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The Moduli Spaces of Vector Bundles over an Algebraic Curve

S. Ramanan

§ 1. Statement of Results

Let X be a complete, nonsingular algebraic curve/$\mathbb{C}$ of genus $g \geqq 2$, and L a line bundle over X of degree d. We denote by $U(n, d)$ (resp. $U_L(n,d)$) the moduli-space of S-equivalence classes of semi-stable vector bundles of rank n and degree d (resp. with determinant isomorphic to L) over X. (The existence of these varieties has been proved in [10] and for a general introduction, see [4].) If n and d are coprime, the variety $U_L(n, d)$ is complete and nonsingular and Seshadri has proved that $Pic(U_L(n, d))$ is free cyclic (see Proposition 3.4 for a proof similar to his). Let u be the ample generator of this group. Our first result is the computation of the canonical class of $U_L(n, d)$ in terms of u.

Theorem 1. *The canonical class of* $U_L(n, d)$ *is* u^{-2}.

The proof consists in the construction of a map from a projective space P into $U_L(n, d)$ and the study of the pull-back of the tangent bundle. The determinant of the pull-back is computed using the base change and the explicit construction of the family of bundles defining the map of P into $U_L(n, d)$. Similar ideas have been used by Tjurin [11, 12], where, in particular, Lemma 2.1 has been proved.

An interesting result which comes out as an application is the following.

Theorem 2. *If n and d are not coprime, there does not exist a Poincaré family on any Zariski open subset of* $U_L(n, d)$.

Basically the proof is similar to that in [4] where this is proved in the case genus 2, $n = 2$, $d = 0$, but the geometric interpretation in [§ 7, 4] is replaced here by a direct computation. It may also be mentioned here that it has since been shown that in the degree 0 case, even topological obstructions exist for the existence of Poincaré family. See Newstead [8].

A refinement of the procedure mentioned with regard to Theorem 1 would actually yield a map of a projective bundle $P(M)$ over the Jacobian into $U_L(2, 1)$. A study now of the 2^{nd} chern class of the universal bundle on

$U_L(2, 1) = X$ gives information on the third cohomology group of $U_L(2, 1)$. This enables one to obtain a somewhat more satisfactory proof of the following theorem of Mumford and Newstead [2].

Theorem 3. *The Kunneth component of the* 2^{nd} *chern class of the Poincaré bundle on* $U_L(2, 1) = X$ *in* $H^3(U_L(2, 1), \mathbb{Z}) \otimes H^1(X, \mathbb{Z})$ *gives a unimodular element.*

Although our proof of the theorem is more direct, we have unfortunately to use the fact due to Newstead [7] that $b_3(U_L(2, 1)) = 2g$, for which only a purely topological proof is known. See however [5].

In fact, a more systematic study of the map $P(M) \to U_L(2, 1)$ should throw more light on the multiplicative structure of the cohomology ring of $U_L(2, 1)$. To illustrate this we have carried out this computation only in the case $g = 3$. We determine the cohomology ring in this case fully, identify the chern classes and by straightforward, if laborious substitution, evaluate $\chi(U_L(2, 1), \Theta)$, where Θ is the tangent bundle. In fact, it has been recently shown (Narasimhan and Ramanan [5]) that $H^0(U_L(2, 1), \Theta) = 0$, and on the other hand as a simple corollary of Theorem 1, we have $H^i(U_L(2, 1), \Theta) = 0$ for $i \geqq 2$. Thus this yields a comparison of the moduli of X and that of $U_L(2, 1)$. For a complete statement of these, see Theorem 4 in § 5.

Finally, I would like to thank C. S. Seshadri for letting me know his proof of the computation of the Picard group and M. S. Narasimhan and D. Mumford for many discussions and illuminating remarks. Also, my thanks are due to P. E. Newstead for keeping me informed of his work on related topics, and to the referee for a careful reading of the manuscript and for several improvements in the exposition, especially in connection with Proposition 3.4 and Lemma 3.5.

Notation. X will denote a nonsingular, complete algebraic curve. If E is a vector bundle $\neq 0$ on X, $\mu(E)$ will denote the rational number $\deg E/rk E$.

§ 2. Computation of First Chern Class of $U_L(n, d)$

Lemma 2.1. *Let n and d be integers which are coprime, and let $l: 0 < l < n$ be the unique integer such that $ld \equiv 1(n)$. Then there exists $e: 0 \leqq e < n$ with $ld - en = 1$. If V, W are stable vector bundles on X of ranks l and $(n-l)$, and degrees e and $(d-e)$ respectively, then any nontrivial extension*

$$0 \to V \to E \to W \to 0$$

gives rise to a stable vector bundle E.

Proof. Let F be any proper subbundle of E. Then we have to show that $\deg F/rk F < d/n$. The map $F \to W$ can be factored through a surjection $F \to W'$ and a generic inclusion $W' \to W$ [6, § 4]. In other words, we have a commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \to & V' & \to & F & \to & W' & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & V & \to & E & \to & W & \to & 0 \end{array}$$

where the rows are exact. By stability of V and W, we have i) $\mu(W') \leqq \mu(W)$ and ii) $\mu(V') \leqq \mu(V)$. Moreover, equality occurs in i) only if $W'=0$ or $W'=W$, and in ii) only if $W'=0$ and $V=V'=F$, or $W'=W$ and $V'=0$. In the first case, the inequality to be proved is $e/l<d/n$, which is clear. In the second case, F maps isomorphically on W which contradicts the assumption that the extension is non-trivial.

Now, assuming that equality does not occur in both i) and ii), we have

$$\mu(F) = (\deg(V') + \deg W')/rkF < 1/rkF\{e\,rk\,V'/l + (d-e)\,rk\,W'/(n-l)\}$$

$$= \frac{1}{l(n-l)\,rkF}\{e(n-l)(rkF - rk\,W') + l(d-e)\,rk\,W'\}$$

$$= e/l + rk\,W'/l(n-l)(rkF).$$

Using the fact that $rk\,W' \leqq n-l$ and also $rk\,W' \leqq rkF$ we obtain

$$rk\,W'/l(n-l)\,rkF = \frac{rk\,W'\{n-l+l\}}{n\,l(n-l)\,rkF} \leqq 1/ln + 1/n \cdot rkF.$$

Hence we have

$$\mu(F) < e/l + 1/ln + 1/n \cdot rkF$$

$$= \frac{ne+1}{ln} + \frac{1}{n \cdot rkF} = d/n + 1/n \cdot rkF$$

i.e. $n \deg F - d \cdot rkF < 1$. This shows that $n \deg F - d \cdot rkF \leqq 0$ and since n and d are coprime that $n \deg F - d \cdot rkF < 0$. q.e.d.

It is well-known that extensions $0 \to V \to E \to W \to 0$ are classified by $H^1(X, \mathrm{Hom}(W, V))$. Moreover, vector bundles given by non-zero elements of the same one-dimensional subspace of $H^1(X, \mathrm{Hom}(W, V))$ are isomorphic [3, Lemma 3.3] and hence one might expect to construct a family of vector bundles on X parametrised by $P = PH^1(X, \mathrm{Hom}(W, V))$.

Lemma 2.2. *Let τ be the hyperplane bundle on P. Then there exists a canonical isomorphism*

$$H^1(P\times X, \operatorname{Hom}(p_X^* W, p_X^* V\otimes p_P^*\tau))\approx \operatorname{End} H^1(X, \operatorname{Hom}(W, V)).$$

Proof. In fact, since the bundle $\operatorname{Hom}(p_X^* W, p_X^* V\otimes p_P^*\tau)$ is clearly isomorphic to $p_P^*\tau\otimes p_X^*\operatorname{Hom}(W, V)$, on applying the Kunneth formula, and noting that $H^1(P,\tau)=0$, we get

$$\text{L.H.S. in the Lemma} \approx H^0(P,\tau)\otimes H^1(X, \operatorname{Hom}(W, V)).$$

But since $H^0(P,\tau)$ is canonically dual to $H^1(X, \operatorname{Hom}(W, V))$, the lemma is proved.

Lemma 2.3. *There exists a family of vector bundles $(E_p)_{p\in P}$ on X parametrised by P so that for each $p\in P$, the E_p is isomorphic to the bundle obtained as the extension of W by V given by p.*

Proof. Consider the bundle E on $P\times X$, obtained as follows. Using the isomorphism of Lemma 2.2, we see that there is a distinguished element γ in $H^1(P\times X, \operatorname{Hom}(p_X^* W, p_X^* V\otimes p_P^*\tau))$ corresponding to the identity endomorphism of $H^1(X, \operatorname{Hom}(W, V))$. This then gives rise to an extension

$$0\to p_X^* V\otimes p_P^*\tau\to E\to p_X^* W\to 0.$$

We have only to show that $E|_{p\times X}$ is of the desired type. Clearly $E_p = E|_{p\times X}$ is an extension of W by V. The element in $H^1(X, \operatorname{Hom}(W, V))$ corresponding to this extension is the image of γ given by the restriction

$$H^1(P\times X, \operatorname{Hom} p_X^* V\otimes p_P^*\tau)\to H^1(X, \operatorname{Hom}(W, V))$$

or, what is the same,

$$H^1(P\times X, p_P^*\tau\otimes p_X^*\operatorname{Hom}(W, V))\to H^1(X, \operatorname{Hom}(W, V)).$$

Identifying the first of these vector spaces with $H^0(P,\tau)\otimes H^1(X, \operatorname{Hom}(W, V))$ and noting that the image of the identity in $H^1(X, \operatorname{Hom}(W, V))$ gives an element of the one-dimensional space p, we obtain that E_p corresponds to the extension p of W by V.

We shall also need later the following generalisation of Lemma 2.3.

Lemma 2.4. *Let $(E_s)_{s\in S}$, $(F_t)_{t\in T}$ be two families of vector bundles on X parametrised by S, T respectively. Assume that $\dim H^1(X, \operatorname{Hom}(F_t, E_s))$ is independent of $s\in S$, $t\in T$. Let $\pi: P(V)\to S\times T$ be the projective bundle associated to the vector bundle V whose fibre at (s,t) is $H^1(X, \operatorname{Hom}(F_t, E_s))$. Let, moreover, $H^i(S\times T, (p_{S\times T})_* \operatorname{Hom}(F, E)\otimes V^*)=0$ for $i=1,2$. Then there exists a family W of vector bundles on X parametrised by $P(V)$*

given as an extension

$$0 \to \pi^* p^*_{S\times X} E \otimes p^*_{P(V)} \tau \to W \to \pi^* p^*_{T\times X} F \to 0$$

which restricts to each fibre of $P(V)$ *as the corresponding extension defined in Lemma 2.3.*

Proof. Similar to [3, Proposition 3.1].

Let d and $n > 0$ be two integers which are coprime. Let $0 < l < n$ and e be so chosen that $ld - en = 1$. Let V, W be stable vector bundles on X of ranks l and $(n-l)$ and degrees e and $(d-e)$ respectively. Then by Lemma 2.3, there is a family of vector bundles on X parametrised by $P = PH^1(X, \mathrm{Hom}(W, V))$. Moreover this is a family E of stable vector bundles of $\deg d$ and rank n and with determinant $= \det W \cdot \det V$ by Lemma 2.1. By the universal property of the variety $U(n, d)$, this gives rise to a morphism $\varphi : P \to U_L(n, d)$, where $L = \det V \cdot \det W$. We would like to study this map φ.

It is known (when n and d are coprime) that there exists a "universal" vector bundle U on $U_L(n, d) \times X$ so that $(\varphi \times 1_X)^* U$ and E coincide on $p \times X$ for each $p \in P$.

Lemma 2.5. *Let* E, F *be two vector bundles on* $T \times X$ *such that* $E|_{t\times X} \approx F|_{t\times X}$ *for each* $t \in T$. *If* $H^0(X, \mathrm{End}\, E|t \times X)$ *is* 1 *dimensional for each* $t \in T$, *then there exists a line bundle* L *on* T *so that* $E \approx F \otimes p^*_T L$.

Proof. Since $H^0(X, \mathrm{Hom}(F, E)|_{t\times X})$ is 1-dimensional for such $t \in T$, the direct image $(p_T)_* \mathrm{Hom}(F, E)$ is an invertible sheaf on T and let L be the associated line bundle. It is obvious that $\mathrm{Hom}(F, E) \otimes p^*_T L^*$ has a trivial line bundle as direct image under p_T. In particular, this gives rise to an element of $H^0(T \times X, \mathrm{Hom}(F \otimes p^*_T L, E))$ which is non-zero on each $t \times X$, $t \in T$. By assumption any non-zero map $F \otimes p^*_T L|_{t\times X} \to E|_{t\times X}$ is an isomorphism. Hence $F \otimes p^*_T L \approx E$.

Applying Lemma 2.5 to our situation, we obtain a map $\varphi : P \to U_L(n, d)$ such that $(\varphi \times 1_x)^* U = E \otimes p^*_P \tau^m$ for some $m \in \mathbb{Z}$. If θ is the tangent bundle of $U_L(n, d)$, then we are interested in studying the bundle $\varphi^* \theta$ on P. In order to do this, we shall interpret θ in terms of U.

Lemma 2.6. *The vector bundle associated to the first direct image* $R_1(p)$ *of* $\mathrm{ad}' U$ *on* $U_L(n, d)$ *is isomorphic to the tangent bundle* θ, *where* $\mathrm{ad}' U$ *denotes the bundle of endomorphisms of trace* 0 *on* U.

Proof. Note that $p_*(\mathrm{ad}' U) = 0$ since $H^0(X, \mathrm{ad}' E) = 0$ if E is stable [6, Corollary to Proposition 4.3]. This shows that $R_1(p)(\mathrm{ad}' U)$ is locally free with fibres of the form $H^1(X, \mathrm{ad}' E)$. From the general theory of deformations, we then have a bundle homomorphism $\theta \to R_1(p)(\mathrm{ad}' U)$.

Since the given family of vector bundles is injective and complete, this homomorphism is also an isomorphism.

Lemma 2.7. *If* $E=(E_t)_{t\in T}$ *is a family of stable vector bundles on* X, *of rank* n *and determinant* $\approx L$, *and* $\varphi: T\to U_L(n,d)$ *the induced morphism, then* $\varphi^*\theta$ *is isomorphic to the first direct image of* ad$'E$ *on* T.

Proof. By Lemma 2.6, we have $\varphi^*\theta\approx\varphi^*(R^1(p)(\mathrm{ad}'U))$. But by the base change theorem [1, Corollary 1, § 7.3], there is a canonical isomorphism, $\varphi^*(R^1(p))(\mathrm{ad}'U)\approx R^1(p_T)(\varphi\times 1_X)^*(\mathrm{ad}'U)$. On the other hand, by Lemma 2.5, we see that $(\varphi\times 1_X)^*(\mathrm{ad}'U)\approx\mathrm{ad}'E$.

2.8. *Proof of Theorem 1.* We apply the Lemma 2.7 for the family $P=PH^1(X,\mathrm{Hom}(W,V))$. Let us first compute the class $[\mathrm{ad}'E]$ in $K(P\times X)$. Clearly,

$$[\mathrm{ad}'E]=(p_X^![V]\,p_P^![\tau]+p_X^![W])(p_X^![V^*]\,p_P^![\tau^*]+p_X^![W^*])-1\,.$$

Since $R_i(p_P)(\mathrm{ad}'E)=0$ for $i\neq 1$, we have

$$\begin{aligned}[R_1(p_P)(\mathrm{ad}'E)]&=-p_![\mathrm{ad}'E]=-p_!\{p_X^![V\otimes V^*]+p_X^![W\otimes W^*]\\&\qquad+p_X^![V\otimes W^*]\,p_P^![\tau]+p_X^![W\otimes V^*]\,p_P^![\tau^*]-1\}\\&=-\{\chi(V\otimes V^*)+\chi(W\otimes W^*)\\&\qquad-\chi(1)+\chi(V\otimes W^*)\tau+\chi(W\otimes V^*)\tau^*\}\,.\end{aligned}$$

Hence $\det(R_1(p_P)\,\mathrm{ad}'E)=H^s$ where $s=-\chi(V\otimes W^*)+\chi(W\otimes V^*)$ and H is the positive generator of $Pic\,P$.

We have

$$\deg(V\otimes W^*)=e(n-l)-(d-e)\,l=-1$$

and

$$\chi(V\otimes W^*)=-1+l(n-l)(1-g)\,.$$

Also

$$\chi(W\otimes V^*)=1+l(n-l)(1-g)\,.$$

Hence

$$\det(R_1(p)\,\mathrm{ad}'E)=H^2\,,$$

i.e.

$$\varphi^*(\det(\theta))=H^2\,.$$

If we show that $\varphi^*: Pic(U_L(n,d))\to Pic\,P$ is an isomorphism, Theorem 1 in the introduction would have been proved. Note, however that since $Pic\,U_L(n,d)$ is isomorphic to $\mathbb{Z}$ (see Proposition 3.4), this proves at least that $\det\theta=u^\lambda$ where u is the positive generator of $Pic\,U_L(n,d)$ and $\lambda=1$ or 2. To complete the proof of Theorem 1, we have only to show that φ^* is surjective on *Pic*. Let now L' be a vector bundle on X of rank r and degree f.

Then

$$(\varphi\times 1_X)^![U\otimes p_X^!L']=[E]\cdot p_P^![\tau^m]\cdot p_X^![L']\,.$$

Also

$$\begin{aligned}\varphi^!(p)_![(U\otimes p_X^! L')] &= (p_P)_!(\varphi\times 1_X)^!(U\otimes P_X^! L')\\ &= \tau^m(p_P)_!([E])\, p_X^![L']\\ &= \tau^m(p_P)_!\{p_X^![V]\, p_P^![\tau]\cdot p_X^![L'] + p_X^![W]\, p_X^![L']\}\\ &= \tau^m\{\tau\chi(X, V\otimes L') + \chi(X, W\otimes L')\}\,.\end{aligned}$$

In particular

$$\varphi^! \det p_!(U\otimes p_X^! L') = u^s\,,$$

where

$$s = (m+1)\,\chi(V\otimes L') + m\cdot\chi(W\otimes L')\,.$$

Let us now take L' to be of degree $f = -\{d+n(1-g)\}$ and of rank n. Then

$$\varphi^! \det p_!(U\otimes p_X^! L') = u^s$$

where

$$\begin{aligned}s &= (m+1)\{en - l(d+n(1-g)) + \ln(1-g)\}\\ &\quad + m\{(d-e)\,n - (n-l)\,(d+n(1-g)) + (n-l)\,m(1-g)\}\\ &= m(dn - n(d+n(1-g))) + n^2(1-g) + en - l(d+n(1-g)) + \ln(1-g)\\ &= (en - ld) = -1 \text{ showing } \varphi^* \text{ is surjective on } Pic. \qquad \text{q.e.d.}\end{aligned}$$

Remark 2.9. Let U be a universal bundle on $U_L(n,d)\times X$. Then the line bundle $\det U$ restricted to $U_L(n,d)\times x$, $x\in X$ does not depend on x and can be computed as follows. Let $(\varphi\times 1_X)^* U = E\otimes\tau^m$. Clearly $E|P\times\{x\} = l\tau + (n-l).\ 1$. Hence, denoting the above bundle on $U_L(n,d)$ by C, we get

$$\begin{aligned}\varphi^* C &= \det(\varphi^!(l\tau + (n-l)\cdot 1)\,\tau^m)\\ &= \tau^{mn+l}\,.\end{aligned}$$

Since φ^* is an isomorphism on Pic, we have $C = u^{mn+l}$. On the other hand, since U may be tensored with a line bundle on $U_L(n,d)$ without affecting its "universality", we may normalise U by requiring $C(U) = u^l$. In this case, we have $\varphi^* U \approx E$.

Definition 2.10. Let n and d be coprime. The unique bundle U on $U_L(n,d)\times X$, with the property that $U|_{t\times X}$ is in the equivalence class t and $\det U|_{U_L(n,d)\times\{x\}}$ is equivalent to the line bundle u^l, $0\leqq l<n$ where $ld\equiv 1(n)$ is called *the universal family* of bundles on X parametrized by $U_L(n,d)$.

Remark 2.11. Let E and F be two vector bundles so that $\det E\otimes\det F = L$ and rank $E + \operatorname{rank} F = n$. Consider the family of vector bundles on X parametrised by $PH^1(X, \operatorname{Hom}(F,E))$ given by extensions as in Lemma 2.3.

Let 0 be the open subset corresponding to stable bundles. Then we have a map $\varphi: 0 \to U_L(n, d)$.

The same computations as in 2.7 then show that

$\varphi^*(\det T(U_L(n,d)) = H^s)$ where $s = \chi(F \otimes E^*) - \chi(E \otimes F^*) = 2\deg(F \otimes E^*)$

and H is the restriction of the hyperplane bundle to 0. Since $\det T(U_L(n, d)) = h^2$, this shows that $\varphi^*(h) = H^r$, where $r = \deg(F \otimes E^*)$.

§ 3. Non-Existence of Poincaré Families when *n* and *d* are not Coprime

The aim of this section is to prove Theorem 2 of the introduction.

Lemma 3.1. *Let E (resp. F) be a vector bundle of rank n and degree d (resp. rank $(n+1)$ and degree d'). Then there exists a line bundle M so that E admits an injective map into $F \otimes M$.*

Proof. Choose M so that the sections of the bundle $\mathrm{Hom}(E, F \otimes M)$ generate the fibre at every point $x \in X$. The set H_x of homomorphisms $E \to F \otimes M$ that are not of maximal rank at x is simply the inverse image by the surjective map

$$H^0(X, \mathrm{Hom}(E, F \otimes M)) \to \mathrm{Hom}(E_x, F_x \otimes M_x)$$

of the set of homomorphisms $E_x \to F_x \otimes M$ which are not of maximal rank. The latter being of codimension $\geqq 2$, we have: the set $H = \bigcup_{x \in X} H_x$ of homomorphisms which are not injective at some $x \in X$ is of codimension $\geqq 1$ and the Lemma is proved.

Remark 3.2. 1) Lemma 3.1 is in fact a theorem of Atiyah when E is the trivial bundle of rank M.

2) If we assume E and F are stable, then M can be chosen to depend only on n, d and d' and not on E and F.

3) In addition, we may also assume that

i) $H^1(X, \mathrm{Hom}(E, F \otimes M)) = 0$.

ii) $H^1(X, \mathrm{Hom}(F \otimes M, F \otimes M/iE)) = 0$ and $H^0\left(X, \mathrm{Hom}\left(F \otimes M, \frac{F \otimes M}{iE}\right)\right)$ generates the fibre at every point $x \in X$, where $i: E \to F \otimes M$ is an injection.

In fact, by taking M to be sufficiently positive,

i) can obviously be satisfied. On the other hand,

$$F \otimes M/iE \approx M^{n+1} \otimes \det F \otimes \det E^{-1}.$$

Hence

$$\mathrm{Hom}(F \otimes M, F \otimes M/iE) \approx F^* \otimes M^n \otimes \det F \otimes \det E^{-1}$$

and again by choosing M sufficiently positive, ii) can also be fulfilled.

Let us now assume that $E \in U_L(n, d)$ and F a stable point of $U_{L'}(n+1, d')$ where $n+1$ and d' are coprime. Choose M to satisfy Lemma 3.1 and also 3) Remarks 3.2. Consider the family of vector bundles on $Y \times X$, where $Y = PH^1(X, \mathrm{Hom}(M^{n+1} \otimes L' \otimes L^{-1}, E))$ given by Lemma 2.3. Denoting by H the hyperplane bundle on Y, this family W is given by an extension

$$0 \to p_X^* E \otimes p_Y^* H \to W \to p_X^*(M^{n+1} \otimes L' \otimes L^{-1}) \to 0 .$$

Let 0 be the open subset of Y corresponding to stable bundles. Then we have just shown that 0 is non-empty. Noting that $Pic\, Y \to Pic\, 0$ is surjective, we will denote the image of H by this restriction map on $Pic\, 0$ also by H. (We will presently see that this restriction map is an isomorphism.)

On the other hand, we consider on $U_{L'}(n+1, d')$ two projective bundles P_1, P_2 constructed as follows:

Let U be the universal bundle on $U_{L'} \times X$ and p_1, p_2 be projections of $U_{L'} \times X$ onto $U_{L'}$ and X. Clearly

$$V_1 = (p_1)_* (\mathrm{Hom}(p_2^* M \otimes U, p_2^* M^{n+1} \otimes L' \otimes L^{-1}))$$

and

$$V_2 = (p_1)_* (\mathrm{Hom}(p_2^* E, p_2^* M \otimes U))$$

are both vector bundles by our choice of M. Let moreover 0_1 (resp. 0_2) be the open subset of $P_1 = P(V_1)$ (resp. $P_2 = P(V_2)$) corresponding to surjections $M \otimes F \to M^{n+1} \otimes L' \otimes L^{-1}$ (resp. injections $E \to M \otimes F$). We now wish to define two maps $\varphi = 0 \to 0_1$ and $\psi : 0 \to 0_2$. If we denote by π_1, π_2 the projections of $0_1, 0_2$ on $U_{L'}$, then we first define $\pi_1 \circ \varphi = \pi_2 \circ \psi$ as follows. The bundle $W \otimes p_X^* M^{-1}$ is a family of stable vector bundles parametrised by 0 of rank$(n+1)$ and determinant $\approx L'$. This induces by the universal property of $U_{L'}(n+1, d')$ a morphism $\lambda : 0 \to U_{L'}(n+1, d')$. It is not true that $(\lambda \times 1_X)^* U \approx W \otimes p_X^* M^{-1}$ in general. However by Lemma 2.5, there exists a line bundle N on 0 so that $(\lambda \times 1_X)^* U \approx W \otimes p_X^* M^{-1} \otimes p_0^* N$. Then we have

$$\lambda^* V_1 \approx (p_0)_* (\mathrm{Hom}(W, p_X^* M^{n+1} \otimes L' \otimes L^{-1})) \otimes N^{-1}$$

and

$$\lambda^* V_2 \approx (p_0)_* (\mathrm{Hom}(p_X^* E, W)) \otimes N .$$

It is obvious that the right sides have canonical subbundles isomorphic respectively to N^{-1} and $N \otimes H$. Hence we have

Lemma 3.3. *Let λ be the map of 0 into $U_{L'}(n+1, d')$ so that $(\lambda \times 1_X)^* U = W \otimes p_X^* M^{-1} \otimes p_0^* N$. There exist maps $\varphi : 0 \to P_1$ and $\psi : 0 \to P_2$ such that*

i) $\varphi^*(\tau_1) \approx N$.

ii) $\psi^*(\tau_2) \approx H^{-1} \otimes N^{-1}$ *and*

iii) $\pi_1 \circ \varphi = \pi_2 \circ \psi = \lambda$.

Proof. The Lemma follows from the preceding remarks and the universal properties of P_1 and P_2.

Proposition 3.4. i) *$Pic\, Y \to Pic\, 0$ is an isomorphism.*

ii) *$Pic\, U_{L'}(n+1, d') \approx \mathbb{Z}$.*

Proof. Since $Pic\, Y \to Pic\, 0$ is surjective, we have rank $Pic\, 0 \leqq 1$. We have only to show rank $Pic\, 0 \geqq 1$. On the other hand, $Pic\, U_{L'}$ has rank $\geqq 1$ since $U_{L'}$ is complete and, since $U_{L'}$ is simply connected, it is enough to show that rank $Pic\, U_{L'} \leqq 1$. Both i) and ii) will hence be proved if we can show that rank $Pic\, 0 \geqq$ rank $Pic\, U_{L'}$. Note that the map $\psi : 0 \to P_2$ maps 0 isomorphically onto 0_2, as any injection $E \to F \otimes M$ with $F \in U_{L'}(n+1, d')$ gives rise to a stable extension of $M^{n+1} \otimes L' \otimes L^{-1}$ by E. If we then show that the complement of 0_2 in P_2 is irreducible, it would follow that

$$\text{rank } Pic\, P_2 \leqq \text{rank } Pic\, 0_2 + 1\,.$$

On the other hand,

$$\text{rank } Pic\, P_2 = \text{rank } Pic\, U_{L'} + 1\,.$$

Thus, in order to complete the proof of Proposition 3.4, it is sufficient to prove.

Lemma 3.5. *$D_2 = P_2 - 0_2$ is irreducible.*

Proof. The morphism $\pi_2 : D_2 \to U_{L'}$ has fibres $D_{2,F}$ say. It is sufficient to show that $D_{2,F}$ is irreducible and of constant dimension. Denote by $D_{2,x}$, the variety of homomorphisms $E \to F \otimes M$, which are not injective at $x \in X$. By the same argument, it is enough to show that $D_{2,x}$ is irreducible and of constant dimension. Since by assumption

$$H^0(X, \operatorname{Hom}(E, F \otimes M)) \to \operatorname{Hom}(E_x, F_x \otimes M_x)$$

is surjective, our assertion is a consequence of the fact that the set of homomorphisms $E_x \to F_x \otimes M_x$ which are not injective is irreducible.

We now consider the map $\varphi : 0 \to P_1$. Crucial to us is the computation of the image of $Pic\, P_1$ in $Pic\, 0$.

Lemma 3.6. *The image $Pic\, P_1 \to Pic\, 0 = \mathbb{Z}$ is the group generated by n and d.*

Proof. Since $Pic\, P_1$ is generated by τ_1 and $\pi^* h$, we will first compute $\varphi^* \pi^* h = \lambda^* h$. By Theorem 1, $\det(T(U_{L'})) \approx h^2$ and it is enough to compute $\lambda^*(T(U_{L'}))$. Let now $\deg M = m$. By Remark 2.10, we have $\lambda^*(H) = H^r$, where $r = -\deg(E \otimes M^{-(n+1)} \otimes L'^{-1} \otimes L) = -d - n\{-(n+1)(m) + d - d'\}$ $= (n+1)(nm-d) + nd'$.

On the other hand, we have now to compute $\varphi^* \tau_1 = N$, by Lemma 3.3. Let l, e be integers so that $ld' - e(n+1) = 1$. Then by the normalisation 2.9, $\det U|_{U_{L'} \times \{x\}} \approx h^1$. Moreover, if $N = H^{n'}$, we have

$$\det(W \otimes p_X^* M^{-1} \otimes p_0^* N)|_{0 \times \{x\}} = H^{n(n'+1)+n'}.$$

Hence $\lambda^*(u^l) = H^{rl} = H^{n(n'+1)+n'}$; or $n'(n+1) + n = rl$.

But

$$\begin{aligned} rl - n &= l(n+1)(nm-d) + n(ld'-1) \\ &= l(n+1)(mn-d) + ne(n+1). \end{aligned}$$

Hence $n' = -l(mn-d) - en$. Thus the image in $\mathbb{Z}$ by $\varphi^* : Pic\, P_1 \to Pic\, 0 = \mathbb{Z}$ is generated by $\lambda^* u = r = (n+1)(mn-d) + nd'$, and $\varphi^*(\tau_1) = -l(mn-d) - en$. It is easy to see that they generate the same subgroup of $\mathbb{Z}$ as d and n.

3.7. Proof of Theorem 2. If there exists a Poincaré family on any Zariski open subset V of $U_L(n, d)$, then we may construct a vector bundle on V whose fibre at $E \in V$ is $H^1(X, \mathrm{Hom}(M^{n+1} \otimes L' \otimes L^{-1}, E))$. Let $0'$ be the open subset of stable extensions in the associated projective bundle. It is clear that $0 \subset 0'$ and the map $\varphi : 0 \to P_1$ extends to an isomorphism of $0'$ onto an open subset of 0_1.

Now it is easy to see that the condition that the above projective bundle comes from a vector bundle leads to the condition $Pic\, 0' \to Pic\, 0$ is surjective. We have the diagram

$$\begin{array}{ccc} Pic\, 0' & \rightarrow & Pic\, 0 \\ \nwarrow & & \nearrow \\ & Pic\, 0_1 & \end{array}$$

$0'$ being an open subset of 0_1, $Pic\, 0_1 \to Pic\, 0'$ is also surjective. But the map $Pic\, 0_1 \to Pic\, 0$ can only be surjective if n and d are coprime by Lemma 3.6.

§ 4. Remarks on the Third Betti Number of $U_L(2, 1)$

It has been proved (among other things) by Newstead [7] that the third integral cohomology group of $U_L(2, 1)$ is free of rank $= 2g$. In this section we show that in fact there is a canonical isomorphism of $H^3(U_L(2, 1), \mathbb{Z})$ with $H^1(X, \mathbb{Z})$ leading to a result of Mumford and Newstead [2]. Unfortunately however we need the fact mentioned above in this approach.

Consider the Poincaré bundle D on $J \times X$. Let W be the bundle of extensions on $P \times X$ where P is the projective bundle on J associated to the first direct image of $p_X^* L^{-1} \otimes D^2$. We have the following exact sequence

$$0 \to \pi^* D \otimes p_P^* \tau \to W \to \pi^*(p_X^* L \otimes D^{-1}) \to 0$$

where $\pi : P \times X \to J \times X$ is the projection and τ is the tautological bundle. By Lemma 2.1, all the bundles in this family are stable and hence we have a morphism $\varphi : P \to U_L(2, 1)$. We wish now to investigate the induced map $\varphi^* : H^3(U_L, \mathbb{Z}) \to H^3(P, \mathbb{Z})$.

Lemma 4.1. *φ^* maps $H^3(U_L, \mathbb{Z})$ isomorphically onto $\tau \cdot \pi^* H^1(J, \mathbb{Z})$.*

Proof. Since $H^3(U_L, \mathbb{Z})$ is a free group on $2g$ generators, and so also is $\tau \cdot \pi^* H^1(J, \mathbb{Z})$, we have only to show that φ^* is surjective. Let U be the universal bundle on $U_L \times X$. Clearly we have $(\varphi \times 1_X)^* U = W \otimes p_P^* \nu$, where ν is a line bundle on P. Moreover, it is easy to see that $(\varphi \times 1_X)^* U$ and W restrict to isomorphic bundles on each fibre in the fibration $\pi : P \to J$. This follows from the normalisation of U and the obvious fact that $\det W|_{P \times (x)} = \tau$. Hence ν is trivial on each fibre of P and consequently we have

$$(\varphi \times 1_X)^* U = W \otimes \pi^* \mu, \qquad \text{where } \mu \text{ is a line bundle on } J.$$

We shall now compute the 2nd chern class c_2 of $(\varphi \times 1_X)^* U = W \otimes \pi^* \mu$. Clearly

$$c_2(W \otimes \pi^* \mu) = c_2(W) + \pi^* c_1(\mu) \cdot c_1(W) + \pi^* c_1(\mu)^2.$$

Note that $c_1(W) = (\tau) + p_X^* c_1(L)$.

If $c_{3,1}(U)$ denotes the $(3, 1)$ component of $c_2(U)$ in the Kunneth decomposition of $H^3(U_L \times X)$, we have

$$c_{3,1}(W \otimes \pi^* \mu) = c_{3,1}(W).$$

By definition of W we have

$$\begin{aligned} c_2(W) &= (\pi^* c_1(D) + \tau)\,(p_X^* c_1(L) - \pi^* c_1(D)) \\ &= -\pi^* c_1(D)^2 + \pi^* c_1(D)\,(p_X^* c_1(L) - \tau) + p_X^* c_1(L)\,\tau. \end{aligned}$$

Since the Poincaré bundle is trivial on $J \times (x)$ for some $x \in X$, and on $1 \times X$, it is clear that $c_1(D) = c_{1,1}(D)$.

Hence

$$c_{3,1}(W) = -\pi^* c_1(D) \cdot \tau.$$

Thus finally, we have $(\varphi \times 1_X)^* (c_{3,1}(U)) = -\tau \cdot \pi^* c_1(D)$. Since $c_{3,1}(U) \in H^3(U_L, \mathbb{Z}) \otimes H^1(X, \mathbb{Z})$, this defines a map $H_1(X, \mathbb{Z}) \to H^3(U_L, \mathbb{Z})$ and let G be the image. On the other hand, it is well-known that the map $H_1(X, \mathbb{Z}) \to H^1(J, \mathbb{Z})$ given by $c_1(D) = c_{1,1}(D)$ is an isomorphism. Thus we have the diagram

(4.2)
$$\begin{array}{ccc} H_1(X,\mathbb{Z}) & \xrightarrow{c_3} & H^3(U_L,\mathbb{Z}) \\ \downarrow{\scriptstyle c_{1,1}} & & \downarrow{\scriptstyle \varphi^*} \\ H^1(J,\mathbb{Z}) & & H^3(P,\mathbb{Z}) \\ & {\scriptstyle \pi^*}\searrow \quad \nearrow{\scriptstyle \tau} & \\ & H^1(P,\mathbb{Z}). & \end{array}$$

The above computations show that this diagram is commutative. In particular, we have $\varphi^*(G) \supset \tau\pi^* H^1(J,\mathbb{Z})$. This shows that $rk\, \varphi^*(G) \geqq 2g$. Hence $rk\, G \geqq 2g$. Moreover, $\varphi^* G$ is injective. Since G is a subgroup of max rank in $H^3(U_L,\mathbb{Z})$, it is clear that $\ker \varphi^* \cap G = 0$ implies that $\ker \varphi^* = 0$. Thus we have proved that φ^* is injective and $\varphi^*(H^3(U_L,\mathbb{Z})) \supset \tau \cdot \pi^* H^1(J,\mathbb{Z})$. Since the latter is a direct summand in $H^3(P,\mathbb{Z})$, this implies that φ^* is an isomorphism onto $\tau \cdot \pi^*(J,\mathbb{Z})$. q.e.d.

Moreover, from the diagram 4.2, we conclude that $c_{3,1}(U)$, interpreted as a map of $H_1(X,\mathbb{Z}) \to H^3(U_L,\mathbb{Z})$ is the same as the map $c_{1,1}: H_1(X,\mathbb{Z}) \to H^1(J,\mathbb{Z})$, on identifying $H^1(J,\mathbb{Z})$ with the subgroup $\varphi^*(\tau \cdot \pi^*(H^1(J,\mathbb{Z})))$. Since $c_{1,1}$ is an isomorphism, Theorem 3 is proved.

§ 5. Cohomology Algebra of $U_L(2.1)$ when Genus = 3

In this article, we study the map φ of § 4 in greater detail in the case $g = 3$ and indicate how this leads to the determination of the rational cohomology algebra of $U_L(2,1)$ in this case. More precisely we have

Theorem 4. *Let X be a complete non-singular curve of genus 3. Denote by V the vector space $H^1(X,\mathbb{Q})$ and by θ the intersection pairing considered as an element of $\lambda^2(V)$. Let A be the free graded commutative algebra generated by* a) *a one-dimensional vector space $\mathbb{Q}h$, of degree 2* b) *V, where every element is supposed of degree 3* c) *a one dimensional vector space $\mathbb{Q}v$, of degree 4.*

Let I be the ideal in A generated by

i) $3h^3 - 10hv - 4\theta$,

ii) $(h^2 - 2v)\, V$,

iii) $(h^2 - 3v)\, v$,

iv) $\{hx, x \in \lambda^2 V \text{ with } x\theta^2 = 0\}$,

v) $\{y \in \lambda^3 V \text{ with } y\theta = 0\}$.

Then

1) $H^*(U_L(2,1),\mathbb{Q})$ *is isomorphic to* A/I.

2) *The chern classes of* $U_L(2,1)$ *are given by*

$$c_1=2h,\ c_2=8v,\ c_3=2h^3,\ c_4=7h^2(3h^2-8v)/3,\ c_5=c_6=0.$$

2′) *Equivalently,*

$$(ch)_1=2h_1,\ (ch)_2=2(h^2-4v),\ (ch)_3=7h^3/3-8hv,$$
$$(ch)_4=(h^2-4v)^2/6,\ (ch)_5=h^5/60,\ (ch)_6=0.$$

3) $h^6[U_L(2,1)]=224$, *where* $[U_L(2,1)]$ *denotes the fundamental cycle.*

4) $\chi(U_L(2,1),\Theta)=-3$, *where* Θ *is the tangent bundle.*

Lemma 5.1. *Let* A/I *be as in Theorem 4. If* $b'_i=\dim(A/I)_i$, *the* i^{th} *graded component of* A/I, *then we have* $b'_i=b'_{12-i}$ *for* $i\leqq 12$ *and* $b'_i=0$ *for* $i>12$. *Also* $b'_0=1$, $b'_1=0$, $b'_2=1$, $b'_3=6$, $b'_4=2$, $b'_5=6$, $b'_6=16$. *Moreover, in* A/I, *we have the relations* a) $h^3v=9/28\,h^5$, b) $hv^2=3/28\,h^5$, c) $v^3=1/28\,h^6$.

Lemma 5.2. *Assertion* 2) *and* 2′) *in Theorem 4 are equivalent.*

Lemma 5.3. *Assertions* 1) *and* 2) *of Theorem 4 imply* 3) *and* 4).

Proofs of these lemmas are straightforward verifications. To prove Lemma 5.3, we first note that since $U_L(2,1)$ is unirational (even rational) [4, § 2], $H^i(U_L(2,1),\mathcal{O})=0$ for $i\geqq 1$, and hence $\chi(U_L(2,1),\mathcal{O})=1$. On the other hand the top Todd class $\mathcal{T}$ can be computed by 2) as a multiple of h^6, using Lemma 5.1. Finally, since $\mathcal{T}[U_L(2,1)]=\chi(U_L(2,1))=1$, 3) would follow from 1) and 2). As for 4), this is a routine calculation using again, Hirzebruch's Riemann-Roch formula.

Lemma 5.4. *The cohomology ring* P *over* J *defined in* § 4 *has the following structure. It is generated by* $H^*(J^1)$ *and an element* $\tau\in H^2$ *which satisfies the (only) relation*

$$\sum_{k=0}^{g}\tau^k\frac{(4\theta)^{g-k}}{(g-k)!}=0,$$

where θ *is a nondegenerate element of* $H^2(J)$.

Proof. If D is the Poincaré bundle on $J\times X$, then let us denote by θ the element $-p_*(c_1(D)^2/2)$. We have

$$ch\,R^1(p_{J^1})(p_X^*L^{-1}\otimes D^2)=-ch(p_{J^1})_!\,[p_X^*L^{-1}\otimes v^2]$$

and is easily seen using Grothendieck-Riemann-Roch theorem to be $4\theta+g$. From this our assertion is trivially deduced.

Lemma 5.5. *Under the map* $\varphi^*: H^*(U_L(2,1),\mathbb{Q}) \to H^*(P,\mathbb{Q})$,

a) *The chern character of* $U_L(2,1)$ *is mapped on the element*

$$3g-3+\sum_{i\geq 1} 2(g-1)\,\tau^{2i}/(2i)!+\sum_{i\geq 0}\{8\theta\cdot\tau^{2i}/(2i)!+2\tau^{2i+1}/(2i+1)!\}.$$

b) *The generator of* $H^2(U_L)$ *maps on* $\tau+4\theta$.

c) $H^3(U_L)$ *maps isomorphically on* $\tau\cdot H^1(P)$.

d) *The second chern class of the Poincaré bundle on* $U_L(2,1)\times(x)$, $x\in X$ *is mapped on* $2\theta(\tau+2\theta)$.

Proof. a) is a simple application of Lemma 2.7 and Grothendieck-Riemann-Roch theorem.

b) Is then a consequence of a) and Theorem 1.

c) Has been proved in § 4.

To prove d), one observes that $(ch\,\Theta)_2$ can be directly computed to be $(g-1)(c_1^2-4c_2)$ where c_1, c_2 are chern classes of the Poincaré bundle on $U_L(2,1)\times x$. Hence $\varphi^*(c_1^2-4c_2)=\tau^2$ by a). Since $\varphi^*(c_1)=\tau+4\theta$, it follows that $\varphi^*(c_2)=2\theta(\tau+2\theta)$.

Lemma 5.6. *The elements* $\varphi^*h, \varphi^*H^3(U_L(2,1))$, φ^*c_2 *satisfy the relations* i), ..., v) *of Theorem 4'.*

Proof. This is simple substitution using Lemma 5.4.

5.7. *Proof of Theorem 4*. Now, let B be the subalgebra of $H^*(P(M),\mathbb{Q})$ generated by the image of $\varphi^*(H^i(U_L(2,1)))$, $i\leq 4$. One can check directly that $\dim B_i\geq b_i(U_L(2,1))$ for $i\leq 10$ using [7]. This shows that $\varphi^*: H^i(U_L(2,1))\to B_i$ is an isomorphism for $i\leq 10$. In particular, φ^* is injective in this range. Since the relations i), ..., v) of Theorem 4 are in the gradation ≤ 10, this implies that the elements h, $H^3(U_L(2,1))$ and c_2 satisfy the same relations. Moreover, these elements generate the cohomology groups $H^i(U_L(2,1))$ for $i\leq 10$ and hence the whole cohomology ring $H^*(U_L(2,1))$. Lastly, this gives a surjective map $A/I\to H^*(U_L(2,1),\mathbb{Q})$. Since by Lemma 5.1 $\dim(A/I)_i=\dim H^i(U_L(2,1),\mathbb{Q})$, this is an isomorphism.

The computation of the chern classes is now straightforward using Lemma 5.5.

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Dr. S. Ramanan
School of Mathematics
Tata Institute of Fundamental Research
Homi Bhabha Road
Bombay 5 BR (India)

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