The normal velocity components $V_{1,\,n}$ and $V_{2,\,n}$ as well as pressure acting on the inner and outer circumferential areas pass through the shaft center, and thus they do not contribute to torque about the origin. Then only the tangential velocity components contribute to torque, and the application of the angular momentum equation $\sum M = \sum_{\text{out}} r\dot{m}V - \sum_{\text{in}} r\dot{m}V$ to the control volume gives

$$T_{\text{shaft}} = \dot{m}(r_2 V_{2,\,t} - r_1 V_{1,\,t}) \qquad \textbf{(6–57)}$$

which is known as **Euler's turbine formula.** When the angles α_1 and α_2 between the direction of absolute flow velocities and the radial direction are known, it becomes

$$T_{\text{shaft}} = \dot{m}(r_2 V_2 \sin \alpha_2 - r_1 V_1 \sin \alpha_1) \qquad \textbf{(6–58)}$$

In the idealized case of the tangential fluid velocity being equal to the blade angular velocity both at the inlet and the exit, we have $V_{1,\,t} = \omega r_1$ and $V_{2,\,t} = \omega r_2$, and the torque becomes

$$T_{\text{shaft, ideal}} = \dot{m}\omega(r_2^2 - r_1^2) \qquad \textbf{(6–59)}$$

where $\omega = 2\pi\dot{n}$ is the angular velocity of the blades. When the torque is known, the shaft power can be determined from $\dot{W}_{\text{shaft}} = \omega T_{\text{shaft}} = 2\pi\dot{n}T_{\text{shaft}}$.

EXAMPLE 6–8 Bending Moment Acting at the Base of a Water Pipe

Underground water is pumped to a sufficient height through a 10-cm-diameter pipe that consists of a 2-m-long vertical and 1-m-long horizontal section, as shown in Fig. 6–37. Water discharges to atmospheric air at an average velocity of 3 m/s, and the mass of the horizontal pipe section when filled with water is 12 kg per meter length. The pipe is anchored on the ground by a concrete base. Determine the bending moment acting at the base of the pipe (point *A*) and the required length of the horizontal section that would make the moment at point *A* zero.

SOLUTION Water is pumped through a piping section. The moment acting at the base and the required length of the horizontal section to make this moment zero is to be determined.

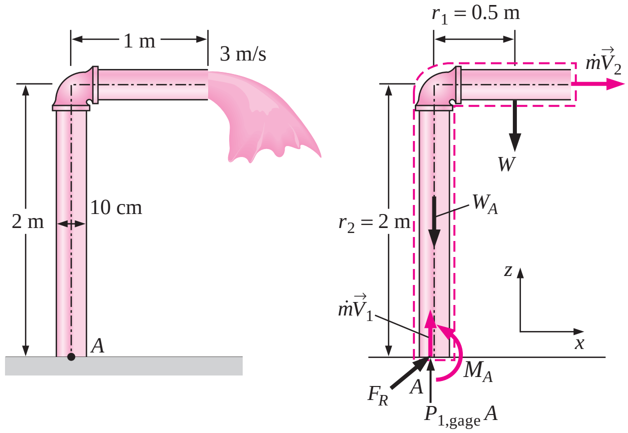

FIGURE 6–37
Schematic for Example 6–8 and the free-body diagram.

Assumptions **1** The flow is steady. **2** The water is discharged to the atmosphere, and thus the gage pressure at the outlet is zero. **3** The pipe diameter is small compared to the moment arm, and thus we use average values of radius and velocity at the outlet.

Properties We take the density of water to be 1000 kg/m^3.

Analysis We take the entire L-shaped pipe as the control volume, and designate the inlet by 1 and the outlet by 2. We also take the *x*- and *z*-coordinates as shown. The control volume and the reference frame are fixed.

The conservation of mass equation for this one-inlet, one-outlet, steady-flow system is $\dot{m}_1 = \dot{m}_2 = \dot{m}$, and $V_1 = V_2 = V$ since A_c = constant. The mass flow rate and the weight of the horizontal section of the pipe are

$$\dot{m} = \rho A_c V = (1000 \text{ kg/m}^3)[\pi(0.10 \text{ m})^2/4](3 \text{ m/s}) = 23.56 \text{ kg/s}$$

$$W = mg = (12 \text{ kg/m})(1 \text{ m})(9.81 \text{ m/s}^2)\left(\frac{1 \text{ N}}{1 \text{ kg} \cdot \text{m/s}^2}\right) = 118 \text{ N}$$

To determine the moment acting on the pipe at point *A*, we need to take the moment of all forces and momentum flows about that point. This is a steady-flow problem, and all forces and momentum flows are in the same plane. Therefore, the angular momentum equation in this case can be expressed as

$$\sum M = \sum_{\text{out}} r\dot{m}V - \sum_{\text{in}} r\dot{m}V$$

where r is the average moment arm, V is the average speed, all moments in the counterclockwise direction are positive, and all moments in the clockwise direction are negative.

The free-body diagram of the L-shaped pipe is given in Fig. 6–37. Noting that the moments of all forces and momentum flows passing through point *A* are zero, the only force that yields a moment about point *A* is the weight *W* of the horizontal pipe section, and the only momentum flow that yields a moment is the outlet stream (both are negative since both moments are in the clockwise direction). Then the angular momentum equation about point *A* becomes

$$M_A - r_1 W = -r_2 \dot{m} V_2$$

Solving for M_A and substituting give

$$\begin{aligned} M_A &= r_1 W - r_2 \dot{m} V_2 \\ &= (0.5 \text{ m})(118 \text{ N}) - (2 \text{ m})(23.56 \text{ kg/s})(3 \text{ m/s})\left(\frac{1 \text{ N}}{1 \text{ kg} \cdot \text{m/s}^2}\right) \\ &= \mathbf{-82.5 \text{ N} \cdot \text{m}} \end{aligned}$$

The negative sign indicates that the assumed direction for M_A is wrong and should be reversed. Therefore, a moment of 82.5 N · m acts at the stem of the pipe in the clockwise direction. That is, the concrete base must apply a 82.5 N · m moment on the pipe stem in the clockwise direction to counteract the excess moment caused by the exit stream.

The weight of the horizontal pipe is $w = W/L = 118$ N per m length. Therefore, the weight for a length of L m is Lw with a moment arm of $r_1 = L/2$. Setting $M_A = 0$ and substituting, the length L of the horizontal pipe that will cause the moment at the pipe stem to vanish is determined to be

$$0 = r_1 W - r_2 \dot{m} V_2 \quad \rightarrow \quad 0 = (L/2)Lw - r_2 \dot{m} V_2$$

or

$$L = \sqrt{\frac{2r_2\dot{m}V_2}{w}} = \sqrt{\frac{2 \times 141.4\ \text{N}\cdot\text{m}}{118\ \text{N/m}}} = \mathbf{2.40\ m}$$

Discussion Note that the pipe weight and the momentum of the exit stream cause opposing moments at point *A*. This example shows the importance of accounting for the moments of momentums of flow streams when performing a dynamic analysis and evaluating the stresses in pipe materials at critical cross sections.

EXAMPLE 6–9 Power Generation from a Sprinkler System

A large lawn sprinkler with four identical arms is to be converted into a turbine to generate electric power by attaching a generator to its rotating head, as shown in Fig. 6–38. Water enters the sprinkler from the base along the axis of rotation at a rate of 20 L/s and leaves the nozzles in the tangential direction. The sprinkler rotates at a rate of 300 rpm in a horizontal plane. The diameter of each jet is 1 cm, and the normal distance between the axis of rotation and the center of each nozzle is 0.6 m. Estimate the electric power produced.

SOLUTION A four-armed sprinkler is used to generate electric power. For a specified flow rate and rotational speed, the power produced is to be determined.

Assumptions **1** The flow is cyclically steady (i.e., steady from a frame of reference rotating with the sprinkler head). **2** The water is discharged to the atmosphere, and thus the gage pressure at the nozzle exit is zero. **3** Generator losses and air drag of rotating components are neglected. **4** The nozzle diameter is small compared to the moment arm, and thus we use average values of radius and velocity at the outlet.

Properties We take the density of water to be 1000 kg/m³ = 1 kg/L.

Analysis We take the disk that encloses the sprinkler arms as the control volume, which is a stationary control volume.

The conservation of mass equation for this steady-flow system is $\dot{m}_1 = \dot{m}_2 = \dot{m}_{total}$. Noting that the four nozzles are identical, we have $\dot{m}_{nozzle} = \dot{m}_{total}/4$ or $\dot{V}_{nozzle} = \dot{V}_{total}/4$ since the density of water is constant. The average jet exit velocity relative to the nozzle is

$$V_{jet} = \frac{\dot{V}_{nozzle}}{A_{jet}} = \frac{5\ \text{L/s}}{[\pi(0.01\ \text{m})^2/4]}\left(\frac{1\ \text{m}^3}{1000\ \text{L}}\right) = 63.66\ \text{m/s}$$

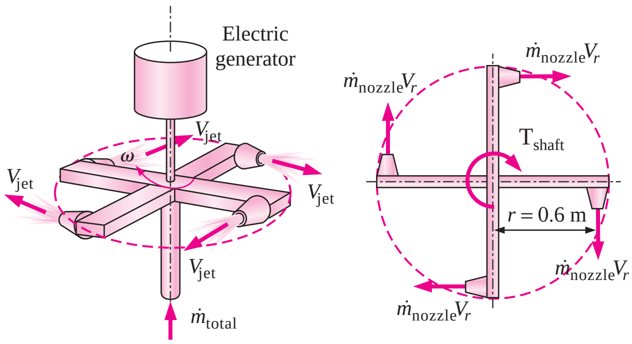

FIGURE 6–38
Schematic for Example 6–9 and the free-body diagram.

The angular and tangential velocities of the nozzles are

$$\omega = 2\pi\dot{n} = 2\pi(300 \text{ rev/min})\left(\frac{1 \text{ min}}{60 \text{ s}}\right) = 31.42 \text{ rad/s}$$

$$V_{\text{nozzle}} = r\omega = (0.6 \text{ m})(31.42 \text{ rad/s}) = 18.85 \text{ m/s}$$

That is, the water in the nozzle is also moving at a velocity of 18.85 m/s in the opposite direction when it is discharged. Then the average velocity of the water jet relative to the control volume (or relative to a fixed location on earth) becomes

$$V_r = V_{\text{jet}} - V_{\text{nozzle}} = 63.66 - 18.85 = 44.81 \text{ m/s}$$

Noting that this is a cyclically steady-flow problem, and all forces and momentum flows are in the same plane, the angular momentum equation can be approximated as $\sum M = \sum_{\text{out}} r\dot{m}V - \sum_{\text{in}} r\dot{m}V$, where r is the moment arm, all moments in the counterclockwise direction are positive, and all moments in the clockwise direction are negative.

The free-body diagram of the disk that contains the sprinkler arms is given in Fig. 6–38. Note that the moments of all forces and momentum flows passing through the axis of rotation are zero. The momentum flows via the water jets leaving the nozzles yield a moment in the clockwise direction and the effect of the generator on the control volume is a moment also in the clockwise direction (thus both are negative). Then the angular momentum equation about the axis of rotation becomes

$$-\text{T}_{\text{shaft}} = -4r\dot{m}_{\text{nozzle}}V_r \qquad \text{or} \qquad \text{T}_{\text{shaft}} = r\dot{m}_{\text{total}}V_r$$

Substituting, the torque transmitted through the shaft is determined to be

$$\text{T}_{\text{shaft}} = r\dot{m}_{\text{total}}V_r = (0.6 \text{ m})(20 \text{ kg/s})(44.81 \text{ m/s})\left(\frac{1 \text{ N}}{1 \text{ kg}\cdot\text{m/s}^2}\right) = 537.7 \text{ N}\cdot\text{m}$$

since $\dot{m}_{\text{total}} = \rho\dot{V}_{\text{total}} = (1 \text{ kg/L})(20 \text{ L/s}) = 20 \text{ kg/s}$.

Then the power generated becomes

$$\dot{W} = 2\pi\dot{n}\text{T}_{\text{shaft}} = \omega\text{T}_{\text{shaft}} = (31.42 \text{ rad/s})(537.7 \text{ N}\cdot\text{m})\left(\frac{1 \text{ kW}}{1000 \text{ N}\cdot\text{m/s}}\right) = \mathbf{16.9 \text{ kW}}$$

Therefore, this sprinkler-type turbine has the potential to produce 16.9 kW of power.

Discussion To put the result obtained in perspective, we consider two limiting cases. In the first limiting case, the sprinkler is stuck and thus the angular velocity is zero. The torque developed will be maximum in this case since $V_{\text{nozzle}} = 0$ and thus $V_r = V_{\text{jet}} = 63.66$ m/s, giving $\text{T}_{\text{shaft, max}} = 764$ N · m. But the power generated will be zero since the shaft does not rotate.

In the second limiting case, the shaft is disconnected from the generator (and thus both the torque and power generation are zero) and the shaft accelerates until it reaches an equilibrium velocity. Setting $\text{T}_{\text{shaft}} = 0$ in the angular momentum equation gives $V_r = 0$ and thus $V_{\text{jet}} = V_{\text{nozzle}} = 63.66$ m/s. The corresponding angular speed of the sprinkler is

$$\dot{n} = \frac{\omega}{2\pi} = \frac{V_{\text{nozzle}}}{2\pi r} = \frac{63.66 \text{ m/s}}{2\pi(0.6 \text{ m})}\left(\frac{60 \text{ s}}{1 \text{ min}}\right) = 1013 \text{ rpm}$$

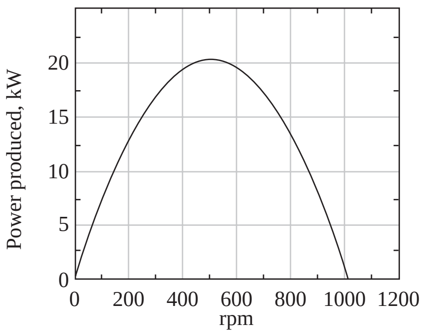

FIGURE 6–39
The variation of power produced with angular speed.

At this rpm, the velocity of the jet will be zero relative to an observer on earth (or relative to the fixed disk-shaped control volume selected).

The variation of power produced with angular speed is plotted in Fig. 6–39. Note that the power produced increases with increasing rpm, reaches a maximum (at about 500 rpm in this case), and then decreases. The actual power produced will be less than this due to generator inefficiency (Chap. 5).

SUMMARY

This chapter deals mainly with the conservation of momentum for finite control volumes. The forces acting on the control volume consist of *body forces* that act throughout the entire body of the control volume (such as gravity, electric, and magnetic forces) and *surface forces* that act on the control surface (such as the pressure forces and reaction forces at points of contact). The sum of all forces acting on the control volume at a particular instant in time is represented by $\sum \vec{F}$ and is expressed as

$$\underbrace{\sum \vec{F}}_{\text{total force}} = \underbrace{\sum \vec{F}_{\text{gravity}}}_{\text{body force}} + \underbrace{\sum \vec{F}_{\text{pressure}} + \sum \vec{F}_{\text{viscous}} + \sum \vec{F}_{\text{other}}}_{\text{surface forces}}$$

Newton's second law can be stated as *the sum of all external forces acting on a system is equal to the time rate of change of linear momentum of the system*. Setting $b = \vec{V}$ and thus $B = m\vec{V}$ in the Reynolds transport theorem and utilizing Newton's second law gives the *linear momentum equation* for a control volume as

$$\sum \vec{F} = \frac{d}{dt}\int_{\text{CV}} \rho \vec{V}\, dV + \int_{\text{CS}} \rho \vec{V}(\vec{V}_r \cdot \vec{n})\, dA$$

It reduces to the following special cases:

Steady flow:
$$\sum \vec{F} = \int_{\text{CS}} \rho \vec{V}(\vec{V}_r \cdot \vec{n})\, dA$$

Unsteady flow (*algebraic form*):
$$\sum \vec{F} = \frac{d}{dt}\int_{\text{CV}} \rho \vec{V}\, dV + \sum_{\text{out}} \beta \dot{m}\vec{V} - \sum_{\text{in}} \beta \dot{m}\vec{V}$$

Steady flow (*algebraic form*):
$$\sum \vec{F} = \sum_{\text{out}} \beta \dot{m}\vec{V} - \sum_{\text{in}} \beta \dot{m}\vec{V}$$

No external forces:
$$0 = \frac{d(m\vec{V})_{\text{CV}}}{dt} + \sum_{\text{out}} \beta \dot{m}\vec{V} - \sum_{\text{in}} \beta \dot{m}\vec{V}$$

where β is the momentum-flux correction factor. A control volume whose mass m remains constant can be treated as a solid body, with a net force or thrust of $\vec{F}_{\text{body}} = m_{\text{body}}\vec{a} = \sum_{\text{in}} \beta \dot{m}\vec{V} - \sum_{\text{out}} \beta \dot{m}\vec{V}$ acting on it.

Newton's second law can also be stated as *the rate of change of angular momentum of a system is equal to the net torque acting on the system*. Setting $b = \vec{r} \times \vec{V}$ and thus $B = \vec{H}$ in the general Reynolds transport theorem gives the *angular momentum equation* as

$$\sum \vec{M} = \frac{d}{dt}\int_{\text{CV}} (\vec{r} \times \vec{V})\rho\, dV + \int_{\text{CS}} (\vec{r} \times \vec{V})\rho(\vec{V}_r \cdot \vec{n})\, dA$$

It reduces to the following special cases:

Steady flow:
$$\sum \vec{M} = \int_{\text{CS}} (\vec{r} \times \vec{V})\rho(\vec{V}_r \cdot \vec{n})\, dA$$

Unsteady flow (algebraic form):
$$\sum \vec{M} = \frac{d}{dt}\int_{\text{CV}} (\vec{r} \times \vec{V})\rho\, dV + \sum_{\text{out}} \vec{r} \times \dot{m}\vec{V} - \sum_{\text{in}} \vec{r} \times \dot{m}\vec{V}$$

Steady and uniform flow:
$$\sum \vec{M} = \sum_{\text{out}} \vec{r} \times \dot{m}\vec{V} - \sum_{\text{in}} \vec{r} \times \dot{m}\vec{V}$$

Scalar form for one direction:
$$\sum M = \sum_{\text{out}} r\dot{m}V - \sum_{\text{in}} r\dot{m}V$$

No external moments:
$$0 = \frac{d\vec{H}_{\text{CV}}}{dt} + \sum_{\text{out}} \vec{r} \times \dot{m}\vec{V} - \sum_{\text{in}} \vec{r} \times \dot{m}\vec{V}$$

A control volume whose moment of inertia I remains constant can be treated as a solid body, with a net torque of

$$\vec{M}_{\text{body}} = I_{\text{body}}\vec{\alpha} = \sum_{\text{in}} \vec{r} \times \dot{m}\vec{V} - \sum_{\text{out}} \vec{r} \times \dot{m}\vec{V}$$

acting on it. This relation can be used to determine the angular acceleration of spacecraft when a rocket is fired.

The linear and angular momentum equations are of fundamental importance in the analysis of turbomachinery and are used extensively in Chap. 14.