THE AREA OF A RANDOM TRIANGLE IN A SQUARE.

JOHAN PHILIP

Abstract. We determine the distribution function for the area of a random triangle in a unit square. The result is not new, [8], [12]. The method presented here is worked out to shed more light on the problem.

1. Introduction

We shall denote the square by K and the random triangle by T and shall consider the random variable $X = area(T)/area(K)$. It is well known that an affine transformation will preserve the ratio X. This follows from the fact that the area scaling is constant for an affine transformation. The scale equals the determinant of the homogeneous part of the transformation. This means that our results hold when K is a parallelogram.

Various aspects of our problem have been considered in the field of geometric probability, see e.g. [11]. J. J. Sylvester considered the problem of a random triangle T in an arbitrary convex set K and posed the following problem: Determine the shape of K for which the expected value $\kappa = E(X)$ is maximal and minimal. A first attempt to solve the problem was published by M. W. Crofton in 1885. Wilhelm Blaschke [3] proved in 1917 that $\frac{35}{48\pi^2} \leq \kappa \leq \frac{1}{12}$, where the minimum is attained only when K is an ellipse and the maximum only when K is a triangle. The upper and lower bounds of κ only differ by about 13%. It has been shown, [2] that $\kappa = \frac{11}{144}$ for K a square.

A. Reńyi and R. Sulanke, [9] and [10], consider the area ratio when the triangle T is replaced by the convex hull of n random points. They obtain asymptotic estimates of κ for large n and for various convex K. R. E. Miles [7] generalizes these asymptotic estimates for K a circle to higher dimensions. C. Buchta and M. Reitzner, [4], has given values of κ (generalized to three dimensions) for $n \geq 4$ points in a tetrahedron. H. A. Alikoski [2] has given expressions for κ when T is a triangle and K a regular r-polygon.

Here, we shall deduce the distribution function for X. We have done this before in a simpler way than in this paper, [8]. We hope that the

1991 *Mathematics Subject Classification.* Primary: 60D05: Secondary: 52A22.
Key words and phrases. Area, random triangle, Sylvester problem.

method presented here shall be applicable if the square K is replaced by a pentagon or hexagon.

2. Notation and formulation.

As K, we will take the unit square $(0 \le x \le 1, 0 \le y \le 1)$. We use a constant probability density in K for generating three random points in K. The coordinates of the points will be denoted (x_k, y_k) for $1 \le k \le 3$. Each x_k and y_k is evenly distributed in $(0,1)$ and they are independent. Let T be the convex hull of the three points. We shall determine the probability distribution of the random variable $X = area(T)$.

Our method will be to shrink the square around its midpoint until one of its sides hits a triangle point. The shrunk square is denoted B. The random variable X that we study will be written as the product of two random variables

$$V = area(B) \text{ and } W = area(T)/area(B).$$

We have six independent variables x_k and y_k, $(1 \le k \le 3)$. One of them stops the shrinking and determines V. The remaining five variables determine W. It follows from the independence of the x_k and y_k that V and W are independent.

We shall determine the distributions of V and W and combine them to get the distribution of $X = VW$.

3. The distribution of V.

V is the area of the shrunken square B. The size of B is determined by the largest of the six variables $|x_k - \frac{1}{2}|$ and $|y_k - \frac{1}{2}|$, $(1 \le k \le 3)$. Each of these variables has the distribution function $K(t) = 2\,t$, $(0 \le t \le \frac{1}{2})$. The largest of the six has the distribution function $K(t_{max})^6 = (2\,t_{max})^6$. The area of B is $v = (2\,t_{max})^2$. We get

$$G(v) = \text{Prob}(V < v) = (2\,t_{max})^6 = v^3, \quad 0 \le v \le 1. \tag{1}$$

4. The distribution of W.

W is the area of a random triangle having one vertex on the boundary of a square (=B) and the other two vertices in the interior of the square. Since the area ratio W is independent of the size of B, we will take B as the original unit square K. Without loss of generality, we will number the three triangle vertices so that vertex one is the one sitting on the boundary and we let this boundary be the x-axis, so that vertex one is $(x, 0)$. The position of the second vertex is (x_2, y_2). Let l_0 be the line through vertices one and two. It contains one side of the triangle. Our calculations will be divided into three cases depending on where l_0 intersects B.

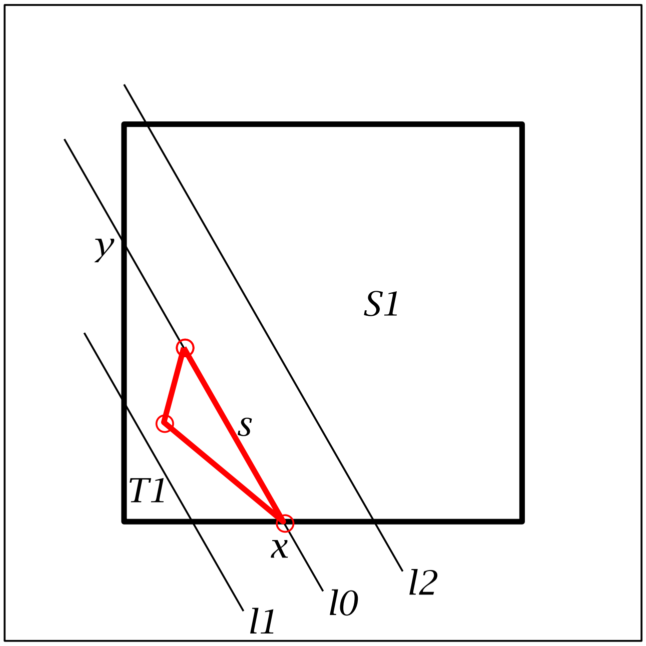

FIGURE 1. Case 1. The line l_0 trough vertices one and two intersects the left side of the square.

4.1. **Case 1.** Case 1, depicted in Figure 1, occurs when l_0 itersects the square side along the y-axis in the point $(0, y)$, $0 \le y \le 1$. The equation for l_0 is

$$l_0 : \eta = -\frac{y}{x}\xi + y.$$

Let $s = \sqrt{(x - x_2)^2 + y_2{}^2}$ be the distance between vertices one and two. For fixed x and y, the maximal value of s is $r_1 = \sqrt{x^2 + y^2}$.

The area of the triangle T will be less than w if the distance between l_0 and the third vertex is less than $2\,w/s$. To avoid the factor 2 in numerous places below, we shall use the double area $u = 2w$ in the calculations. The lines l_1 and l_2 have the distance u/s to l_0.

$$l_1 : \eta = -\frac{y}{x}\xi + y - \frac{u\,r_1}{s\,x}.$$

$$l_2 : \eta = -\frac{y}{x}\xi + y + \frac{u\,r_1}{s\,x}.$$

This means that the conditional probability $P(W \le u/2\,|\,x,\,y,\,s)$ equals the area between the lines l_1 and l_2 in the unit square in Figure 1. We shall use the formula $1 - T_1 - S_1$ (see Figure 1) for this area and we shall average T_1 and S_1 over x, y, and s to get the contribution to $P(W \le u/2)$ from Case 1. In fact, when we consider all possible directions of l_0 in all our cases, it follows from a symmetry argument

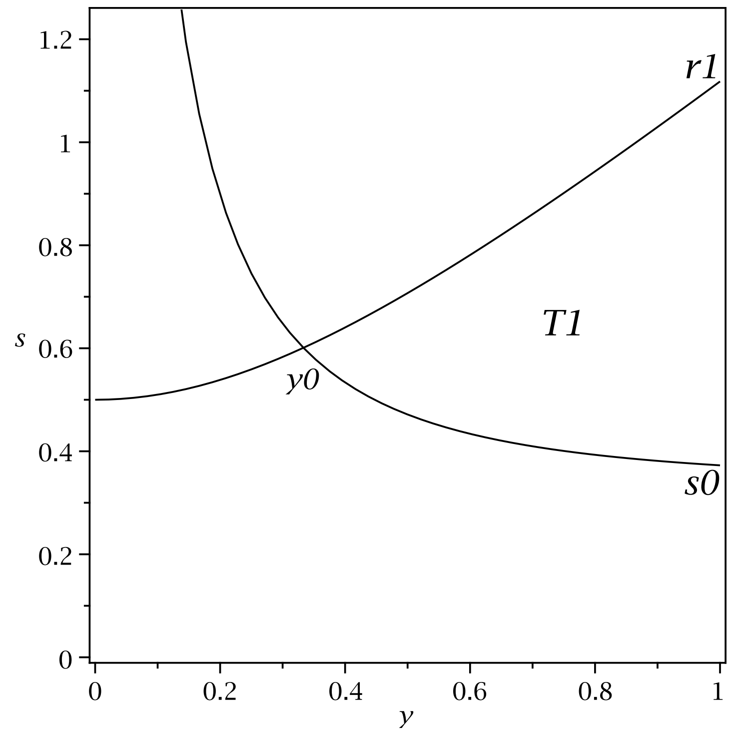

FIGURE 2. Area to integrate s and y over in Case 1 when $u = 1/6$ and $x = 1/2$.

that the areas to the left of l_1 will be the same as those to the right of l_2. This implies that it suffices to calculate the areas to the left and then double the result. Thus, we shall average $2\,T_1$ over x, y, and s and neglect S_1.

From the equation of l_1, we get

$$2\,T_1 = \frac{x}{y}\left(y - \frac{u\,r_1}{s\,x}\right)^2 \text{ if } s > \frac{u\,r_1}{x\,y} \text{ , otherwise } 0 \;. \tag{2}$$

We shall determine the densities of x, y, and s. Obviously, x is evenly distributed over $(0,1)$. The area to the left of l_0 is $xy/2$, so for fixed x, the density is the differential $\frac{x}{2}\,dy$. For fixed x and y consider the small triangle with vertices in $(x,0)$, $(0,y)$, and $(0,y+dy)$. The fraction of the small triangle below s is $(\frac{s}{r_1})^2$ and the density is the differential $\frac{2s}{{r_1}^2}\,ds$. Notice that the integral of the combined density $\rho_1 = x\,s/{r_1}^2$ over the whole range of (x,y,s) is not 1 but $\frac{1}{4}$, which is the probability for Case 1.

Figure 2 shows the range in (s,y)-space to integrate over for fixed u and x. The increasing curve is the upper bound r_1 for s and the decreasing curve is its lower bound $s_0 = \frac{u\,r_1}{x\,y}$. The intersection of the lower and upper s-bounds is the lower bound $y_0 = u/x$ for y. We have $y_0 < 1$ when $x > u$.

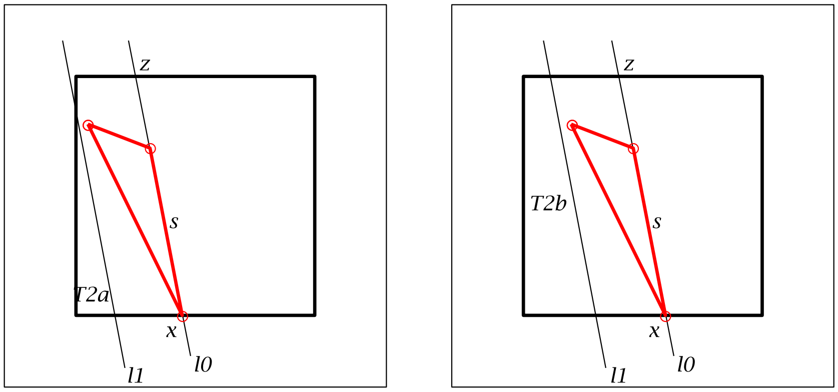

FIGURE 3. Case 2. The line l_0 through vertices one and two intersects the top side of the square, while l_1 intersects the left side in Case 2a and the top side in Case 2b. The figure is drawn with $z < x$.

The contribution from Case 1 is the weighted average of $2\,T_1$:

$$h_1(u) = \int_u^1 x dx \int_{u/x}^1 {r_1}^{-2} dy \int_{u\,r_1/x\,y}^{r_1} \frac{x}{y} \left(y - \frac{u\,r_1}{s\,x} \right)^2 s ds. \tag{3}$$

Maple is helpful in solving integrals of this kind and delivers the result

$$h_1(u) = -\frac{1}{3}\,u^3 + \frac{5}{4}\,u^2 - u + \frac{1}{12} - \frac{1}{2}\,u^2 \log(u)\,(1 - \log(u)). \tag{4}$$

4.2. **Cases 2.** Case 2 occurs when l_0 itersects the top side of the square. It has two subcases a, and b, depicted in Figures 3a and 3b. Case 2a occurs when l_1 intersects the left side of the square and Case 2b when l_1 intersects the top side.

We let z stand for the x-coordinate of the intersection between the top side and l_0.

In Cases 2, the maximal value of s is $r_2 = \sqrt{1 + (x - z)^2}$. The area to the left of l_0 is $\frac{x+z}{2}$, so the z-density is $\frac{1}{2}$ and like in Case 1, the s-density is $2\,s/{r_2}^2$. The combined density is $\rho_2 = s/{r_2}^2$.

We have the two subcases $z < x$ and $z > x$. By symmetry, we can do the calculations for $z < x$ and then double the result.

The equations for l_0 and l_1 are

$$l_0 : \eta_0 = -\frac{1}{x - z}\,(\xi - x),$$
$$l_1 : \eta_1 = -\frac{1}{x - z}\,(\xi - x) - \frac{u\,r_2}{s\,(x - z)}.$$

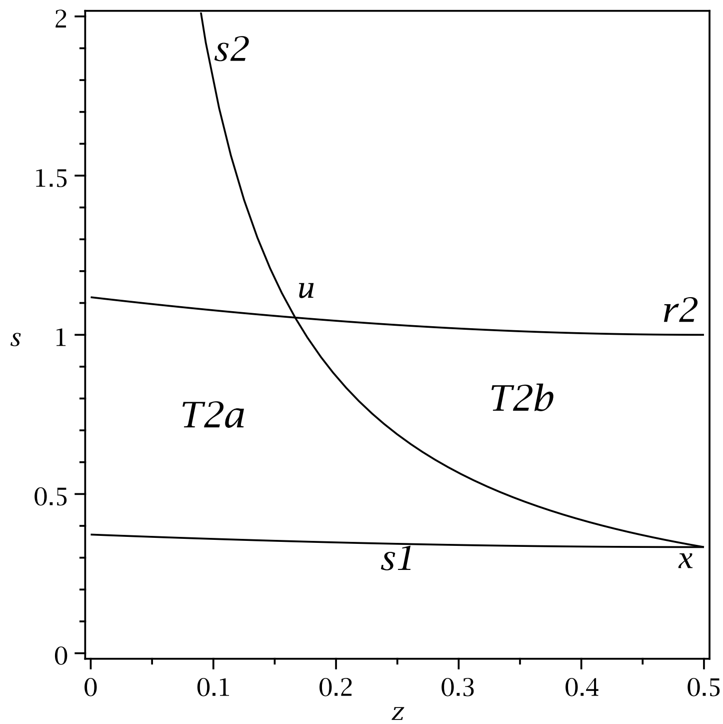

FIGURE 4. Areas to integrate s and z over in Case 2 when $u = 1/6$ and $x = 1/2$.

We have Case 2a when $0 < \eta_1 < 1$, which occurs when

$$\frac{u\,r_2}{x} = s_1 < s < s_2 = \frac{u\,r_2}{z}.$$

Case 2b occurs when $\eta_1 > 1$ i.e. when $s > s_2$.

The expression for $2T_{2a}$ and $2T_{2b}$ are obtained from the equation of l_1:

$$(5) \qquad \begin{aligned} 2\,T_{2a} &= \frac{(x - u\,r_2/s)^2}{x - z}\,, \\ 2\,T_{2b} &= x + z - 2\,u\,r_2/s\,. \end{aligned}$$

Figure 4 shows the range in (s, z)-space to integrate over for fixed u and x. The lower and upper bounds for s are $s_1 = \frac{u\,r_2}{x}$ and r_2. The curve $s_2 = \frac{u\,r_2}{z}$ is upper bound for Case 2a and lower bound for Case 2b. r_2 and s_2 intersect at $z = u$.

The contributions from Cases 2a and 2b are the doubled weighted averages of $2\,T_{2a}$ and $2\,T_{2b}$:

$$
\begin{aligned}
h_{2a}(u) =& 2\int_u^1 dx\, \left(\int_0^u {r_2}^{-2} dz \int_{u\,r_2/x}^{r_2} 2\,T_{2a}\, s\,ds \right.\\
&\left. + \int_u^x {r_2}^{-2} dz \int_{u\,r_2/x}^{u\,r_2/z} 2\,T_{2a}\, s\,ds \right), \\
h_{2b}(u) =& 2\int_u^1 dx \int_u^x {r_2}^{-2} dz \int_{u\,r_2/z}^{r_2} 2\,T_{2b}\, sds.
\end{aligned}
\tag{6}
$$

Evaluation of the integrals gives

$$
\begin{aligned}
h_{2a}(u) =& \frac{1}{6}(2u + u^2 - 3u^3) + \frac{1}{6}(9 + 2u)\, u^2 \, \log(u) \\
&- \frac{1}{6}(1 - 5u - 2u^2)(1 - u)\, \log(1 - u) \\
&+ u^2\, (\log(u)\, \log(1 - u) + \mathrm{dilog}(u)), \\
h_{2b}(u) =& \frac{1}{4}(1 - 5u - 2u^2)(1 - u) - \frac{3}{2}\, u^2\, \log(u).
\end{aligned}
\tag{7}
$$

Here, dilog is *Maple's* dilog function. See Appendix A.

4.3. **Case 3.** This case, depicted in Figures 5 a-d, occurs when l_0 intersects the right side of the square in the point $(1, y)$, $0 \le y \le 1$. When dealing with Case 3, we shall use the variable $x_1 = 1 - x$. The density ρ_3 is obtained by replacing x by x_1 in ρ_1. The equations for l_0 and l_1 are

$$
\begin{aligned}
l_0 &: \eta = \frac{y}{x_1}\, (\xi - 1) + y., \\
l_1 &: \eta = \frac{y}{x_1}\, (\xi - 1) + y + \frac{u\, r_3}{s\, x_1}.
\end{aligned}
$$

Here, s ranges from 0 to $r_3 = \sqrt{x_1^2 + y^2}$. Depending on the values of x_1, y, and u/s, the area to the left of l_1 takes four different shapes as demonstrated in Figure 5. The four areas are:

$$
\begin{aligned}
2\,T_{3a} &= \frac{1}{x_1\, y} \left(x_1 + y - x_1 y - \frac{r_3\, u}{s}\right)^2, \\
2\,T_{3b} &= 2 - 2x_1 + \frac{x_1}{y} - \frac{2\, r_3\, u}{s\, y}, \\
2\,T_{3c} &= 2 - 2y + \frac{y}{x_1} - \frac{2\, r_3\, u}{s\, x_1}, \\
2\,T_{3d} &= 2 - \frac{1}{x_1\, y} \left(x_1\, y + \frac{r_3\, u}{s}\right)^2.
\end{aligned}
\tag{8}
$$

T_{3a} occurs for the largest values of u/s, i.e. for the smallest s. $T_{3a} = 0$ when $s < s_3 = \frac{r_3\, u}{x_1 + y - x_1\, y}$. When s increases, l_1 moves to the right and if

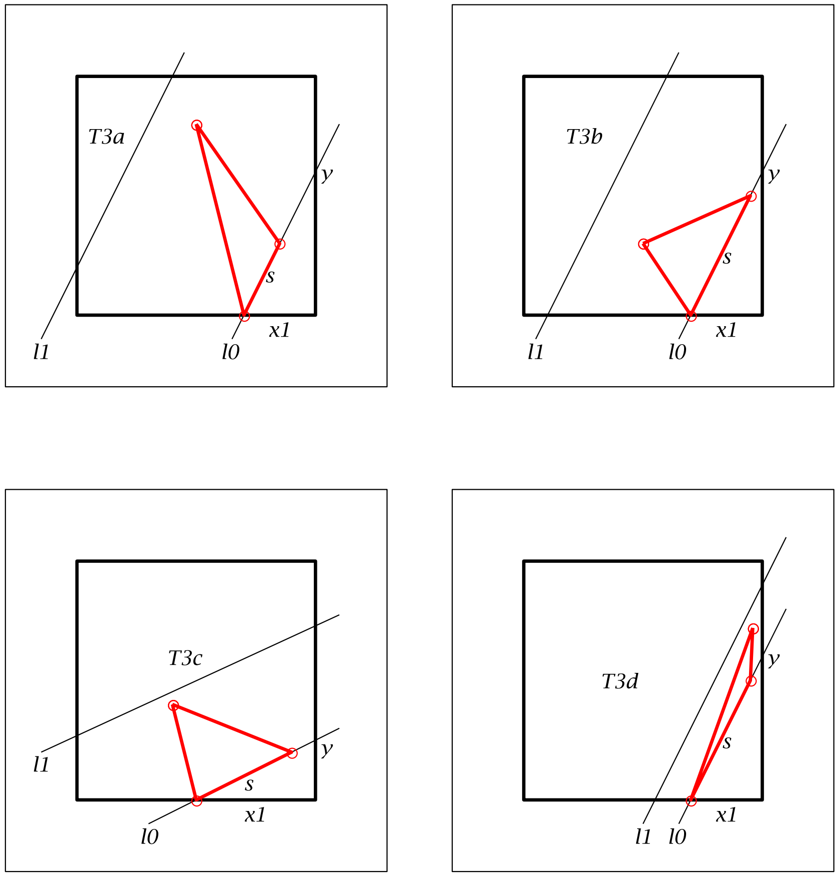

FIGURE 5. The four subcases of Case 3

$y > x_1$ it passes the origin so that T_{3b} replaces T_{3a}. This happens when s passes $s_4 = \frac{r_3\,u}{y(1-x_1)}$. If, on the other hand, $y < x_1$, l_1 passes the point $(1,1)$ and T_{3c} replaces T_{3a} when s passes $s_5 = \frac{r_3\,u}{x_1(1-y)}$. T_{3d} occurs when $s > max(s_4, s_5)$. The upper bound for s is r_3, so the various subcases occur only where s_4 and s_5 are smaller than r_3.

Figure 7 displaying the boundaries in (y, s)-space for $u = \frac{1}{8}$ and $x_1 = \frac{1}{2}$ gives an idea of the situation. The corresponding boundaries for $u = \frac{3}{8}$ and $x_1 = \frac{1}{2}$ are given in Figure 8.

The intersections of the curves in the figures are at:

$$y_3 = \frac{u - x_1}{1 - x_1} \text{ between } r_3 \text{ and } s_3$$

$$y_4 = \frac{u}{1 - x_1} \text{ between } r_3 \text{ and } s_4$$

$$y_5 = 1 - \frac{u}{x_1} \text{ between } r_3 \text{ and } s_5.$$

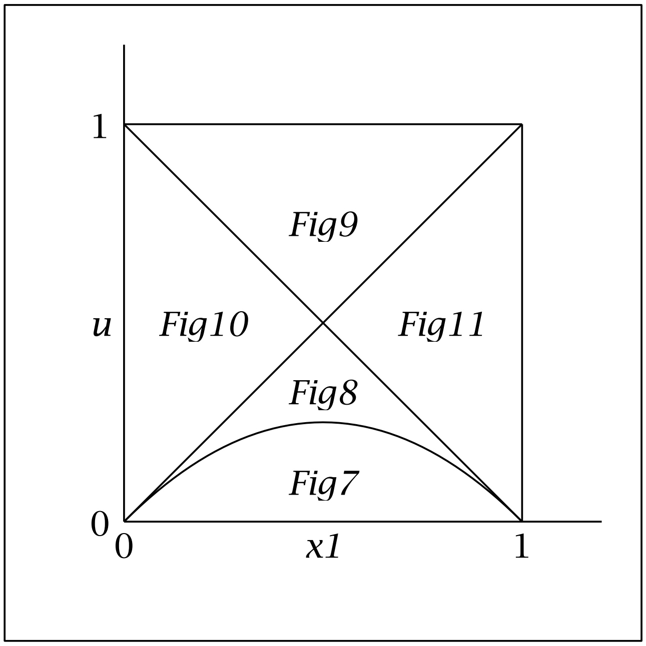

FIGURE 6. Areas in (u,x)-space, where the various subcases occur .

We always have $y_3 \leq y_4$, and we have $y_3 > 0$ when $u > x_1$, and $y_4 < 1$ when $x_1 < 1 - u$, and $y_4 < y_5$ when $u < x_1(1 - x_1)$. This means that we have different subcases depending on the values of u and x_1. We have drawn the boundaries $x_1 = u$, $x_1 = 1 - u$, and $u = x_1(1 - x_1)$ in Figure 6 and indicated where the configurations in Figures 7 and 8 occur.

We shall show that, even though the areas to integrate s and y over are very different in Figures 7 and 8, the resulting integrals are the same, meaning that we don't have to carry out the integrations in Figure 7. Later, we shall show that we don't have to calculate the integrals in Figure 8 either because the integrals in Figure 9 give the same result.

When going from Figure 7 to 8, i.e. when increasing u past $x_1(1-x_1)$, s_3, s_4, and s_5 rise and y_5 becomes smaller then y_4. The intersection of s_4 and s_5 passes r_3 so that the area in Figure 7 marked T_{3d} disappers and the area marked 0 in Figure 8 is created. Denote the area where T_{3d} is valid in Figure 7 by A and consider the integral of T_{3b}. This integral goes in Figure 8 from s_4 to r_3 in the s-direction and from y_4 to 1 in the y-direction. In Figure 7, it goes over the same area minus A. A corresponding argument holds for T_{3c}. The missing integral of T_{3a} over the area marked 0 in Figure 8 is

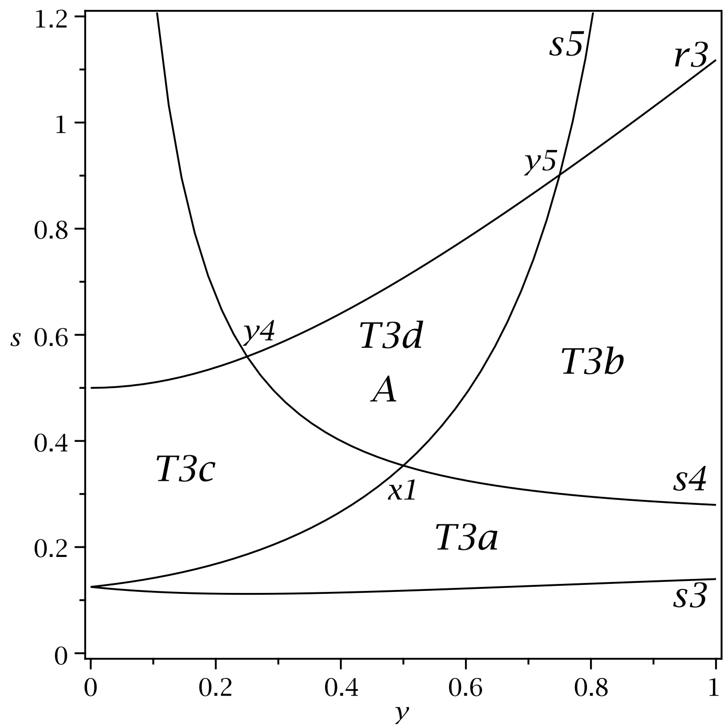

FIGURE 7. Areas in (y,s)-space, where the different T_{3i} occur when $u = \frac{1}{8}$ and $x_1 = \frac{1}{2}$.

$$(9)\quad \int_{y_5}^{x_1} dy \int_{r_3}^{s_5} T_{3a}\,\rho_3 ds + \int_{x_1}^{y_4} dy \int_{r_3}^{s_4} T_{3a}\,\rho_3 ds = \\ = \int_{x_1}^{y_5} dy \int_{s_5}^{r_3} T_{3a}\,\rho_3 ds + \int_{y_4}^{x_1} dy \int_{s_4}^{r_3} T_{3a}\,\rho_3 ds.$$

The latter two integrals are integration of T_{3a} over A. This means that the difference between Figure 8 and 7 is integration over A of $\Delta = T_{3a} + T_{3d} - T_{3b} - T_{3c}$. Insertion of the T_{3i} from equation (8) shows that $\Delta = 0$, implying that integration in Figure 7 gives the same result as in Figure 8.

Now, consider the area in Figure 6 marked Fig10 and also Figure 10 as well as Figure 8. These Figures show that T_{3b} exists when $y_4 < 1$, i.e. when $x_1 < 1 - u$. The contribution from T_{3b} to the distribution function is

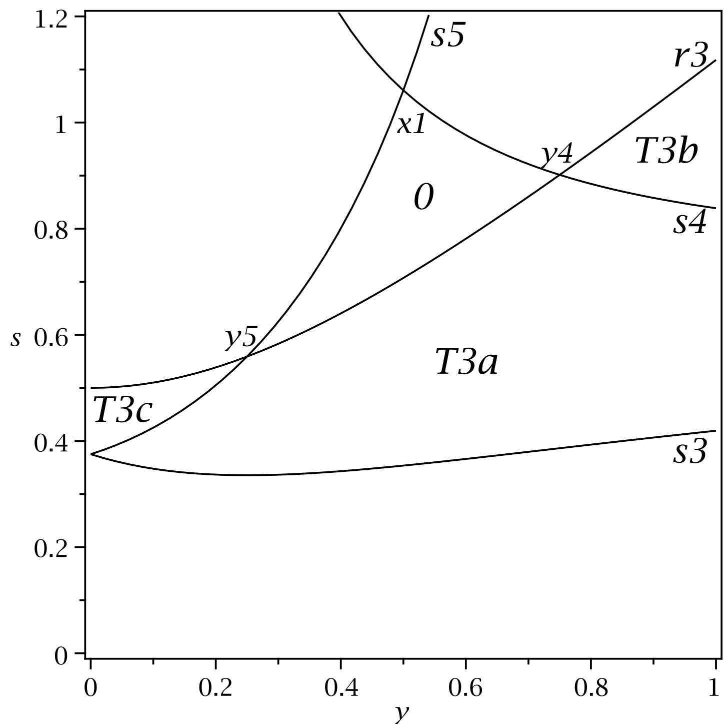

FIGURE 8. Areas in (y,s)-space, where the different T_{4i} occur when $u = \frac{3}{8}$ and $x_1 = \frac{1}{2}$.

(10)

$$h_{3b}(u) = \int_0^{1-u} dx_1 \int_{y_4}^{1} dy \int_{s_4}^{r_3} 2T_{3b}\,\rho_3 ds$$
$$= \frac{1}{18}(1-u)(-4+41\,u+5\,u^2) + \left(-\frac{1}{6}+u+\frac{3}{2}\,u^2\right)\log{(u)}.$$

The Figures 8, and 11 show that T_{3c} exists when $y_5 > 0$, i.e. when $x_1 > u$. The contribution from T_{3c} to the distribution function is

$$h_{3c}(u) = \int_u^1 dx_1 \int_0^{y_5} dy \int_{s_5}^{r_3} T_{3c}\,\rho_3 ds = \frac{1}{2}\left(1-u+u\,\log{(u)}\right)^2. \tag{11}$$

The contribution from T_{3a} is more complicated since it exists for all x_1. T_{3a} is present in Figures 8, 9, 10, and 11. We shall show that the contribution is the same for $u < \frac{1}{2}$ and $u > \frac{1}{2}$. For $u < \frac{1}{2}$, we have $u < 1-u$ and shall integrate over the areas in Figures 10, 8, and 11. Omitting the integrand $2T_{3a}\,\rho_3$ and the differentials and just writing the integration boundaries, we have for $u < \frac{1}{2}$:

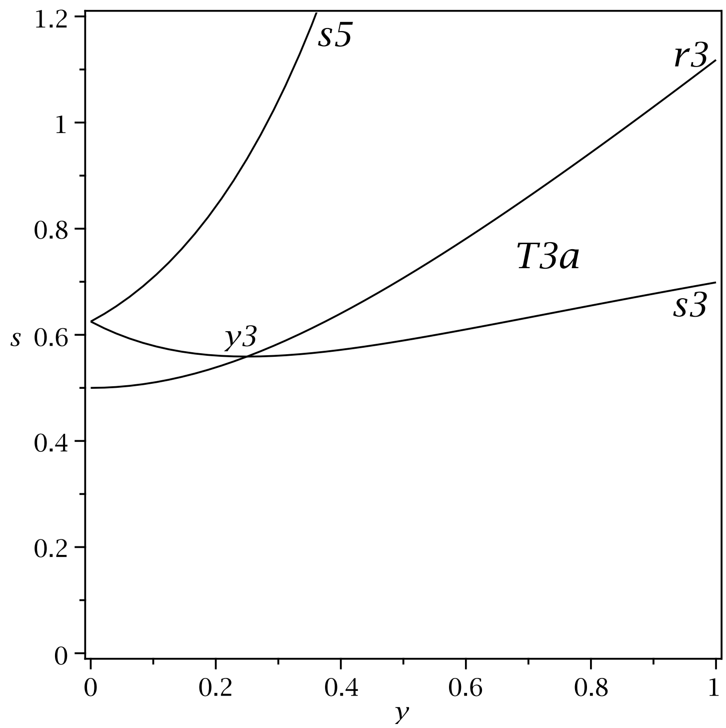

FIGURE 9. Areas in (y,s)-space, where T_{3a} is valid when $u = \frac{5}{8}$ and $x_1 = \frac{1}{2}$. Note that $1 - u < x_1 < u$.

$$(12)\quad I_1 = \int_0^u \Big(\int_{y_3}^{y_4} \int_{s_3}^{r_3} + \int_{y_4}^{1} \int_{s_3}^{s_4} \Big) + \int_u^{1-u} \Big(\int_0^{y_5} \int_{s_3}^{s_5} + \int_{y_5}^{y_4} \int_{s_3}^{r_3} + \int_{y_4}^{1} \int_{s_3}^{s_4} \Big) + \int_{1-u}^{1} \Big(\int_0^{y_5} \int_{s_3}^{s_5} + \int_{y_5}^{1} \int_{s_3}^{r_3} \Big)$$

For $u > \frac{1}{2}$, we have $1 - u < u$ and shall integrate over the areas in Figures 10, 9, and 11 and get:

$$(13)\quad I_2 = \int_0^{1-u} \Big(\int_{y_3}^{y_4} \int_{s_3}^{r_3} + \int_{y_4}^{1} \int_{s_3}^{s_4} \Big) + \int_{1-u}^{u} \int_{y_3}^{1} \int_{s_3}^{r_3} + \int_u^1 \Big(\int_0^{y_5} \int_{s_3}^{s_5} + \int_{y_5}^{1} \int_{s_3}^{r_3} \Big)$$

We shall show that the integrals I_1 and I_2 are the same.

First, notice that in both I_1 and I_2, the x-integration of $\int_{y_4}^{1} \int_{s_3}^{s_4}$ goes from 0 to $1 - u$ and that of $\int_0^{y_5} \int_{s_3}^{s_5}$ goes from u to 1.

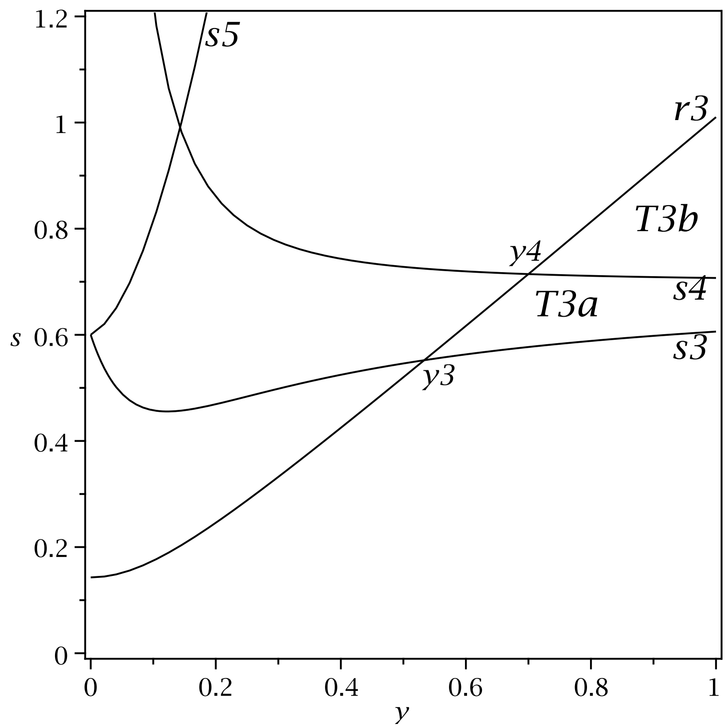

FIGURE 10. Areas in (y,s)-space, where T_{3a} and T_{3b} are valid when $u = \frac{3}{5}$ and $x_1 = \frac{1}{7}$. Note that $0 < x_1 < 1 - u$.

The three remaining integrals in I_1 and I_2 are all of type $\int_{s_3}^{r_3}$ in the s-direction. In the x_1- and y- directions, they are

$$I_1^* = \int_0^u \int_{y_3}^{y_4} + \int_u^{1-u} \int_{y_5}^{y_4} + \int_{1-u}^1 \int_{y_5}^1,$$

and

$$I_2^* = \int_0^{1-u} \int_{y_3}^{y_4} + \int_{1-u}^u \int_{y_3}^1 + \int_u^1 \int_{y_5}^1.$$

I_2^* can be split up into:

$$I_2^* = \int_0^u \int_{y_3}^{y_4} + \int_u^{1-u} \int_{y_3}^{y_4} - \int_u^{1-u} \int_{y_3}^1 + \int_u^{1-u} \int_{y_5}^1 + \int_{1-u}^1 \int_{y_5}^1.$$

Comparing terms, it is easily seen that I_1^* and I_2^* are the same. This implies that the contribution h_3 to the distribution of u from Case 3, has the same formula for $u < \frac{1}{2}$ and $u > \frac{1}{2}$.

We shall use the expression for $u > \frac{1}{2}$ to calculate $h_3 = h_{3a} + h_{3b} + h_{3c}$, where h_{3b} and h_{3c} are given in (10) and (11) and h_{3a} is the contribution from T_{3a} which we take from I_2 in (13). With the notation

$$k = k(u, x_1, y, s) = 2T_{3a}\,\rho_3,$$

we have.

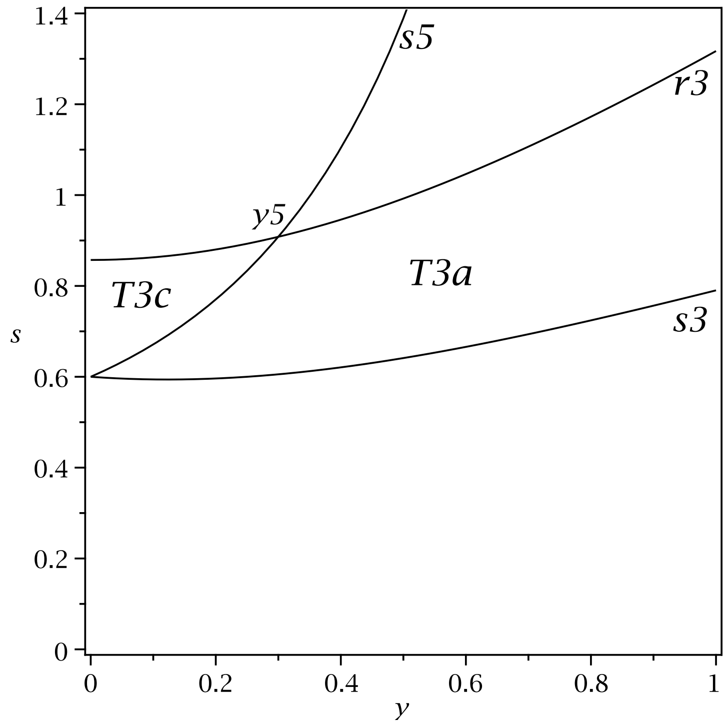

FIGURE 11. Areas in (y,s)-space, where T_{3a} and T_{3c} are valid when $u = \frac{3}{5}$ and $x_1 = \frac{6}{7}$. Note that $u < x_1 < 1$.

$$
\begin{aligned}
h_{3a}(u) &= \int_0^{1-u} dx_1 \left(\int_{y_3}^{y_4} dy \int_{s_3}^{r_3} kds + \int_{y_4}^{1} dy \int_{s_3}^{s_4} kds \right) \\
&+ \int_{1-u}^{u} dx_1 \int_{y_3}^{1} dy \int_{s_3}^{r_3} kds \\
&+ \int_{u}^{1} dx_1 \left(\int_{0}^{y_5} dy \int_{s_3}^{s_5} kds + \int_{y_5}^{1} dy \int_{s_3}^{r_3} kds \right) \\
&= \frac{1}{36}(1-u)(5 - 97\,u - 22\,u^2) \\
&+ \frac{1}{3}(1-u)(-1 + 5\,u + 2\,u^2)\log(1-u) \\
&+ \left(\frac{1}{6} - 2\,u + \frac{2}{3}u^3 \right) \log(u) - u^2 \log(u)^2 \\
&+ 2\,u^2 (\log(u)\log(1-u) + \mathrm{dilog}(u)),
\end{aligned} \tag{14}
$$

4.4. **Combination of cases.** By combining the calculated h_{nx}, we get the distribution function for twice the area of a random triangle in a unit square, when one of the triangle vertices sits on the boundary of the square:

$$H(u) = 1 - h_1 - h_{2a} - h_{2b} - h_{3a} - h_{3b} - h_{3c}$$
$$\begin{aligned} &= \frac{u}{3}(14 - 11\,u - 4\,u^2\,\log(u)) \\ &+ \frac{2}{3}(1-u)(1-5\,u-2\,u^2)\,\log(1-u) \\ &- 4\,u^2\,(\log(u)\,\log(1-u) + \mathrm{dilog}(u)), \quad 0 \le u \le 1. \end{aligned} \tag{15}$$

5. Combination of the V- and W-distributions.

Let $F(x)$ be the distribution function for the triangle area X. We have $X \le x$ when $VW = UV/2 \le x$. Putting $x = y/2$, this happens when $UV \le y$ and we get

$$\begin{aligned} F(y/2) &= \int_0^1 G(y/u)\,dH(u) = \\ &= [G(y/u)H(u)]_0^1 - \int_y^1 H(u)\frac{d}{du}G(y/u)\,du = \\ &= G(y) - \int_y^1 H(u)\frac{d}{du}G(y/u)\,du, \quad 0 \le y \le 1. \end{aligned} \tag{16}$$

The partial intergration in (16) is used to avoid integrating to the lower bound $u = 0$. To write the result, we need the ν function

$$\nu(x) = -\int_0^x \frac{\log|1-t|}{t}\,dt. \tag{17}$$

This function is the real part of the dilogarithm function $\mathrm{Li}_2(x)$ discussed by Euler in 1768 and named by Hill, [5]. $\nu(x)$ is well defined on the whole real axis. Some properties of $\nu(x)$ are given in Appendix A.

We will not carry out the integration (16) in detail, but will just give the result

$$\begin{aligned} F(x) &= \frac{4x}{3}(10-17x) - \frac{16x^3}{3}(17 - 3\log(2x))\log(2x) \\ &+ \frac{2}{3}(1-16x-68x^2)(1-2x)\log(1-2x) + 16x^2(3+2x)\left(\nu(2x) - \frac{\pi^2}{6}\right) \\ &\qquad 0 \le x \le 1/2. \end{aligned} \tag{18}$$

Figure 12 shows the density function is $f(x) = dF/dx$:

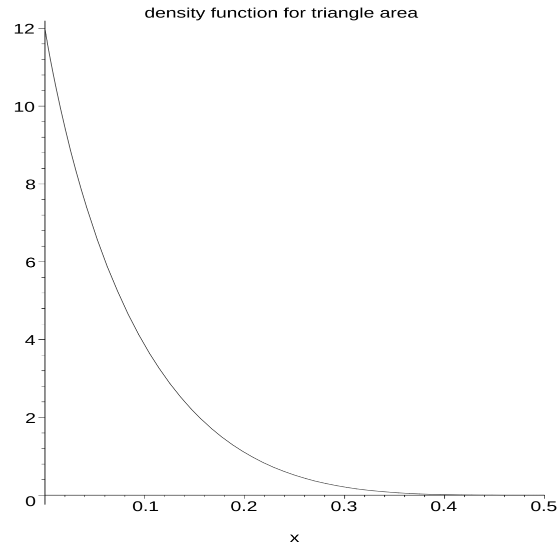

FIGURE 12. Density function for the area of an random triangle in a square.

$$f(x) = 12 \Bigg[1 - 2x - 4x^2(5 - \log(2x)) \log(2x) - (1+10x)(1-2x)\log(1-2x) + 8x(1+x)\left(\nu(2x) - \frac{\pi^2}{6}\right)\Bigg], \quad 0 \le x \le 1/2. \tag{19}$$

The first moments and the standard deviation of the triangle area are

$$\alpha_1 = \int_0^{\frac{1}{2}} x\, dF(x) = \frac{11}{144} \approx .076389, \tag{20}$$

$$\alpha_2 = \int_0^{\frac{1}{2}} x^2\, dF(x) = \frac{1}{96}, \tag{21}$$

$$\sigma = \sqrt{\alpha_2 - {\alpha_1}^2} = \frac{\sqrt{95}}{144} \approx .067686. \tag{22}$$

6. Concluding comment.

We have not shown any integral calculations in detail. In principle, they are elementary, which doesn't mean that they don't require a substantial effort. As indicated, the calculations have been done in Maple. The calculations would not have been possible without some tool for handling the large number of terms that come out of the integrations. This doesn't mean that Maple performs the integrations automatically. Often, we had to split up the integrands in parts and treat each part in a special way. We had to do some partial integrations manually.

We will supply any interested reader with a Maple file describing the calculations.

Appendix A

The dilogarithm function $\mathrm{Li}_2(x)$ is defined in [6] for complex x as

$$\mathrm{Li}_2(x) = -\int_0^x \frac{\log(1-t)}{t}\, dt. \tag{23}$$

When x is real and greater than unity, the logarithm is complex. A branch cut from 1 to ∞ can give it a definite value. In this paper, we are only interested in real x and the real part of Li_2

$$\nu(x) = \mathrm{Re}(\mathrm{Li}_2(x)) = -\int_0^x \frac{\log|1-t|}{t}\, dt. \tag{24}$$

We have the series expansion

$$\nu(x) = \mathrm{Re}(\mathrm{Li}_2(x)) = \sum_{k=1}^{\infty} \frac{x^k}{k^2}\ , \quad |x| \le 1. \tag{25}$$

Although the series is only convergent for $|x| \le 1$, the integrals in (23) and (24) are not restricted to these limits and the ν function is defined and is real on the whole real axis. We use this function for $0 \le x \le 1$.

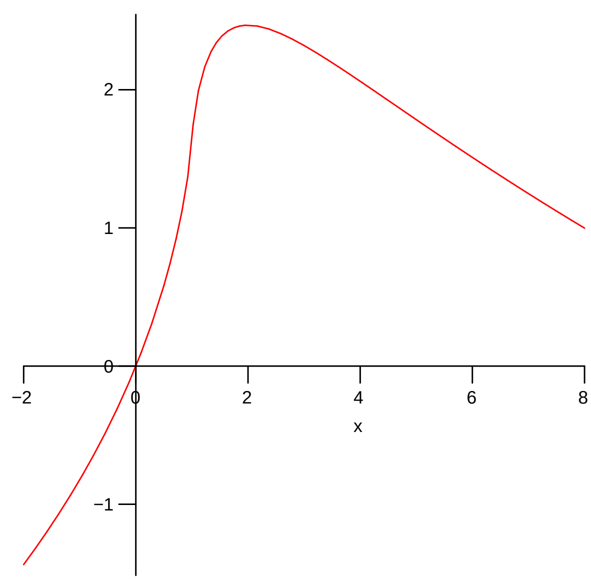

Figure 13. The function $\nu(x)$.

The definition of the dilogarithm function has varied a little from author to author. Maple has the function polylog$(2, x)$ which is defined by the series expansion (25) for $|x| \le 1$ otherwise by analytic continuation. Maple also has a function $\operatorname{dilog}(x) = \operatorname{Li}_2(1-x)$ defined on the whole real axis. Maple's dilog function is the same as the dilog function given in [1], page 1004.

$\nu(x)$ is increasing from $\nu(0) = 0$ via $\nu(1) = \pi^2/6$ to $\nu(2) = \pi^2/4$.

The integrals involving $\nu(x)$ needed for calculating the moments of various distributions take rational values like

$$\int_0^1 x\, d\nu(x) = 1,$$
$$\int_0^1 x^2\, d\nu(x) = \frac{3}{4},$$
$$\int_0^1 x^3\, d\nu(x) = \frac{11}{18}.$$

References

[1] M. Abramowitz and I. Segun, *Handbook of Mathematical Functions* Dover Publications, Inc., New York, 1965
[2] H. A. Alikoski, Über das Sylvestersche Vierpunktproblem *Ann. Acad. Sci. Fenn.* 51 (1938), no. 7, pp. 1-10.
[3] W. Blaschke, *Vorlesungen über Differentialgeometrie Vol 2.* Springer, Berlin 1923.
[4] C. Buchta and M. Reitzner, *The convex hull of random points in a tetrahedron: Solution of Blaschke's problem and more general results J. reine angew. Math.* 536 (2001), 1-29.
[5] C. J. Hill, Journal für Math. (Crelle) 3:107, 1828
[6] L. Lewin, *Polylogarithms and Associated functions* North Holland, New York, Oxford, 1981.
[7] R. E. Miles, Isotropic Random Simplices *Advances in Appl. Probability* 3 (1971), pp. 353-382.
[8] J. Philip, The Area of a Random Convex Polygon. TRITA MAT 04 MA 07 (2004). `www.math.kth.se/~johanph/area12.pdf`.
[9] A. Rényi and R. Sulanke, Über die konvexe Hülle von n zufällig gewählte Punkten *Z. Wahrscheinlichkeitstheorie* 2(1963), pp. 75-84.
[10] A. Rényi and R. Sulanke, Über die konvexe Hülle von n zufällig gewählte Punkten, II *Z. Wahrscheinlichkeitstheorie* 3(1964), pp. 138-147.
[11] L. Santaló, *Integral Geometry and Geometric Probability* Encyclopedia of mathematics and its Applications, Addison-Wesley 1976
[12] M. Trott, The Area of a Random Triangle in a Square, 1.10.1 in *The Mathematica GuideBook for Symbolics*, Springer-Verlag, pp. 298-311, 2006

Department of Mathematics, Royal Institute of Technology, S-10044 Stockholm Sweden

E-mail address: `johanph@kth.se`