjugation.

Definition 6.3. A C^k-differential form on a domain G is a function

$$\omega : G \longrightarrow \mathcal{L}(\mathbb{C}), z \mapsto \omega_z,$$

such that the function $G \longrightarrow \mathbb{C}, z \mapsto \omega_z(h)$, is a C^k-function for all $h \in \mathbb{C}$. We denote $\mathcal{D}^k(G)$ the set of all C^k-differential forms on G, with the convention $\mathcal{D}(G) := \mathcal{D}^0(G)$.

Remark 6.4.

1. Let us try a physical interpretation of differential forms: Differential forms can be regarded as rules associating to any small displacement vector $h \in \mathbb{C}$ at a point $z \in G$ the variation $\omega_z(h) \in \mathbb{C}$ of some (complex) "magnitude"; and h being small we may assume that $\omega_z(h)$ is $\mathbb{R}$-linear in $h \in \mathbb{C}$.

2. Differential forms can be added, and multiplied with functions $f \in C^k(G)$ as follows
$$(f\omega)_z := f(z)\omega_z.$$

Definition 6.5. The total differential of a function $F \in C^1(G)$ is defined as the continuous differential form

$$dF : G \longrightarrow \mathcal{L}(\mathbb{C})$$

associating to a point $z \in G$ the differential of F at z, i.e.

$$dF_z := DF(z).$$

On the other hand, given a differential form $\omega \in \mathcal{D}(G)$, any function $F \in C^1(G)$ with $dF = \omega$ is called a primitive function for ω.

In the following example we see that differential forms are determined by two complex valued coefficient functions:

Example 6.6.

1. The total differential of an $\mathbb{R}$-linear map $F = L : \mathbb{C} \longrightarrow \mathbb{C}$ is the map L itself:
$$dF \equiv L.$$
In particular
$$dx \equiv \mathrm{Re},\ dy \equiv \mathrm{Im},$$
while
$$dz \equiv \mathrm{id}_{\mathbb{C}},\ d\overline{z} \equiv \overline{\mathrm{id}}_{\mathbb{C}}.$$

As a consequence every differential form $\omega \in \mathcal{D}^k(G)$ has a unique representation

$$\omega = f dx + g dy$$

with functions $f, g \in C^k(G)$ as well as

$$\omega = h_1 dz + h_2 d\overline{z}$$

with

$$h_1 = \frac{1}{2}(f - ig), \ h_2 = \frac{1}{2}(f + ig).$$

To see that use the relations

$$dz = dx + idy, \ d\overline{z} = dx - idy$$

and

$$dx = \frac{1}{2}(dz + d\overline{z}), \ dy = \frac{1}{2i}(dz - d\overline{z}).$$

2. The total differential of a function $F \in C^1(G)$ then takes the form

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = \frac{\partial F}{\partial z}dz + \frac{\partial F}{\partial \overline{z}}d\overline{z}.$$

Remark 6.7. If $dF = d\widetilde{F}$ on a domain G, then $\widetilde{F} = F + c$ with some $c \in \mathbb{C}$, i.e. two primitive functions differ only by a constant. To prove that we consider $d(\widetilde{F} - F) = 0$. So it is sufficient to show that $dF = 0$ implies $F \equiv c \in \mathbb{C}$. It is clear that $F \equiv F(z_0)$ on any disc $D_r(z_0) \subset G$, since F is constant on the line segments $[z_0, x + iy_0]$ and $[x + iy_0, z]$. Now consider some value $c \in F(G)$, Take

$$U := \{z \in G; f(z) = c\}, \ V := \{z \in G; f(z) \neq c\}.$$

Then G is the disjoint union $G = U \cup V$, the set V is open, since F is continuous, but U is as well by the above argument. Since G is connected and $U \neq \emptyset$, we conclude $G = U$.

Definition 6.8. A differential form $\omega \in \mathcal{D}(G)$ is called

1. *integrable* or *exact*, if it admits a primitive function $F \in C^1(G)$, i.e. $\omega = dF$,

2. *locally integrable*, if for every point $z_0 \in G$ there is an open disc $D = D_r(z_0)$, such that $\omega|_D$ admits a primitive function $F \in C^1(D)$.

Example 6.9. Let $G \subset \mathbb{C}^*$ be a domain. Then the differential form

$$\omega := \frac{dz}{z} \in \mathcal{D}(G)$$

is locally integrable, but not necessarily integrable: Indeed, it is integrable iff there is a branch of the logarithm $\log : G \longrightarrow \mathbb{C}$. We have already seen that $\omega = d\log$. On the other hand, if $\omega = dF$, then

$$d(ze^{-F}) = zde^{-F} + e^{-F}dz = -ze^{-F}dF + e^{-F}dz = 0$$

and thus $ze^{-F(z)} \equiv c$ with some $c \in \mathbb{C}^*$. If then $e^d = c$, we see that $\log(z) := F(z) - d$ is a branch of the logarithm on G. The above integrability criterion applies to discs $D \subset \mathbb{C}^*$: If $D = D_r(z_0)$, then $D \subset \mathbb{C}^* \setminus \mathbb{R}_{\geq 0}(-z_0)$, the latter domain admitting a branch of the logarithm. Thus ω is locally integrable on any domain $G \subset \mathbb{C}^*$.

Differential forms can be integrated along paths:

Definition 6.10. Let $G \subset \mathbb{C}$ be a domain.

1. A path in G is a continuous map $\gamma : I = [a, b] \longrightarrow G$ from a closed interval $I = [a, b]$ to G. The point $\gamma(a) \in G$ is called the start point, $\gamma(b) \in G$ the end point of γ. We call γ a *closed path* or a *loop* if $\gamma(a) = \gamma(b)$.

2. A smooth path in $G \subset \mathbb{C}$ is a C^1-map $\gamma : I = [a, b] \longrightarrow G$, i.e. both component functions α, β of $\gamma = \alpha + i\beta$ have continuous first derivatives. We denote $\dot{\gamma} : I \longrightarrow \mathbb{C}$ the derivative of γ, i.e. $\dot{\gamma} = \dot{\alpha} + i\dot{\beta}$, and call $\dot{\gamma}(t)$ the tangent vector on γ at $t \in I$.

3. The length $L(\gamma)$ of a smooth path is defined as

$$L(\gamma) := \int_a^b |\dot{\gamma}(t)| dt,$$

its trace is the set

$$|\gamma| := \gamma(I).$$

4. A piecewise smooth path γ in G is a sequence of smooth paths $\gamma_1, ..., \gamma_r$, such that the end point of $\gamma_i, i = 1, ..., r-1$, coincides with the start point of γ_{i+1}. Its length is
$$L(\gamma) := \sum_{i=1}^{r} L(\gamma_i),$$
its trace
$$|\gamma| := \bigcup_{i=1}^{r} |\gamma_i|.$$

Some standard paths: By $[z_1, z_2]$ we mean the smooth path
$$[0,1] \longrightarrow \mathbb{C}, t \mapsto z_1 + t(z_2 - z_1).$$
Furthermore, given a coordinate rectangle $R = [a,b] + i[c,d]$ we denote ∂R the piecewise smooth loop consisting of the smooth pieces $[a+ic, b+ic]$, $[b+ic, b+id], [b+id, a+id], [a+id, a+ic]$.

We denote $\partial D_r(z_0)$ the smooth loop
$$[0, 2\pi] \longrightarrow \mathbb{C}, t \mapsto z_0 + re^{it},$$
i.e. the (counterclockwise) circle with center z_0 and radius $r > 0$.

As a general rule if a path surrounds a domain, one calls it *positively oriented* if one has that domain always on ones left hand side when following the path.

Given a path $\gamma : I \longrightarrow G$ its inverse path is
$$\gamma^{-1} : -I \longrightarrow G, t \mapsto \gamma(-t),$$
the inverse of a piecewise smooth path $\gamma = (\gamma_1, ..., \gamma_r)$ is $\gamma^{-1} := (\gamma_r^{-1}, ..., \gamma_1^{-1})$.

Definition 6.11. 1. Let $\gamma : [a,b] \longrightarrow G$ be a smooth path. For a differential form ω we define
$$\int_\gamma \omega := \int_a^b \omega_{\gamma(t)}(\dot{\gamma}(t))dt.$$

2. If γ is a piecewise smooth path, say $\gamma = (\gamma_1, ..., \gamma_r)$, we set

$$\int_\gamma \omega := \sum_{k=1}^{r} \int_{\gamma_k} \omega.$$

Remark 6.12. 1. If $\gamma = \alpha + i\beta$ and $\omega = f dx + g dy$, we have

$$\int_\gamma \omega = \int_a^b (f(\gamma(t))\dot{\alpha}(t) + g(\gamma(t))\dot{\beta}(t))dt.$$

2. For $\omega = h_1 dz + h_2 d\overline{z}$ we have

$$\int_\gamma \omega = \int_a^b \left(h_1(\gamma(t))\dot{\gamma}(t) + h_2(\gamma(t))\overline{\dot{\gamma}(t)} \right) dt.$$

3. If $\tau : [c, d] \longrightarrow [a, b], s \mapsto t = \tau(s)$, is a C^1-map with $\tau(c) = a, \tau(d) = b$, and $\widetilde{\gamma} := \gamma \circ \tau$, then

$$\int_{\widetilde{\gamma}} \omega = \int_\gamma \omega$$

as a consequence of the chain rule. Furthermore

$$L(\widetilde{\gamma}) = L(\gamma)$$

if $\frac{d\tau}{ds} \geq 0$. So, the value of the integral of a differential form along a path does not change under a reparameterization! But if you are hesitating while going from $\gamma(a)$ to $\gamma(b)$ and even go backwards a little bit before continuing to $\gamma(b)$ then the length of your way is bigger: $L(\widetilde{\gamma}) > L(\gamma)$.

4. We have

$$\int_{\gamma^{-1}} \omega = -\int_\gamma \omega$$

for any piecewise smooth path γ.

5. The fundamental theorem of calculus together with the chain rule

$$\frac{d}{dt}(F \circ \gamma)(t) = dF_{\gamma(t)}\left(\dot{\gamma}(t)\right)$$

for two variables gives

$$\int_\gamma dF = F(\gamma(b)) - F(\gamma(a)),$$

so the path integral of a differential form admitting a primitive function depends only on the start and end point of the path. In particular

$$\int_\lambda dF = 0$$

for any (piecewise smooth) loop λ.

Example 6.13. 1. Let $D = D_r(z_0)$ and $\overline{D} := \{z \in \mathbb{C}; |z - z_0| \leq r\}$. We want to prove that

$$\int_{\partial D} \frac{dz}{z-a} = \begin{cases} 2\pi i, & , \quad \text{if } a \in D \\ 0, & , \quad \text{if } a \notin \overline{D} \end{cases} .$$

If $a \in \overline{D}$, we decompose

$$\mathbb{C} \setminus \{a\} = G_1 \cup G_2$$

into two parts, where $\omega = \frac{dz}{z-a}$ admits a primitive function $F_i : G_i \longrightarrow \mathbb{C}$. Take

$$G_1 := \mathbb{C} \setminus (a + \mathbb{R}_{\leq 0}), G_2 := \mathbb{C} \setminus (a + \mathbb{R}_{\geq 0}).$$

On G_1 the function $F_1(z) := \text{Log}(z - a)$ is a primitive function of $dz/(z - a)$, on G_2 we can take $F_2(z) := \text{Log}(a - z)$. Now let $\gamma_1 : I \longrightarrow G_1$ be the arc of ∂D on the right hand side of the line $a + i\mathbb{R}$, and $\gamma_2 : J \longrightarrow G_2$ the complementary arc (assuming ∂D to start and end at the lower point of $\partial D \cap (a + i\mathbb{R})$). Denote $a + b$ the start point of γ_1, and $a + c$ the end point of γ_1. Then

$$\int_{\partial D} \frac{dz}{z-a} = \int_{\gamma_1} \frac{dz}{z-a} + \int_{\gamma_2} \frac{dz}{z-a}$$

$$= (F_1(a + c) - F_1(a + b)) + (F_2(a + b) - F_2(a + c))$$

$$= (\text{Log}(c) - \text{Log}(b)) + (\text{Log}(-b) - \text{Log}(-c))$$

$$= (\text{Log}(-b) - \text{Log}(b)) + (\text{Log}(c) - \text{Log}(-c)) = \pi i + \pi i = 2\pi i.$$

Finally, if $a \notin \overline{D}$ the differential form $\frac{dz}{z-a}$ has a primitive function on $\mathbb{C} \setminus (a + \mathbb{R}_{\leq 0}(z_0 - a)) \supset \overline{D}$, hence the integral vanishes.

2. As a consequence of the above discussion the differential form

$$\frac{dz}{z-a} \in \mathcal{D}(\mathbb{C} \setminus \{a\})$$

is locally integrable, but does not admit a primitive function: There are loops with non zero integral.

Proposition 6.14. *For a differential form $\omega \in \mathcal{D}(G)$ the following statements are equivalent*

1. *ω is locally integrable.*
2. *For every coordinate rectangle $R = [a, b]+i[c, d] \subset G$ we have $\int_{\partial R} \omega = 0$.*
3. *$\omega|_D$ is integrable on any disc $D = D_r(z_0) \subset G$.*

Proof. "1) $\Longrightarrow$ 2)": Let $R \subset G$ be a coordinate rectangle. For every $n \in \mathbb{N}$ we may subdivide R into n^2 rectangles $R_{ij}, 1 \le i, j \le n$, similar to R, but of $1/n$ the size of R. Then

$$\int_{\partial R} \omega = \sum_{i,j} \int_{\partial R_{ij}} \omega$$

for all $\omega \in \mathcal{D}(G)$, since any edge of an R_{ij} inside R shows up twice, but with opposite orientation. For n sufficiently large any R_{ij} is contained in a disc $D_{ij} \subset G$, where ω admits a primitive function, and thus $\int_{\partial R_{ij}} \omega = 0$.
"2) $\Longrightarrow$ 3)": We construct a primitive function on $D = D_r(z_0) \subset G$ as follows: We have

$$\int_{[z_0,x+iy_0]} \omega + \int_{[x+iy_0,x+iy]} \omega = \int_{[z_0,x_0+iy]} \omega + \int_{[x_0+iy,x+iy]} \omega =: F(z).$$

With other words, if $\omega = fdx + gdy$, then

$$F(x+iy) = \int_{x_0}^{x} f(s+iy_0)ds+\int_{y_0}^{y} g(x+it)dt = \int_{y_0}^{y} g(x_0+it)dt+\int_{x_0}^{x} f(s+iy)ds.$$

To check $\frac{\partial F}{\partial y} = g$ consider the first expression for F: The first term does not depend on y and for the second apply the fundamental theorem of calculus. For $\frac{\partial F}{\partial x} = f$ the analogous argument applies to the second expression.
"2) $\Longrightarrow$ 3)": Obvious. $\square$

The rectangle criterion for local integrability is not very convenient to be checked, but it is of interest since it may be used to show the following extension property:

Proposition 6.15. *If a differential form $\omega \in \mathcal{D}(G)$ is locally integrable on $G \setminus \{z_0\}$, then it is locally integrable on G as well.*

Proof. We show that $\int_{\partial R} \omega = 0$ for every coordinate rectangle $R \subset G$. For $R \not\ni z_0$ that is clear. With the notation of the proof of Prop. 6.14 we have $z_0 \in R_{ij}$ for at most 4 such rectangles, and the ω-integrals over the boundaries of these rectangles tend to zero for $n \mapsto \infty$: We have

$$\left| \int_{\partial R_{ij}} \omega \right| \leq M \cdot L(\partial R_{ij}) = \frac{M}{n} L(\partial R)$$

with $M = \max\{|f(z)|, |g(z)|; z \in R\}$, if $\omega = f dx + g dy$. Hence $\int_{\partial R} \omega = 0$. $\square$

The following result is interesting from a theoretical point of view; the reader mainly interested in a straight forward introduction to complex analysis may skip it. Indeed we shall prove it once more soon under slightly stronger assumptions.

Theorem 6.16. *For $f \in \mathcal{O}(G)$ the differential form $\omega = f(z)dz$ is locally integrable.*

Proof. 1) Let $R \subset D_\varepsilon(z_0) \subset G$ be a coordinate rectangle contained in a small disc $D_\varepsilon(z_0)$. We are hunting for an estimate of

$$\left| \int_{\partial R} f(z) dz \right|.$$

Write

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + h(z)(z - z_0)$$

with a continuous function $h : G \longrightarrow \mathbb{C}$. For $z \neq z_0$ we may simply solve for $h(z)$, and define $h(z_0) := 0$. Let

$$M(\varepsilon) := \max\{|h(z)|; |z - z_0| \leq \varepsilon\}.$$

We have

$$(f(z_0) + f'(z_0)(z - z_0))dz = d\left(f(z_0)z + f'(z_0)\frac{1}{2}(z - z_0)^2 \right)$$

and thus

$$\int_{\partial R} f(z)dz = \int_{\partial R} h(z)(z - z_0)dz,$$

whence

$$\left| \int_{\partial R} f(z)dz \right| \leq L(\partial R)M(\varepsilon)\varepsilon \leq 8\varepsilon M(\varepsilon)\varepsilon = 8\varepsilon^2 M(\varepsilon).$$

2) Consider a coordinate rectangle $R \subset G$. Assume $A := |\int_{\partial R} f(z)dz| \neq 0$. We shall find a decreasing sequence of rectangles R_n shrinking to a point $z_0 \in R$: Each R_n is similar to R and has 2^{-n} the size of R, and we have

$$\left| \int_{R_n} f(z)dz \right| \geq 4^{-n}A.$$

Take $R_0 := R$. If R_n is found with the given estimate, we consider the subdivision of R into four coordinate rectangles having the barycenter of R_n as one of its vertices. The integral of ω over ∂R_n is the sum over the ω-integrals along the boundaries of the subdividing rectangles, hence at least the absolute value of one of these integrals has to be $\geq (1/4) \cdot 4^{-n}A = 4^{-(n+1)}A$. Take as R_{n+1} the corresponding rectangle. We have then

$$\bigcap_{n=0}^{\infty} R_n = \{z_0\}$$

for some point $z_0 \in \mathbb{C}$. Then $R_n \subset D_\varepsilon(z_0)$ for $\varepsilon = 2^{-n}D$, where $D > 0$ denotes the length of the diagonal of R. Thus, comparing the upper bound of the first part with the lower bound just found, we obtain

$$8 \cdot 2^{-2n}D^2 \cdot M(2^{-n}D) \geq 4^{-n}A,$$

resp.

$$M(2^{-n}D) \geq \frac{A}{8D^2},$$

but the left hand side converges to 0, a contradiction. □

Local integrability of differential forms $\omega \in \mathcal{D}^1(G)$ can also be expressed in terms of derivatives:

Definition 6.17. The differential form $\omega = fdx + gdy \in \mathcal{D}^1(G)$ is said to be *closed* or to *satisfy the (local) integrability condition* if

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}.$$

Example 6.18. We leave it to the reader to check that $\omega = h_1 dz + h_2 d\overline{z}$ is closed iff

$$\frac{\partial h_1}{\partial \overline{z}} = \frac{\partial h_2}{\partial z}$$

holds. So the differential form $fdz \in \mathcal{D}^1(G)$ is closed if and only if f is holomorphic.

Proposition 6.19. *A differential form $\omega \in \mathcal{D}^1(G)$ is locally integrable if and only if it is closed.*

Proof. Let F be a primitive function on the disc $D \subset G$. Since $F \in C^2(D)$, we have

$$\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right).$$

On the other hand, if $D = D_r(z_0)$ and $\omega = fdx + gdy$ we define

$$F(x+iy) := \int_{x_0}^{x} f(s+iy_0)ds + \int_{y_0}^{y} g(x+it)dt.$$

Then

$$\frac{\partial F}{\partial y}(z) = g(x+iy)$$

according to the fundamental theorem of calculus, the first term not depending on y. On the other hand

$$\frac{\partial F}{\partial x}(z) = f(x+iy_0) + \int_{y_0}^{y} \frac{\partial g}{\partial x}(x+it)dt$$

$$= f(x+iy_0) + \int_{y_0}^{y} \frac{\partial f}{\partial y}(x+it)dt = f(x+iy_0) + (f(x+iy) - f(x+iy_0)) = f(x+iy).$$

$\square$

Corollary 6.20. *For a function $f \in C^1(G)$ the following statements are equivalent:*

1. *The differential form $f(z)dz$ is closed.*
2. *The differential form $f(z)dz$ is locally integrable.*
3. $f \in \mathcal{O}(G)$.

We remark that the equivalence of the second and the third statement holds for any $f \in C(G)$. The implication "3) $\Longrightarrow$ 2)" then is nothing but Th. 6.16, while "2) $\Longrightarrow$ 3)" is known as *Morera's theorem*: It follows from the fact that the derivative of a holomorphic function again is holomorphic, see Th.7.4.

On an open disc every closed differential form has a primitive function. In the remaining part of this section we are looking for more domains enjoying that property. We define:

Definition 6.21. A domain $G \subset \mathbb{C}$ is called simply connected if every locally integrable differential form $\omega \in \mathcal{D}(G)$ admits a primitive function $F \in C^1(G)$.

Example 6.22.
1. Open discs are simply connected.
2. The union $G = \bigcup_{n=1}^{\infty} G_n$ of an increasing sequence $G_1 \subset G_2 \subset ...$ of simply connected domains $G_n, n \in \mathbb{N}$, is simply connected.
3. If $G_1, G_2 \subset \mathbb{C}$ are simply connected and the intersection $G_1 \cap G_2$ is connected, then $G := G_1 \cup G_2$ is simply connected as well: A locally integrable form $\omega \in \mathcal{D}(G)$ has on G_i a primitive function $F_i \in C^1(G_i)$ for $i = 1, 2$. So $d(F_1 - F_2) = 0$ on the domain $G_1 \cap G_2$, hence $F_1 - F_2 \equiv c \in \mathbb{C}$, and we may define a primitive function $F \in C^1(G)$ by $F|_{G_1} := F_1$ and $F|_{G_2} := F_2 + c$.
4. A starshaped domain G is simply connected: A domain $G \subset \mathbb{C}$ is called *starshaped*, if there is a point $z_0 \in G$, such that for all $z \in G$ the line segment from z_0 to z is contained in G as well: For a locally integrable form ω the integral
$$F(z) := \int_{[z_0,z]} \omega$$
defines a primitive function.

We mention without proof the following characterization of simply connected domains.

Theorem 6.23. *For a domain $G \subset \mathbb{C}$ the following statements are equivalent:*

1. *G is simply connected.*
2. *The domain G "has no holes": The complement $\mathbb{C} \setminus G$ can not be written*
$$\mathbb{C} \setminus G = A \dot{\cup} K$$
as the disjoint union of a compact set $K \neq \emptyset$ and a closed set $A \subset \mathbb{C}$.
3. *Every continuous loop $\gamma : I \longrightarrow G$ can within G be contracted to a point, i.e. there is a continuous map*
$$H : I \times I \longrightarrow G$$
with $H(t, 0) = \gamma(t), H(t, 1) \equiv z_0 \in G$ and $H(0, s) = H(1, s)$ for all $s \in I$.

Remark 6.24. We remark, that the third characterization of simply connected domains is usually taken as definition. In that form it also applies to arbitrary topological spaces. - The above definition Def. 6.21 is less intuitive, but sometimes it is easier to handle, e.g. the proof of Rem. 6.22.3 becomes quite easy then.

We conclude this section by looking at some nonsimply connected domains, namely annuli

$$A_{\varrho,r}(a) := \{z \in \mathbb{C}; \varrho < |z - a| < r\}$$

with inner radiur ϱ and outer radius r, where $0 \leq \varrho < r \leq \infty$.

Proposition 6.25. *Let $\varrho < s < r$. A locally integrable differential form $\omega \in \mathcal{D}(A_{\varrho,r}(a))$ is integrable if and only if*

$$\int_{\partial D_s(a)} \omega = 0.$$

Proof. "$\Longrightarrow$": Obvious.
"$\Longleftarrow$": We may assume $a = 0$. The dissected annulus

$$A^* := A_{\varrho,r}(0) \setminus (-r, -\varrho)$$

is simply connected: To see that one can note that the restricted exponential

$$\exp : (\ln \varrho, \ln r) + i(-\pi, \pi) \longrightarrow A^*$$

is a homeomorphism (a bijective map, continuous in both directions) from a rectangle to our dissected annulus, and simple connectedness is preserved under such maps - use the third characterization of simply connected domains in Th.6.23. Alternatively we may decompose A^* into starshaped segements: For $n \in \mathbb{N}$ let

$$A_n := (\varrho, r) \exp\left(\pi i(-\frac{1}{n}, \frac{1}{n})\right).$$

If $\cos(\frac{\pi}{n})r > \varrho$, then A_n is starshaped with respect to $z_0 := \varrho \cos^{-1}(\frac{\pi}{n})$ and

$$A^* = \bigcup_{\nu=-n+1}^{n-1} \exp\left(\frac{\pi i \nu}{n}\right) A_n$$

is the union of such segments, such that the intersection of one of them with the union of the previous ones is connected.

So on both A^* and $-A^*$ there is a primitive function F resp. $\tilde{F}$ of ω. Since

$$A^* \cap (-A^*) = A_+ \cup A_-$$

with the connected sets $A_+ = \{z \in A_{\varrho,r}(0); \mathrm{Im}(z) > 0\}, A_- := -A_+$, we have $(F - \tilde{F})|_{A_\pm} \equiv c_\pm$. Thus integrating separately over the right and the left half arc of $\partial D_s(0)$ we obtain:

$$0 = \int_{\partial D_s(0)} \omega = F\left(\frac{i}{s}\right) - F\left(-\frac{i}{s}\right) + \tilde{F}\left(-\frac{i}{s}\right) - \tilde{F}\left(\frac{i}{s}\right) = c_+ - c_-.$$

So F may be extended to a function $\mathcal{O}(A_{\varrho,r}(0))$ by defining it on $-A^*$ as $\tilde{F} + c$ with $c := c_+ = c_-$. $\square$

7 The Cauchy formula

As an immediate consequence of the previous section let us note:

Theorem 7.1 (Cauchy's integral theorem). *Let $G \subset \mathbb{C}$ be a simply connected domain. Then for $f \in \mathcal{O}(G)$ we have*

$$\int_\lambda f(z)dz = 0$$

for every piecewise smooth loop λ in G.

Proof. The differential form $\omega = fdz$ is locally integrable, since $f \in \mathcal{O}(G)$, and G being simply connected it has a primitive function F (which is itself holomorphic) and thus the integral of $\omega = dF$ over any piecewise smooth loop vanishes. □

If we cannot guarantee that an integral of the above form vanishes, it often will be useful to have an estimate instead:

Proposition 7.2. *Let $f \in C(G)$ and γ be a piecewise smooth path in G. Then we have*

$$\left| \int_\gamma f(z)dz \right| \leq ||f||_\gamma \cdot L(\gamma).$$

Here

$$||f||_\gamma := \max\{|f(z)|; z \in |\gamma|\}$$

denotes the maximum of $|f|$ along the trace of γ.

Proof. We may assume that γ is smooth, say $\gamma : [a, b] \longrightarrow G$. For a complex valued, continuous function $g : [a, b] \longrightarrow \mathbb{C}$ we have

$$\left| \int_a^b g(t)dt \right| \leq \int_a^b |g(t)|dt,$$

since

$$\left| \sum_{i=1}^{s} g(t_i)(t_i - t_{i-1}) \right| \leq \sum_{i=1}^{s} |g(t_i)| \cdot (t_i - t_{i-1}),$$

where $t_0 = a < t_1 < ... < t_s = b$, and, with shrinking interval lengths $t_i - t_{i-1}$, the sum on the right hand side converges to the corresponding integral, and the same is true for the left hand side.

Now

$$\left| \int_\gamma f(z)dz \right| = \left| \int_a^b f(z)\dot\gamma(t)dt \right| \leq \int_a^b |f(z)\dot\gamma(t)|dt$$

$$\leq ||f||_\gamma \int_a^b |\dot\gamma(t)|dt = ||f||_\gamma \cdot L(\gamma).$$

□

The next theorem is behind a lot of nice results about holomorphic functions:

Theorem 7.3 (**Cauchy's integral formula**). *Let G be a domain and $f \in \mathcal{O}(G)$, furthermore D an open disc with $\overline{D} \subset G$. Then we have for any point $a \in D$ (a need not be the center!):*

$$f(a) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)dz}{z-a} \ .$$

So the values of a holomorphic function on D are already determined by its values on the boundary circle ∂D.

Proof. Since f is $\mathbb{C}$-differentiable at $a \in G$, we have

$$\omega := \frac{f(z) - f(a)}{z-a} dz \in \mathcal{D}(G).$$

Since $\frac{f(z)-f(a)}{z-a} \in \mathcal{O}(G \setminus \{a\})$, the differential form ω is locally integrable in $G \setminus \{a\}$, hence locally integrable in G according to Prop. 6.15. Since $D = D_r(z_0)$ satisfies $\overline{D} \subset G$, we have even $D_\varrho(z_0) \subset G$ for some $\varrho > r$ and thus $\omega = dF$ on $D_\varrho(z_0)$ resp. $\int_{\partial D} \omega = 0$. Consequently

$$\frac{1}{2\pi i} \int_{\partial D} \frac{f(z)dz}{z-a} = \frac{1}{2\pi i} \int_{\partial D} \frac{f(a)dz}{z-a} = f(a)$$

according to Example 1. □

With the traditional choice of letters the Cauchy formula reads

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)d\zeta}{\zeta - z} \ ,$$

where $z \in D$ is arbitrary. The letter z suggesting a variable we obtain:

Theorem 7.4 (**Generalized Cauchy's integral formula**). *A holomorphic function has complex derivatives of every order, indeed for $z \in D \subset G$ we have*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\partial D} \frac{f(\zeta)d\zeta}{(\zeta - z)^{(n+1)}} \ .$$

Proof. We use induction on $n \in \mathbb{N}$. For $n = 0$ nothing has to be shown. Now assume f has partial derivatives up to order n and the generalized Cauchy formula holds for n. We shall use

Remark 7.5. Let γ be a smooth path and

$$K : D \times |\gamma| \longrightarrow \mathbb{C},\ (z, \zeta) \mapsto K(z, \zeta),$$

be a continuous function, $\mathbb{R}$-differentiable with respect to z and its partial derivatives

$$\frac{\partial K}{\partial z}, \frac{\partial K}{\partial \overline{z}} : D \times |\gamma| \longrightarrow \mathbb{C}$$

being continuous as well. Then the function

$$D \longrightarrow \mathbb{C}, z \mapsto \int_\gamma K(z, \zeta) d\zeta$$

is $\mathbb{R}$-differentiable and

$$\frac{\partial}{\partial z}\left(\int_\gamma K(z, \zeta) d\zeta\right) = \int_\gamma \frac{\partial K}{\partial z}(z, \zeta) d\zeta.$$

as well as

$$\frac{\partial}{\partial \overline{z}}\left(\int_\gamma K(z, \zeta) d\zeta\right) = \int_\gamma \frac{\partial K}{\partial \overline{z}}(z, \zeta) d\zeta.$$

Taking $\gamma = \partial D$ and

$$K(z, \zeta) := \frac{f(\zeta)}{(\zeta - z)^{n+1}}$$

with

$$\frac{\partial K}{\partial z}(z, \zeta) = n \frac{f(\zeta)}{(\zeta - z)^{(n+2)}}$$

we obtain with Rem. 7.5 the case $n + 1$. □

We can do even better:

Theorem 7.6. *Let $f \in \mathcal{O}(G)$ and $D_r(z_0) \subset G$ be an open disc. Then the Taylor series of f represents f on $D_r(z_0)$, i.e.:*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \text{ where } a_n = f^{(n)}(z_0)/n!,$$

for all $z \in D_r(z_0)$, the right hand side converging uniformly on every closed disc $\overline{D}_\varrho(z_0), \varrho < r$.

A function (of one or several real or complex) variables is called (real or complex) analytic if near any point of its domain of definition it can be written as a power series. So Th. 7.6 tells us that holomorphic functions are complex analytic functions.

Proof. We may assume that $\overline{D}_r(z_0) \subset G$. We write

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) - (z - z_0)} =$$

$$= \frac{1}{\zeta - z_0} \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} = \frac{1}{\zeta - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0} \right)^n .$$

Note that, for $|z - z_0| \leq \varrho, \zeta \in \partial D_r(z_0)$, we have

$$\left| \frac{z - z_0}{\zeta - z_0} \right| \leq \frac{\varrho}{r} < 1,$$

so the geometric series converges uniformly on $\overline{D}_\varrho \times \partial D_r(z_0)$. So we may interchange summation and integration and obtain

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{2\pi i} \int_{\partial D_r(z_0)} \frac{f(\zeta) d\zeta}{(\zeta - z_0)^{n+1}} \cdot (z - z_0)^n .$$

□

Corollary 7.7 (Weak Identity Theorem for holomorphic functions). *Let $f, g \in \mathcal{O}(G)$ be holomorphic functions on the domain G. If $f|_D = g|_D$ for some open disc $D \subset G$, then already $f = g$. Or, equivalently, the restriction map*

$$\mathcal{O}(G) \longrightarrow \mathcal{O}(D), f \mapsto f|_D$$

is injective for every open disc $D \subset G$.

Proof. We may assume $g \equiv 0$. So let $f|_D \equiv 0$. Then the nonempty set

$$U := \{z \in G; \exists\, r > 0 : \ f|_{D_r(z)} \equiv 0\}$$

is obviously open, but

$$V := \{z \in G; \exists\, n \in \mathbb{N} : \ f^{(n)}(z) \neq 0\}$$

is as well, and $G = U \cup V$, since a point, where all derivatives vanish, belongs to U. Since G is connected and U non-empty, we conclude $U = G$. □

Remark 7.8. Assume $(a_n)_{n\in\mathbb{N}}$ is a sequence of complex numbers such that the sequence $(|a_n|\varrho^n)_{n\in\mathbb{N}}$ is bounded by $M > 0$. Then the power series

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n$$

converges uniformly on every closed disc $\overline{D}_s(z_0)$ with $s < \varrho$, since there we have

$$|a_n(z-z_0)^n| \le R_n := M\left(\frac{s}{\varrho}\right)^n$$

with $\sum_n R_n < \infty$ (geometric series). In particular it defines a continuous function $f : D_\varrho(z_0) \longrightarrow \mathbb{C}$. As a consequence we see that a complex analytic function is holomorphic: The function

$$\frac{f(z)-f(z_0)}{z-z_0} = \sum_{n=1}^{\infty} a_n(z-z_0)^{n-1}$$

is continuous at z_0. But even better: The above power series defines a function $f \in \mathcal{O}(D_\varrho(z_0))$, as we shall see in the next section.

Corollary 7.9. *Assume $\overline{D}_\varrho(z_0) \subset G$. For the coefficients a_n of the Taylor series of $f \in \mathcal{O}(G)$ at $z_0 \in G$ we have*

$$a_n = \frac{1}{2\pi i}\int_{\partial D_\varrho(z_0)} \frac{f(\zeta)d\zeta}{(\zeta-z_0)^{n+1}},$$

in particular

$$|a_n| \le \frac{M(f;\varrho,z_0)}{\varrho^n}$$

where $M(f;\varrho,z_0) := \max\{|f(\zeta)|; |\zeta-z_0| = \varrho\}$.

Proof. Th. 7.4 with $z = z_0$ and Prop. 7.2 with $L(\partial D_\varrho(z_0)) = 2\pi\varrho$. □

8 Basic properties of holomorphic functions

In this section we present the most important consequences of Cauchy's integral formula resp. the fact that holomorphic functions are analytic.

Definition 8.1. By an *entire function* one means a holomorphic function $f \in \mathcal{O}(\mathbb{C})$.

Theorem 8.2 (Liouville)**.** *A bounded entire function $f \in \mathcal{O}(\mathbb{C})$ is constant.*

Proof. We have

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

for all $z \in \mathbb{C}$, the coefficients a_n satisfying the estimate

$$|a_n| \le \frac{M(f;r,0)}{r^n} \le \frac{M}{r^n},$$

where $M > 0$ is an upper bound for f, i.e. $|f(z)| \le M$ for all $z \in \mathbb{C}$. Since the right hand side holds for any $r > 0$, we conclude $a_n = 0$ for $n > 0$. $\square$

Theorem 8.3 (Fundamental Theorem of Algebra)**.** *Every complex polynomial*

$$f(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$$

of degree n can be factorized as a product of n linear factors:

$$f(z) = a_n \prod_{\nu=1}^{n} (z - b_\nu).$$

Note that the zeros b_ν need not be pairwise distinct!

Proof. We do induction on n, the case $n = 1$ being trivial. Clearly we may assume $a_n = 1$. If f has no zero, the function $g(z) := 1/f(z)$ is a bounded entire function: Since g as a continuous function is bounded on every disc $\overline{D}_r(0)$, it suffices to show that $|f(z)| \ge R > 0$ in case $|z| > r$ for a suitable $r > 0$. Indeed,

$$|f(z)| = \left| z^n \Big(1 + \sum_{\nu=1}^{n} \frac{a_{n-\nu}}{z^\nu}\Big) \right| = |z^n| \left| 1 + \sum_{\nu=1}^{n} \frac{a_{n-\nu}}{z^\nu} \right|$$

$$\geq |z^n|(1 - \left|\sum_{\nu=1}^{n} \frac{a_{n-\nu}}{z^\nu}\right|) \geq |z|^n(1 - \sum_{\nu=1}^{n} \frac{|a_{n-\nu}|}{|z^\nu|})$$

$$= |z|^n(1 - \sum_{\nu=1}^{n} \frac{|a_{n-\nu}|}{|z|^\nu}) \geq |z|^n(1 - \sum_{\nu=1}^{n} \frac{|a_{n-\nu}|}{|z|}) \geq \frac{|z|^n}{2} \ ,$$

if $|z| \geq r := \max(1, 2\sum_{\nu=1}^{n} |a_{n-\nu}|)$.

Now Th. 8.2 says that the bounded entire function g and thus as well f is constant, a contradiction.

So, there is a zero $z_0 \in \mathbb{C}$. Then let us write

$$f(z) = f((z-z_0)+z_0) = \sum_{\nu=1}^{n} c_\nu (z-z_0)^\nu = (z-z_0) \cdot \sum_{\nu=1}^{n} c_\nu (z-z_0)^{\nu-1} = (z-z_0)h(z)$$

and apply the induction hypothesis to the polynomial $h(z)$. □

Corollary 8.4. *Every real polynomial*

$$f(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$$

of degree n can be factorized as a product of monic linear or quadratic polynomials:

$$f(z) = a_n \prod_{\nu=1}^{r} f_\nu(z),$$

with the quadratic polynomials having no real zeros.

Proof. We do induction on $n = \deg f$. For $n = 1$ everything is clear. For $n > 1$ take $z_0 \in \mathbb{C}$ with $f(z_0) = 0$. If $z_0 \in \mathbb{R}$, wee may write $f(z) = (z - z_0)g(z)$ with a polynomial g with real coefficients; otherwise $\overline{z}_0$ is an other zero of f and $f(z) = q(z)g(z)$ with $q(z) = (z - z_0)(z - \overline{z}_0) = z^2 - 2\mathrm{Re}(z_0)z + |z_0|^2$ and a polynomial $g(z)$ with real coefficients. Finally apply the induction hypothesis to $g(z)$. □

For algebraically minded readers we remark:

Theorem 8.5. *1. Let $K \supset \mathbb{C}$ be a field the arithmetic operations of which extend the addition and multiplication of complex numbers. If then K, as a complex vector space, has finite dimension, then $K = \mathbb{C}$.*

2. *Let $K \supsetneqq \mathbb{R}$ be a field the arithmetic operations of which extend the addition and multiplication of real numbers. If then K, as a real vector space, has finite dimension, then $K = \mathbb{R} + \mathbb{R}i$ with an element $i \in K$, such that $i^2 = -1$.*

Proof. 1.) Take an element $c \in K$. Since $\dim K < \infty$, the powers $c^\nu, \nu \in \mathbb{N}$, are not linearly independent. So there is a nontrivial relation

$$\sum_{\nu=0}^{n} a_\nu c^\nu = 0$$

with complex coefficients $a_\nu \in \mathbb{C}$. Of course we may assume $a_n = 1$. But, according to the fundamental theorem of algebra:

$$f(z) := \sum_{\nu=0}^{n} a_\nu z^\nu = \prod_{\nu=1}^{n}(z - b_\nu).$$

The above identity holds for all $z \in \mathbb{C}$ and thus may be understood as an identity for the coefficients of the polynomials on the right and left hand side (when writing the RHS as a linear combination of powers of the variable z), hence holds even for all $z \in K$. Taking $z = c$ we obtain

$$0 = f(c) = \prod_{\nu=1}^{n}(c - b_\nu).$$

Since K as a field does not contain zero divisors we obtain that $c = b_\nu \in \mathbb{C}$ for some ν.
2.) Take $c \in K \setminus \mathbb{R}$. As in the first part we see that $f(c) = 0$ for some real polynomial $f(z)$. Now by Cor.8.4 we see, that we may assume that f is a quadratic polynomial without real zeros. We leave it to the reader to check that there is an element $i \in \mathbb{R} + \mathbb{R}c$ with $i^2 = -1$. Using the first part we conclude that $K = \mathbb{R} + \mathbb{R}i$. □

Th. 8.3 may also be formulated by saying that

$$f(\mathbb{C}) = \mathbb{C}$$

holds for polynomial functions of degree > 0. For transcendent functions (=entire nonpolynomial functions) that is not true any longer, since

$$\exp(\mathbb{C}) = \mathbb{C}^*.$$

But we can prove

Theorem 8.6 (Weak Casorati-Weierstraß). *For any entire function $f \in \mathcal{O}(\mathbb{C}) \setminus \mathbb{C}$ we have*

$$\overline{f(\mathbb{C})} = \mathbb{C},$$

i.e. given any point $w_0 \in \mathbb{C}$ there are points of the form $f(z), z \in \mathbb{C}$, arbitrarily close to w_0. With other words

$$f(\mathbb{C}) \cap D \neq \emptyset$$

for every open disc $D \subset \mathbb{C}$.

Proof. Assume $D_r(z_0) \cap f(\mathbb{C}) = \emptyset$. Then

$$g(z) := \frac{1}{f(z) - w_0}$$

is a bounded entire function: $|g(z)| \leq 1/r$. Hence g as well as f are constant functions. □

But much more is true, the exponential function is more or less typical. We state without proof:

Theorem 8.7 (Weak theorem of Picard). *For a non-constant entire function $f \in \mathcal{O}(\mathbb{C})$ we have*

$$f(\mathbb{C}) = \mathbb{C}$$

or

$$f(\mathbb{C}) = \mathbb{C} \setminus \{a\}$$

with some $a \in \mathbb{C}$.

Let us now come to the local behaviour of holomorphic functions. It is completely determined by the following invariant:

Definition 8.8. The *multiplicity* of a holomorphic function $f \in \mathcal{O}(G) \setminus \mathbb{C}$ at $z_0 \in G$ is the number $n \in \mathbb{N}_{>0}$, such that

$$f(z) = a_0 + (z - z_0)^n h(z), \quad h(z_0) \neq 0$$

with a holomorphic function $h \in \mathcal{O}(G)$, or equivalently

$$f(z) = a_0 + \sum_{\nu=n}^{\infty} a_\nu (z - z_0)^\nu, a_n \neq 0,$$

near $z_0 \in G$.

Note that $f \in \mathcal{O}(G) \setminus \mathbb{C}$ implies, according to Th.7.7, that $f|_D \notin \mathbb{C}$ for all discs $D \subset \mathbb{C}$. Hence the multiplicity is well defined for every point $z_0 \in G$.

Lemma 8.9. *The zeros of a holomorphic function $f \in \mathcal{O}(G), f \not\equiv 0$, are isolated, i.e. given any zero $z_0 \in G$ of f, there is an open disc $D = D_r(z_0) \subset G$, such that f has no zeros on the punctured disc $D^* := D \setminus \{z_0\}$.*

Proof. We use the notation of Def.8.8. We have $a_0 = 0$, and the function $h \in \mathcal{O}(G)$ is continuous, hence $h(z) \neq 0$ on some disc $D_r(z_0)$ and thus $f(z) = (z - z_0)^n h(z) \neq 0$ for $z \in D_r(z_0)^*$. □

Theorem 8.10 (Identity theorem). *Let $A \subset G$ be a subset with a point z_0 of accumulation within G and $f, g \in \mathcal{O}(G)$. If $f|_A = g|_A$, then $f = g$.*

Proof. The zeros of the function $f - g \in \mathcal{O}(G)$ are not isolated, since by continuity, $f(z_0) - g(z_0) = 0$. As a consequence of the previous lemma we have $f - g \equiv 0$. □

In order to understand the local mapping properties of a holomorphic function we rewrite the presentation given in Def.8.8:

Theorem 8.11. *Let $f : G \longrightarrow \mathbb{C}$ be a holomorphic function of multiplicity n at $z_0 \in G$. Then there is an open disc $D = D_r(z_0)$ together with a holomorphic function $g \in \mathcal{O}(D)$ with $g(z_0) = 0, g'(z_0) \neq 0$, such that*

$$f(z) = a_0 + g(z)^n.$$

Proof. Write $f(z) - a_0 = (z - z_0)^n h(z)$. Choose $g(z) := (z - z_0)\sqrt[n]{h(z)}$. Here $\sqrt[n]{z} := \exp(\log(z)/n)$ with some branch of the logarithm $\log : V \longrightarrow \mathbb{C}$ defined on a neighbourhood V of $h(z_0) \neq 0$. Finally choose $D_r(z_0) \subset G$ with $h(D_r(z_0)) \subset V$. □

Definition 8.12. A bijective map $f : G \longrightarrow G'$ between domains G, G' is called biholomorphic if both f and f^{-1} are holomorphic.

Here is a geometric reformulation of Th.8.11:

Theorem 8.13. *Let $f : G \longrightarrow \mathbb{C}$ be a holomorphic function, $z_0 \in G$ a point, where f is of multiplicity n. Then there is an open neighbourhood $U \subset G$ of $z_0 \in G$, such that there is a factorization*

$$f|_U = p_n \circ g : U \xrightarrow{g} D_\varrho(0) \xrightarrow{p_n} D_r(w_0),$$

where $w_0 := f(z_0)$, g is biholomorphic, $r = \varrho^n$ and $p_n(\zeta) = w_0 + \zeta^n$.

In particular, a bijective holomorphic map is biholomorphic.

Proof. We use the notation of Th.8.11. We have $g'(z_0) \neq 0$. Since the Jacobian of g at z_0 has determinant $|g'(z_0)|^2 \neq 0$, the inverse function theorem tells us, that there is an open neighbourhood U_0 of z_0, such that $g(U_0)$ is open and $g|_{U_0} : U_0 \longrightarrow g(U_0)$ is bijective with an $\mathbb{R}$-differentiable inverse $g(U_0) \longrightarrow U_0$. Indeed, it is even holomorphic, as is easily checked, hence biholomorphic. Now take $\varrho > 0$ with $D_\varrho(0) \subset g(U_0)$ and $U := (g|_{U_0})^{-1}(D_\varrho(0))$. □

Corollary 8.14 (**Open mapping theorem**). *A non-constant holomorphic function $f : G \longrightarrow \mathbb{C}$ on a domain G is an open map, i.e. the image $f(U)$ of an open set $U \subset G$ is again open.*

Proof. Since we may replace f with $f|_U$, it suffices to show the $f(G)$ is open. And that is an immediate consequence of Th.8.13: Take $w_0 = f(z_0) \in f(G)$. As a nonconstant holomorphic function f has finite multiplicity at z_0 and thus, with the notation of Th.8.13, $D_r(w_0) \subset f(G)$. □

Corollary 8.15 (**Maximum principle**). *Let G be a domain and $f \in \mathcal{O}(G)$. If there is a point $z_0 \in G$ with $|f(z_0)| \geq |f(z)|$ for all $z \in G$, then $f \equiv f(z_0)$. With other words: A non-constant holomorphic function on a domain does not attain its maximum.*

Proof. Since $f(G) \subset \mathbb{C}$ is open, for any point $w_0 = f(z_0) \in f(G)$ there is an open disc $D_r(w_0) \subset f(G)$, in particular $w = (1+\varepsilon)w_0 \in f(G)$ for any sufficiently small $\varepsilon > 0$, and thus $|f(z_0)| < |f(z)|$, if $w = f(z)$. □

Another fundamental property of the class of all holomorphic functions is that it is stable under locally uniform limits:

Definition 8.16. A sequence of functions $f_n \in C(G)$ is said to converge locally uniformly to $f \in C(G)$, if every point $z_0 \in G$ is the center of an open disc $D = D_r(z_0) \subset G$, such that the restricted functions $f_n|_D$ converge uniformly to $f|_D$. Or equivalently, if for any compact set $K \subset G$ the restrictions $f_n|_K$ converge uniformly to $f|_K$.

Example 8.17. Assume $(a_n)_{n\in\mathbb{N}}$ is a sequence of complex numbers such that the sequence $(|a_n|\varrho^n)_{n\in\mathbb{N}}$ is bounded. Then the power series

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n$$

converges locally uniformly on the open disc $D_\varrho(z_0)$. Hence, by the next result, it defines a holomorphic function $f \in \mathcal{O}(D_\varrho(z_0))$.

Proposition 8.18. *Assume the sequence of functions $(f_n)_{n\in\mathbb{N}} \subset \mathcal{O}(G)$ converges locally uniformly to the function $f : G \longrightarrow \mathbb{C}$. Then*

1. *$f \in \mathcal{O}(G)$ and*
2. *for all $k \in \mathbb{N}$ the sepuence $(f_n^{(k)})_{n\in\mathbb{N}}$ of k-th derivatives converges locally uniformly to $f^{(k)}$.*

Note that a locally uniform limit of real analytic functions need not be a real analytic function. Indeed, every continuous function on $\mathbb{R}$ is the locally uniform limit of polynomials! Second: The sequence of real analytic functions $f_n(x) = \frac{1}{n}\sin(nx)$ converges even uniformly to 0, but their derivatives do not converge locally uniformly to 0.

On the other hand, in Prop.8.18 we can not replace locally uniform convergence by pointwise convergence: The pointwise limit of holomorphic functions need not even be continuous, but examples for that phenomenon are not that easy at hand, cf. Ex.8.21.

Proof. We show that $f|_D \in \mathcal{O}(D)$ for every open disc with $\overline{D} \subset G$. Indeed f_n converges uniformly on $\overline{D}$ to $f|_{\overline{D}} \in C(\overline{D})$. Hence

$$f(z) = \lim_{n\to\infty} f_n(z) = \lim_{n\to\infty} \frac{1}{2\pi i}\int_{\partial D} \frac{f_n(\zeta)d\zeta}{\zeta - z} = \frac{1}{2\pi i}\int_{\partial D} \frac{f(\zeta)d\zeta}{\zeta - z},$$

since (f_n) converges uniformly to f on $\gamma := \partial D$, cf. the below lemma 8.20. Finally use Rem. 7.5.

In order to see that the $f_n^{(k)}$ converge uniformly on $\overline{D} = \overline{D}_r(z_0)$ to $f^{(k)}$ we use the generalized Cauchy formula

$$h^{(k)}(z) = \frac{k!}{2\pi i}\int_{\partial D_{r+\varepsilon}(z_0)} \frac{h(\zeta)d\zeta}{(\zeta - z)^{k+1}}$$

for $h = f_n - f$ and obtain the estimate

$$||f_n^{(k)} - f^{(k)}||_{\overline{D}} \le \frac{k!||f_n - f||_{\partial D_{r+\varepsilon}(z_0)}}{\varepsilon^{k+1}} \cdot (r + \varepsilon).$$

□

Remark 8.19. In the above proof we have only used, that the sequence $(f_n)_{n\in\mathbb{N}}$ converges uniformly on the boundary circle ∂D: Indeed, if a sequence of functions $f_n \in \mathcal{O}(G)$ converges uniformly on ∂D to a function $f \in C(\partial D)$, then they converge as well on $\overline{D}$ uniformly to a continuous function $\hat{f} \in C(\overline{D}) \cap \mathcal{O}(D)$ extending f. This is due to the fact that

$$||f_n - f_m||_{\overline{D}} = ||f_n - f_m||_{\partial D}$$

as a consequence of the maximum principle.

Lemma 8.20. *Let γ be a piecewise smooth path. If the continuous functions $f_n : |\gamma| \longrightarrow \mathbb{C}$ converge uniformly to the function $f : |\gamma| \longrightarrow \mathbb{C}$, then*

$$\lim_{n\to\infty} \int_\gamma f_n(z)dz = \int_\gamma f(z)dz.$$

Proof. We have

$$\left|\int_\gamma f_n(z)dz - \int_\gamma f(z)dz\right| = \left|\int_\gamma (f_n(z) - f(z))dz\right| \le ||f_n - f||_\gamma \cdot L(\gamma).$$

□

We conclude this section with an example of a pointwise convergent sequence of holomorphic functions with a non-continuous limit function:

Example 8.21. We start with an increasing sequence of compact sets K_n exhausting the complex plane $\mathbb{C}$, while their interiors only yield the plane without the real line, i.e. $\bigcup_{n=1}^\infty \mathring{K}_n = \mathbb{C} \setminus \mathbb{R}$. Let

$$K_n := K_n^+ \cup K_n^-$$

with

$$K_n^+ := \overline{D}_n(0) \cap (\mathbb{R} + i\mathbb{R}_{\ge 1/n}).$$

and

$$K_n^- := \overline{D}_n(0) \cap (\mathbb{R} + i\mathbb{R}_{\le 0}).$$

Now the function $f : \mathbb{C} \longrightarrow \mathbb{C}$ with

$$f(z) = \begin{cases} 0 & , \quad \text{if } \operatorname{Im}(z) > 0 \\ 1 & , \quad \text{if } \operatorname{Im}(z) \le 0 \end{cases}$$

is holomorphic in a suitable neighbourhood $U_n \supset K_n$ for every $n \in \mathbb{N}_{\geq 1}$, and Runge's theorem Th.10.1 provides a polynomial $p_n : \mathbb{C} \longrightarrow \mathbb{C}$ with $||f - p_n||_{K_n} < \frac{1}{n}$. Then the sequence $(p_n)_{n\geq 1}$ converges pointwise to the non-continuous function f. We remark that the behaviour of the sequence $(p_n)_{n\geq 1}$ near points on the real line becomes more and more "chaotic", necessarily we have

$$||p_n||_K \to \infty$$

for any compact set K having an interior point on the real line.

9 Laurent series and residues

One of the most impressive applications of complex analysis is the help it provides in computing certain real integrals. Indeed, they are reduced to complex loop integrals

$$\int_\lambda f(z)dz$$

for loops λ in the domain G of holomorphy of the function f.

First of all, the integral does not change, if we replace λ with a loop $\tilde{\lambda}$ homotopic to λ in G.

Definition 9.1. Two continuous loops $\lambda, \tilde{\lambda} : [a, b] \longrightarrow G$ are called homotopic in G, if there is a continuous map

$$H : [a, b] \times I \longrightarrow G$$

with $H(t, 0) = \lambda(t), H(t, 1) = \tilde{\lambda}(t)$ and $H(a, s) = H(b, s)$ for all $s \in I := [0, 1]$.

Remark 9.2. For a locally integrable differential form $\omega \in \mathcal{D}(G)$ we may define

$$\int_\gamma \omega$$

for any path $\gamma : [a, b] \longrightarrow G$ as follows: Let $t_\nu := a + \nu\frac{b-a}{n}, \nu = 0, ..., n.$

Then, for $n \gg 0$ the images $\gamma([t_{\nu-1}, t_\nu])$ are contained in open sets, where ω admits a primitive function $F_\nu, \nu = 1, ..., n.$ Then we set

$$\int_\gamma \omega := \sum_{\nu=1}^{n} F_\nu(\gamma(t_\nu)) - F_\nu(\gamma(t_{\nu-1})).$$

We leave it to the reader to check that the given definition does not depend on $n \in \mathbb{N}$.

Proposition 9.3. *Let $\lambda, \tilde{\lambda}$ be loops in G. If they are homotopic in G, then*

$$\int_{\tilde{\lambda}} \omega = \int_{\lambda} \omega$$

for any locally integrable differential form $\omega \in \mathcal{D}(G)$.

Proof. Denote $H : R := [a, b] \times I \longrightarrow G$ a homotopy between the loops λ and $\tilde{\lambda}$. Let $R = \bigcup_{1 \le i,j \le n} R_{ij}$ be the decomposition of R into n^2 congruent rectangles of size $\frac{1}{n}$ the size of R. For sufficiently big $n \in \mathbb{N}$ we have

$$\int_{\lambda} \omega - \int_{\tilde{\lambda}} \omega = \int_{H(\partial R)} \omega = \sum_{i,j} \int_{H(\partial R_{ij})} \omega = 0,$$

since the "vertical edges" $H(b \times [0, 1])$ and $H(a \times [1, 0])$ of the "rectangle" $H(\partial R)$ are inverse one to the other and for $n \gg 0$ any "rectangle" $H(\partial R_{ij})$ is contained in an open set, where ω admits a primitive function. $\square$

Definition 9.4. A *cycle* $\underline{\lambda}$ in a domain G is a finite sequence of piecewise smooth loops $\lambda_1, ..., \lambda_s$ in G. It is called *nullhomologous* in G if and only if

$$\int_{\underline{\lambda}} \omega := \sum_{j=1}^{s} \int_{\lambda_j} \omega = 0$$

holds for all locally integrable differential forms $\omega \in \mathcal{D}(G)$.

Remark 9.5. Loops are considered to be cycles of length $s = 1$.

Example 9.6. 1. A loop λ, such that $|\lambda| \subset G_0 \subset G$ with a simply connected domain G_0 is nullhomologous in G.

2. A cycle $\underline{\lambda} = (\tilde{\lambda}, \lambda^{-1})$ with (in G) homotopic loops $\tilde{\lambda}, \lambda$ is nullhomologous in G.

3. Let $G = \mathbb{C} \setminus \{\pm\frac{i}{2}\}$, $D := D_1(0)$ and λ_+ resp. λ_- the loops consisting of the upper resp. lower half circle (in counterclockwise orientation) and $[-1, 1]$ resp. $[1, -1]$. Then

$$\int_{\partial D} \omega = \int_{\lambda_+} \omega + \int_{\lambda_-} \omega$$

holds for any differential form $\omega \in \mathcal{D}(G)$ (not only locally integrable ones). Thus

$$(\partial D, \lambda_+^{-1}, \lambda_-^{-1})$$

is nullhomologous in G.

The framework in which we consider loop integrals is the following:

1. A domain $G \subset \mathbb{C}$ is given as well as
2. a function $f \in \mathcal{O}(G \setminus S)$, where $S = S_f \subset G$ is a finite set, and
3. a piecewise smooth loop λ in $G \setminus S$ nullhomologous in G.

We shall see that the integral over a loop nullhomologous in G is determined by local contributions arising from the points in $S = S_f$. First of all we have to take into account how many times the loop λ winds around a point $a \in S$.

Definition 9.7. Let $\underline{\lambda}$ be a cycle in $\mathbb{C}$, $a \in \mathbb{C} \setminus |\underline{\lambda}|$. Then

$$\mathrm{ind}_{\underline{\lambda}}(a) := \frac{1}{2\pi i} \int_{\underline{\lambda}} \frac{dz}{z-a} \in \mathbb{Z}$$

is called the *index of a with respect to the cycle $\underline{\lambda}$* or the *winding number of $\underline{\lambda}$ with respect to a.*

The index $\mathrm{ind}_{\underline{\lambda}}(a)$ is an integer: If $\gamma : I \longrightarrow \mathbb{C}$ is a smooth path with $a \notin |\gamma|$, the function

$$I \longrightarrow \mathbb{C}, t \mapsto \frac{1}{2\pi i} \frac{1}{\gamma(t) - a}$$

has a primitive function $F : I \longrightarrow \mathbb{C}$ satisfying $e^{2\pi i F(t)} = \gamma(t) - a$. Now if $\lambda = (\gamma_1, ..., \gamma_r)$ with $\gamma_j : I_j = [a_j, b_j] \longrightarrow \mathbb{C}$, we may choose $F_j : I_j \longrightarrow \mathbb{C}$ such that $F_{j+1}(a_{j+1}) = F_j(b_j)$ for $j = 1, ..., r-1$. Then we have

$$\mathrm{ind}_\lambda(a) = F_r(b_r) - F_1(a_1), e^{2\pi i F_r(b_r)} = e^{2\pi i F_1(a_1)},$$

whence the result follows. The generalization from loops to cycles is immediate.

We shall need a global version of Cauchy's integral formula:

Theorem 9.8. *Let $\underline{\lambda}$ be a nullhomologous cycle in G. Then for a holomorphic function $f \in \mathcal{O}(G)$ and $a \in G \setminus |\underline{\lambda}|$ the following Cauchy formula holds*

$$\frac{1}{2\pi i}\int_{\underline{\lambda}} \frac{f(\zeta)d\zeta}{\zeta - a} = \operatorname{ind}_{\underline{\lambda}}(a) \cdot f(a).$$

Proof. The differential form

$$\omega := \frac{f(\zeta) - f(a)}{\zeta - a} d\zeta \in \mathcal{D}(G)$$

is locally integrable, hence $\int_{\underline{\lambda}} \omega = 0$ and thus

$$\int_{\underline{\lambda}} \frac{f(\zeta)d\zeta}{\zeta - a} = f(a)\int_{\underline{\lambda}} \frac{d\zeta}{\zeta - a} = 2\pi i \cdot \operatorname{ind}_{\underline{\lambda}}(a) f(a).$$

□

The local contributions from points $a \in S$ are called residues:

Definition 9.9. Let $a \in \mathbb{C}$. The *residue* of a holomorphic function $f \in \mathcal{O}(D_r(a)^*)$ at a is defined as

$$\operatorname{Res}_a(f) := \frac{1}{2\pi i}\int_{\partial D_\varrho(a)} f(z)dz,$$

where $0 < \varrho < r$. Note that the right hand side does not depend on $\varrho > 0$ because of Ex.9.6.2: The loops $\partial D_\varrho(a)$ and $\partial D_{\varrho'}(a)$ with $0 < \varrho, \varrho' < r$ are homotopic in $D_r(a)^*$.

Remark 9.10. If $f \in \mathcal{O}(D_r(a)^*)$ admits a continuous extension to $D_r(a)$, then we have

$$\operatorname{Res}_a(f) = 0$$

as a consequence of Prop. 6.15 applied to the differential form $\omega = f(z)dz$.

Theorem 9.11 (Residue Theorem). *Let λ be a nullhomologous loop in G, $S \subset G \setminus |\lambda|$ a finite set and $f \in \mathcal{O}(G \setminus S)$. Then*

$$\int_\lambda f(z)dz = 2\pi i \sum_{a\in S} \operatorname{ind}_\lambda(a) \cdot \operatorname{Res}_a(f).$$

Proof. The key point in the proof of Th.9.11 is a "generalized partial fraction decomposition": We write

$$f(z) = g(z) + h(z) + \sum_{a \in S} \frac{c_a}{z-a},$$

where

1. $g \in \mathcal{O}(G)$ and
2. $h \in \mathcal{O}(\mathbb{C} \setminus S)$ is the derivative of a function $H \in \mathcal{O}(\mathbb{C} \setminus S)$,
3. $c_a = \mathrm{Res}_a(f)$ for $a \in S$.

If $G = \mathbb{C}$ and $f(z) = \frac{p(z)}{q(z)}$ is a rational function and S denotes the set of zeros of the polynomial $q(z)$, then $g(z)$ is the polynomial part of the partial fraction decomposition of f and $h(z)$ the sum of all singular terms of multiplicity > 1.

Given the decomposition we obtain the residue formula as follows: For sufficiently small $\varepsilon > 0$ integration over $D_\varepsilon(a)$ yields $c_a = \mathrm{Res}_a(f)$, while integration over λ gives the desired formula: We have $\int_\lambda g(z)dz = 0$, since λ is nullhomologous in G and $\int_\lambda h(z)dz = 0$ because of $hdz = dH$.

First we derive the above decomposition in the case where $G = D_r(a)$ and $S = \{a\}$, thereby obtaining a new interpretation of the residue $\mathrm{Res}_a(f)$.

Indeed the situation we consider is slightly more general. Denote

$$A_{\varrho,r}(a) := \{z \in \mathbb{C}; \varrho < |z-a| < r\}, \quad 0 \leq \varrho < r \leq \infty.$$

the annulus with center a, inner radius ϱ and outer radius r.

Theorem 9.12. *Given a function $f \in \mathcal{O}(A_{\varrho,r}(a))$ there are unique functions $f_+ \in \mathcal{O}(D_r(0))$ and $f_- \in \mathcal{O}(D_{\varrho^{-1}}(0))$ (where "$0^{-1} = \infty$" and $D_\infty(0) = \mathbb{C}$) with $f_-(0) = 0$, such that*

$$f(z) = f_+(z-a) + f_-\left(\frac{1}{z-a}\right).$$

In particular there are complex numbers $a_n, n \in \mathbb{Z}$, such that

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-a)^n := \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{\ell=1}^{\infty} a_{-\ell}(z-a)^{-\ell}.$$

A series of the above type is called a Laurent series, the first part is called the power series part of the Laurent series, the second one its principal part. Indeed

$$a_n = \frac{1}{2\pi i} \int_{\partial D_s(a)} \frac{f(z)dz}{(z-a)^{n+1}}, \ \varrho < s < r.$$

Corollary 9.13. *Let $G \subset \mathbb{C}$ be a domain, $a \in G$. If the function $f \in \mathcal{O}(G \setminus \{a\})$ has the Laurent expansion $f(z) = \sum_{n=-\infty}^{\infty} a_n(z-a)^n$ in $D_r(a)^*$, then its residue at a satisfies*

$$\mathrm{Res}_a(f) = a_{-1}.$$

Let us first finish the proof of Th.9.11. In the case of $G = D_r(a), S = \{a\}$, the decomposition of Th. 9.11 is given by considering $D_r(a)^* = A_{0,r}(a)$ and taking

$$g(z) = f_+(z-a), \ h(z) = \sum_{n=-2}^{-\infty} a_n(z-a)^n, \ c_a = a_{-1}.$$

Note that $g \in \mathcal{O}(D_r(a))$ and $h \in \mathcal{O}(\mathbb{C} \setminus \{a\})$. In the general case of Th.9.11 let $c_a := \mathrm{Res}_a(f)$ for $a \in S$ and denote $h_a(z)$ the principal part of the Laurent series of f in some annulus $A_{0,r}(a)$ minus $c_a/(z-a)$. Then we set

$$h(z) := \sum_{a \in S} h_a(z), g(z) := f(z) - h(z) - \sum_{a \in S} \frac{c_a}{z-a}.$$

Obviously all the h_a are derivatives and thus h as well. This finishes the proof of Th. 9.11. $\square$

Proof of Th.9.12. Fix $z \in A_{\varrho,r}(a)$, choose $\varepsilon > 0$ with $\varrho + \varepsilon < |z| < r - \varepsilon$. According to Ex.9.6.2 the cycle $\underline{\lambda} = (\partial D_{r-\varepsilon}(a), \partial D_{\varrho+\varepsilon}(a)^{-1})$ is nullhomolgous and $\mathrm{ind}_{\underline{\lambda}}(z) = 1$, hence

$$f(z) = \frac{1}{2\pi i} \int_{\underline{\lambda}} \frac{f(\zeta)d\zeta}{\zeta - z}.$$

Define

$$f_+(z) := \frac{1}{2\pi i} \int_{\partial D_{r-\varepsilon}(a)} \frac{f(\zeta)d\zeta}{\zeta - z}$$

and

$$f_-(z) := -\frac{1}{2\pi i}\int_{\partial D_{\varrho+\varepsilon}(a)} \frac{f(\zeta)d\zeta}{\zeta - z}.$$

The power series expansion for $f_+(z)$ is obtained as in the proof of Th. 7.6, while for f_- we write

$$-\frac{f(\zeta)}{\zeta - z} = \frac{1}{z-a}\cdot\frac{f(\zeta)}{1-\frac{\zeta-a}{z-a}} = \sum_{n=0}^{\infty}\frac{f(\zeta)(\zeta-a)^n}{(z-a)^{n+1}}$$

and integrate with respect to ζ over $\partial D_{\varrho+\varepsilon}(a)$. Note that the integrals do not depend on the choice of ε. $\square$

The discussion in the lectures

1. how to compute explicitly a residue $\mathrm{Res}_a(f)$, and
2. how to relate certain real integrals to loop integrals in the complex plane,

will be quite close to that one in the text book, pp. 314 - 354; so we don't give any comments here.

Instead we discuss briefly isolated singularities of holomorphic functions:

Definition 9.14. Let $G \subset \mathbb{C}$ be a domain. A point $a \in \mathbb{C} \setminus G$ is called an isolated boundary point of G if $D_r(a)^* \subset G$ for sufficiently small $r > 0$.

Note that for an isolated boundary point a of G the union $G_a := G \cup \{a\}$ is a domain as well. There are three essentially different possible behaviours of functions $f \in \mathcal{O}(G)$ near an isolated boundary point:

Definition 9.15. Let $f \in \mathcal{O}(G)$ be a holomorphic function on the domain G. An isolated boundary point $a \in \mathbb{C}$ of G is called

1. a *removable singularity* of f if there is a function $\widehat{f} \in \mathcal{O}(G_a)$ extending f.
2. a *pole* of f if a is not a removable singularity, but there is some $n \in \mathbb{N}_{>0}$, auch that the function $g(z) = (z-a)^n f(z)$ has at a a removable singularity.
3. an *essential singularity* of f otherwise.

Theorem 9.16. *Let $f \in \mathcal{O}(G)$ be a holomorphic function on the domain G and $a \in \mathbb{C}$ an isolated boundary point of G with $D_r(a)^* \subset G$, write $f(z) = \sum_{n=-\infty}^{\infty} a_n(z-a)^n$. Then the point a is*

1. *a removable singularity iff $f|_{D_\varrho(a)^*}$ is bounded for some $\varrho > 0$ iff $a_n = 0$ for $n < 0$,*

2. *a pole iff $\lim_{z\to a} f(z) = \infty$ iff $a_n = 0$ for finitely many, but at least one $n < 0$,*

3. *an essential singularity iff $f(D_\varrho(a)^*)$ is dense in $\mathbb{C}$ for all $\varrho < r$ iff $a_n \neq 0$ for infinitely many $n < 0$.*

Proof. Obviously a is a removable singularity of f if the principal part of its Laurent series vanishes, and in that case f is bounded near a. On the other hand, if f is bounded, then $(z-a)f(z)dz \in \mathcal{D}(D_\varrho(a))$ is integrable on $D_\varrho(a)^*$ resp. $D_\varrho(a)$ and thus $(z-a)f(z) = F'(z)$ with a function $F \in \mathcal{O}(D_\varrho(a))$. We must have $F'(a) = 0$ — otherwise f would not be bounded near a — and thus $f(z)$ is holomorphic near a as well.

Second, the principal part f_- is a nonzero polynomial in $(z-a)^{-1}$ iff $(z-a)^n f(z)$ extends to G_a for some $n \in \mathbb{N}$. In that case choosing n minimal yields $a_{-n} \neq 0$, i.e. $f(z) = g(z)/(z-a)^n, g(a) \neq 0$. Hence $\lim_{z\to a} f(z) = \infty$. On the other hand, if that is true, we have $f(D_\varepsilon(a)^*) \subset \mathbb{C} \setminus \overline{D}_1(0)$ for some $\varepsilon > 0$, hence $1/f$ is a holomorphic function bounded near a, and thus holomorphic near a. So f itself has at most a pole at a.

Third, assume f_- is not a polynomial in $(z-a)^{-1}$ and $f(D_\varrho(a)^*) \cap D_\varepsilon(b) = \emptyset$. Then $g := \frac{1}{b-f}$ is bounded on $D_\varrho(a)^*$, hence holomorphic near a. But then $f = b - \frac{1}{g}$ has at most a pole at a.

□

The third point of Th.9.16 indicates a quite irregular behaviour of a holomorphic function near an essential singularity. We mention without proof the famous Picard Theorem:

Theorem 9.17. *Let $a \in \partial G$ be an isolated boundary point of G and an essential singularity of the function $f \in \mathcal{O}(G)$. Then for all $b \in \mathbb{C}$ with possibly one exception we have*

$$|f^{-1}(b) \cap D_\varepsilon(a)^*| = \infty$$

for all $\varepsilon > 0$.

The notion of a rational function, being a quotient of polynomials, has a natural generalization:

Definition 9.18. Let G be a domain. A meromorphic function on G is a function $f : G \longrightarrow \widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ such that

1. the set $P := f^{-1}(\infty) \subset G$ has no points of accumulation in G,
2. $f|_{G\setminus P} \in \mathcal{O}(G \setminus P)$, and
3. the points $a \in f^{-1}(\infty)$ are poles of f.

We denote

$$\mathcal{M}(G) := \{f : G \longrightarrow \widehat{\mathbb{C}} \text{ meromorphic}\}$$

the set of all meromorphic functions on G.

Remark 9.19. Meromorphic functions can be added, multiplied, and, if $\not\equiv 0$, inverted: Given f, g we define $f + g$ and fg first as holomorphic functions on $G \setminus (f^{-1}(\infty) \cup g^{-1}(\infty))$, the points in $f^{-1}(\infty) \cup g^{-1}(\infty)$ being either poles or removable singularities. Accordingly we extend it to G. If $f \not\equiv 0$, its zeros form a set without accumulation points in G, they become the poles of $\frac{1}{f}$, while its poles become the zeros of $\frac{1}{f}$. So the set $\mathcal{M}(G)$ is a field!

The local behaviour of a meromorphic function $f \in \mathcal{M}(G) \setminus \{0\}$ near a point $a \in G$ is determined by its order:

Definition 9.20. Let $f \in \mathcal{M}(G) \setminus \{0\}$ be a meromorphic function. The order

$$\mathrm{ord}_a(f) := \ell \in \mathbb{Z}$$

of f at $a \in G$ is defined as the integer ℓ such that

$$f(z) = \sum_{n=\ell}^{\infty} a_n (z-a)^n$$

with $a_\ell \neq 0$, or equivalently

$$f(z) = (z-a)^\ell g(z)$$

with a function g holomorphic near a and $g(a) \neq 0$.

So the poles of a meromorphic function are treated as zeros of negative order. Furthermore for function holomorphic at $a \in G$ its multiplicity there is nothing but $\mathrm{ord}_a(f - f(a))$.

For a function $f \in \mathcal{M}(G) \setminus \{0\}$, the meromorphic function $\frac{f'}{f}$ is called the logarithmic derivative of f. It has only simple poles, namely at the zeros and poles of f.

Proposition 9.21. *1. For $f \in \mathcal{M}(G) \setminus \{0\}$ one has*

$$\mathrm{Res}_a(f'/f) = \mathrm{ord}_a(f).$$

2. If G is simply connected, then every function $g \in \mathcal{M}(G)$ with only simple poles and integral residues is the logarithmic derivative of some function $f \in \mathcal{M}(G)$.

Proof. 1) We write

$$f(z) = (z-a)^\ell h(z), \;\; h(a) \neq 0$$

and apply

$$\frac{(gh)'}{gh} = \frac{g'}{g} + \frac{h'}{h}.$$

We obtain

$$\frac{f'}{f} = \frac{\ell}{z-a} + \frac{h'}{h},$$

whence the result, the second term being holomorphic at a.

2) Fix a point $z_0 \in G_0 := G \backslash P$, where P is the set of poles of the meromorphic function g. Then for a path $\gamma_z : [a,b] \longrightarrow G_0$ from z_0 to z the integral

$$\int_{\gamma_z} g(\zeta) d\zeta$$

depends, as a consequence of the residue theorem, up to an integer multiple of $2\pi i$ only on z_0 and z. Thus we may define a function $f \in \mathcal{O}(G_0)$ by

$$f(z) := \exp\left(\int_{\gamma_z} g(\zeta) d\zeta \right).$$

$\square$

Definition 9.22. A continuous loop $\lambda : [a, b] \longrightarrow \mathbb{C}$ is called *simple* or a *simply closed path* or a *Jordan curve* if $\lambda|_{[a,b)}$ is injective.

Though intuitive, the following remark is nontrivial:

Remark 9.23. 1. The Jordan curve theorem tells us, that the complement of the trace of a simple loop λ is the union of two disjoint connected open sets:

$$\mathbb{C} \setminus |\lambda| = \operatorname{int}(\lambda) \cup \operatorname{ext}(\lambda),$$

called the interior resp.the exterior of λ, where $\operatorname{int}(\lambda)$ is bounded and $\operatorname{ext}(\lambda)$ is not.

2. We have

$$\operatorname{ind}_\lambda|_{\operatorname{ext}(\lambda)} \equiv 0$$

and either

$$\operatorname{ind}_\lambda|_{\operatorname{int}(\lambda)} \equiv 1 \ \text{ or } \operatorname{ind}_\lambda|_{\operatorname{int}(\lambda)} \equiv -1.$$

In the first case we say that λ is positively oriented.

3. A simple loop $\lambda : [a, b] \longrightarrow G$ is nullhomologous in G iff $\operatorname{int}(\lambda) \subset G$. The condition is necessary: For a point $a \in \operatorname{int}(\lambda) \setminus G$ the function $f(z) = \frac{1}{z-a}$ is holomorphic on G and satisfies $\int_\lambda f(z)dz \neq 0$. On the other hand sufficiency follows from Th.10.1 applied to $K := |\lambda| \cup \operatorname{int}(\lambda)$: Any function $f \in \mathcal{O}(G)$ is on K the uniform limit of a sequence of polynomials $p_n : \mathbb{C} \longrightarrow \mathbb{C}$. But $\int_\lambda f(z)dz = \lim_{n\to\infty} \int_\lambda p_n(z)dz = \lim_{n\to\infty} 0 = 0$.

The above remark is mainly of theoretical interest; in actual computations the statements are easily verified. So we do not comment on the demanding proof. —

The power function $f_n \in \mathcal{M}(\mathbb{C})$ with $f_n(z) = z^n$ has at 0 a zero of order n and transforms the simple closed loop $\lambda = \partial D_r(0)$ into the loop $\lambda_n := f_n \circ \lambda$ winding around the origin n times:

$$\operatorname{ind}_{f_n \circ \lambda}(0) = n.$$

In the next theorem that formula is generalized to meromorphic functions defined on a neighbourhood of $|\lambda| \cup \operatorname{int}(\lambda)$, with no zeros or poles on $|\lambda|$.

Theorem 9.24 (**Argument Principle**). *Let $f \in \mathcal{M}(G)$ and λ be a simple positively oriented loop in G with* $\mathrm{int}(\lambda) \subset G$*, such that no poles and no zeros of f lie on $|\lambda|$. Then*

$$\mathrm{ind}_{f\circ\lambda}(0) = \frac{1}{2\pi i}\int_\lambda \frac{f'(z)dz}{f(z)} = \sum_{a\in \mathrm{int}(\lambda)} \mathrm{ord}_a(f),$$

i.e. the winding number of the image loop $f \circ \lambda$ around the origin equals the number of the zeros and poles of f inside the loop λ counted with multiplicities.

Proof. Apply Th.9.11 and Prop.9.9. □

Theorem 9.25 (**Rouché's Theorem**). *Let $f, h \in \mathcal{O}(G)$ and λ be a nullhomologous simple loop in G such that no zeros of f lie on $|\lambda|$. If $|h(z)| < |f(z)|$ for $z \in |\lambda|$, then f and $g := f + h$ have the same number of zeros (counted with multiplicities) in* $\mathrm{int}(\lambda)$.

Proof. The paths $f \circ \lambda$ and $g \circ \lambda$ are homotopic in $\mathbb{C}^*$ with the homotopy

$$(t, s) \mapsto f(\lambda(t)) + sh(\lambda(t)).$$

Hence

$$\frac{1}{2\pi i}\int_\lambda \frac{f'(z)dz}{f(z)} = \mathrm{ind}_{f\circ\lambda}(0) = \mathrm{ind}_{g\circ\lambda}(0) = \frac{1}{2\pi i}\int_\lambda \frac{g'(z)dz}{g(z)}.$$

□

10 Construction of holomorphic and meromorphic functions

In this section we discuss briefly two famous theorems dealing with the distribution of zeros resp. singularities of holomorphic resp. meromorphic functions on a domain G. First we need:

Theorem 10.1 (**Theorem of Runge**). *Let $K \subset \mathbb{C}$ be a compact set with connected complement $\mathbb{C} \setminus K$. Then every holomorphic function $f : U \longrightarrow \mathbb{C}$ defined on an open neighbourhood $U \supset K$ (it need not be connected!) can on K be uniformly approximated by polynomials: Given $\varepsilon > 0$, there is a polynomial function $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ with $||f - p||_K < \varepsilon$.*

Remark 10.2. 1. Th.10.1 is trivial if $K = \overline{D}_r(a)$ is a closed disc. Then we may take $p(z) = \sum_{\nu=0}^{n} c_\nu (z-a)^\nu$, where $f(z) = \sum_{\nu=0}^{\infty} c_\nu (z-a)^\nu$ is the Taylor expansion of f around a and $n \gg 0$.

2. A compact set K is called polynomially convex if $\mathbb{C} \setminus K$ is connected. To any compact set K we can associate its polynomially convex hull $\hat{K}$: It is defined as
$$\hat{K} := \mathbb{C} \setminus E,$$
where E is the unbounded connected component of $\mathbb{C} \setminus K$. So $\hat{K}$ is the union of K and the holes in K, i.e. the bounded connected components of $\mathbb{C} \setminus K$.

3. By Mergelyan's theorem we know that Runge's theorem holds even for functions $f \in C(K) \cap \mathcal{O}(\mathring{K})$, i.e. functions continuous on K and holomorphic in the interior $\mathring{K}$ of K.

Sketch of proof. First we need that there is a cycle $\underline{\lambda}$ in $U \setminus K$, such that $\mathrm{ind}_{\underline{\lambda}}(z) = 1$ for all $z \in K$. As a consequence
$$f(z) = \frac{1}{2\pi i} \int_{\underline{\lambda}} \frac{f(\zeta)d\zeta}{\zeta - z}$$
holds for $z \in K$ and thus $f|_K$ may be uniformly approximated by functions
$$\sum_{\nu=1}^{r} \frac{c_\nu}{a_\nu - z},$$
where $a_1, ..., a_r \in |\underline{\lambda}|$. Namely: The above integral is a finite sum of integrals
$$f_\gamma(z) := \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)d\zeta}{\zeta - z}$$
over smooth paths $\gamma : [a, b] \longrightarrow U \setminus K$, and their Riemann sums are functions on K of the above type and approximate the function f_γ uniformly on K. So
$$f = \sum_\gamma f_\gamma$$
is as well the uniform limit of linear combinations of degree one Laurent monomials.

Now for $b \in \mathbb{C} \setminus K$ denote

$$\mathcal{L}_b(K)$$

the set of all functions on K being the uniform limit of polynomials in $\frac{1}{z-b}$. We show that

$$\frac{1}{a-z} \in \mathcal{L}_b(K)$$

holds for all $a \in \mathbb{C} \setminus K$, or, equivalently,

$$\mathcal{L}_a(K) \subset \mathcal{L}_b(K).$$

In particular we obtain

$$f|_K \in \mathcal{L}_b(K).$$

Now choose a closed disc $\overline{D} := \overline{D}_r(0) \supset K$ and $b \notin \overline{D}$. Denote $q(z)$ a polynomial in $\frac{1}{z-b}$ with $||f - q||_K < \varepsilon/2$ and take $p(z)$ as the n-th Taylor polynomial of q around 0. Then $||q - p||_{\overline{D}} < \varepsilon/2$ for $n \gg 0$ and thus $||f - p||_K < \varepsilon$.

It remains to show that all degree one Laurent monomials $\frac{1}{a-z}$ with $a \in \mathbb{C} \setminus K$ belong to $\mathcal{L}_b(K)$. We consider the set

$$V_b := \left\{ a \in \mathbb{C} \setminus K; \frac{1}{a-z} \in \mathcal{L}_b(K) \right\}.$$

Since $b \in V_b$, we have $V_b \neq \emptyset$. Furthermore

$$a \in V_b \Longrightarrow D_r(a) \subset V_b$$

with $r = \operatorname{dist}(a, K)$. As a consequence a path in $\mathbb{C} \setminus K$ starting at b never leaves V_b, i.e. $V_b = \mathbb{C} \setminus K$.

Indeed let $c \in D_r(a)$. Then, for $|z - a| > |c - a|$ we have

$$\frac{1}{c-z} = -\sum_{\nu=0}^{\infty} \frac{(c-a)^\nu}{(z-a)^{\nu+1}} \in \mathcal{L}_a(K) \subset \mathcal{L}_b(K),$$

since the right hand side converges uniformly on $\mathbb{C} \setminus D_\varrho(0)$ for any $\varrho > |c - a|$. $\square$

Theorem 10.3 (Theorem of Mittag-Leffler). *Let $G \subset \mathbb{C}$ be a domain. Given a sequence $(a_n)_{n\in\mathbb{N}} \subset G$ without accumulation point in G and polynomials*

$$p_n(z) = \sum_{k=1}^{r_n} \frac{c_{nk}}{(z-a_n)^k}$$

in $\frac{1}{z-a_n}$ without constant term, there is a meromorphic function having the points a_n as its poles with $p_n(z)$ as the principal part of its Laurent series around the point $a_n \in G$.

Proof. The general idea is to write

$$f(z) = \sum_{n=0}^{\infty} (p_n(z) + g_n(z)),$$

where the functions $g_n \in \mathcal{O}(G)$ are chosen in such a way, that the above sequence converges uniformly on every compact set K: That makes sense, since there is some $n_K \in \mathbb{N}$, such that all p_n with $n \geq n_K$ are holomorphic around K. The above sum is the canonical form of $f(z)$ in most of all cases; nevertheless in the proof we construct a, for technical reasons, slightly different series.

We sketch the proof for a simply connected domain G. Given a compact set $K \subset G$ we have as well $\hat{K} \subset G$ for its polynomially convex hull $\hat{K}$, cf. Rem.10.2. To see that we have to show $F \subset G$ for all bounded connected components F of $\mathbb{C} \setminus K$, cf. Th.6.23. Given such F, we write

$$\mathbb{C} \setminus G = A \,\dot{\cup}\, L$$

as the disjoint union of the compact set

$$L := (\mathbb{C} \setminus G) \cap F$$

and the closed set

$$A = (\mathbb{C} \setminus G) \setminus L.$$

Since G is simply connected, it follows $L = \emptyset$ and thus $F \subset G$. Using that remark one easily constructs an exhaustion of G by polynomially convex compacts sets $K_\nu \subset G$, where we assume that even $K_\nu \subset \mathring{K}_{\nu+1}$. Let

$$q_\nu(z) := \sum_{a_n \in K_{\nu+1} \setminus K_\nu} p_n(z).$$

The function q_ν being holomorphic in a neighbourhood of K_ν, Runge's theorem Th.10.1 provides a polynomial $h_\nu : \mathbb{C} \longrightarrow \mathbb{C}$ satisfying

$$||q_\nu - h_\nu||_{K_\nu} < 2^{-\nu}.$$

Finally set

$$f(z) = \sum_{\nu=0}^{\infty} (q_\nu(z) - h_\nu(z)).$$

$\square$

Example 10.4. 1. Take $G = \mathbb{C}$, $a_n = -n$ (where $n \in \mathbb{N}$) and $p_n := \frac{1}{z+n}$. Then we can choose $g_0 = 0$ and $g_n(z) = -\frac{1}{n}$ for $n \geq 1$, the resulting function is

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z+n} - \frac{1}{n} \right).$$

2. Take the above example, but replace the indexing set $\mathbb{N}$ with $\mathbb{Z}$. We obtain the function

$$\pi \cot(\pi z) = \frac{1}{z} + \sum_{n \in \mathbb{Z}^*} \left(\frac{1}{z+n} - \frac{1}{n} \right),$$

where $\mathbb{Z}^* := \mathbb{Z} \setminus \{0\}$.

3. Consider a lattice

$$\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2,$$

where the complex numbers $\omega_1, \omega_2 \in \mathbb{C}$ are linearly independent over $\mathbb{R}$. We use Λ itself as indexing set, i.e. $a_\omega = \omega$, take

$$p_\omega(z) = \frac{1}{(z-\omega)^2}$$

and $g_\omega(z) = \frac{1}{\omega^2}$. The resulting function

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda^*} \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2},$$

where $\Lambda^* := \Lambda \setminus \{0\}$, is called Weierstraß' $\wp$-function. It is Λ-periodic, i.e. satisfies

$$\wp(z+\omega) = \wp(z)$$

for all lattice points $\omega \in \Lambda$.

Theorem 10.5 (Weierstraß factorization theorem). *Let $G \subset \mathbb{C}$ be a domain. Given a sequence $(a_n)_{n\in\mathbb{N}} \subset G$ without accumulation point in G and positive integers $\ell_n \in \mathbb{N}_{>0}$, there is a holomorphic function, whose zeros are exactly the points $a_n \in G$ with corresponding zero order* $\mathrm{ord}_{a_n}(f) = \ell_n$.

Proof. For a simply connected domain G we can argue as follows: First we apply Th.10.3 with $p_n(z) := \frac{\ell_n}{z}$ and obtain a function $g \in \mathcal{M}(G)$ with only simple poles and integer residues. Then Prop.9.21.2 provides a function $f \in \mathcal{O}(G)$ with g as its logarithmic derivative. □

Example 10.6. We are looking for a function, which has simple zeros exactly at the points in $\mathbb{Z}_{\leq 0}$. We remember the proof of Prop.9.21.2 and apply the same strategy with $z_0 = 0$: The summand

$$p_n(z) + g_n(z) = \frac{1}{z+n} - \frac{1}{n}$$

of Ex.10.4.1 is the logarithmic derivative of

$$\left(1 + \frac{z}{n}\right) e^{-z/n}.$$

Thus the function

$$f(z) = z \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}$$

is a solution of our problem. It satisfies the functional equation

$$zf(z+1) = e^{-C} f(z)$$

with "Euler's constant"

$$C := \lim_{m\to\infty} \left(\sum_{n=1}^{m} \frac{1}{n} - \ln(m+1) \right).$$

Hence the Gamma function

$$\Gamma(z) := \frac{e^{-Cz}}{f(z)}$$

satisfies

$$\Gamma(z+1) = z\Gamma(z)$$

as well as $\Gamma(0) = 1$.

Corollary 10.7. *Every meromorphic function $f \in \mathcal{M}(G)$ can be written $f = g/h$ as the quotient of holomorphic functions $g, h \in \mathcal{O}(G)$. With other words: The field $\mathcal{M}(G)$ of meromorphic functions on G is the field of fractions of the integral domain $\mathcal{O}(G)$, i.e.*

$$\mathcal{M}(G) = \mathbb{Q}(\mathcal{O}(G)).$$

Proof. Let $(a_n)_{n\in\mathbb{N}}$ be the set of poles of f, $\ell_n := -\mathrm{ord}_{a_n}(f)$. Denote $h \in \mathcal{O}(G)$ a holomorphic function, which has a zero of order ℓ_n at a_n for every $n \in \mathbb{N}$ (and, w.l.o.g., no further zeros). Take $g := fh$. □

11 The Dirichlet Problem

Let us start with the following question: Given a function $u : G \longrightarrow \mathbb{R}$, when can we find another function $v : G \longrightarrow \mathbb{R}$, such that $f = u + iv : G \longrightarrow \mathbb{C}$ is holomorphic?

Theorem 11.1. *Let $G \subset \mathbb{C}$ be a simply connected domain. Then a real valued function $u \in C^2(G)$ is the real part of some holomorphic function $f = u + iv \in \mathcal{O}(G)$ if and only if u is harmonic, i.e.*

$$\Delta u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Before we give the easy proof, we introduce for systematic reasons the following conjugation operation on differential forms:

Remark 11.2. For a differential form $\omega \in \mathcal{D}(G)$ we define its conjugate form $\omega^* \in \mathcal{D}(G)$ by

$$(\omega^*)_z(h) := \overline{\omega}_z(-ih).$$

Indeed for

$$\omega = f dx + g dy = h_1 dz + h_2 d\overline{z}$$

we find

$$\omega^* = -\overline{g} dx + \overline{f} dy = i(-\overline{h}_2 dz + \overline{h}_1 d\overline{z}).$$

Proof of Th.11.1. The equation $\Delta u = 0$ means that the differential form

$$(du)^* = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \ \in \mathcal{D}^1(G)$$

is closed. Since G is simply connected that is equivalent to the existence of a primitive function $v \in C^2(G)$, i.e. such that

$$(du)^* = dv,$$

the latter equality being nothing but the Cauchy-Riemann equations for $f = u + iv$. $\square$

Remark 11.3. We note that

$$\Delta u = 4 \cdot \frac{\partial}{\partial z}(\frac{\partial u}{\partial \overline{z}}) = 4 \cdot \frac{\partial}{\partial \overline{z}}(\frac{\partial u}{\partial z}).$$

In particular a real polynomial is harmonic if and only if it does not contain mixed terms $z^\mu \overline{z}^\nu$ (with both $\mu, \nu > 0$). With other words, harmonic real polynomials are those which can be written $f(z) + g(\overline{z})$ with complex polynomials $f, g : \mathbb{C} \longrightarrow \mathbb{C}$. Furthermore,

$$du + i(du)^* = 2\frac{\partial u}{\partial z} dz$$

for a real valued function $u : G \longrightarrow \mathbb{R}$.

Here is an example showing what may happen on non simply connected domains:

Example 11.4. We consider the annulus $A_{\varrho,r}(a)$.

1. The harmonic function $\ln|z-a|$ is not the real part of some function $f \in \mathcal{O}(A_{\varrho,r}(a))$. But up to that failure everything is fine:

2. In the annulus $A_{\varrho,r}(a)$ any real valued harmonic function u can be written
$$u(z) = c \ln|z-a| + \mathrm{Re}(f(z))$$
with a holomorphic function $f \in \mathcal{O}(A_{\varrho,r}(a))$ and a unique $c \in \mathbb{R}$. For the proof we may assume $a = 0$. Take
$$c := \frac{1}{2\pi} \int_{\partial D_s(0)} (du)^* \in \mathbb{R}.$$
Now we apply Prop.6.25 to the differential form
$$\omega = du + i(du)^* - \frac{cdz}{z} = \left(2\frac{\partial u}{\partial z} - \frac{c}{z}\right) dz$$

- it is closed since $(du)^*$ is - and find a function $f \in \mathcal{O}(A_{\varrho,r}(0))$ with

$$df = \omega.$$

Taking real parts finally yields

$$d\,\mathrm{Re}(f) = du - c\mathrm{Re}\left(\frac{dz}{z}\right),$$

and thus, w.l.o.g.,

$$\mathrm{Re}(f(z)) = u - c\ln|z|.$$

Harmonic functions play an important rôle in physics: They can be regarded as the potential of a stationary electric field or the temperature in a steady state-heat flow. In such a situation the following problem occurs in a natural way:

Dirichlet Problem: Let $G \subset \mathbb{C}$ be a domain and $u_0 : \partial G \longrightarrow \mathbb{R}$ a continuous function. Find a continuous function $u : \overline{G} \longrightarrow \mathbb{R}$, harmonic in G, such that

$$u|_{\partial G} = u_0 \ .$$

In this section we solve the Dirichlet problem on an open disc and then give some comments on the general case.

We start with an idea originating from the above relationship between harmonic and holomorphic functions and the Cauchy formula: For a holomorphic function f defined on a domain containing the closed disc $\overline{D}$ the Cauchy formula

$$f(z) = \frac{1}{2\pi i}\int_{\partial D}\frac{f(\zeta)d\zeta}{\zeta - z}$$

describes the values $f(z), z \in D$, in terms of the boundary values $f|_{\partial D}$. Now, if we could rewrite the right hand side of the Cauchy formula in terms of $u = \mathrm{Re}(f)$ only, we would obtain a good candidate for a solution $u : D \longrightarrow \mathbb{R}$ of the Dirichlet problem by applying the revised Cauchy formula to the given function $u_0 : \partial D \longrightarrow \mathbb{R}$ and then taking $u = \mathrm{Re}(f)$. Of course, then it remains to be shown that for $z \to w \in \partial D$ one has $u(z) \to u_0(w)$.

We study the above situation for $D = D_r(0)$. In some slightly bigger disc $D_{r+\varepsilon}(0)$ we have

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Now we express the coefficients $a_n, n > 0$, in terms of $u = \mathrm{Re}(f)$ only, using the Fourier expansion of the function $\vartheta \mapsto u(re^{i\vartheta})$ which is obtained from the above power series as follows: In the equation

$$u(z) = \mathrm{Re}(a_0) + \frac{1}{2}\left(\sum_{n=1}^{\infty} a_n z^n + \overline{a}_n \overline{z}^n\right)$$

we substitute $z = re^{i\vartheta}$ and obtain the Fourier series

$$u(re^{i\vartheta}) = \mathrm{Re}(a_0) + \frac{1}{2}\left(\sum_{n=1}^{\infty} a_n r^n e^{in\vartheta} + \sum_{n=1}^{\infty} \overline{a}_n r^n e^{-in\vartheta}\right).$$

Its coefficients satisfy

$$\mathrm{Re}(a_0) = \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\vartheta})d\vartheta$$

and

$$a_n r^n = \frac{1}{\pi}\int_0^{2\pi} u(re^{i\vartheta})e^{-in\vartheta}d\vartheta$$

for $n > 0$. Substituting this in the Taylor series expansion of f and interchanging summation and integration we obtain the desired integral formula:

Theorem 11.5 (Schwarz' integral formula). *Let $f = u + iv \in \mathcal{O}(G)$ and $G \supset \overline{D}_r(0)$. Then for $z \in D := D_r(0)$ we have*

$$\begin{aligned} f(z) &= iv(0) + \frac{1}{2\pi i}\int_{\partial D} u(\zeta)\frac{\zeta + z}{\zeta - z}\frac{d\zeta}{\zeta} \\ &= iv(0) + \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\vartheta})\frac{re^{i\vartheta} + z}{re^{i\vartheta} - z}d\vartheta. \end{aligned}$$

Proof. For fixed $z \in D_r(0)$ the series

$$u(re^{i\vartheta}) + 2\sum_{n=1}^{\infty} u(re^{i\vartheta}) \left(\frac{z}{re^{i\vartheta}}\right)^n$$

converges uniformly on the interval $[0, 2\pi]$, thus integration and summation may be interchanged and we obtain

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = iv(0) + \mathrm{Re}(a_0) + \sum_{n=1}^{\infty} a_n z^n$$

$$= iv(0) + \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\vartheta})d\vartheta + \sum_{n=1}^{\infty} \frac{1}{\pi}\int_0^{2\pi} u(re^{i\vartheta})e^{-in\vartheta}d\vartheta \cdot \left(\frac{z}{r}\right)^n$$

$$= iv(0) + \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\vartheta}) \left(1 + 2\sum_{n=1}^{\infty} \left(\frac{z}{re^{i\vartheta}}\right)^n\right) d\vartheta$$

$$= iv(0) + \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\vartheta}) \left(\frac{2}{1 - \frac{z}{re^{i\vartheta}}} - 1\right) d\vartheta$$

$$= iv(0) + \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\vartheta})\frac{re^{i\vartheta} + z}{re^{i\vartheta} - z}d\vartheta.$$

□

Finally, taking real parts of both sides we obtain:

Theorem 11.6 (Poisson integral formula). *Let $u : G \longrightarrow \mathbb{R}$ be a harmonic function, $G \supset \overline{D}$ with $D := D_r(0)$. Then*

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\vartheta})\frac{r^2 - |z|^2}{|re^{i\vartheta} - z|^2}d\vartheta$$

$$= \frac{1}{2\pi i}\int_{\partial D} u(\zeta)\frac{r^2 - |z|^2}{|\zeta - z|^2}\frac{d\zeta}{\zeta}.$$

holds for all $z \in D_r(0)$.

And here is the solution of the Dirichlet problem on a disc:

Theorem 11.7. *Let $D := D_r(0)$ and $u_0 : \partial D \longrightarrow \mathbb{R}$ be a bounded function, continuous with, may be, finitely many exceptions. Then there is a unique bounded harmonic function $u : D \longrightarrow \mathbb{R}$, such that*

$$\lim_{D \ni z \to w} u(z) = u_0(w)$$

for all points $w \in \partial D$, where u_0 is continuous. Indeed

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u_0(re^{i\vartheta}) \frac{r^2 - |z|^2}{|re^{i\vartheta} - z|^2} d\vartheta$$

holds for $z \in D$.

Remark 11.8. If u_0 is continuous, it is bounded, ∂D being compact. Furthermore the function $\widehat{u} : \overline{D} \longrightarrow \mathbb{R}$ with $\widehat{u}|_D = u, \widehat{u}|_{\partial D} = u_0$ is continuous and thus bounded as well. So the adjective "bounded" can be omitted in that case.

But we need it in case we admit discontinuities: Take $u_0(re^{i\vartheta}) = \vartheta$ for $0 \le \vartheta < 2\pi$. Then, given a bounded solution $u : D \longrightarrow \mathbb{R}$ another unbounded solution is $u(z) + g(z)$ with

$$g(z) := \frac{|z|^2 - 1}{|z-1|^2} = \mathrm{Re}\left(\frac{z+1}{z-1}\right).$$

Proof. Uniqueness: Let $r_n = r - r/n$. For fixed $z \in D$ the uniformly bounded functions

$$[0, 2\pi] \in \vartheta \mapsto u(r_n e^{i\vartheta}) \frac{r_n^2 - |z|^2}{|r_n e^{i\vartheta} - z|^2}$$

converge almost everywhere to

$$[0, 2\pi] \in \vartheta \mapsto u_0(re^{i\vartheta}) \frac{r^2 - |z|^2}{|re^{i\vartheta} - z|^2}.$$

So, by the Lebesgue dominated convergence theorem we know

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(r_n e^{i\vartheta}) \frac{r_n^2 - |z|^2}{|r_n e^{i\vartheta} - z|^2} d\vartheta \to \frac{1}{2\pi} \int_0^{2\pi} u_0(re^{i\vartheta}) \frac{r^2 - |z|^2}{|re^{i\vartheta} - z|^2} d\vartheta.$$

Existence: The function given by the Poisson formula is harmonic as the real part of the holomorphic function f given by the Schwarz formula. We have to show that $D \ni z_n \to w \in \partial D$ implies $u(z_n) \to u_0(w)$, if u_0 is continuous at w. We consider the sequence of functions

$$\varphi_n : \partial D \longrightarrow \mathbb{R}, \zeta \mapsto \frac{1}{2\pi} \frac{r^2 - |z_n|^2}{|\zeta - z_n|^2}.$$

They form a "Dirac sequence" with respect to $w \in \partial D$, i.e.

1. $\varphi_n \geq 0$,
2. $\int_0^{2\pi} \varphi_n(re^{i\vartheta}) d\vartheta = 1 \ \forall n \in \mathbb{N}$,
3. $\varphi_n \to 0$ uniformly on $\partial D \setminus D_\delta(w)$ for every $\delta > 0$.

The first condition is satisfied because of $|z_n| < r$, and the second one follows from Poissons formula with $u \equiv 1$. Finally

$$\varphi_n(\zeta) \leq \frac{1}{2\pi} \frac{r^2 - |z_n|^2}{(\delta - |w - z_n|)^2}$$

for $\zeta \in \partial D \setminus D_\delta(w)$, with the right hand side tending to 0 for $n \to \infty$. We have used the following estimate for the denominator:

$$|\zeta - z_n| = |(\zeta - w) - (z_n - w)| \geq |\zeta - w| - |z_n - w| \geq \delta - |z_n - w|.$$

So we have

$$u(z_n) = \int_{\partial D} u_0(\zeta) \varphi_n(\zeta) \frac{d\zeta}{i\zeta}.$$

Now take some $\varepsilon > 0$. Since u_0 is continuous at w, there is a $\delta > 0$ such that

$$|u_0(\zeta) - u_0(w)| < \frac{\varepsilon}{2}$$

for $\zeta \in \overline{D}_\delta(w) \cap \partial D$. Choose $n_0 \in \mathbb{N}$ such that

$$|\varphi_n(\zeta)| \leq \frac{\varepsilon}{4||u_0||}, \quad \forall \zeta \in \partial D \setminus D_\delta(w)$$

and $n \geq n_0$. The second condition for the φ_n implies

$$u(z_n) - u_0(w) = \int_0^{2\pi} (u_0(re^{i\vartheta}) - u_0(w)) \varphi_n(re^{i\vartheta}) d\vartheta$$

$$= \left(\int_{\gamma_1} + \int_{\gamma_2} \right) (u_0(\zeta) - u_0(w))\varphi_n(\zeta) \frac{d\zeta}{i\zeta}$$

with $\gamma_1 = \partial D \cap \overline{D}_\delta$ and γ_2 the arc complementary to γ_1. Now

$$\left| \int_{\gamma_i} (u_0(\zeta) - u_0(w))\varphi_n(\zeta) \frac{d\zeta}{i\zeta} \right| \leq \frac{\varepsilon}{2}$$

for both $i = 1$ and $i = 2$. For $i = 2$ we use

$$|u_0(\zeta) - u_0(w)| \leq 2||u_0||$$

and the second condition for the functions φ_n, while for $i = 1$ the above estimate $|u_0(\zeta) - u_0(w)| < \frac{\varepsilon}{2}$ and $\int_{\gamma_1} \varphi_n(\zeta) \frac{d\zeta}{i\zeta} \leq 1$ applies. □

After having discussed the Dirichlet problem for harmonic functions on a disc one might wonder what can be said about the corresponding problem for holomorphic functions:

Theorem 11.9. *Let $D := D_r(0)$. For a continuous function $f_0 : \partial D \longrightarrow \mathbb{C}$ the following statements are equivalent*

1. *There is a continuous function $f : \overline{D} \longrightarrow \mathbb{C}$ holomorphic in D extending f_0, i.e. $f|_{\partial D} = f_0$.*
2. *We have $\int_{\partial D} f_0(z) z^n dz = 0$ for all $n \in \mathbb{N}$.*

Proof. "1) $\Longrightarrow$ 2)": We have

$$\int_{\partial D} f_0(z) z^n dz = \lim_{\varrho \to r} \int_{\partial D_\varrho(0)} f(z) z^n dz = \lim_{\varrho \to r} 0 = 0.$$

"1) $\Longrightarrow$ 2)": Let $f_0 = u_0 + iv_0$. We apply Th.11.7 to u_0 and v_0 and set $f = u + iv$ with the solutions u, v of the Dirichlet problem. We have to show $f|_D \in \mathcal{O}(D)$, or equivalently that

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f_0(\zeta) d\zeta}{\zeta - z}$$

holds for

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} f_0(\zeta) \frac{r^2 - |z|^2}{|\zeta - z|^2} \frac{d\zeta}{\zeta}.$$

For $\zeta \in \partial D$ we find

$$\frac{r^2 - |z|^2}{|\zeta - z|^2 \zeta} - \frac{1}{\zeta - z} = \frac{\overline{z}}{r^2 - \overline{z}\zeta} = \frac{\overline{z}}{r^2} \sum_{n=0}^{\infty} \left(\frac{\overline{z}\zeta}{r^2} \right)^n,$$

the last series converging uniformly on ∂D because of $|z| < r$. Thus the difference between the above integrals is

$$\frac{\overline{z}}{r^2} \int_{\partial D} f(\zeta) \left(\sum_{n=0}^{\infty} \left(\frac{\overline{z}\zeta}{r^2} \right)^n \right) d\zeta = \frac{\overline{z}}{r^2} \sum_{n=0}^{\infty} \frac{\overline{z}^n}{r^{2n}} \int_{\partial D} f(\zeta) \zeta^n d\zeta = 0.$$

□

We conclude this section by giving some comments on the general case. First of all we note

Remark 11.10.

1. A harmonic function, being locally the real part of a complex analytic function, is a real analytic function.
2. Harmonic functions on a domain G, being real analytic functions, satisfy the weak identity theorem Th. 7.7. In particular, such a function is either constant, or nonconstant on every open disc $D \subset G$.
3. A nonconstant harmonic function $u : G \longrightarrow \mathbb{R}$ is an open function: Such a function is locally the composition $u = \mathrm{Re} \circ f$ of two open maps, a nonconstant holomorphic function f and the projection $\mathrm{Re} : \mathbb{C} \longrightarrow \mathbb{R}$ on the real line.
4. Let G be a bounded domain. A continuous function $u : \overline{G} \longrightarrow \mathbb{R}$, harmonic on G, attains its maximum and minimum on ∂G ($u|_G$ being an open map or constant).
5. For $u : \overline{G} \longrightarrow \mathbb{R}$ as in the previous point we have: $u|_{\partial G} \equiv 0 \Longrightarrow u \equiv 0$.

In order to formulate a very general sufficient criterion for the solvability of the Dirichlet problem we extend the notion of a connected set to closed sets:

Definition 11.11. A closed subset $A \subset \mathbb{C}$ is called connected if it can not be written as the disjoint union $A = A_1 \dot{\cup} A_2$ of two nonenpty closed subsets $A_i \subset A$.

Remark 11.12. 1. Any interval $[a, b] \subset \mathbb{R}$ is connected.

2. The trace $|\gamma|$ of a path γ is connected.

3. A path connected closed set is connected, but in contrast to the case of open sets, a connected closed set need not be path connected: The set

$$A := [-i, i] \cup \left\{ x + i \sin\left(\frac{1}{x}\right) ; x \in \mathbb{R}_{>0} \right\}$$

is connected, but not path connected.

Theorem 11.13. *Let $G \subset \mathbb{C}$ be a bounded domain, such that for every boundary point $a \in \partial G$ there is a compact connected set $K \subset \mathbb{C} \backslash G$ containing a as well as points different from a. Then the Dirichlet problem is solvable on G, i.e. for every continuous function $u_0 : \partial G \longrightarrow \mathbb{R}$ there is a (unique) continuous function $u : \overline{G} \longrightarrow \mathbb{R}$, harmonic in G, such that*

$$u|_{\partial G} = u_0 \ .$$

Before we give a sketch of the proof let us note:

1. The uniqueness of the solution of the Dirichlet problem follows from Rem 11.10.5.

2. For a bounded simply connected domain G the complement $\mathbb{C} \setminus G$ is connected, hence, if $\overline{G} \subset D := D_r(0)$, we could choose $K := \overline{D} \setminus G$ for every boundary point $a \in G$.

3. The above sufficient condition on G is not satisfied, if G has isolated boundary points. And indeed, in that case there are always boundary values for which the Dirichlet problem is not solvable: If a is an isolated boundary point of G, we may write $\partial G = \{a\} \dot{\cup} A$ with the closed set $A := \partial G \setminus \{a\}$, and there is no $u : \overline{G} \longrightarrow \mathbb{R}$, harmonic in G, with $u(a) = 1, u|_A \equiv 0$. That is a consequence of the below proposition and the maximum principle applied to $G_0 := G \cup \{a\}$ (with $\partial G_0 = A$).

Proposition 11.14. *Let $D := D_r(a)$. A continuous function $u : D \longrightarrow \mathbb{R}$, harmonic on $D^* := D \setminus \{a\}$, is harmonic on D.*

Proof. According to Rem.11.4 we can write

$$u(z) = c \ln|z-a| + \mathrm{Re}(f(z))$$

with a function $f \in \mathcal{O}(D^*)$ and $c \in \mathbb{R}$. We have to show that a is a removable singularity of f, whence even $c = 0$, and we are done. Assume that this is not the case. Then the holomorphic function $g(z) = e^{f(z)}$ on D^* has an essential singularity at a, since $g(D_\varepsilon(0)^*)$ is dense in $\mathbb{C}$ for every $\varepsilon > 0$. If f itself has an essential singularity at a, that is clear, since $\exp(\mathbb{C}) = \mathbb{C}^*$ is dense; if f has a pole at a, then $f(D_\varepsilon(0)^*) \supset A_{r,\infty}(0)$ for some $r > 0$ - the function $1/f$ is holomorphic at a and in particular open. Thus $g(D_\varepsilon(0)^*) \supset \exp(A_{r,\infty}(0)) = \mathbb{C}^*$.

On the other hand

$$|g(z)| = |e^{f(z)}| = e^{u(z)} \cdot |z-a|^{-c}.$$

Hence, for $n > c$ we see that $h(z) = (z-a)^n g(z)$ satisfies $\lim_{z \to a} h(z) = 0$, i.e. a is a removable singularity of h and at most a pole of g. Contradiction. $\square$

Sketch of the proof of Th.11.13. First of all one introduces a new class of continuous functions containing the harmonic functions as "upper extremal functions". We start with the observation that harmonic functions satisfy the mean value property: As a consequence of the Poisson integral formula, given a harmonic function $u : G \longrightarrow \mathbb{R}$ and a closed disc $\overline{D}_r(a) \subset G$ the functional value at its center equals the mean value over its boundary circle:

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{i\vartheta}) d\vartheta.$$

On the other hand one can show, that any continuous function satisfying the mean value property is already harmonic.

Now a continuous function $u : G \longrightarrow \mathbb{R}$ is called *subharmonic* if for any closed disc $\overline{D}_r(a) \subset G$ we have

$$u(a) \leq \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{i\vartheta}) d\vartheta.$$

We denote $\mathcal{S}(G)$ the set of all subharmonic functions on G. Here are some of their most important properties:

1. We have
$$\mathcal{S}(G) \cap C^2(G) = \{u \in C^2(G); \Delta u \geq 0\}.$$
Indeed, if Δu is taken as a distribution, then
$$\mathcal{S}(G) = \{u \in C(G); \Delta u \geq 0\}.$$
Here for a distribution $v : C_0^\infty(G, \mathbb{R}) \longrightarrow \mathbb{R}$ we use the convention
$$v \geq 0 :\Longleftrightarrow v(C_0^\infty(G, \mathbb{R}_{\geq 0})) \subset \mathbb{R}_{\geq 0}.$$
In particular subharmonicity is a local property.
2. $g, h \in \mathcal{S}(G) \Longrightarrow g + h, \max(g, h), \lambda g \in \mathcal{S}(G), \text{where } \lambda \in \mathbb{R}_{\geq 0}.$
3. Subharmonic functions satisfy the maximum principle: If $g \in \mathcal{S}(G)$ attains its maximum at a point $z_0 \in G$, then $g \equiv g(z_0)$.

Now given a continuous function $u_0 : \partial G \longrightarrow \mathbb{R}$ one considers the following family of subharmonic functions
$$\mathcal{S}_{\leq u_0} := \left\{g \in \mathcal{S}(G); \forall a \in \partial G : \overline{\lim}_{z \to a} g(z) \leq u_0(a)\right\}.$$
Now the candidate for a solution is the function $u : G \longrightarrow \mathbb{R}$ defined by
$$u(z) := \sup\{h(z); h \in \mathcal{S}_{\leq u_0}\} \leq ||u_0||_{\partial G},$$
the inequality being a consequence of the maximum principle. In order to see that u is a harmonic function one needs the following observation: Given any open disc $D = D_r(z_0)$ with $\overline{D} \subset G$ the "Poisson modification"
$$P_D(h) : G \longrightarrow \mathbb{R}$$
of a subharmonic function $h : G \longrightarrow \mathbb{R}$ is defined as the unique continuous function satisfying
$$P_D(h)|_{G \setminus D} = h, \ \Delta(P_D(h)|_D) = 0,$$
i.e. outside D the function $P_D(h)$ coincides with h, while on D it is the unique solution of the Dirichlet problem with the boundary values $h|_{\partial D}$. It is again a subharmonic function: $P_D(h) \in \mathcal{S}(G)$, lying above h, i.e.
$$P_D(h) \geq h.$$

Since the family $\mathcal{S}_{\le u_0}$ is stable under Poisson modifications, we see that u is harmonic, using the characterization of harmonic functions by the mean value property.

On the other hand, in order to prove

$$\lim_{z\to a} u(z) = u_0(a)$$

for all boundary points $a \in \partial G$, we need that there are sufficiently many functions in the family $\mathcal{S}_{\le u_0}$. That is the case if the domain G admits at any boundary point $a \in \partial G$ arbitrarily steep *barrier functions*, i.e. given a disc $D = D_\varepsilon(a)$ there should exist a function $\beta \in \mathcal{S}(G)$ satisfying

1. $\beta \le 0$,
2. $\lim_{z\to a} \beta(z) = 0$,
3. $z \in G \setminus D \Longrightarrow \beta(z) \le -1$.

In the remark 12.17 of the next section we shall explain how such barrier functions are constructed using the condition of Th.11.13. $\square$

12 Conformal maps

In the previous section we have seen that Dirichlet's problem is solvable under quite general assumptions, but we have no recipe how to find the solution explicitly except on the unit disc. And even there the evaluation of the Poisson integral is not always that easy either. Sometimes for domains of another shape a good guess can lead at once to the unique bounded solution. Because of the below remark 12.1, a biholomorphic transformation of the domain of interest could, may be, be helpful in such a situation.

Remark 12.1. Let $\varphi : G \longrightarrow G'$ be a biholomorphic map. Then a function $u : G' \longrightarrow \mathbb{R}$ is harmonic iff $u \circ \varphi$ is. This is a consequence of the fact that harmonic functions are locally the real part of holomorphic functions and that compositions of holomorphic functions are again holomorphic.

Example 12.2. 1. Let $H := \{z \in \mathbb{C}; \mathrm{Im}(z) > 0\}$ be the upper half plane, and $u_0 : \partial H = \mathbb{R} \longrightarrow \mathbb{R}$ the function

$$u_0(x) := \begin{cases} 0 & , \quad \text{if } x < 0 \\ 1 & , \quad \text{if } x > 0 \end{cases},$$

where it does not matter how we define $u_0(0)$, since the values of u_0 at points of discontinuity do not matter in the Dirichlet problem. Then

$$u(z) := 1 - \frac{1}{\pi}\mathrm{Arg}(z)$$

is a bounded solution, while

$$u(z) := 1 - \frac{1}{\pi}\mathrm{Arg}(z) + y$$

is an unbounded solution.

2. Now consider the unit circle $D = D_1(0)$ and $\tilde{u}_0 : \partial D \longrightarrow \mathbb{R}$ with

$$\widetilde{u}_0(z) := \begin{cases} 1 & , \quad \text{if } \mathrm{Im}(z) < 0 \\ 0 & , \quad \text{if } \mathrm{Im}(z) > 0 \end{cases} .$$

   Let us look at the following biholomorphic transformation

$$f : D \longrightarrow H, z \mapsto -i\frac{z-1}{z+1}.$$

   It extends to a homeomorphism $\overline{D}\backslash\{-1\} \longrightarrow \overline{H}$, thereby transforming the Dirichlet problem of the previous point to the given one. Consequently, the function $\widetilde{u} := u \circ f$ is a solution of our Dirichlet problem on the unit disc.

Now, in order to find the good transformation for a given Dirichlet problem on $G \subset \mathbb{C}$, one should understand how biholomorphic maps (defined in an open neighbourhood of $\overline{G}$, say) act on curves as possible pieces of ∂G. First of all they preserve the angle between two intersecting curves.

Definition 12.3. The (oriented) angle $\vartheta = \angle(z, w) \in [0, 2\pi)$ between two nonzero complex numbers is defined by

$$\frac{w}{|w|} = e^{i\vartheta} \cdot \frac{z}{|z|}.$$

The angle between two regular paths γ, δ (i.e. smooth paths with nonvanishing tangent vector everywhere) at a common point $\gamma(s) = \delta(t)$ is the angle $\angle(\dot{\gamma}(s), \dot{\delta}(t))$ between their tangent vectors $\dot{\gamma}(s), \dot{\delta}(t)$.

Lemma 12.4. *A nonsingular $\mathbb{R}$-linear map $A : \mathbb{C} \longrightarrow \mathbb{C}$ is angle preserving, i.e.*

$$\angle(A(\zeta), A(\eta)) = \angle(\zeta, \eta)$$

holds for all $\zeta, \eta \in \mathbb{C}^*$, if and only if

$$A(z) = az$$

with some $a \in \mathbb{C}^*$.

Proof. The multiplication with some nonzero complex number preserves obviously angles. On the other, if A is angle preserving, the map $B := a^{-1}A, a := A(1)$, is angle preserving as well and satisfies $B(1) = 1$. Then necessarily $B(i) = \mu i$ with some $\mu \in \mathbb{R}_{>0}$. Now

$$\angle(1, 1 + \mu i) = \angle(B(1), B(1 + i)) = \angle(1, 1 + i) = \frac{\pi}{4}.$$

implies $\mu = 1$. So $B = \mathrm{id}_{\mathbb{C}}$ and $A(z) = az$. □

Definition 12.5. A C^1-map $f : G \longrightarrow \mathbb{C}$ is called conformal (or angle preserving) at $z \in G$, if its differential

$$Df(z) : \mathbb{C} \longrightarrow \mathbb{C}$$

is invertible, and given any regular paths γ, δ with $z = \gamma(s) = \delta(t)$, the angle between $f \circ \gamma$ and $f \circ \delta$ at $f(z)$ equals the angle between γ and δ at z.

Lemma 12.6. *A C^1-map $f : G \longrightarrow \mathbb{C}$ is conformal at $z \in G$ iff f is $\mathbb{C}$-differentiable there with nonzero derivative: $f'(z) \neq 0$.*

Proof. The tangent vector of $f \circ \gamma$ at t is $f'(\gamma(t)) \cdot \dot{\gamma}(t)$. □

As next we discuss in detail an example where a biholomorphic transformation leads to the solution of the Dirichlet problem:

Example 12.7. Th.11.13 tells us that the Dirichlet problem is solvable on the domain

$$G := D \setminus [0, 1),$$

the unit disc with a prick removed. Note that its boundary $\partial G = \partial D \cup [0, 1]$ is not a Jordan curve. We shall show how we can obtain a solution by finding a biholomorphic map

$$f : G \longrightarrow D$$

between G and the entire unit disc D. Its inverse extends to a continuous map $f^{-1} : \overline{D} \longrightarrow \overline{G} = \overline{D}$ inducing a surjection of the boundaries

$$f^{-1} : \partial D \longrightarrow \partial G,$$

such that every point of the open prick $(0, 1)$ has two inverse images. Now given a continuous function

$$u_0 : \partial G \longrightarrow \mathbb{R},$$

the Poisson formula provides a function

$$\tilde{u} : \overline{D} \longrightarrow \mathbb{R}$$

with $\tilde{u}|_{\partial D} = u_0 \circ f^{-1}$. Finally define

$$u : \overline{G} \longrightarrow \mathbb{R} \text{ by } u|_G = \tilde{u} \circ f, u|_{\partial G} = u_0.$$

The map $f : G \longrightarrow D$ is found as the composition of several maps, which stepwise simplify the domain G. First of all we unfold the prick: A branch of the square root

$$\sqrt{} : G \longrightarrow D^+, \quad -\frac{1}{4} \mapsto \frac{i}{2}$$

maps G biholomorphically to the upper half disc D^+, the intersection of the unit disc with the upper half plane. As next we want to get rid of one of the two corners of D^+: We shift D^+ one length unit to the right hand side, apply inversion and shift half a length unit to the left hand side, i.e. consider the map

$$g : D^+ \longrightarrow \mathbb{R}_{>0} + i\mathbb{R}_{<0}, z \mapsto \frac{1}{z+1} - \frac{1}{2} \quad ;$$

it transforms D^+ to the fourth quadrant. In order to see that, we investigate where ∂D^+ is mapped: The path $(-1, 1]$ is mapped through $(0, 2]$ and $(\infty, 1/2]$ to $(\infty, 0]$, and since biholomorphic maps preserve oriented angles, $g(D^+)$ lies on the left hand side when travelling from ∞ to the origin. As next there is a 90°-turn to the left and then one follows the upper part $\partial_+ D$ of the unit circle, so $g(\partial D^+)$ includes a 90°-turn to the left as well followed by $g(\partial_+ D)$, which is an unbounded line segment, since $-1 \in \partial_+ D$ is a pole, so $g(\partial_+ D) = i[0, -\infty)$. It follows that $g(D^+)$ is the part of the plane to the left hand side of $g(\partial D^+)$, i.e. the fourth quadrant.

In the third step we map the fourth quadrant to the lower half plane using the square function

$$\mathbb{R}_{>0} + i\mathbb{R}_{<0} \longrightarrow \mathbb{R} + i\mathbb{R}_{<0}, z \mapsto z^2.$$

Finally we arrive with

$$h : \mathbb{R} + i\mathbb{R}_{<0} \longrightarrow D, z \mapsto \frac{z+i}{z-i}$$

at the unit disc - note that $|h(z)| < 1$ iff $|z+i| < |z-i|$, i.e. iff the distance fron z to $-i$ is less than the distance of z to i, and this condition characterizes obviously the lower half plane.

All the biholomorphic maps from a domain G to itself form the *automorphism group* of G.

Definition 12.8. Let G be a domain. An *automorphism* of G is a biholomorphic map $f : G \longrightarrow G$. The set

$$\mathrm{Aut}(G) := \{f : G \longrightarrow G,\ f \text{ biholomorphic}\}$$

of all such automorphisms is called the *automorphism group* of G.

Indeed the automorphisms of a domain G form a group in the sense of abstract algebra: Two of them can be composed in order to give a new automorphism (the result depending on the order of the factors), and every automorphism admits an inverse automorphism.

Proposition 12.9. *The automorphism group of the complex plane is the group of affine linear maps:*

$$\mathrm{Aut}(\mathbb{C}) = \{f : \mathbb{C} \longrightarrow \mathbb{C}; f(z) = az + b, \text{where } a \in \mathbb{C}^*, b \in \mathbb{C}\}.$$

Proof. The affine linear maps $f(z) = az + b$ are obviously automorphisms of $\mathbb{C}$. Now let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be any automorphism. For a polynomial $f : \mathbb{C} \longrightarrow \mathbb{C}$ we have $|f^{-1}(w)| = \deg f$ for all but finitely many $w \in \mathbb{C}$ - indeed for $w \notin f(N(f'))$. With other words: f is affine linear. For a transcendent function $f : \mathbb{C} \longrightarrow \mathbb{C}$ we consider $g \in \mathcal{O}(\mathbb{C}^*), g(z) = f(z^{-1})$. It has an essential singularity at the origin. In particular for the unit disc $D := D_1(0)$ we obtain with Th.9.16.3 that

$$\mathbb{C} \setminus f(D) = f(\mathbb{C} \setminus D) \supset g(D^*)$$

is dense, a contradiction, since $f(D) \subset \mathbb{C}$ is open. $\square$

In order to have more automorphisms at our disposal we enlarge the complex plane $\mathbb{C}$, adding a point at infinity, which finally turns out to be as good as any other point!

Definition 12.10. The extended plane

$$\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$$

is the complex plane with a point at infinity, denoted ∞, added. A subset $U \subset \widehat{\mathbb{C}}$ is called open, if either $U \subset \mathbb{C}$ is open or otherwise $K := \widehat{\mathbb{C}} \setminus U$ is compact (closed and bounded).

Remark 12.11. There are two ways to investigate the properties of $\widehat{\mathbb{C}}$. The first one appeals to intuition, i.e. three dimensional geometry, the second one is very effective in avoiding cumbersome and lengthy calculations: It uses linear algebra for the complex vector space $\mathbb{C}^2$ in order to understand the "symmetries" of the extended plane, where the point ∞ ceases to play a distinguished rôle.

1. The **Riemann sphere** is nothing but the two dimensional unit sphere

   $$\mathbb{S}^2 := \left\{(w,t) \in \mathbb{C} \times \mathbb{R}; |w|^2 + t^2 = 1\right\}.$$

   The map

   $$\mathbb{S}^2 \setminus \{(0,1)\} \longrightarrow \mathbb{C}, (w,t) \mapsto \frac{w}{1-t}$$

   is called stereographic projection. Indeed, it associates to $(w,t) \in \mathbb{S}^2 \setminus \{(0,1)\}$ the point $z \in \mathbb{C}$, such that $(z,0)$ is the intersection point of the line through the north pole $(0,1) \in \mathbb{S}^2$ and (w,t) with the plane $\mathbb{C} \times 0$. Its inverse is

   $$z \mapsto \frac{1}{|z|^2+1} \cdot (2z, |z|^2 - 1).$$

   So we can view $\widehat{\mathbb{C}}$ as the Riemann sphere with $\infty \in \widehat{\mathbb{C}}$ corresponding to the north pole $(0,1) \in \mathbb{S}^2$.

2. The **complex projective line** $\mathbb{P}_1(\mathbb{C})$: The points in $\widehat{\mathbb{C}}$ are in one-to-one correspondence to the one dimensional (complex) vector subspaces of $\mathbb{C}^2$: Except $\mathbb{C} \times 0$ they are all of the form $\mathbb{C}(z,1)$ with some $z \in \mathbb{C}$. Hence

   $$z \mapsto L_z := \mathbb{C}(z,1), \ \infty \mapsto L_\infty := \mathbb{C} \times 0$$

defines a bijection

$$\widehat{\mathbb{C}} \longrightarrow \mathbb{P}_1(\mathbb{C})$$

between the extended plane $\widehat{\mathbb{C}}$ and the "complex projective line"

$$\mathbb{P}_1(\mathbb{C}) := \{L \subset \mathbb{C}^2; L \text{ a one dimensional complex vector subspace}\}.$$

Now let us define holomorphic maps of the extended plane to itself:

Definition 12.12. Denote $\iota : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}, z \mapsto \frac{1}{z}, 0 \mapsto \infty, \infty \mapsto 0$, the extended inversion.

1. A map $f : G \longrightarrow \hat{\mathbb{C}}$ defined on a domain $G \subset \mathbb{C}$ is called holomorphic if either $f \in \mathcal{M}(G)$ or $f \equiv \infty$.
2. A map $f : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$ is called holomorphic if both $f|_{\mathbb{C}}$ and $(f \circ \iota)|_{\mathbb{C}}$ are holomorphic.

Proposition 12.13. *The holomorphic maps $f : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}, f \not\equiv \infty$, correspond to rational functions.*

Proof. Obviously rational functions define holomorphic maps from the extended plane to itself. On the other hand, if $f : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}, f \not\equiv \infty, 0$, is holomorphic, the meromorphic function $f|_{\mathbb{C}} \in \mathcal{M}(\mathbb{C})$ has only finitely many zeros and poles, due to the fact that 0 is an isolated singularity of $f \circ \iota$. Denote them $a_1, ..., a_r$ and $n_1, ..., n_r \in \mathbb{Z}$ their multiplicities, Then $f(z) = \prod_{\nu=1}^{r}(z - a_\nu)^{n_\nu} \cdot g(z)$ with an entire function $g \in \mathcal{O}(\mathbb{C})$ without zeros. Furthermore $g \circ \iota \in \mathcal{M}(\mathbb{C})$, s.th. g defines a holomorphic map $\hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$. If $g(\infty) \neq \infty$, the function $g|_{\mathbb{C}}$ is bounded, hence constant according to Liouville, if $g(\infty) = \infty$ consider $\frac{1}{g}$ instead. □

And here are some automorphisms of the extended plane:

Definition 12.14. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an invertible matrix: $\det A = ad - bc \neq 0$. Then the holomorphic map

$$\mu_A : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$$

with

$$\mu_A(z) = \frac{az + b}{cz + d}$$

is called a *Möbius transformation.*

Remark 12.15. 1. We have $\mu_B = \mu_A$ iff $B = \lambda A$ for some $\lambda \in \mathbb{C}^*$.

2. We have $\mu_{AB} = \mu_A \circ \mu_B$. In particular, with $B = A^{-1}$, one obtains: Möbius transformations are automorphisms of the extended plane $\widehat{\mathbb{C}}$.

3. In terms of the projective line the Möbius transformation μ_A has a simple description:

$$\mu_A : \mathbb{P}_1(\mathbb{C}) \longrightarrow \mathbb{P}_1(\mathbb{C}), L \mapsto A(L).$$

Indeed, if

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

we have

$$A(\mathbb{C}(z,1)) = \mathbb{C}(az+b, cz+d) = \mathbb{C}\left(\frac{az+b}{cz+d}, 1\right).$$

4. Given two triples z_1, z_2, z_3 and w_1, w_2, w_3 of pairwise distinct points in $\widehat{\mathbb{C}}$ there is a unique Möbius transformation $\mu : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ with $\mu(z_i) = w_i$ for $i = 1, 2, 3$. In order to see that, we consider $\mathbb{P}_1(\mathbb{C})$ instead of $\widehat{\mathbb{C}}$.

Existence: Given pairwise different one dimensional subspaces $L_i = \mathbb{C}u_i$ and $\widetilde{L}_i = \mathbb{C}v_i$ we have to find a matrix A with $A(L_i) = \widetilde{L}_i$ for $i = 1, 2, 3$. Since u_1, u_2 resp. v_1, v_2 are bases of $\mathbb{C}^2$ and we may replace u_1, u_2 resp. v_1, v_2 with nonzero scalar multiples, we may assume $u_3 = u_1 + u_2$ as well es $v_3 = v_1 + v_2$. Now choose A with $Au_i = v_i, i = 1, 2$.

Uniqueness: If both A, B satisfy $A(L_i) = \widetilde{L}_i = B(L_i)$, the matrix $C = B^{-1}A$ has u_1, u_2 as eigenvectors. The corresponding eigenvalues agree – otherwise any eigenvector of C is either a multiple of u_1 or u_2 – and thus $C = \lambda E$.

Indeed every automorphism of $\hat{\mathbb{C}}$ is a Möbius transformation:

Theorem 12.16. *The automorphisms of the extended plane are the Möbius transformations:*

$$\mathrm{Aut}(\widehat{\mathbb{C}}) = \{\mu_A; A \in \mathbb{C}^{2,2}, \det A \neq 0\}.$$

Proof. Let $f : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ be an automorphism. If $f(\infty) = \infty$, the restriction $f|_{\mathbb{C}}$ is an automorphism of the complex plane: We have $f(z) = az + b$ and thus $f = \mu_A$ with

$$A := \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}.$$

If $f(\infty) = c \in \mathbb{C}$ and

$$B := \begin{pmatrix} 0 & 1 \\ 1 & -c \end{pmatrix},$$

such that

$$\mu_B(z) = \frac{1}{z - c},$$

we may apply the above argument to $\mu_B \circ f$ and obtain $f = \mu_{B^{-1}A}$. $\square$

Sometimes problems in $\mathbb{C}$ become easier to handle within $\widehat{\mathbb{C}} \supset \mathbb{C}$, since there we may always assume that some point of interest is the point ∞, the action of the Möbius transformations being transitive. Here is one such case:

Remark 12.17. The construction of barrier functions: Let us use the geometry of $\widehat{\mathbb{C}}$ in order to construct barrier functions at boundary points $a \in \partial G$ satisfying the condition of Th.11.13. We may assume $a = \infty$, since we may replace G with $\mu(G)$ with the Möbius transformation $\mu(z) = \frac{1}{z-a}$. Hence, given $r > 0$, we have to look for a function $\beta \in \mathcal{S}(G)$ satisfying

1. $\beta \leq 0$,
2. $\lim_{z \to \infty} \beta(z) = 0$,
3. $z \in G, |z| \leq r \Longrightarrow \beta(z) \leq -1$.

Now let $K \subset \widehat{\mathbb{C}} \backslash G$ be a connected compact set containing ∞ and more points. Its complement is the disjoint union of simply connected domains, called the *connected components* of $\widehat{\mathbb{C}} \backslash K$, and G, being connected, is contained in one of them. So there is in particular a branch of the logarithm $\log \in \mathcal{O}(G)$. Being injective, it defines a biholomorphic map $\log : G \longrightarrow \log(G)$. Replacing G with $\log(G)$, our problem takes the following form: Given $s \in \mathbb{R}$, we should hunt for a function $\beta \in \mathcal{S}(G)$ satisfying

1. $\beta \leq 0$,

2. $\lim_{\mathrm{Re}(z)\to\infty} \beta(z) = 0$,

3. $z \in G, \mathrm{Re}(z) \le s \Longrightarrow \beta(z) \le -1$.

Since $\exp|_G$ is injective, the line $s+i\mathbb{R}$ through $s \in \mathbb{R}$ parallel to the imaginary axis intersects G in at most countably many intervals

$$(s + i\mathbb{R}) \cap G = \bigcup_{\nu\in I}^{\infty}(s + ia_\nu, s + ib_\nu),$$

whose lengths add up to at most 2π, i.e.

$$\sum_{\nu\in I}(b_\nu - a_\nu) \le 2\pi,$$

assuming $a_\nu < b_\nu$. For $z \in G, \mathrm{Re}(z) \le s$, we set

$$\beta(z) \equiv -1,$$

while for $z \in G, \mathrm{Re}(z) > s$, we define

$$\beta(z) := \sum_{\nu\in I} \beta_\nu(z),$$

where

$$\beta_\nu(z) := \frac{1}{\pi}\angle(s + ia_\nu - z, s + ib_\nu - z) = \mathrm{Im}(\mathrm{Log}\left(\frac{s + ib_\nu - z}{s + ia_\nu - z}\right)).$$

Then we have

$$-1 < \beta(z) < 0$$

and thus

$$\lim_{z\to w} \beta_\nu(z) = -1,\ \forall\ w \in (s + ia_\nu, s + ib_\nu),$$

implies that

$$\lim_{z\to w} \beta(z) = -1,\ \forall\ w \in G,\ \mathrm{Re}(w) = s,$$

as well. On the other hand, some elementary geometry gives

$$\beta(x + iy) \ge -2 \arctan\left(\frac{\pi}{x - s}\right),\ \forall z = x + iy \in G, x > s,$$

so condition 2) is also satisfied.

The geometry of Möbius transformations is based on the fact that they preserve the following class of subsets of the extended plane $\widehat{\mathbb{C}}$:

Definition 12.18. A *generalized circle* in $\widehat{\mathbb{C}}$ is either

1. a usual circle $\partial D_r(z_0)$ in $\mathbb{C} \subset \widehat{\mathbb{C}}$, or
2. a set $L \cup \{\infty\}$ with a line $L \subset \mathbb{C}$.

Remark 12.19. We comment on generalized circles both from the point of view of $\widehat{\mathbb{C}}$ as the complex projective line and as the Riemann sphere.

1. Generalized circles consist of the points in $\mathbb{P}_1(\mathbb{C})$ corresponding to the one dimensional subspaces $L \subset \mathbb{C}^2$, which are contained in the zero cone
$$NC(\sigma) := \{u \in \mathbb{C}^2; \sigma(u, u) = 0\}$$
of a nondegenerate indefinite hermitian form $\sigma : \mathbb{C}^2 \times \mathbb{C}^2 \longrightarrow \mathbb{C}$. Indeed, such a form looks as follows
$$\sigma(u, v) = u^T H \overline{v}$$
with a matrix $H \in \mathbb{C}^{2,2}$ satisfying
$$H = \overline{H}^T, \det H < 0.$$
That means
$$H = \begin{pmatrix} a & \overline{c} \\ c & b \end{pmatrix}, \quad a, b \in \mathbb{R},\ c \in \mathbb{C},\ ab - |c|^2 < 0.$$
Thus the equation $\sigma(u, u) = 0$ takes for $u = (z, 1)$ the form
$$az\overline{z} + \overline{c}z + c\overline{z} + b = 0.$$
For $a = 0$ we have $c \neq 0$, and $z \mapsto \overline{c}z + c\overline{z}$ with $c \neq 0$ is the general expression for a surjective linear map $\mathbb{C} \longrightarrow \mathbb{R}$. Hence the equation $\overline{c}z + c\overline{z} = -b$ with $b \in \mathbb{R}$ describes a line in $\mathbb{C}$, and every line is obtained in that way.

   If $a \neq 0$, we may even assume $a = 1$ and rewrite our equation
$$r^2 := |c|^2 - b = z\overline{z} + \overline{c}z + c\overline{z} + c\overline{c} = |z - (-c)|^2,$$

describing a circle of radius $r > 0$ with center $-c$. Obviously any circle can be obtained in that way.

The above explicit discussion shows in particular that the hermitian form $\sigma = \sigma_{(a,b,c)}$ is determined by its zero cone $NC(\sigma) \subset \mathbb{C}^2$ up to a nonzero real constant.

2. Generalized circles correspond via the stereographic projection to ordinary circles in the Riemann sphere. Such a circle is the intersection of $\mathbb{S}^2$ with a plane having a distance < 1 to the origin. It consists of the points $(w, t) \in \mathbb{S}^2$ satisfying an equation:

$$\overline{c}w + c\overline{w} + \gamma t = d, \text{ where } c \in \mathbb{C}, \gamma, d \in \mathbb{R}, d^2 < 4|c|^2 + \gamma^2.$$

The left hand side of the equation is the general expression for an $\mathbb{R}$-linear map $\mathbb{C} \times \mathbb{R} \longrightarrow \mathbb{R}$, while the inequality for the coefficients guarantees, that the plane has distance < 1 to the origin. Now we substitute the expression of Rem. 12.11 for the inverse of the stereographic projection and obtain:

$$(d + \gamma)|z|^2 - 2\overline{c}z - 2c\overline{z} + (d - \gamma) = 0,$$

where $|2c|^2 > (d + \gamma)(d - \gamma)$.

As a consequence we obtain:

Proposition 12.20. *Möbius transformations map generalized circles to generalized circles, and given two such circles there is a Möbius transformation mapping the first one to the second one.*

Proof. If a generalized circle arises from the hermitian form σ, its image with respect to μ_A belongs to the form $\widetilde{\sigma}(u, v) = \sigma(A^{-1}u, A^{-1}v)$.

On the other hand given two forms $\sigma, \widetilde{\sigma}$ choose a σ-ON-basis u_1, u_2 and a $\widetilde{\sigma}$-ON-basis $\widetilde{u}_1, \widetilde{u}_2$ (the first vectors having square length 1, the second ones -1). Then $A \in \mathbb{C}^2$ with $Au_i = \widetilde{u}_i$ satisfies $\widetilde{\sigma}(u, v) = \sigma(A^{-1}u, A^{-1}v)$ and thus $A(NC(\sigma)) = NC(\widetilde{\sigma})$. □

Remark 12.21. Reflections at generalized circles: Given a generalized circle with corresponding hermitian form σ we can define a reflection (not a Möbius transformation!) at that circle: Replacing $\widehat{\mathbb{C}}$ with $\mathbb{P}_1(\mathbb{C})$ it is defined as follows

$$\mathbb{P}_1(\mathbb{C}) \ni L \mapsto L^{\perp}.$$

Here $L^\perp \subset \mathbb{C}^2$ denotes the σ-orthogonal complement of L. In particular $L^\perp = L$ for $L \in NC(\sigma)$. Indeed for the corresponding reflection $\varrho : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ on $\widehat{\mathbb{C}}$, we have

$$\varrho(z) = \mu_A(\overline{z})$$

with a suitable Möbius transformation μ_A. (We remark, that $\mathbb{C}^2 \longrightarrow \mathbb{C}^2, u \mapsto A\overline{u}$, is not a reflection of $\mathbb{C}^2$ on some real hyperplane, instead it has two eigenvalues 1 and two eigenvalues -1.)

In order to see that, let us first consider the generalized circle $\mathbb{R} \cup \{\infty\}$. We may take

$$H = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

Then

$$(\mathbb{C}(z_1, z_2))^\perp = \mathbb{C}(\overline{z}_1, \overline{z}_2)$$

and thus

$$\varrho(z) = \overline{z}.$$

In the general case take a Möbius transformation μ_B mapping $\mathbb{R} \cup \{\infty\}$ to the given generalized circle. Then

$$\varrho(z) = \mu_{B\overline{B}^{-1}}(\overline{z}).$$

We leave it to the reader to show the following: If $Z \subset \widehat{\mathbb{C}}$ is a generalized circle, then the reflection $\varrho_Z : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$

1. is the ordinary reflection at the line $Z \setminus \{\infty\}$, if $\infty \in Z$.
2. is the map
$$z \mapsto r^2 \frac{z - z_0}{|z - z_0|^2} + z_0, z_0 \mapsto \infty, \ \infty \mapsto z_0,$$
if $Z = D_r(z_0)$.

As a byproduct of the above discussion we obtain that the zero cones $NC(\sigma) \subset \mathbb{C}^2$ are the sets

$$\mathbb{S}^1 \cdot P = \{\lambda u; \lambda \in \mathbb{S}^1, u \in P\},$$

where $\mathbb{S}^1$ denotes the unit circle and $P \subset \mathbb{C}^2$ is a totally real plane, i.e. a two dimensional real subspace, which is not a complex subspace, indeed the eigenspace of $B\overline{B}^{-1}$ for the eigenvalue 1.

Dynamics of Möbius transformations: A Möbius transformation $\mu_A \neq \mathrm{id}_{\widehat{\mathbb{C}}}$ has either one or two fixed points, corresponding to diagonalizable resp. nondiagonalizable $A \in \mathbb{C}^{2,2}$. In the first case $\mu_A : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ is similar to a map

$$\mu(z) = az$$

with $a \in \mathbb{C} \setminus \{0, 1\}$, while in the second case it is similar to a translation

$$\mu(z) = z + a$$

with $a \in \mathbb{C}^*$. In particular $\widehat{\mathbb{C}}$ is the union of the fixed points of μ_A and a family of mutually disjoint μ_A-invariant curves: These curves are

1. generalized circles if A is diagonalizable and both eigenvalues have the same absolute value. For $\mu(z) = az$ our family is

$$\partial D_r(0), r > 0.$$

2. generalized circles with one point, the fixed point of μ_A, removed, if A is nondiagonalizable. For $\mu(z) = z + a$ that family is just the family

$$\mathbb{R}a + tb, \ t \in \mathbb{R}$$

of lines parallel to $\mathbb{R}a$. Here $b \in \mathbb{C} \setminus \mathbb{R}a$ denotes any complex number not on $\mathbb{R}a$.

3. open segments of a generalized circle with the two fixed points of μ_A as boundary points, if A is diagonalizable and the eigenvalues of A are linearly dependent over $\mathbb{R}$. For $\mu(z) = az$ with $a \in \mathbb{R}$ we obtain the family

$$\mathbb{R}_{>0}e^{i\vartheta}, 0 \leq \vartheta < 2\pi.$$

4. "spirals" from one fixed point to the other one, if A is diagonalizable and the eigenvalues of A are linearly independent over $\mathbb{R}$ and have different absolute value. In the case $\mu(z) = az$ with $a = se^{i\varphi}$ they look as follows (depending on the choise ϑ)

$$S_r := \left\{ z = rs^t \exp(it\varphi); t \in \mathbb{R} \right\}, r \in \mathbb{R}_{>0}.$$

Here are the automorphisms of the unit circle:

Theorem 12.22. *For $D := D_1(0)$ we have*

$$\mathrm{Aut}(D) = \{\mu_A|_D; A^T H\overline{A} = H\},$$

where

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

More classically: The restrictions of the Möbius transformations

$$\mu(z) = e^{i\vartheta}\frac{z+z_0}{1+\overline{z}_0 z},\ |z_0| < 1,$$

are the automorphisms of the unit disc.

We remark that the equation $A^T H\overline{A} = H$ is equivalent to the fact that the column vectors of $A = (u_1, u_2)$ constitute a σ-ON-basis for $\sigma(u,v) := u^T H\overline{v}$, such that $\sigma(u_1,u_1) = 1, \sigma(u_2,u_2) = -1$, are their respective "square lengths".

Proof. A Möbius transformation μ_A maps D into itself iff $A : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ satisfies $\sigma(Au, Av) = \lambda\sigma(u,v)$ with some $\lambda \in \mathbb{R}_{>0}$ for $\sigma(u,v) := u^T H\overline{v}$. Since A is only determined up to a scalar multiple, we may assume that $\lambda = 1$, i.e. A is a σ-isometry, equivalently $A^T H\overline{A} = H$. On the other hand if we don't want A to be a σ-isometry, we know nevertheless that the second column vector should have negative square length, so its second component does not vanish, and we may even assume that it equals 1. Denoting its first component a, its square length is $|a|^2 - 1 < 0$, i.e. $|a| < 1$. The first column vector should be σ-orthogonal to the second one and have up to sign the same square length. So we arrive at

$$A = \begin{pmatrix} e^{i\vartheta} & a \\ e^{i\vartheta}\overline{a} & 1 \end{pmatrix};$$

finally take $z_0 := e^{-i\vartheta}a$.

It remains to show that every automorphism $f : D \longrightarrow D$ is the restriction of a Möbius transformation. With

$$B := \begin{pmatrix} 1 & -f(0) \\ -\overline{f(0)} & 1 \end{pmatrix}$$

we obtain an automorphism

$$g := \mu_B \circ f$$

fixing the origin: $g(0) = 0$. Schwarz' lemma, i.e. lemma 12.23, may be applied to both g and g^{-1} and gives that $g(z) = e^{i\vartheta}z$ is a rotation, hence

$$f = \mu_A \text{ with } A = B^{-1} \begin{pmatrix} e^{i\vartheta} & 0 \\ 0 & 1 \end{pmatrix} .$$

□

Lemma 12.23 (Schwarz' lemma). *Let $D = D_1(0)$ be the open unit disc, and $g \in \mathcal{O}(D)$ a function with $g(0) = 0, |g(z)| \leq 1$ for all $z \in D$. Then*

1. *either g is a rotation: $g(z) = e^{i\vartheta}z$*
2. *or $|g(z)| < |z|$ for all $z \in D \setminus \{0\}$.*

Proof. We consider the function

$$h(z) := \begin{cases} g(z)/z & , \text{ if } z \neq 0 \\ g'(0) & , \text{ if } z = 0 \end{cases} .$$

It is holomorphic on D and for $r < 1$ the maximum principle gives

$$||h||_{D_r(0)} = ||h||_{\partial D_r(0)} \leq 1/r,$$

since $|g(z)| \leq 1$ for all $z \in D$. Taking the limit $r \to 1$ we find $|h(z)| \leq 1$ for all $z \in D$. Now the maximum principle gives that either $h(z) \equiv e^{i\vartheta}$ - so $g(z) = e^{i\vartheta}$ is a rotation - or $|h(z)| < 1$ for all $z \in D$ resp. $|g(z)| < |z|$ for $z \in D^*$ □

Finally we want to discuss the following question: Given two domains G, G', does there exist a biholomorphic map $\varphi : G \longrightarrow G'$?

Here is a necessary criterion:

Proposition 12.24. *Assume $f : G \longrightarrow G'$ is a diffeomorphism, i.e. bijective and C^1-differentiable in both directions. Then G is simply connected iff G' is.*

Proof. The statement follows immediately from Th.6.23, but we want to give here a proof based on our definition of simple connectedness: We show that f induces a bijection $f^* : \mathcal{D}(G') \longrightarrow \mathcal{D}(G)$ preserving both total differentials and locally integrable forms: First of all we define for any C^1-map $f : G \longrightarrow G'$ a pull back map on the level of functions:

$$C^1(G') \longrightarrow C^1(G), g \mapsto g \circ f$$

and on the level of differential forms:

$$f^* : \mathcal{D}(G') \longrightarrow \mathcal{D}(G), \omega = gdx + hdy \mapsto f^*(\omega) := (g \circ f)du + (h \circ f)dv,$$

where $f = u + iv$. Then the chain rule implies

$$d(F \circ f) = f^*(dF).$$

In particular, for a diffeomorphism f we may apply that to both f and f^{-1}: So differential forms with primitive function on G' correspond under f^* to differential forms with primitive function on G. This holds as well for locally integrable differential forms: Indeed, a differential form on a domain G is locally integrable iff every point $z \in G$ admits an open neighbourhood U, such that $\omega|_U$ has a primitive function – this finally justifies the adjective "locally integrable". The implication "$\Longrightarrow$" is clear. For "$\Longleftarrow$" we have to show, according to Prop.6.14, that $\int_{\partial R} \omega = 0$ for any coordinate rectangle $R \subset G$, if ω admits locally a primitive function: Take a sufficiently fine subdivision of R into n^2 congruent rectangles R_{ij}, of size $\frac{1}{n}$ the size of R. If n is sufficiently big, every R_{ij} is contained in a neighbourhood U_{ij}, where ω has a primitive function. Consequently

$$\int_{\partial R} \omega = \sum_{i,j} \int_{\partial R_{ij}} \omega = \sum_{i,j} 0 = 0.$$

$\square$

On the other hand:

Remark 12.25. There is no biholomorphic map $f : \mathbb{C} \longrightarrow D := D_1(0)$ between the complex plane $\mathbb{C}$ and the open unit disc D. Namely, $f \in \mathcal{O}(\mathbb{C})$ would be a bounded holomorphic function, but such functions are constant, according to Liouville. We leave it to the reader to determine a diffeomorphism $f : \mathbb{C} \longrightarrow D$

But this is the only disappointment:

Theorem 12.26 (**Riemann mapping theorem**). *Let $G \subsetneqq \mathbb{C}$ be a simply connected domain. Then there is a biholomorphic map $f : G \longrightarrow D = D_1(0)$ from G onto the open unit disc D.*

Proof. Our proof is based on the following:

Lemma 12.27. *Given a holomorphic function $f \in \mathcal{O}(G)$ on a simply connected domain $G \subset \mathbb{C}$ without zeros there is a square root $\sqrt{f} \in \mathcal{O}(G)$.*

Proof. The differential form $\frac{df}{f}$ is closed, and hence, G being simply connected, has a primitive function $F \in \mathcal{O}(G)$. Then the function $g(z) := f(z)e^{-F(z)}$ satisfies $g' = f'e^{-F} - fF'e^{-F} = 0$, thus $g \equiv c \in \mathbb{C}^*$. Since F is only determined up to an additive constant, we may assume that $c = 1$. So $f = e^F$ and

$$\sqrt{f} := \exp\left(\frac{F}{2}\right)$$

is a square root of f. □

Construction of a biholomorphic map $h : G \longrightarrow h(G)$ **with** $0 \in h(G) \subset D$: Take a point $a \notin G$ and apply in a first step the translation

$$G \longrightarrow G_1 := G - a, z \mapsto z - a.$$

Then take a square root

$$f : G_1 \longrightarrow G_2 := f(G_1)$$

of id_{G_1}, i.e. $f(z)^2 = z$ for all $z \in G_1$. Since $z \mapsto f(z)^2 = z$ is injective, for an open disc $D_r(b) \subset G_2$ we have $D_r(-b) \cap G_2 = \emptyset$. Then

$$g : G_2 \longrightarrow G_3 := g(G_2), z \mapsto \frac{r}{z + b}$$

maps G_2 biholomorphically onto $G_3 \subset D$. Finally pick $z_0 \in G_3$ and compose with

$$\mu : G_3 \longrightarrow G_4 := \mu(G_3), z \mapsto \frac{z - z_0}{1 - \overline{z}_0 z}.$$

Now given a domain $G \subsetneqq D$ we try to enlarge G by applying dilatations:

Definition 12.28. Let $G \subset \mathbb{C}$ be a domain, $0 \in G \subset D$. A dilatation of G is a biholomorphic map $\kappa : G \longrightarrow \kappa(G) \subset D$, such that

$$\kappa(0) = 0,\ |\kappa(z)| > |z|,\ \forall z \in G^* := G \setminus \{0\}.$$

We shall prove:

Proposition 12.29. *For a simply connected domain $G \subset \mathbb{C}$ with $0 \in G \subset D := D_1(0)$ the following statements are equivalent:*

1. *$G = D$.*
2. *There is no dilatation $\kappa : G \longrightarrow \kappa(G) \subset D$.*

Proof. "$\Longrightarrow$": According to Schwarz' lemma, i.e. Lemma 12.23, a holomorphic map $f : D \longrightarrow D$ with $f(0) = 0$ is either a rotation or satisfies $|f(z)| < |z|$ for $z \in D^*$.
"$\Longleftarrow$": We subdivide our argument into four steps:

1. For $G \subsetneqq D$ we construct a dilatation $\kappa : G \longrightarrow D$ as a partial right inverse of a contraction $g : D \longrightarrow D$, i.e. a holomorphic map satisfying $g(0) = 0$ and $|g(z)| < |z|$ for $z \in D^*$. More precisely, we want

   $$g \circ \kappa = \mathrm{id}_G \text{ and } \kappa(0) = 0.$$

   As a consequence
   $$|z| = |g(\kappa(z))| < |\kappa(z)|$$
   for $z \in G^*$, i.e. $\kappa : G \longrightarrow D$ is a dilatation.

2. According to Schwarz' lemma any holomorphic map $g : D \longrightarrow D, g(0) = 0$, which is not a rotation, is a contraction.

3. For $a \in D$ we consider the following automorphism of the unit disc:

   $$\mu_a(z) := \frac{z - a}{\bar{a}z - 1}.$$

   It satisfies $\mu_a(a) = 0, \mu_a(0) = a$, indeed it is involutive: $\mu_a \circ \mu_a = \mathrm{id}_D$. With $j(z) := z^2$ the map

   $$g_a := \mu_{a^2} \circ j \circ \mu_a : D \longrightarrow D$$

   satisfies $g_a(0) = 0$, but it is not a rotation, since g_a is not injective. Hence it is a contraction.

4. Now, if $a^2 \in D \setminus G$, there is a function $h \in \mathcal{O}(G)$ with
$$h^2 = \mu_{a^2}, h(0) = a.$$
Then $\kappa := \mu_a \circ h$ is what we are looking for (with $g = g_a$).

$\square$

Finally let us give the proof of Riemanns mapping theorem: We may assume $0 \in G \subset D$ and consider the following family of functions:
$$\mathcal{F} := \{f \in \mathcal{O}(G); f(0) = 0, f : G \longrightarrow f(G) \subset D \text{ is biholomorphic}\}.$$
Fix a point $a \in G^* := G \setminus \{0\}$ and set
$$\varrho := \sup\{|f(a)|; f \in \mathcal{F}\}.$$
Obviously $\varrho \leq 1$. Take a series of functions $(f_n)_{n\in\mathbb{N}} \subset \mathcal{F}$ with $\lim_{n\to\infty} |f_n(a)| = \varrho$. According to Th.12.30 we may assume that it converges uniformly to some function $f \in \mathcal{O}(G)$. Then $f(0) = 0$ and $f(a) \neq 0$ implies that $f : G \longrightarrow f(G)$ is biholomorphic. Now assume $f(G) \neq D$. Then there is a dilatation $\kappa : f(G) \longrightarrow D$, and since then $\kappa \circ f \in \mathcal{F}$ as well, we find
$$\varrho \geq |\kappa(f(a))| > |f(a)| = \varrho,$$
a contradiction. $\square$

We have used the following two facts about locally uniform convergence of holomorphic functions:

Theorem 12.30 (Theorem of Montel). *Let $G \subset \mathbb{C}$ be a domain. For a family (i.e. a subset) $\mathcal{F} \subset \mathcal{O}(G)$ the following statements are equivalent*

1. *Every sequence $(f_n)_{n\in\mathbb{N}} \subset \mathcal{F}$ admits a subsequence $(g_\nu)_{\nu\in\mathbb{N}}$, i.e. $g_\nu = f_{n_\nu}$, converging locally uniformly to some function $g \in \mathcal{O}(G)$.*

2. *The family $\mathcal{F} \subset \mathcal{O}(G)$ is "uniformly bounded", i.e. for every compact set $K \subset G$ there is a constant $M_K \in \mathbb{R}_{>0}$, such that*
$$||f||_K \leq M_K, \ \forall f \in \mathcal{F}.$$
Here $||f||_K := \sup\{|f(z)|; z \in K\}$.

For the next result recall that a holomorphic function $f \in \mathcal{O}(G)$ is injective if and only if $f : G \longrightarrow f(G)$ is biholomorphic.

Theorem 12.31. *Let $G \subset \mathbb{C}$ be a domain and $(f_n)_{n\in\mathbb{N}} \subset \mathcal{O}(G)$ be a sequence converging locally uniformly to some function $f \in \mathcal{O}(G)$. If all functions f_n are injective, so is f or $f \equiv c \in \mathbb{C}$.*

We conclude this section with a discussion of the relationship between the Dirichlet problem and Riemanns mapping theorem. First of all:

Theorem 12.32. *Let $D = D_1(0)$ be the unit disk, and $g : D \longrightarrow G$ be a biholomorphic map from the unit disk to a bounded domain $G \subset \mathbb{C}$. Then the following statements are equivalent*

1. *The map $g : D \longrightarrow G$ extends to a continuous map $\widehat{g} : \overline{D} \longrightarrow \overline{G}$.*
2. *$\partial G = |\gamma|$, i.e. the boundary of G is the trace of a continuous loop $\gamma : I \longrightarrow \mathbb{C}$.*

Furthermore the extension $\hat{g}$ is a homeomorphism if we can choose γ as a Jordan curve, i.e. with $I = [a, b]$ and $\gamma|_{[a,b)}$ injective.

Example 12.33. The domain $G := D \setminus [0, 1)$ satisfies the criterion, but

$$G = D \setminus \left(\left[\frac{1}{2}, 1\right) \cup \bigcup_{n=1}^{\infty} \left[\frac{1}{2}, 1\right) \exp\left(\frac{2\pi i}{n}\right) \right)$$

does not.

Remark 12.34.

1. From Riemanns mapping theorem, Th.12.26, and Th.12.32 it follows that the Dirichlet problem is solvable for domains G whose boundary is the trace of a path: Assume $u_0 : \partial G \longrightarrow \mathbb{R}$ is continuous. Now there is a solution $h : \overline{D} \longrightarrow \mathbb{R}$ of the Dirichlet problem with $h|_{\partial D} = u_0 \circ \widehat{g}$. Define then $u : \overline{G} \longrightarrow \mathbb{R}$ by $u|_G = h \circ g^{-1}$ and $u|_{\partial G} = u_0$.
2. Assume the Dirichlet problem is solvable for the simply connected domain G. Then there is a natural candidate for a biholomorphic map $f : G \longrightarrow D$. We fix a point $a \in G$ and denote $u : \overline{G} \longrightarrow \mathbb{R}$ the solution of the Dirichlet problem with

$$u(z) = \ln |z - a|, z \in \partial G.$$

(Note that the logarithm itself, though harmonic, is not the solution of the Dirichlet problem, since it has a singularity at $a \in G$). Since G is simply connected there is a harmonic function $v : G \longrightarrow \mathbb{R}$, such that $h := u + iv$ is holomorphic. Now set $f(z) := (z - a)e^{-h(z)}$. If v can be (continuously) extended to $\overline{G}$, then we see that $|f(z)| = 1$ for $z \in \partial G$. On the other hand: Since there is exactly one zero on G, Rouchés theorem implies, that every value $w \in D$ is attained exactly once.