# NOTES FOR MATH 520: COMPLEX ANALYSIS

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## 1. Complex numbers

1.1. **Definition of C.** As a *set*, $\mathbf{C} = \mathbf{R}^2 = \{(x, y) | \ x, y \in \mathbf{R}\}$. In other words, elements of $\mathbf{C}$ are pairs of real numbers.

**C as a field:** $\mathbf{C}$ can be made into a *field*, by introducing addition and multiplication as follows:

(1) (Addition) $(a, b) + (c, d) = (a + c, b + d)$.
(2) (Multiplication) $(a, b) \cdot (c, d) = (ac - bd, ad + bc)$.

$\mathbf{C}$ is an *Abelian (commutative) group* under $+$:

(1) (Associativity) $((a, b) + (c, d)) + (e, f) = (a, b) + ((c, d) + (e, f))$.
(2) (Identity) $(0, 0)$ satisfies $(0, 0) + (a, b) = (a, b) + (0, 0) = (a, b)$.
(3) (Inverse) Given $(a, b)$, $(-a, -b)$ satisfies $(a, b) + (-a, -b) = (-a, -b) + (a, b)$.
(4) (Commutativity) $(a, b) + (c, d) = (c, d) + (a, b)$.

$\mathbf{C} - \{(0, 0)\}$ is also an Abelian group under multiplication. It is easy to verify the properties above. Note that $(1, 0)$ is the identity and $(a, b)^{-1} = (\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2})$.

(Distributivity) If $z_1, z_2, z_3 \in \mathbf{C}$, then $z_1(z_2 + z_3) = (z_1 z_2) + (z_1 z_3)$.

Also, we require that $(1, 0) \neq (0, 0)$, i.e., the additive identity is not the same as the multiplicative identity.

1.2. **Basic properties of C.** From now on, we will denote an element of $\mathbf{C}$ by $z = x + iy$ (the standard notation) instead of $(x, y)$. Hence $(a + ib) + (c + id) = (a + c) + i(b + d)$ and $(a + ib)(c + id) = (ac - bd) + i(ad + bc)$.

$\mathbf{C}$ has a subfield $\{(x, 0)|x \in \mathbf{R}\}$ which is isomorphic to $\mathbf{R}$. Although the polynomial $x^2 + 1$ has no zeros over $\mathbf{R}$, it does over $\mathbf{C}$: $i^2 = -1$.

**Alternate descriptions of C:**

1. $\mathbf{R}[x]/(x^2 + 1)$, the quotient of the ring of polynomials with coefficients in $\mathbf{R}$ by the ideal generated by $x^2 + 1$.

2. The set of matrices of the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, $a, b \in \mathbf{R}$, where the operations are standard matrix addition and multiplication.

**HW 1.** *Prove that the alternate descriptions of* $\mathbf{C}$ *are actually isomorphic to* $\mathbf{C}$.

**Fundamental Theorem of Algebra:** $\mathbf{C}$ is *algebraically closed*, i.e., any polynomial $a_n x^n + a_{n-1}x^{n-1} \dots a_0$ with coefficients in $\mathbf{C}$ has a root in $\mathbf{C}$.

This will be proved later, but at any rate the fact that $\mathbf{C}$ is algebraically closed is one of the most attractive features of working over $\mathbf{C}$.

**Example:** Find a square root of $a + ib$, i.e., $z = x + iy$ such that $z^2 = a + ib$. Expanding, we get $x^2 - y^2 = a$ and $2xy = b$. Now, $(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2 = a^2 + b^2$, so taking square roots (over $\mathbf{R}$), $x^2 + y^2 = \sqrt{a^2 + b^2}$. (Here we take the positive square root.) Then

$$x^2 = \frac{a + \sqrt{a^2 + b^2}}{2}, \quad y^2 = \frac{-a + \sqrt{a^2 + b^2}}{2}.$$

Now just take square roots.

1.3. **$\mathbf{C}$ as a vector space over $\mathbf{R}$.** We will now view $\mathbf{C}$ as a vector space over $\mathbf{R}$. An $\mathbf{R}$-vector space is equipped with addition and scalar multiplication so that it is an Abelian group under addition and satisfies:

(1) $1z = z$,
(2) $a(bz) = (ab)z$,
(3) $(a + b)z = az + bz$,
(4) $a(z + w) = az + aw$.

Here $a, b \in \mathbf{R}$ and $z, w \in \mathbf{C}$. The addition for $\mathbf{C}$ is as before, and the scalar multiplication is inherited from multiplication, namely $a(x + iy) = (ax) + i(ay)$.

$\mathbf{C}$ is geometrically represented by identifying it with $\mathbf{R}^2$. (This is sometimes called the *Argand diagram.*)

1.4. **Complex conjugation and absolute values.** Define *complex conjugation* as an $\mathbf{R}$-linear map $\mathbf{C} \to \mathbf{C}$ which sends $z = x + iy$ to $\overline{z} = x - iy$.

**Properties of complex conjugation:**

(1) $\overline{\overline{z}} = z$.
(2) $\overline{z + w} = \overline{z} + \overline{w}$.
(3) $\overline{z \cdot w} = \overline{z} \cdot \overline{w}$.

Given $z = x + iy \in \mathbf{C}$, $x$ is called the *real part* of $\mathbf{C}$ and $y$ the *imaginary part.* We often denote them by $\mathrm{Re}\, z$ and $\mathrm{Im}\, z$.

$$\text{Re } z = \frac{z + \overline{z}}{2}, \ \text{Im } z = \frac{z - \overline{z}}{2i}.$$

Define $|z| = \sqrt{x^2 + y^2}$. Observe that, under the identification $z = x + iy \leftrightarrow (x, y)$, $|z|$ is simply the (Euclidean) norm of $(x, y)$.

**Properties of absolute values:**

(1) $|z|^2 = z\overline{z}$.
(2) $|zw| = |z||w|$.
(3) (Triangle Inequality) $|z + w| \leq |z| + |w|$.

The first two are staightforward. The last follows from computing

$$|z + w|^2 = (z + w)(\overline{z} + \overline{w}) = |z|^2 + |w|^2 + 2\text{Re } z\overline{w} \leq |z|^2 + |w|^2 + 2|z\overline{w}| = (|z| + |w|)^2.$$

## 2. Day 2

### 2.1. Some point-set topology.

**Definition 2.1.** *A* topological space $(X, \mathcal{T})$ *consists of a set* $X$*, together with a collection* $\mathcal{T} = \{U_\alpha\}$ *of subsets of* $X$*, satisfying the following:*

(1) $\emptyset, X \in \mathcal{T}$,
(2) *if* $U_\alpha, U_\beta \in \mathcal{T}$*, then* $U_\alpha \cap U_\beta \in \mathcal{T}$,
(3) *if* $U_\alpha \in \mathcal{T}$ *for all* $\alpha \in I$*, then* $\cup_{\alpha \in I} U_\alpha \in \mathcal{T}$*. (Here* $I$ *is an indexing set, and is not necessarily finite.)*

$\mathcal{T}$ *is called a* topology *for* $X$*, and* $U_\alpha \in \mathcal{T}$ *is called an* open set *of* $X$*.*

**Example:** $\mathbf{R}^n = \mathbf{R} \times \mathbf{R} \times \cdots \times \mathbf{R}$ ($n$ times). If $x = (x_1, \dots, x_n)$, we write $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$. $U \subset \mathbf{R}^n$ is open iff $\forall x \in U \ \exists \, \delta > 0$ and $B(x, \delta) = \{y \in \mathbf{R}^n | |y - x| < \delta\} \subset U$.

In particular, the topology on $\mathbf{C}$ is the topology on $\mathbf{R}^2$.

The complement of an open set is said to be *closed*.

**Definition 2.2.** *A map* $\phi : X \to Y$ *between topological spaces is* continuous *if* $U \subset Y$ *open* $\Rightarrow \phi^{-1}(U) = \{x \in X | f(x) \in U\}$ *open.*

Restricting to the case of $\mathbf{R}^n$, we say that $f(x)$ has limit $A$ as $x$ tends to $a$ and write $\lim_{x \to a} f(x) = A$ if for all $\varepsilon > 0$ there exists $\delta > 0$ so that $0 < |x - a| < \delta \Rightarrow |f(x) - A| < \varepsilon$.

Let $\Omega \subset \mathbf{C}$ be an open set and $f : \Omega \to \mathbf{C}$ be a (complex-valued) function. Then $f$ is *continuous at* $a$ if $\lim_{x \to a} = f(a)$.

**HW 2.** *Prove that* $f$ *is a continuous function iff* $f$ *is continuous at all* $a \in \Omega$.

**HW 3.** *Prove that if* $f, g : \Omega \to \mathbf{C}$ *are continuous, then so are* $f + g$*,* $fg$ *and* $\frac{f}{g}$ *(where the last one is defined over* $\Omega - \{x | g(x) = 0\}$*).*

### 2.2. Analytic functions.

**Definition 2.3.** *A function* $f : \Omega \to \mathbf{C}$ *(here* $\Omega$ *is open) is* differentiable at $a \in \Omega$ *if the derivative*

$$f'(a) \stackrel{def}{=} \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

*exists. If* $f$ *is differentiable at all* $a \in \Omega$*, then* $f$ *is said to be* analytic *or* holomorphic *on* $\Omega$*.*

Suppose $f, g : \Omega \to \mathbf{C}$ are analytic. Then so are $f + g, fg, \frac{f}{g}$ (where the last one is defined over $\Omega - \{x | g(x) = 0\}$.

**Example:** $f(z) = 1$ and $f(z) = z$ are analytic functions from $\mathbf{C}$ to $\mathbf{C}$, with derivatives $f'(z) = 0$ and $f'(z) = 1$. Therefore, all polynomials $f(z) = a_n z^n + \cdots + a_1 z + a_0$ are analytic, with $f'(z) = n a_n z^{n-1} + \cdots + a_1$.

**Fact:** An analytic function is continuous.

*Proof.* Suppose $f : \Omega \to \mathbf{C}$ is analytic with derivative $f'(z) = \lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$. Then $\lim_{h\to 0}(f(z+h) - f(z)) = f'(z) \lim_{h\to 0} h = 0$. $\square$

2.3. **The Cauchy-Riemann equations.** Write $f(z) = u(z) + iv(z)$, where $u, v : \Omega \to \mathbf{R}$ are *real-valued* functions. Suppose $f$ is analytic. We compare two ways of taking the limit $f'(z)$:

First take $h$ to be a real number approaching 0. Then

$$f'(z) = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.$$

Next take $h$ to be purely imaginary, i.e., let $h = ik$ with $k \in \mathbf{R}$. Then

$$f'(z) = \lim k \to 0 \frac{f(z+ik) - f(z)}{ik} = -i\frac{\partial f}{\partial y} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}.$$

We obtain:

$$\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y},$$

or, equivalently,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

The equations above are called the *Cauchy-Riemann equations.*

Assuming for the time being that $u, v$ have continuous partial derivatives of all orders (and in particular the mixed partials are equal), we can show that:

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \; \Delta v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

Such an equation $\Delta u = 0$ is called *Laplace's equation* and its solution is said to be a *harmonic function.*

## 3. Day 3

3.1. **Geometric interpretation of the Cauchy-Riemann equations.** Let $f : \Omega \subset \mathbf{C} \to \mathbf{C}$ be a holomorphic function, i.e., it has a complex derivative $f'(z) = \lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$ at all $z \in \Omega$. If we write $z = x + iy$ and view $f(z)$ as a function $(u(x, y), v(x, y)) : \Omega \subset \mathbf{R}^2 \to \mathbf{R}^2$, then $u, v$ satisfy the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

Recall from multivariable calculus that the *Jacobian* $J(x, y)$ is a linear map $\mathbf{R}^2 \to \mathbf{R}^2$ given by the matrix $\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$. Using the Cauchy-Riemann equations, we can write

$$J(x, y) = \begin{pmatrix} \frac{\partial u}{\partial x} & -\frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial x} & \frac{\partial u}{\partial x} \end{pmatrix}.$$

Namely, $J(x, y)$ is of the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, which looks suspiciouly like the second alternative description of $\mathbf{C}$ from Day 1.

The Jacobian $J(x, y)$ maps the tangent vector $(1, 0)$ based at $(x, y)$ to the vector $(\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x})$ based at $(u(x, y), v(x, y))$. Likewise it maps $(0, 1)$ to $(\frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}) = (-\frac{\partial v}{\partial x}, \frac{\partial u}{\partial x})$. The thing to notice is that $J(x, y)$ maps the pair $(1, 0), (0, 1)$ of orthogonal vectors to the pair $(\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}), (-\frac{\partial v}{\partial x}, \frac{\partial u}{\partial x})$ of orthogonal vectors. Moreover, we can show the following:

**HW 4.** *Prove that every linear transformation $\phi : \mathbf{R}^2 \to \mathbf{R}^2$ given by a matrix of the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ ($a, b$ not both zero) is* conformal, *namely it the angle between any two vectors $v, w \in \mathbf{R}^2$ is the same as the angle between vectors $\phi(v), \phi(w)$. (Hint: First show that $\langle v, w\rangle = C\langle \phi(v), \phi(w)\rangle$, where $\langle \cdot, \cdot\rangle$ is the standard Euclidean inner product and $C$ is a constant which does not depend on $v, w$.)*

Thus, $f : \Omega \to \mathbf{C}$ is often called a *conformal mapping.*

**Example:** Consider $f(z) = z^2$. Then $u(x, y) = x^2 - y^2$, $v(x, y) = 2xy$. First we look at the level curves $u = u_0$ and $v = v_0$. $x^2 - y^2 = u_0$ and $2xy = v_0$ are both mutually orthogonal families of hyperbolas. (Notice that since $f$ is conformal, $f^{-1}$, where defined and differentiable, is also conformal.) Next, consider $x = x_0$. Then $u = x_0^2 - y^2$, $v = 2x_0 y$, and we obtain $v^2 = 4x_0^2(x_0^2 - u)$. If $y = y_0$, then $v^2 = 4y_0^2(y_0^2 + u)$. They give orthogonal families of parabolas.

**Theorem 3.1.** *$f(z) = u(z) + iv(z)$ is analytic with continuous derivative $f'(z)$ iff $u, v$ have continuous first-order partial derivatives which satisfy the Cauchy-Riemann equations.*

*Proof.* We already proved one direction. Suppose $u, v$ have continuous first-order partials satisfying the Cauchy-Riemann equations. Then

$$u(x+h, y+k) - u(x,y) = \frac{\partial u}{\partial x}h + \frac{\partial u}{\partial y}k + \varepsilon_1,$$
$$v(x+h, y+k) - v(x,y) = \frac{\partial v}{\partial x}h + \frac{\partial v}{\partial y}k + \varepsilon_2,$$

where $\frac{\varepsilon_1}{h+ik} \to 0$ and $\frac{\varepsilon_2}{h+ik} \to 0$ as $(h,k) \to 0$. Now,

$$f(x+h, y+k) - f(x,y) = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)(h+ik) + \varepsilon_1 + i\varepsilon_2.$$

Therefore,

$$\lim_{h+ik\to 0} \frac{f(x+h,y+k) - f(x,y)}{h+ik} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.$$

□

3.2. **Harmonic functions.** Recall that if $f = u + iv$ is analytic, then $\Delta u = \Delta_v = 0$, i.e., $u, v$ are harmonic. If $u, v$ satisfy the Cauchy-Riemann equations, then $v$ is said to be the *conjugate harmonic function* of $u$.

**Remark:** If $v$ is the conjugate harmonic function of $u$, then $-u$ is the conjugate harmonic function of $v$.

**Example:** Compute the conjugate of $u = 2xy$. Then $\frac{\partial u}{\partial x} = 2y$ and $\frac{\partial u}{\partial y} = 2x$ and we verify that $\delta u = 0$. Next, $\frac{\partial v}{\partial x} = -2x$ and $\frac{\partial v}{\partial y} = 2y$. Then $v(x,y) = -x^2 + \phi(y)$ and $\phi'(y) = 2y$. Therefore, $\phi(y) = y^2 + c$, and $v(x,y) = -x^2 + y^2 + c$. $u + iv = -iz^2$.

## 4. Day 4

4.1. **More on analytic and harmonic functions.** Continuing our discussion from last time:

**Proposition 4.1.** *If $u : \mathbf{R}^2 \to \mathbf{R}$ is a harmonic function and $u$ is of class $C^\infty$, then there is a harmonic function $v : \mathbf{R}^2 \to \mathbf{R}$ satisfying $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$.*

This follows immediately from the following lemma:

**Lemma 4.2.** *Suppose $\frac{\partial v}{\partial x} = f$, $\frac{\partial v}{\partial y} = g$, and $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$, then $v$ exists.*

*Proof.* Define $v(x) = \int_0^x f(t,y)dt + \phi(y)$. Then clearly $\frac{\partial v}{\partial x} = f$. Now,

$$\frac{\partial v}{\partial y} = \int_0^x \frac{\partial f}{\partial y}(t,y)dt + \phi'(y) = \int_0^x \frac{\partial g}{\partial x}(t,y)dt + \phi'(y) = g(x,y) - g(0,y) + \phi'(y).$$

If we set $\phi(y) = \int_0^y g(0,t)dt$, then we're done. $\square$

**Example:** Consider $f(z) = \overline{z} = x - iy$. This is not analytic, as we can check the Cauchy-Riemann equations: $\frac{\partial u}{\partial x} = 1$, $\frac{\partial v}{\partial y} = -1$, and they are not identical!!

One formal way of checking is to write:

$$\frac{\partial f}{\partial z} \overset{def}{=} \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right), \ \frac{\partial f}{\partial \overline{z}} \overset{def}{=} \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

**Claim.** *$f$ is analytic iff $\frac{\partial f}{\partial \overline{z}} = 0$.*

The proof is immediate from the Cauchy-Riemann equations.

Observe: $\partial_z(z) = 1$ and $\partial_{\overline{z}}(z) = 0$, whereas $\partial_z(\overline{z}) = 0$ and $\partial_{\overline{z}}(\overline{z}) = 1$.

**Claim.** *If $p(z,\overline{z})$ is a polynomial in two variables $z, \overline{z}$, then $p(z,\overline{z})$ is analytic iff there are no terms involving $\overline{z}$.*

**HW 5.** *Prove the claim!*

4.2. **Geometric representation of complex numbers.** View $\mathbf{C}$ as $\mathbf{R}^2$.

**Addition:** Since $z_1 + z_2$ corresponds to $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$, the addition is standard vector addition.

**Multiplication:** Write $z$ in polar form $r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$, where $r > 0$ (hence $|z| = r$). Then:

$$z_1 z_2 = r_1(\cos\theta_1 + i\sin\theta_1) r_2(\cos\theta_2 + i\sin\theta_2) = r_1 r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)).$$

(1) The norm ($r$) of a product is the product of the lengths of the factors.
(2) The argument ($\theta$) of a product is the sum of the arguments of the factors.

We will often write $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$, whatever this means. This will be explained later when we actually define $e^z$, but for the time being it's not unreasonable because you expect:

$$z_1z_2 = r_1e^{i\theta_1}r_2e^{i\theta_2} = r_1r_2e^{i(\theta_1+\theta_2)},$$

using rules of exponentiation.

**Remark:** Notice that we're using properties of trigonometric functions when they haven't been defined yet.

Ignoring such rigorous considerations for the time being, we will compute powers and roots of complex numbers.

1. If $z = ae^{i\theta}$, then $z^n = a^ne^{in\theta}$. This is often called de Moivre's formula, and can be used for computing $\cos n\theta$ and $\sin n\theta$ in terms of $\sin\theta$ and $\cos\theta$.

2. If $z = ae^{i\theta}$, then its $n$th roots are

$$a^{\frac{1}{n}}e^{i\left(\frac{\theta}{n}+k\frac{2\pi}{n}\right)},$$

for $k = 0, 1, \ldots, n-1$. Also, when $a = 1$, the solutions to $z^n = 1$ are called $n$th roots of unity. If we write $\omega = \cos\frac{\theta}{n} + i\sin\frac{\theta}{n}$, then the other roots of unity are given by $1, \omega, \omega^2, \ldots, \omega^n$.

**Example:** Consider the analytic function $f(z) = z^2$. $f$ maps rays $\theta = \theta_0$ to $\theta = 2\theta_0$. Hence $f$ is a 2:1 map away from the origin. The unit circle $|z| = 1$ winds twice around itself under the map $f$. [Describe how the lines $y = \mathit{const}$ get mapped to parabolas under $f$.]

## 5. Polynomials and rational functions

5.1. **Polynomials.** Let $P(z) = a_0 + a_1 z + \cdots + a_n z^n$ be a polynomial in $z$ with coefficients in $\mathbf{C}$. By the Fundamental Theorem of Algebra (which we will prove later), $P(z)$ admits a factorization $(z - \alpha_1)P_1(z)$. By repeated application of the Fundamental Theorem, we obtain a complete factorization:

$$P(z) = a_n(z - \alpha_1)(z - \alpha_2)\dots(z - \alpha_n),$$

where $\alpha_1, \dots, \alpha_n$ are not necessarily distinct. The factorization is unique except for the order of the factors. (Why?)

If exactly $h$ of the $\alpha_i$'s coincide, then their common value is a zero of *order* or *multiplicity* $h$. We write $P(z) = a_n(z - \alpha)^h P_h(z)$ with $P_h(\alpha) \neq 0$.

Observe: $P'(\alpha) = \cdots = P^{(h-1)}(\alpha) = 0$ but $P^{(h)}(\alpha) \neq 0$.

5.2. **Rational functions.** Consider the rational function $R(z) = \frac{P(z)}{Q(z)}$ which is the quotient of two polynomials. We assume that $P(z)$ and $Q(z)$ have no common factors. Then the zeros of $Q(z)$ will be called the *poles* of $R(z)$. The *order* of the pole $\alpha$ is the multiplicity of $z - \alpha$ in $Q(z)$.

We will now explain how to extend

$$R : \mathbf{C} - \{\text{poles}\} \to \mathbf{C}$$

to

$$R : \mathbf{C} \cup \{\infty\} \to \mathbf{C} \cup \{\infty\}.$$

$\mathbf{C} \cup \{\infty\}$ is called the *extended complex plane*, obtained by adding the point at $\infty$ to $\mathbf{C}$. At this point $\mathbf{C} \cup \{\infty\}$ is not even a topological space, but later when we discuss Riemann surfaces, we'll explain how $R$ is a *holomorphic* map $S^2 \to S^2$ between Riemann surfaces.

First define $R(\text{pole}) = \infty$. (The reason for this is that as $z \to$ pole, $|R(z)| \to \infty$.)

Next, $R(\infty)$ is defined as follows: if

$$R(z) = \frac{a_0 + a_1 z + \cdots + a_n z^n}{b_0 + b_1 z + \cdots + b_m z^m},$$

then

$$R\left(\frac{1}{z}\right) = z^{m-n}\left(\frac{a_0 z^n + a_1 z^{n-1} + \dots}{b_0 z^m + b_1 z^{m-1} + \dots}\right).$$

(The reason for doing is that as $\frac{1}{z} \to \infty$, $z \to 0$.) If $m > n$, then $R$ has a zero of order $m - n$ at $\infty$; if $m < n$, then $R$ has a pole of order $n - m$ at $\infty$; if $m = n$, then $R(\infty) = \frac{a_n}{b_m}$.

Let $p = max(m, n)$; this is called the *order* of the rational function.

**Observe:** A rational function $R(z)$ of order $p$ has exactly $p$ zeros and $p$ poles. Indeed, if $m \geq n$, then there are $m$ poles in $\mathbf{C}$ and no pole at $\infty$; also there are $n$ zeros in $\mathbf{C}$ and $(m-n)$ zeros at $\infty$.

**Example:** The simplest rational functions are the *fractional linear transformations* $S(z) = \frac{\alpha z+\beta}{\gamma z+\delta}$ with $\det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \neq 0$. Special cases are $S(z) = \frac{1}{z}$ (inversion), and $S(z) = z+1$ (parallel translation).

5.3. **Partial fractions.** We will explain how to write any rational function $R(z)$ as

$$R(z) = G(z) + \sum_j G_j\left(\frac{1}{z-\beta_j}\right),$$

where $G, G_j$ are polynomials and $\beta_j$ are the poles of $R$.

**Example:** $\frac{1}{z^2-1} = \frac{1}{(z-1)(z+1)} = \frac{1/2}{z-1} + \frac{-1/2}{z+1}$.

First write $R(z) = G(z) + H(z)$, where $G(z)$ is a polynomial without a constant term and $H(z)$ has degree of denominaor $\geq$ degree of numerator, i.e., $H(z)$ is finite at $\infty$.

If $\beta_j$ is a pole of $R(z)$, then substituting $z = \beta_j + \frac{1}{\zeta}$ ($\Leftrightarrow \zeta = \frac{1}{z-\beta_j}$), we obtain:

$$R\left(\beta_j + \frac{1}{\zeta}\right) = G_j(\zeta) + H_j(\zeta),$$

where $G_j$ is a polynomial and $H_j(\zeta)$ is finite at $\zeta = \infty$.

Then take $R(z) - (G(z) + \sum G_j(\frac{1}{z-\beta_j}))$. There are no poles besides $\infty$ and $\beta_j$. At each $z = \beta_j$, the only infinite terms cancel out, and the difference is finite. Hence the difference must be a constant. By placing the constant inside $G(z)$ (for example), we have shown that $R(z)$ admits a partial fraction expansion as above.

## 6. Riemann surfaces and holomorphic maps

6.1. **The extended complex plane.** Today we try to make sense of the "extended complex plane" $\mathbf{C} \cup \{\infty\}$, which is also called the *Riemann sphere*. As a topological space, we take $S^2 = \{x_1^2 + x_2^2 + x_3^2 = 1\} \subset \mathbf{R}^3$. (Its topology is the *induced topology* from $\mathbf{R}^3$, namely the topology is $\mathcal{T} = \{W \cap S^2 | \, W \text{ is open in } \mathbf{R}^3\}$.) Then consider the *stereographic projection* from the "north pole" $(0,0,1)$ to the $x_1x_2$-plane, which we think of as $\mathbf{C}$. The straight line passing through $(0,0,1)$ and $(x_1,x_2,x_3) \in S^2$ intersects the $x_1x_2$-plane at $(\frac{x_1}{1-x_3}, \frac{x_2}{1-x_3}, 0)$. (Check this!) This gives us a continuous map

$$\phi : S^2 - \{(0,0,1)\} \to \mathbf{C},$$

$$(x_1,x_2,x_3) \mapsto z = \frac{x_1 + ix_2}{1 - x_3}.$$

It is not hard to see that $\phi$ is 1-1, onto, and inverse is also continuous.

**HW 6.** *Prove that $\phi : S^2 - \{(0,0,1)\} \to \mathbf{C}$ is a* homeomorphism*, i.e., $\phi$ is invertible and $\phi$, $\phi^{-1}$ are both continuous.*

The above stereographic projection misses $(0,0,1)$. If we want a map which misses the "south pole" $(0,0,-1)$, we could also do a stereographic projection from $(0,0,-1)$ to the $x_1x_2$-plane.

**HW 7.** *Compute the stereographic projection from $(0,0,-1)$ to the $x_1x_2$-plane.*

For our purposes, we want to do something slightly different: First rotate $S^2$ by $\pi$ along the $x_1$-axis, and then do stereographic projection from $(0,0,1)$. This has the effect of mapping

$$(x_1,x_2,x_3) \mapsto (x_1,-x_2,-x_3) \mapsto \frac{x_1 - ix_2}{1 + x_3}.$$

6.2. **Riemann surfaces.**

**Definition 6.1.** *A* Riemann surface $\Sigma$*, also called a* 1-dimensional complex manifold*, is a topological space $(\Sigma, \mathcal{T})$ together with a collection $\mathcal{A} = \{U_\alpha\}$ of open sets (called an* atlas *of $\Sigma$) such that:*

1. *$\cup U_\alpha = \Sigma$, i.e., $\mathcal{A}$ is an open cover of $\Sigma$.*
2. *For each $U_\alpha$ there exist a* coordinate chart *$\phi_\alpha : U_\alpha \to \mathbf{C}$, which is a homeomorphism onto its image.*
3. *For every $U_\alpha \cap U_\beta \neq \emptyset$, $\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ is a holomorphic map. (These are called* transition functions*.)*
4. *(Technical condition 1) $\Sigma$ is Hausdorff, i.e., for any $x \neq y \in \Sigma$ there exist open sets $U_x$ and $U_y$ containing $x, y$ respectively and $U_x \cap U_y = \emptyset$.*
5. *(Technical condition 2) $\Sigma$ is second countable, i.e., there exists a countable subcollection $\mathcal{T}_0$ of $\mathcal{T}$ and any open set $U \in \mathcal{T}$ is a union (not necessarily finite) of open sets in $\mathcal{T}_0$.*

**Example:** The extended complex plane $S^2 = \mathbf{C} \cup \{\infty\}$ is a Riemann surface. We have two open sets $U = S^2 - \{(0,0,1)\}$ and $V = S^2 - \{(0,0,-1)\}$, and $U \cup V = S^2$. We defined homeomorphisms

$$\phi : U \to \mathbf{C},\ (x_1, x_2, x_3) \mapsto \frac{x_1 + ix_2}{1 - x_3},$$

and

$$\psi : V \to \mathbf{C},\ (x_1, x_2, x_3) \mapsto \frac{x_1 - ix_2}{1 + x_3}.$$

$U \cap V = S^2 - \{(0,0,1),(0,0,-1)\}$. The transition function is then given by $\psi \circ \phi^{-1} : \mathbf{C} - \{0\} \to \mathbf{C} - \{0\}$, $z = \frac{x_1+ix_2}{1-x_3} \mapsto w = \frac{x_1-ix_2}{1+x_3}$. We compute that $\frac{1}{z} = \frac{1-x_3}{x_1+ix_2}\frac{x_1-ix_2}{x_1-ix_2} = \frac{x_1-ix_2}{1+x_3} = w$, using $x_1^2 + x_2^2 + x_3^2 = 1$. Therefore, the transition function is $z \mapsto \frac{1}{z}$, which is indeed holomorphic!

**HW 8.** *Prove that taking the stereographic projection $\phi$ from $(0,0,1)$ and the stereographic projection $\psi_0$ from $(0,0,-1)$ would not have given us a holomorphic transition function!*

6.3. **Holomorphic maps between Riemann surfaces.** Having defined Riemann surfaces, we now describe the appropriate class of maps between Riemann surfaces.

**Definition 6.2.** *A map $f : \Sigma_1 \to \Sigma_2$ between Riemann surfaces is* holomorphic *if for all $x \in \Sigma_1$ there exist coordinate charts $U \ni x$ and $V \ni f(x)$ s.t. composition*

$$\phi(U) \xrightarrow{\phi^{-1}} U \xrightarrow{f} V \xrightarrow{\psi} \psi(V)$$

*is holomorphic.*

**Example:** Given a rational function $R(z)$, we described it as a function from $S^2 \to S^2$ last time.

**HW 9.** *Prove that the extension of $R(z)$ to the extended complex plane $S^2 = \mathbf{C} \cup \{\infty\}$ is a holomorphic map $S^2 \to S^2$.*

Although the general case is left for HW, I'll explain some simpler cases:

**Case 1:** $R(z) = z^2$. Use coordinates $z_1, w_1 = \frac{1}{z_1}$ for the source and $z_2, w_2 = \frac{1}{z_2}$ for the target. "$R(z) = z^2$" means with respect to coordinates $z_1$ and $z_2$, $z_1 \mapsto z_2 = z_1^2$. This is a perfectly holomorphic function! Now, with respect to $w_1$ and $z_2$, we have: $w_1 \mapsto \frac{1}{w_1^2}$, which is not defined for $w_1 = 0$. Therefore, we switch to coordinates $w_1, w_2$, and write: $w_1 \mapsto w_2 = w_1^2$, which is holomorphic.

**Case 2:** $R(z) = \frac{z-a}{z-b}$, $a \neq b$. $z_1 \mapsto z_2 = \frac{z_1-a}{z_1-b}$ and is holomorphic for $z_1 \neq b$. Near $z_1 = b$ we use coordinates $z_1, w_2$ and write $z_1 \mapsto w_2 = \frac{z_1-b}{z_1-a}$. This is holomorphic for $z_1 \neq a$. Now, $w_1 \mapsto \frac{1/w_1-a}{1/w_1-b} = \frac{1-aw_1}{1-bw_1}$, which is holomorphic near $w_1 = 0$. Moreover, $R(\infty) = R(w_1 = 0) = 1$.

Hopefully in the framework of Riemann surfaces and maps between Riemann surfaces, the ad hoc definitions for extending rational functions to $\mathbf{C} \cup \{\infty\}$ now make more sense!

## 7. Fractional linear transformations

7.1. **Group properties.** Recall that a *fractional linear transformation* is a rational function of the form $S(z) = \frac{az+b}{cz+d}$. From the discussion on Riemann surfaces, $S$ is a holomorphic map of the Riemann sphere $S^2$ to itself.

Let $GL(2, \mathbf{C})$ be the group of $2 \times 2$ complex matrices with nonzero determinant (= invertible $2 \times 2$ complex matrices). $GL(2, \mathbf{C})$ is called the *general linear group* over $\mathbf{C}$. (Verify that $GL(2, \mathbf{C})$ is indeed a group!)

$GL(2, \mathbf{C})$ *acts* on $S^2$ as follows: Given $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbf{C})$, it maps $z \mapsto \frac{az+b}{cz+d}$. It is not hard to see directly that $id(z) = z$ and $(S_1 S_2)(z) = S_1(S_2(z))$.

However, we'll use homogeneous coordinates in order to see that we have a group action. Write $z = \frac{z_1}{z_2}$ and $w = \frac{w_1}{w_2}$, then $w = S(z)$ can be written as:

$$w_1 = az_1 + bz_2,$$

$$w_2 = cz_1 + cz_2.$$

Equivalently,

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$

With this notation, it is clear that the composition $S_1(S_2(z))$ corresponds to the product of the two matrices corresponding to $S_1$ and $S_2$.

Observe that there is some redundancy, namely $S$ and $\lambda S$ give rise to the same transformation on $S^2$, if $\lambda \in \mathbf{C}^* = \mathbf{C} - \{0\}$. Hence we define the *projectivized general linear group* to be $PGL(2, \mathbf{C}) = GL(2, \mathbf{C})/\mathbf{C}^*$, where the equivalence relation is given by $S \sim \lambda S$ for all $S \in GL(2, \mathbf{C})$ and $\lambda \in \mathbf{C}^*$. Another way of describing $PGL(2, \mathbf{C})$ is to take the *special linear group* $SL(2, \mathbf{C})$ consisting of $2 \times 2$ complex matrices with determinant 1, and quotienting by the subgroup $\{\pm id\}$. [$PGL(2, \mathbf{C})$ is also called $PSL(2, \mathbf{C})$.]

**$\mathbf{CP}^1$:** We will now describe *1-dimensional complex projective space* $\mathbf{CP}^1$. As a set, it is $\mathbf{C}^2 - \{(0,0)\}/\sim$, where $(z_1, z_2) \sim (\lambda z_1, \lambda z_2)$ for $\lambda \in \mathbf{C}^* = \mathbf{C} - \{0\}$. We have local coordinate charts $\phi_1 : U_1 = \{z_1 \neq 0\} \to \mathbf{C}$ which maps $(z_1, z_2) \sim (1, \frac{z_2}{z_1}) \mapsto \frac{z_2}{z_1}$ and $\phi_2 : U_2 = \{z_2 \neq 0\} \to \mathbf{C}$ which maps $(z_1, z_2) \sim (\frac{z_1}{z_2}, 1) \mapsto \frac{z_1}{z_2}$.

**HW 10.** *Prove that* $\mathbf{CP}^1$ *can be given the structure of a Riemann surface and that* $\mathbf{CP}^1$ *is* biholomorphic *to* $S^2$. *Here two Riemann surfaces* $X, Y$ *are* biholomorphic *if there is a map* $\phi : X \to Y$ *such that both* $\phi$ *and* $\phi^{-1}$ *are holomorphic.*

In view of the above HW, it is clear that the natural setting for $PGL(2, \mathbf{C})$ to act on $S^2$ is by viewing $S^2$ as $\mathbf{CP}^1$!

**Examples:** $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ is called a *parallel translation*; $\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$ is called a *homothety*, with special case $|k| = 1$ a *rotation*; $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is called an *inversion*.

7.2. **The cross ratio.** We consider fractional linear transformations (FLT's) $S$ which take $z_2, z_3, z_4$ into $1, 0, \infty$ in that order. (We assume that $z_2, z_3, z_4$ are distinct and are not $\infty$.) One such FLT is:

$$S(z) = \frac{z_2 - z_4}{z_2 - z_3} \cdot \frac{z - z_3}{z - z_4}.$$

**Claim:** There is a unique FLT (the one above) which takes $z_2, z_3, z_4$ to $1, 0, \infty$, in that order.

*Proof.* It suffices to prove that there is a unique FLT $S$ which takes $1, 0, \infty$ to $1, 0, \infty$, in that order; the FLT is the identity map $S(z) = z$. If there are two FLTs $S_1, S_2$ sending $z_2, z_3, z_4$ to $1, 0, \infty$, then $S_2 S_1^{-1}$ sends $1, 0, \infty$ to itself, and $S_1 = S_2$.

Let $S(z) = \frac{az+b}{cz+d}$. Then $S(0) = 0$ implies that $\frac{b}{d} = 0$, and hence $b = 0$. Likewise, $S(\infty) = \infty$ implies that $c = 0$. Now $S(z) = \frac{a}{d} z$ and $S(1) = 1$ implies that $S(z) = z$. $\square$

We can generalize the above claim:

**Claim:** There is a unique FLT which takes $z_2, z_3, z_4$ to $z_2', z_3', z_4'$. (Assume both triples are distinct.)

**Definition 7.1.** *The* cross ratio *of a 4-tuple of distinct complex numbers* $(z_1, z_2, z_3, z_4)$ *is* $S(z_1)$*, where* $S$ *is the FLT which maps* $z_2, z_3, z_4$ *to* $1, 0, \infty$*.*

In other words, $(z_1, z_2, z_3, z_4) = \frac{z_2 - z_4}{z_2 - z_3} \cdot \frac{z_1 - z_3}{z_1 - z_4}$.

**Fact:** If $z_1, z_2, z_3, z_4$ are distinct points on the Riemann sphere $S^2$ and $T$ is any FLT, then $(Tz_1, Tz_2, Tz_3, Tz_4) = (z_1, z_2, z_3, z_4)$.

*Proof.* Suppose $S$ maps $z_2, z_3, z_4$ to $1, 0, \infty$. Then $ST^{-1}$ maps $Tz_2, Tz_3, Tz_4$ to $1, 0, \infty$. The fact follows from observing that $Sz_1 = (ST^{-1})Tz_1$. $\square$

Now we come to the key property of FLTs. First we define a *"circle"* to be either a circle in $\mathbf{C}$ or a line in $\mathbf{C}$. A line passes through $\infty$, so is a circle in the Riemann sphere $S^2$.

**Theorem 7.2.** *FLTs take "circles" to "circles".*

*Proof.* We will prove that an FLT $S$ maps the real axis to a "circle". (Why does this quickly imply the theorem?) The image of the real axis satisfies the equation $\operatorname{Im} S^{-1}z = 0$. Equivalently, $S^{-1}z = \overline{S^{-1}z}$. Writing $S^{-1}(z) = \frac{az+b}{cz+d}$, we have:

$$\frac{az+b}{cz+d} = \frac{\overline{a}\,\overline{z} + \overline{b}}{\overline{c}\,\overline{z} + \overline{d}}.$$

Cross multiplying gives:

$$(a\overline{c}-\overline{a}c)|z|^2+(a\overline{d}-\overline{b}c)z+(b\overline{c}-\overline{a}d)\overline{z}+(b\overline{d}-\overline{b}d)=0.$$

If $a\overline{c}-\overline{a}c=0$, then we have Im $((a\overline{d}-\overline{b}c)z-b\overline{d})=0$, which is the equation of a line. Otherwise, we can divide by $r=a\overline{c}-\overline{a}c$ and complete the square as follows:

$$z\overline{z}+\frac{a\overline{d}-\overline{b}c}{r}z+\frac{b\overline{c}-\overline{a}d}{r}\overline{z}=-\frac{b\overline{d}-\overline{b}d}{r},$$

$$\left(z+\frac{b\overline{c}-\overline{a}d}{r}\right)\left(\overline{z}+\frac{a\overline{d}-\overline{b}c}{r}\right)=-\frac{b\overline{d}-\overline{b}d}{r}+\frac{b\overline{c}-\overline{a}d}{r}\frac{a\overline{d}-\overline{b}c}{r},$$

$$\left|z+\frac{a\overline{d}-\overline{b}c}{r}\right|=-\frac{(ad-bc)(\overline{a}\overline{d}-\overline{b}\overline{c})}{r^2}=\left|\frac{ad-bc}{r}\right|.$$

Here we note that $r$ is purely imaginary. The equation is the equation of a circle. $\square$

**Corollary 7.3.** *The cross ratio $(z_1,z_2,z_3,z_4)$ is real iff the four points lie on a "circle".*

## 8. Power series

Today we study the convergence of power series

$$a_0 + a_1 z + a_2 z^2 + \cdots + a_n a^n + \ldots$$

where $a_i$ are complex and $z$ is complex.

### 8.1. Review of series.

**Definition 8.1.** *A sequence* $\{a_n\}_{n=1}^{\infty}$ *has* limit $A$ *if for all* $\varepsilon > 0$ *there is an integer* $N > 0$ *such that* $n \geq N$ *implies that* $|a_n - A| < \varepsilon$. *If* $\{a_n\}$ *has a limit, then the sequence is* convergent *and we write* $\lim_{n\to\infty} a_n = A$.

**Fact:** A sequence $\{a_n\}$ is convergent iff $\{a_n\}$ is *Cauchy*, i.e., for all $\varepsilon > 0$ there is $N$ s.t. $m, n \geq N$ implies $|a_m - a_n| < \varepsilon$.

$\lim_{n\to\infty} \sup a_n$: Let $A_n = \sup\{a_n, a_{n+1}, \ldots\}$. $A_n$ is a nonincreasing sequence and its limit is $\lim_{n\to\infty} \sup a_n$. It may be a finite number or $\pm\infty$. $\lim_{n\to\infty} \inf a_n$ is defined similarly. Note that if $\lim_{n\to\infty} a_n$ exists, then it is the same as $\limsup$ and $\liminf$.

An infinite series $a_1 + a_2 + \cdots + a_n + \ldots$ *converges* if the sequence of partial sums $S_n = a_1 + \cdots + a_n$ converges.

**Absolute convergence:** If $|a_1| + |a_2| + \ldots$ converges, then so does $a_1 + a_2 + \ldots$, and the sequence is said to be *absolutely convergent*.

### 8.2. Uniform convergence.

Consider a sequence of functions $f_n(x)$, all defined on the same set $E$.

**Definition 8.2.** $f_n$ converges to $f$ uniformly on $E$ *if* $\forall\varepsilon > 0$ $\exists N$ *s.t.* $n \geq N \Rightarrow |f_n(x) - f(x)| < \varepsilon$ *for all* $x \in E$.

**Proposition 8.3.** *The limit* $f$ *of a uniformly convergent sequence of continuous functions is continuous.*

*Proof.* Given $\varepsilon > 0$, $\exists N$ s.t. $n \geq N \Rightarrow |f_n(x) - f(x)| < \frac{\varepsilon}{3}$ for all $x \in E$. Also, since $f_n$ is continuous at $x_0$, $\exists\delta$ s.t. $|x - x_0| < \delta \Rightarrow |f_n(x) - f_n(x_0)| < \frac{\varepsilon}{3}$. Adding them up, we have:

$$|f(x) - f(x_0)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

□

**Cauchy criterion:** $f_n$ converges uniformly on $E$ iff $\forall\varepsilon > 0$ $\exists N$ s.t. $m, n \geq N$ implies $|f_m(x) - f_n(x)| < \varepsilon$ for all $x \in E$.

8.3. **Power series.** The convergence of $a_0 + a_1 z + a_2 z^2 + \dots$ is modeled on the convergence of the *geometric series*

$$1 + z + z^2 + z^3 + \dots.$$

Take partial sums

$$S_n(z) = 1 + z + \dots z^{n-1} = \frac{1 - z^n}{1 - z}.$$

Then if $|z| < 1$, then $z^n \to 0$ and the series converges to $\frac{1}{1-z}$. If $|z| > 1$, then $|z|^n \to \infty$, so diverges. Finally, $|z| = 1$ is the hardest. If $z = 1$, then $1 + 1 + \dots$ diverges. If $z \neq 1$, then we have $\frac{e^{in\theta}-1}{e^{i\theta}-1}$, and $e^{in\theta}$ wanders around the unit sphere and does not approach any single point.

**HW 11.** *Carefully treat the case* $|z| = 1$.

**Radius of convergence:** In general, define the *radius of convergence* $R$ as follows:

$$\frac{1}{R} = \lim_{n\to\infty} \sup \sqrt[n]{|a_n|}.$$

**Example:** For the geometric series $1 + z + z^2 + \dots$, $\lim_{n\to\infty} \sup \sqrt[n]{1} = 1$.

**Example:** For the "derivative" $\sum_{n=1}^{\infty} n z^{n-1}$ of the geometric series, we have $\lim_{n\to\infty} \sup \sqrt[n-1]{n} = \lim_{n\to\infty} \sqrt[n]{n}$. (Why can we do this?)

**Claim:** $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

*Proof.* Indeed, if $\sqrt[n]{n} = 1 + \delta$, then $n = (1+\delta)^n = 1 + n\delta + \frac{n(n-1)}{2}\delta^2 + \dots > 1 + \frac{n(n-1)}{2}\delta^2$. Hence $n - 1 > \frac{n(n-1)}{2}\delta^2$ and $\sqrt{\frac{2}{n}} > \delta$. As $n$ goes to $\infty$, $\delta$ goes to zero. □

Now we describe our main theorem:

**Theorem 8.4** (Abel)**.** *Consider the series* $a_0 + a_1 z + a_2 z^2 + \dots$.

1. *The series converges absolutely for every* $z$ *with* $|z| < R$. *If* $0 < \rho < R$, *then the convergence is uniform on* $E = \{|z| \leq \rho\}$.
2. *The series diverges for* $|z| > R$.

*Proof.* The proof is by comparison with the geometric series.

(1) If $|z| < R$, then there is $\rho > 0$ so that $|z| < \rho < R$. Hence $\frac{1}{\rho} > \frac{1}{R}$. By definition of $\lim\sup$, $\sqrt[n]{|a_n|} < \frac{1}{\rho}$ for all sufficiently large $n$. This means that $|a_n| < \frac{1}{\rho^n} \Rightarrow |a_n z^n| < \frac{|z^n|}{\rho^n}$, and

$$|a_n z^n| + |a_{n+1} z^{n+1}| + \dots < \frac{|z|^n}{\rho^n} + \frac{|z|^{n+1}}{\rho^{n+1}} + \dots.$$

The RHS is a geometric series which converges, so the LHS is convergent; hence the original series is absolutely convergent. For uniform convergence, take $\rho < \rho' < R$. By repeating the above with $\rho'$ instead of $\rho$, we obtain

$$|a_n z^n| + |a_{n+1} z^{n+1}| + \cdots < \left(\frac{\rho}{\rho'}\right)^n + \left(\frac{\rho}{\rho'}\right)^{n+1} + \ldots,$$

and the RHS is finite and independent of $z$.

(2) If $|z| > R$, then $\exists \rho > 0$ s.t. $R < \rho < |z| \Rightarrow \frac{1}{R} > \frac{1}{\rho}$. Hence there exist arbitrarily large $n$ s.t. $\sqrt[n]{|a_n|} > \frac{1}{\rho} \Rightarrow |a_n z^n| > \frac{|z|^n}{\rho^n}$. The RH term goes to $\infty$, hence the series diverges. $\square$

**Remark:** We are not making any statements about $|z| = R$.

## 9. More Series

9.1. **Analyticity.** Let $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots = \sum a_n z^n$, with a radius of convergence $R > 0$ defined by $\frac{1}{R} = \lim_{n\to\infty} \sup \sqrt[n]{|a_n|}$.

**Theorem 9.1.** *$f(z)$ is analytic for $|z| < R$ with derivative $f'(z) = \sum n a_n z^{n-1}$. The derivative also has the same radius of convergence $R$.*

*Proof.* We prove the theorem in two steps.

**Step 1:** $\sum n a_n z^{n-1}$ has the same radius of convergence $R$ as $\sum a_n z^n$.

Indeed, $\lim_{n\to\infty} \sup \sqrt[n]{n a_n} = \lim_{n\to\infty} \sqrt[n]{n} \cdot \frac{1}{R}$, and we've already shown that $\lim_{n\to\infty} \sqrt[n]{n} = 1$. (Note we've also shifted terms by one....)

**Step 2:** Write $f(z) = S_n(z) + R_n(z)$, where $S_n(z) = \sum_{i=0}^{n-1} a_i z^i$ is the $n$th partial sum. For the time being, write $g(z) = \sum n a_n z^{n-1}$. Then

$$\left|\frac{f(z) - f(z_0)}{z - z_0} - g(z_0)\right| \leq \left|\frac{S_n(z) - S_n(z_0)}{z - z_0} - S_n'(z_0)\right| + \left|\frac{R_n(z) - R_n(z_0)}{z - z_0}\right| + |(S_n'(z_0) - g(z_0))|.$$

For any $\varepsilon > 0$, $\exists \delta$ s.t. the first term on the right is $< \frac{\varepsilon}{3}$, since $S_n$ is a polynomial, hence analytic. The second term is bounded as follows:

$$\left|\frac{\sum_{k=n}^{\infty} a_k (z^k - z_0^k)}{z - z_0}\right| \leq \sum_{k=n}^{\infty} |a_k (z^{k-1} + z^{k-2} z_0 + \cdots + z_0^{k-1})| \leq \sum_{k=n}^{\infty} a_k k \rho^{k-1},$$

where $|z|, |z_0| < \rho < R$. Since the series converges, by taking $n$ sufficiently large we may bound the second term by $\frac{\varepsilon}{3}$. Finally, the third term is $|\sum_{k=n}^{\infty} a_k k z_0^{k-1}|$, which is bounded by $\frac{\varepsilon}{3}$ in the same way taking $n$ sufficiently large. □

**Corollary 9.2.** *If $f(z) = \sum a_n z^n$ with radius of convergence $R > 0$, then it has derivatives $f', f'', f'''$, etc., and their radius of convergence is also $R$.*

It is also not hard to see (by repeated differentiation) that

$$f(z) = f(0) + f'(0) z + \frac{f''(0)}{2!} z^2 + \frac{f'''(0)}{3!} z^3 + \ldots,$$

namely we have the familiar Taylor series, *provided we assume that $f(z)$ admits a power series expansion!*

9.2. **Abel's Limit Theorem.**

**Theorem 9.3** (Abel's Limit Theorem)**.** *Consider the power series $f(z) = \sum a_n z^n$. Assume WLOG (without loss of generality) that the radius of convergence $R = 1$. If $\sum a_n$ converges (i.e., $f(1)$ exists), then $f(z)$ approaches $f(1)$, provided $z$ approaches $1$ while keeping $\frac{|1-z|}{1-|z|}$ bounded.*

One way of interpreting $\frac{|1-z|}{1-|z|}$ is as follows: Let $z$ be on a circle of radius $r$, where $r$ is very close to but smaller than 1. Then $|1-z|$ is the distance from $z$ to 1, and $1-|z|=1-r$ is the distance from the circle of radius $r$ to 1. When $z$ is close to 1, the ratio $\frac{|1-z|}{1-|z|}$ is very close to $\frac{1}{\cos}$ of the angle made by $z-1$ and $r-1$. Hence there exists $\varepsilon>0$ such that if we write $z=1+\rho e^{i\theta}$ with $0\leq\theta\leq 2\pi$, then $\theta\in(\frac{\pi}{2}+\varepsilon,\frac{3\pi}{2}-\varepsilon)$.

*Proof.* WLOG $\sum a_n=0$. Write $s_n=a_0+a_1+\cdots+a_n$. Rewrite $S_n(z)=a_0+a_1z+a_2z^2+\cdots+a_nz^n$ as:

$$\begin{aligned} S_n(z) &= (1-z)(a_0+(a_0+a_1)z+(a_0+a_1+a_2)z^2+\cdots+(a_0+\cdots+a_{n-1})z^{n-1})+(a_0+\cdots+a_n)z^n \\ &= (1-z)(s_0+s_1z+\cdots+s_{n-1}z^{n-1})+s_nz^n. \end{aligned}$$

Here, we are taking $|z|<1$, and $s_n\to 0$, so $s_nz^n\to 0$. Therefore,

$$f(z)=(1-z)\sum s_nz^n.$$

Now, we write

$$|f(z)|\leq|1-z|\left|\sum_{k=0}^{m-1}s_kz^k\right|+|1-z|\left|\sum_{k=m}^{\infty}s_kz^k\right|.$$

Given $\varepsilon>0$, there exists $k$ sufficiently large (for example, $k\geq m$) such that $|s_k|\leq\varepsilon$. Hence the second term on the RHS is dominated by the sum $|1-z|\frac{\varepsilon}{1-|z|}$ of the geometric series. Using our assumption, this in turn is dominated by $K\varepsilon$ for some predetermined constant $K$. Now the first term on the RHS is a product of a finite number of terms, so can be made arbitrarily close to 0 by taking $z\to 0$. This proves the theorem. $\square$

9.3. **Exponential functions.** Define the exponential function

$$e^z=1+\frac{z}{1!}+\frac{z^2}{2!}+\cdots+\frac{z^n}{n!}+\ldots.$$

The radius of convergence of $e^z$ is $\infty$, i.e., $e^z$ converges on the whole plane, since $\sqrt[n]{\frac{1}{n!}}\to 0$.

**HW 12.** *Prove that* $\sqrt[n]{n!}\to\infty$. *[Hint: given any real number* $x$*, show that* $x^n<n!$ *for* $n$ *sufficiently large.]*

The exponential function is the unique function which is a power series in $z$ and is a solution of the differential equation

$$f'(z)=f(z),$$

with initial condition $f(0)=1$. In fact, if we differentiate $f(z)=a_0+a_1z+\cdots+a_nz^n+\ldots$, we obtain $f'(z)=a_1+2a_2z+\cdots+na_nz^{n-1}+\ldots$, and equating the coefficients we obtain $a_0=1$, $a_1=1$, $a_2=\frac{1}{2}$, and so on. [Why is it legitimate to equate the coefficients?]

## 10. Exponential and trigonometric functions; Arcs, curves, etc.

10.1. **Exponential functions.** Last time we defined $e^z = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + \ldots$. $f(z) = e^z$ is the unique function such that $f'(z) = f(z)$ and $f(0) = 1$.

**Lemma 10.1.** $e^{a+b} = e^a e^b$

*Proof.* $\frac{\partial}{\partial z}(e^z e^{c-z}) = e^z e^{c-z} - e^z e^{c-z} = 0$, and hence $e^z e^{c-z}$ is a constant. By setting $z = 0$, we find that $e^z e^{c-z} = e^c$. Finally, if we write $c = a + b$ and $z = a$, then we're done. □

**Remark:** When restricted to $\mathbf{R}$, $e^z$ is the standard exponential function on $\mathbf{R}$. Hence we can think of $e^z : \mathbf{C} \to \mathbf{C}$ as the extension of $e^x : \mathbf{R} \to \mathbf{R}$.

10.2. **Trigonometric functions.** We define:

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \ \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

Then we compute that

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots,$$
$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots.$$

It is easy to prove the following properties:

(1) $\cos z + i \sin z = e^{iz}$.
(2) $\cos^2 + \sin^2 = 1$.
(3) $\frac{\partial}{\partial z}(\cos z) = -\sin z$.
(4) $\frac{\partial}{\partial z}(\sin z) = \cos z$.

If $z = x + iy$, then we can write $e^z = e^x e^{iy} = e^x(\cos y + i \sin y)$, where $e^x$, $\cos y$, $\sin y$ are all functions of one real variable.

$f(z)$ has *period* $c$ if $f(z + c) = f(z)$ for all $z$. By writing in polar form as above, it is clear that the smallest period of $f(z) = e^z$ is $2\pi i$.

We view $e^z$ geometrically: $e^z$ maps the infinite strip $0 \leq y \leq 2\pi$ to $\mathbf{C} - \{0\}$. (Notice that $e^z$ is never zero.) When $y = 0$, $e^z = e^x$, and the image is $\{x > 0\} \cap \mathbf{R}$. The line $y = \theta$ maps to the ray which makes an angle $\theta$ with the positive $x$-axis. Since $e^{z+2\pi i} = e^z$, the lines $y = \theta$ and $y = \theta + 2\pi$ map to the same ray.

10.3. **The logarithm.** We want to define the logarithm $\log z$ as the inverse function of $e^z$. The problem is that each $w \in \mathbf{C} - \{0\}$ has infinitely many preimages $z$ such that $e^z = w$; they are all given by $exp^{-1}(\{w\}) = \{\log |w| + i(\arg w + 2\pi n), n \in \mathbf{Z}\}$. Here $\log |w|$ is the real logarithm, and $0 \leq \arg w \leq 2\pi$. We will define one branch as follows:

$$\log w := \log |w| + i \arg w.$$

The important thing to remember is that, although we chose $0 \leq \arg w < 2\pi$, there is nothing canonical (natural) about this choice.

Also define $a^b = e^{b \log a}$, when $a, b \in \mathbf{C}$.

10.4. **Arcs and curves.** We now change topics and give the basics of arcs, curves, etc.

**Definition 10.2.** *An* arc *or a* path *in* $\mathbf{C}$ *is a continuous map* $\gamma : [a, b] \to \mathbf{C}$. *(Here* $a < b$.*)* $\gamma(a)$ *is called the* initial point *of the arc and* $\gamma(b)$ *is the* terminal point *of the arc.*

**Remark:** Two arcs are the same iff they agree as maps. It is not sufficient for them to have the same image in $\mathbf{C}$.

$\gamma : [a, b] \to \mathbf{C}$ is:

1. *of class* $C^1$ (called *differentiable* in the book), if $\frac{d\gamma}{dt} = \gamma'(t) = x'(t) + iy'(t)$ exists and is continuous. (Here differentiability at a point means differentiability on some open set containing that point.)
2. *simple* if $\gamma(t_1) = \gamma(t_2) \Rightarrow t_1 = t_2$.
3. *a closed curve* if $\gamma(a) = \gamma(b)$.
4. *a simple closed curve* if $\gamma$ is a closed curve and $\gamma(t_1) = \gamma(t_2) \Rightarrow t_1 = t_2$ away from the endpoints.

Define $-\gamma : [-b, -a] \to \mathbf{C}$ by $-\gamma(t) = \gamma(-t)$. This traces the image of $\gamma$ in the opposite direction, and is called the *opposite arc* of $\gamma$.

10.5. **Conformality revisited.** Suppose $\gamma : [a, b] \to \mathbf{C}$ is an arc and $f : \Omega \subset \mathbf{C} \to \mathbf{C}$ is an analytic function. Let $w = f(\gamma(t))$. If $\gamma'(t)$ exists, then $w'(t)$ exists, and:

$$w'(t) = f'(\gamma(t))\gamma'(t).$$

Suppose $z_0 = \gamma(t_0)$ and $w_0 = f(z_0)$. If $\gamma'(t_0) \neq 0$ and $f'(z_0) \neq 0$, then $w'(t_0) \neq 0$. We also have

$$\arg w'(t_0) = \arg f'(z_0) + \arg \gamma'(t_0).$$

Here, we are considering the argument to be mod $2\pi$. This implies that the angle between $\gamma'(t_0)$ and $w'(t_0)$ is equal to $f'(z_0)$. If $\gamma_0(t_0) = \gamma_1(t_0) = z_0$, then the angle made by $\gamma_0'(t_0)$ and $\gamma_1'(t_0)$ is preserved under composition with $f$. Recall we called this property *conformality.* Also observe that

$$|w'(t_0)| = |f'(z_0)| \cdot |\gamma'(t_0)|,$$

in other words, the scaling factor is also constant.

## 11. Inverse functions and their derivatives

Let $\Omega \subset \mathbf{C}$ be an open set and $f : \Omega \to \mathbf{C}$ be an analytic function. Suppose $f$ is 1-1, $f'(z) \neq 0$ at all $z \in \Omega$, and $f$ is an *open mapping*, i.e., sends open sets to open sets. If $z_0 \in \Omega$ and $w_0 = f(z_0)$, then:

**Claim.** *$f^{-1}$ is analytic and* $(f^{-1})'(w_0) = \frac{1}{f'(z_0)}$.

Observe that $f$ is an open mapping $\Leftrightarrow$ $f^{-1}$ is defined on an open set and $f^{-1}$ is continuous. Hence, given $\varepsilon > 0$, there exists $\delta > 0$ such that $|w - w_0| < \delta$ implies $|f^{-1}(w) - f^{-1}(w_0)| < \varepsilon$.

*Proof.*

$$\lim_{w \to w_0} \frac{f^{-1}(w) - f^{-1}(w_0)}{w - w_0} = \lim_{f^{-1}(w) \to f^{-1}(w_0)} \frac{f^{-1}(w) - f^{-1}(w_0)}{w - w_0} = \lim_{z \to z_0} \frac{z - z_0}{w - w_0} = \frac{1}{f'(z_0)}.$$

Here, by the above discussion, $w \to w_0$ implies $f^{-1}(w) \to f^{-1}(w_0)$. $\square$

**Fact:** The condition that $f$ be an open mapping is redundant.

The Open Mapping Theorem (used in the proof of the inverse/implicit function theorems) states that if $f'(z_0) \neq 0$ and $f$ is in class $C^1$ (differentiable with continuous derivative), then there is an open neighborhood $U \ni z_0$ on which $f$ is an open mapping. [It will turn out that if $f$ is analytic, then $f$ has derivatives of all orders.]

We will not use this fact for the time being – its proof will be given when we discuss integration.

**Example:** Consider $w = \sqrt{z}$. This is naturally a "multiple-valued function", since there is usually more than one point $w$ such that $w^2 = z$. To make it a single-valued function (what we usually call "function"), there are choices that must be made. The choices are usually arbitrary. (The procedure of making these choices is often called "choosing a single-valued branch".)

Let the domain $\Omega$ be $\mathbf{C} - \{x + iy | x \leq 0, y = 0\}$, i.e., the complement of the negative real axis (and the origin). We think of $\Omega$ as obtained from $\mathbf{C}$ by cutting along the negative real axis, hence the name "branch cut". Write $z = re^{i\theta}$, where $-\pi < \theta < \pi$. Then define $w = \sqrt{r}e^{i\theta/2}$. Geometrically, $\Omega$ gets mapped onto the right half-plane $\Omega' = \{x > 0\}$.

We verify that $w = \sqrt{z}$ is continuous: Write $w = u + iv$ and $w_0 = u_0 + iv_0$. If $|z - z_0| < \delta$, then $|w - w_0||w + w_0| < \delta$. Now $w, w_0$ are in the right half-plane, so $|w + w_0| \geq u + u_0 \geq u_0$. Hence $|w - w_0| < \varepsilon$ if $\delta = u_0\varepsilon$.

Now that we know $w = \sqrt{z}$ is continuous, it is analytic with derivative

$$\frac{\partial w}{\partial z} = \frac{1}{\partial z / \partial w} = \frac{1}{2w} = \frac{1}{2\sqrt{z}}.$$

**Example:** Consider $w = \log z$. Again, take the domain to be $\Omega = \mathbf{C} - \{x+iy | x \leq 0, y = 0\}$. Then choose a branch as follows: Write $z = re^{i\theta}$, with $-\pi < \theta < \pi$. Then map $z \mapsto w = \log|r| + i\theta$. The image is the infinite strip $\Omega' = \{-i\pi < y < i\pi\}$.

If $w = \log z$ is continuous, then

$$\frac{\partial w}{\partial z} = \frac{1}{\partial z/\partial w} = \frac{1}{e^w} = \frac{1}{z},$$

as expected from calculus.

It remains to prove the continuity of $w = \log z$. Write $w = u+iv$ and $w_0 = u_0+iv_0$. Define a closed set $A$ to be a box in the range which is bounded to the left by $u = u_0 - \log 2$, to the right by $u = u_0 + \log 2$, below by $v = -\pi$ and above by $v = \pi$, with the disk $|w - w_0| < \varepsilon$ (with $\varepsilon$ small) removed. The continuous function $|e^w - e^{w_0}|$ attains a minimum $\rho$ on $A$ (it's a minimum, not an infimum, since $A$ is closed). Moreover, $\rho > 0$, since the only times when the minimum is zero are when $w = w_0 + 2\pi i n$, which are not in $A$.

Define $\delta = \min(\rho, \frac{1}{2}e^{u_0})$.

We claim that no point of $|z - z_0| < \delta$ maps into $A$: if $|z - z_0| < \delta$ maps into $A$, then $|e^w - e^{w_0}| \geq \rho \geq \delta$, a contradiction.

We also claim that no point of $|z-z_0| < \delta$ maps into $u < u_0 - \log 2$ (to the left of $A$) or into $u > u_0 + \log 2$ (to the right of $A$). If $z$ is mapped to the right of $A$, then $|e^w - e^{w_0}| \geq e^u - e^{u_0} \geq 2e^{u_0} - e^{u_0} = e^{u_0}$. If mapped to the left, then $|e^w - e^{w_0}| \geq e^{u_0} - e^u \geq e^{u_0} - \frac{1}{2}e^{u_0} = \frac{1}{2}e^{u_0}$. In either case, we obtain a contradiction.

This implies that $|z - z_0| < \delta$ has nowhere to go but $|w - w_0| < \varepsilon$, proving the continuity.

**Example:** $w = \sqrt{1 - z^2}$. Define $\Omega$ to be the complement in $\mathbf{C}$ of two half-lines $y = 0$, $x \leq -1$ and $y = 0$, $x \geq 1$. Consider $\Omega^+ = \Omega \cap \{y \geq 0\}$. This is mapped to $\mathbf{C} - \{y = 0 \text{ and } x \geq 1\}$ under $z^2$. Multiplying by $-1$ rotates this region by $\pi$ about the origin to $\mathbf{C} - \{y = 0, x \leq -1\}$. Adding 1 shifts to $\mathbf{C} - \{y = 0, x \leq 0\}$, and squaring gives $\{x > 0\}$. The map $\sqrt{1 - z^2}$ on $\Omega^+$ with the exception of points on the $x$-axis which map to $y = 0, 0 \leq x \leq 1$ in the image. Now $\Omega^- = \Omega \cap \{y \leq 0\}$ is also mapped to the same right half-plane. Observe that the map from $\Omega$ is a 2-1 map everywhere except for $z = 0$ which corresponds to $w = 1$.

**Example:** $w = \cos^{-1} z$. If we write $z = \cos w = \frac{e^{iw} + e^{-iw}}{2}$ and solve for $e^{iw}$ in terms of $z$, we get a quadratic equation

$$e^{2iw} - 2ze^{iw} + 1 = 0$$

with solution $e^{iw} = z \pm \sqrt{z^2 - 1}$. Hence $w = -i\log(z \pm \sqrt{z^2 - 1})$. Since $z \pm \sqrt{z^2 - 1}$ are reciprocals, we can write

$$w = \pm i \log(z + \sqrt{z^2 - 1}).$$

We take the following branch: $w = i\log(z + i\sqrt{1 - z^2})$. Take the domain to be the same $\Omega$ as in the previous example. One can check that $z + i\sqrt{1 - z^2}$ maps to the upper half-plane (minus the $x$-axis): Since $z \pm i\sqrt{1 - z^2}$ are reciprocals, if one of them is real, then $z, i\sqrt{1 - z^2}$

are both real. But if $z$ is on the real axis with $-1 < x < 1$, then $i\sqrt{1-z^2}$ is not real. Hence we can map using log as defined above to the infinite strip $0 < y < i\pi$.

## 12. Line integrals

12.1. **Line integrals.** If $f = (u, v) : [a, b] \to \mathbf{C}$ is a continuous function, then

$$\int_a^b f(t)dt \stackrel{def}{=} \int_a^b u(t)dt + i \int_a^b v(t)dt,$$

where the right-hand side terms are the standard Riemann integrals on $\mathbf{R}$.

**Properties:**

(1) $\int_a^b c \cdot f(t)dt = c \int_a^b f(t)dt$, $c \in \mathbf{C}$.
(2) $\int_a^b (f(t) + g(t))dt = \int_a^b f(t)dt + \int_a^b g(t)dt$.
(3) $|\int_a^b f(t)dt| \leq \int_a^b |f(t)|dt$.

Let $\gamma : [a, b] \to \Omega \subset \mathbf{C}$ be a *piecewise differentiable* arc. Given $f : \Omega \to \mathbf{C}$ continuous, define:

$$\int_\gamma f(z)dz \stackrel{def}{=} \int_a^b f(\gamma(t))\gamma'(t)dt.$$

This is not so unreasonable on a formal level. Indeed, if $z = \gamma(t)$, then $dz = \gamma'(t)dt$.

**Key Property:** If $\phi$ is an increasing, piecewise differentiable function $[\alpha, \beta] \to [a, b]$, then

$$\int_\gamma f(z)dz = \int_{\gamma\circ\phi} f(z)dz.$$

Assuming everything in sight is differentiable (not just piecewise), we have:

$$\int_a^b f(\gamma(t))\gamma'(t)dt = \int_\alpha^\beta f(\gamma(\phi(\tau)))\gamma'(\phi(\tau))\phi'(\tau)d\tau = \int_\alpha^\beta f(\gamma \circ \phi(\tau))(\gamma \circ \phi)'(\tau)d\tau.$$

The first equality follows from the change of variables formula in integration, and the latter follows from the chain rule. Here we write $t = \phi(\tau)$.

**Orientation Change:** We have

$$-\int_\gamma f(z)dz = \int_{-\gamma} f(z)dz.$$

Indeed, writing $\tau = -t$, we have:

$$\begin{aligned}
\int_{-\gamma} f(z)dz &= \int_{-b}^{-a} f(\gamma(-t))(-\gamma'(-t))dt \\
&= \int_{-a}^{-b} -f(\gamma(-t))(-\gamma'(-t))dt \\
&= \int_a^b f(\gamma(\tau))\gamma'(\tau)(-d\tau) = -\int_\gamma f(z)dz.
\end{aligned}$$

The integral $\int f(z)dz$ over $\gamma$ does not depend on the actual parametrization – it only depends on the direction/orientation of the arc. If we change the direction/orientation, then $\int f(z)dz$ acquires a negative sign.

**Notation:** We often write $dz = dx + idy$ and $d\overline{z} = dx - idy$. Then $\int_\gamma f dx = \int_\gamma f(\gamma(t))\frac{dx}{dt}dt$, $\int_\gamma f dy = \int_\gamma f(\gamma(t))\frac{dy}{dt}dt$, and $\int_\gamma f(z)dz = \int_\gamma (u(z) + iv(z))(dx + idy) = \int_\gamma (udx - vdy) + i\int(vdx + udy)$.

**More Notation:** A subdivision of $\gamma$ can be written as $\gamma_1 + \cdots + \gamma_n$, and $\int_{\gamma_1+\cdots+\gamma_n} f dz \stackrel{def}{=} \int_{\gamma_1} f dz + \cdots + \int_{\gamma_n} f dz$.

**Path Integrals:** Define $\int_\gamma f|dz| \stackrel{def}{=} \int_a^b f(\gamma(t))|\gamma'(t)|dt$, with $\gamma : [a, b] \to \Omega$. Then the following are easy to see:

$$\int_{-\gamma} f|dz| = \int_\gamma f|dz|,$$

$$\left|\int_\gamma f dz\right| \leq \int_\gamma |f||dz|.$$

12.2. **Exact differentials.** A differential $\alpha = pdx+qdy$, where $p, q : \Omega \to \mathbf{C}$ are continuous, is called an *exact differential* if there exists $U : \Omega \to \mathbf{C}$ such that $\frac{\partial U}{\partial x} = p$ and $\frac{\partial U}{\partial y} = q$. [It turns out (but we will not make use of the fact) that if $U$ has continuous partials, then it is differentiable and of class $C^1$.]

**Independence of path:** $\int_\gamma \alpha$ *depends only on the endpoints* of $\gamma$ if we have $\int_{\gamma_1} \alpha = \int_{\gamma_2} \alpha$, whenever $\gamma_1$ and $\gamma_2$ have the same initial and terminal points.

**Claim.** *$\int_\gamma \alpha$ depends only on the endpoints iff $\int_\gamma \alpha = 0$ for all closed curves $\gamma$.*

*Proof.* If $\gamma$ is a closed curve, then $\gamma$, $-\gamma$ have the same endpoints. Hence $\int_\gamma \alpha = \int_{-\gamma} \alpha = -\int_\gamma \alpha$, which implies that $\int_\gamma \alpha = 0$. If $\gamma_1$, $\gamma_2$ have the same endpoints, then $\int_{\gamma_1} \alpha = \int_{\gamma_2} \alpha$, and $\int_{\gamma_1-\gamma_2} \alpha = 0$. □

**Theorem 12.1.** *$\alpha = pdx + pdy$, with $p, q$ continuous, is exact iff $\int_\gamma \alpha$ only depends on the endpoints of $\gamma$.*

*Proof.* Suppose $pdx + qdy$ is exact. Then

$$\int_\gamma pdx + qdy = \int_\gamma \left(\frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy\right) = \int_a^b \frac{d}{dt}U(x(t), y(t))dt = U(b) - U(a).$$

Suppose $\int_\gamma \alpha$ only depends on the endpoints of $\gamma$. We assume WLOG that $\Omega$ is *connected*, i.e., if $\Omega = A \sqcup B$ (disjoint union) with $A, B$ open, then $A$ or $B$ is $\emptyset$.

**HW 13.** *Prove that an open set $\Omega \subset \mathbf{C}$ is connected iff it is* path-connected*, i.e., for any $p, q \in \mathbf{C}$ there exists a continuous path $\gamma$ from $p$ to $q$. Moreover, we may take $\gamma$ to be a path which is piecewise parallel either to the $x$-axis or the $y$-axis.*

Pick $z_0 \in \Omega$. Let $\gamma$ be a path from $z_0$ to $z \in \Omega$ which is piecewise of the form $x = a$ or $y = b$. Then define $U(z) = \int_\gamma pdx + qdy$. Since the integral only depends on the endpoints, $U(z)$ is well-defined. If we extend $\gamma$ horizontally to a path from $z_0$ to $z + h$, $h \in \mathbf{R}$, then $U(z+h) - U(z) = \int_z^{z+h} pdx + qdy$, and $\frac{\partial U}{\partial x} = p$. Similarly, $\frac{\partial U}{\partial y} = q$. $\square$

12.3. **Complex differentials.** Given a differentiable function $F : \Omega \to \mathbf{C}$, we write $dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy$; in other words, an exact differential is always of the form $dF$ for some function $F$.

An alternative way of writing $dF$ is $dF = \frac{\partial F}{\partial z} dz + \frac{\partial F}{\partial \bar{z}} d\bar{z}$.

**HW 14.** *Verify this!*

The latter term vanishes if $F$ is analytic. Hence we have the following:

**Fact:** If $F$ is analytic, then $F'(z)dz = \frac{\partial F}{\partial z} dz$ is exact.

**Example:** $\int_\gamma (z-a)^n dz = 0$ for $n \geq 0$, if $\gamma$ is closed. This follows from the fact that $F(z) = \frac{(z-a)^{n+1}}{n+1}$ is analytic with derivative $(z-a)^n$.

**Example:** $\int_C \frac{1}{z-a} dz = 2\pi i$, where $C$ is the circle parametrized by $z = a + \rho e^{i\theta}$, $\theta \in [0, 2\pi]$. This proves that there is no analytic function $F(z)$ on $\mathbf{C} - \{a\}$ such that $F'(z) = \frac{1}{z-a}$. (In other words, it is not possible to define $\log(z-a)$ on all of $\mathbf{C} - \{a\}$.)

## 13. Cauchy's theorem

Let $R$ be the rectangle $[a,b]\times[c,d]$. Define its *boundary* $\partial R$ to be the simple closed curve $\gamma_1+\gamma_2+\gamma_3+\gamma_4$, where $\gamma_1:[a,b]\to\mathbf{C}$ maps $t\mapsto t+ic$, $\gamma_2:[c,d]\to\mathbf{C}$ maps $t\mapsto b+it$, $\gamma_3:[-b,-a]\to\mathbf{C}$ maps $t\mapsto -t+id$, and $\gamma_4:[-d,-c]\to\mathbf{C}$ maps $t\mapsto a-it$. In other words, $\partial R$ is oriented so that it goes around the boundary of $R$ in a counterclockwise manner.

We prove the following fundamental theorem:

**Theorem 13.1** (Cauchy's theorem for a rectangle)**.** *If $f(z)$ is analytic on $R$, then $\int_{\partial R} dz=0$.*

Later, a more general version of the theorem will be given – it will be valid for more general regions.

*Proof.* Define $\eta(R)\stackrel{def}{=}\int_{\partial R} f(z)dz$. Subdivide $R$ into 4 congruent rectangles $R^{(1)}$, $R^{(2)}$, $R^{(3)}$, and $R^{(4)}$. [Draw a line segment from the midpoint of $\gamma_1$ to the midpoint of $\gamma_3$ and a segment from the midpoint of $\gamma_2$ to the midpoint of $\gamma_4$.] Then it is easy to verify that $\partial R=\partial R^{(1)}+\partial R^{(2)}+\partial R^{(3)}+\partial R^{(4)}$ and

$$\eta(R)=\eta(R^{(1)})+\dots\eta(R^{(4)}).$$

Taking $R^{(i)}$ with the largest $|\eta(R^{(i)})|$, we have $|\eta(R^{(i)})|\geq\frac{1}{4}|\eta(R)|$. By continuing the subdivision, there exist rectangles $R\supset R_1\supset R_2\supset\dots$ such that $|\eta(R_n)|\geq\frac{1}{4^n}|\eta(R)|$.

Now, we can take a sequence of points $z_n\in R_n$; by the Cauchy criterion, this sequence converges to some point $z^*$. (Why?) Hence we have $R_n\subset\{|z-z^*|<\delta\}$ for sufficiently large $n$. If $f$ is analytic, then for any $\varepsilon>0$ there exists $\delta>0$ such that $|z-z^*|<\delta\Rightarrow\left|\frac{f(z)-f(z^*)}{z-z^*}-f'(z*)\right|<\varepsilon$, or $|f(z)-f(z^*)-f'(z^*)(z-z^*)|<\varepsilon|z-z^*|$.

Recall from last time that $\int_{\partial R_n} dz=0$ and $\int_{\partial R_n} zdz=0$, since 1 and $z$ have antiderivatives $z$ and $\frac{z^2}{2}$, and hence $dz$ and $zdz$ are exact. Now,

$$\eta(R_n)=\int_{\partial R_n} f(z)dz=\int_{\partial R_n}\left(f(z)-f(z^*)-f'(z^*)(z-z^*)\right)dz.$$

Therefore, $|\eta(R_n)|\leq\varepsilon\int_{\partial R_n}|z-z^*||dz|$. If we write $d$ for the diagonal of $R$, then an upper bound for $|z-z^*|$ is $\frac{1}{2^n}d$, the diagonal of $R_n$. Next, if $L$ is the perimeter of $R$, then $\frac{1}{2^n}L$ is the perimeter of $R_n$. We obtain $|\eta(R_n)|\leq\varepsilon(\frac{1}{2^n}d)(\frac{1}{2^n}L)=\varepsilon dL\frac{1}{4^n}$.. Since we may take $\varepsilon$ to be arbitrarily small, we must have $\eta(R_n)=0$ and hence $\eta(R)=0$. $\square$

**A Useful Generalization:** If $f$ is analytic on $R-\{\text{finite number of interior points}\}$, and $\lim_{z\to\zeta}(z-\zeta)f(z)=0$ for any singular point $\zeta$, then $\int_{\partial R} f(z)dz=0$.

*Proof.* By subdividing and using Cauchy's theorem for a rectangle, we may consider a small $R$ with one singular point $\zeta$ at the center. We may shrink $R$ if necessary so that $|(z-\zeta)f(z)|<\varepsilon$ for $z\in R\Rightarrow|f(z)|<\frac{\varepsilon}{|z-\zeta|}$ for $z\neq\zeta\in R$.

Therefore, $|\eta(R)| \leq \int_{\partial R} \frac{\varepsilon}{|z-\zeta|}|dz|$. It remains to estimate the right-hand term. If $R$ is a square whose sides have length $r$, then $|z-\zeta|$ is bounded below by $\frac{1}{2}r$ on $\partial R$. Hence an upper bound for the right-hand term is $\varepsilon\frac{4r}{1/2r} = 8\varepsilon$. Since $\varepsilon$ was arbitrary, we are done. □

Cauchy's theorem on a rectangle can be used to prove a stronger version of the theorem, valid for any closed curve on an open disk, not just the boundary of a rectangle.

**Theorem 13.2** (Cauchy's theorem for a disk)**.** *If $f(z)$ is analytic on an open disk $D = \{|z-a| < \rho\}$, then $\int_\gamma f(z)dz = 0$ for every closed curve $\gamma$ in $D$.*

*Proof.* Let $\gamma_0$ be a path from $a$ to $z = x + iy$, consisting of a horizontal line segment, followed by a vertical line segment. Then define $F(z) = \int_{\gamma_0} f(z)dz$. By Cauchy's theorem on a rectangle, $F(z) = \int_{\gamma_1} f(z)dz$, where $\gamma_1$ is a path from $a$ to $z$, consisting of a vertical line segment, followed by a horizontal one. Using one of the two types of paths, we compute:

$$\frac{\partial F}{\partial x} = \frac{d}{dh}\int_x^{x+h} f(z)dx = f(z),$$
$$\frac{\partial F}{\partial y} = \frac{d}{dk}\int_y^{y+k} f(z)idy = if(z).$$

Hence $F$ satisfies the Cauchy-Riemann equations. Therefore, $f(z)dz = F'(z)dz$ is exact, and $\int_\gamma f(z)dz = 0$ for all closed curves $\gamma$ (by Theorem 12.1). [Observe that the fact that $F$ satisfies the CR equations and that it has continuous partials implies that $F$ itself is analytic. This follows from Theorem 3.1.] □

## 14. The winding number and Cauchy's integral formula

14.1. **The winding number.** Let $\gamma : [\alpha, \beta] \to \mathbf{C}$ be a piecewise differentiable closed curve. Assume $\gamma$ does not pass through $a$, i.e., $a \notin \mathrm{Im}\ \gamma$, where the image of $\gamma$ is written as $\mathrm{Im}\ \gamma$.

**Definition 14.1.** *The* winding number of $\gamma$ about $a$ *is* $n(\gamma, a) \stackrel{def}{=} \dfrac{1}{2\pi i}\displaystyle\int_\gamma \frac{dz}{z-a}$.

**Properties of the winding number:**

1. $n(-\gamma, a) = -n(\gamma, a)$. *Proof.* Follows from Orientation Change property from Day 12.

2. If $C$ is a circle centered at $a$ and oriented in the counterclockwise direction, then $n(C, a) = \frac{1}{2\pi i}\int_C \frac{dz}{z-a} = 1$. *Proof.* Direct computation.

3. $n(\gamma, a)$ is always an integer.

*Proof.* (See the book for a slightly different proof.) Subdivide $\gamma : [\alpha, \beta] \to \mathbf{C}$ into $\gamma_1 + \cdots + \gamma_n$, where each $\gamma_j : [\alpha_{j-1}, \alpha_j] \to \mathbf{C}$ is contained in a sector $C \leq \arg(z-a) \leq C + \varepsilon$ for small $\varepsilon$. Here $\alpha = \alpha_0 < \alpha_1 < \cdots < \alpha_n = \beta$. (Give proof!)

Now define a single-valued branch of $\log(z-a)$ on each sector so that $\arg(\gamma_{j-1}(\alpha_{j-1})) = \arg(\gamma_j(\alpha_{j-1}))$. Then

$$\int_{\gamma_j} \frac{dz}{z-a} = \int_{\gamma_j} d\log(z-a) = \int_{\gamma_j} d\log|z-a| + i\int_{\gamma_j} d\arg(z-a).$$

Therefore,

$$\int_\gamma \frac{dz}{z-a} = \int_\gamma d\log|z-a| + i\sum_{j=1}^n d\arg(z-a).$$

Since the initial and terminal points of $\gamma$ are the same, the first term on the right is zero and the second is a multiple of $2\pi i$. □

4. If $\mathrm{Im}\ \gamma \subset D^2 = \{|z-b| < \rho\}$, then $n(\gamma, a) = 0$ if $a \notin D^2$.

*Proof.* If $a \notin D^2$, then $\frac{1}{z-a}$ is holomorphic on $D^2$. Therefore, by Cauchy's Theorem, $\int_\gamma \frac{dz}{z-a} = 0$. □

5. As a function of $a$, $n(\gamma, a)$ is constant on each connected component $\Omega$ of $\mathbf{C} - \mathrm{Im}\ \gamma$.

*Proof.* Any two points $a, b \in \Omega$ can be joined by a polygonal path in $\Omega$. We may reduce to the case where $a, b$ are connected by a straight line segment in $\Omega$. (The general case follows by inducting on the number of edges of the polygon.) Observe that the function $\frac{z-a}{z-b}$ is real and negative iff $z$ is on the line segment from $a$ to $b$. (Check this!) Therefore, there is a single-valued branch of $\log \frac{z-a}{z-b}$ which can be defined on the complement of the line segment.

Hence,

$$\int_\gamma \left( \frac{1}{z-a} - \frac{1}{z-b} \right) dz = \int_\gamma d \log \frac{z-a}{z-b} = 0.$$

This proves that $n(\gamma, a) = n(\gamma, b)$. ☐

14.2. **Cauchy Integral Formula.**

**Theorem 14.2.** *Suppose $f(z)$ is analytic on the open disk $D$ and let $\gamma$ be a closed curve in $D$. If $a \notin Im\ \gamma$, then*

$$n(\gamma, a) \cdot f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)dz}{z-a}.$$

*Proof.* Consider $F(z) = \frac{f(z)-f(a)}{z-a}$. Suppose first that $a \in D$. Then $F$ is analytic on $D - \{a\}$ and satisfies $\lim_{z \to a}(z-a)F(z) = 0$. Hence Cauchy's Theorem holds:

$$\int_\gamma \frac{f(z)-f(a)}{z-a} dz = 0.$$

Rewriting this equation, we get:

$$f(a) \cdot n(\gamma, a) = f(a) \int_\gamma \frac{dz}{z-a} = \int_\gamma \frac{f(z)dz}{z-a}.$$

On the other hand, if $a \notin D$, then $F$ is analytic on all of $D$, and we can also apply Cauchy's Theorem. (In this case, both sides of the equation are zero.) ☐

**Application:** When $n(\gamma, a) = 1$, e.g., $\gamma$ is a circle oriented in the clockwise direction, we have

$$f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)dz}{z-a}.$$

We can rewrite this as:

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)d\zeta}{\zeta - z}.$$

(Here we're thinking of $z$ as the variable.)

**Key Observation:** The point of the Cauchy Integral Formula is that the value of $f$ at $z \in D$ can be determined from the values of $f$ on $\partial D$, using some *averaging process*. (Assume $f$ is analytic in a neighborhood of $D$.)

## 15. Higher derivatives, including Liouville's theorem

15.1. **Higher Derivatives.** Let $f : \Omega \to \mathbf{C}$ be an analytic function, and let $D = \{|z - a| < \rho\} \subset \Omega$. Let $\partial D$ be the boundary circle, oriented counterclockwise. Then

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta) d\zeta}{\zeta - z}$$

for all $z \in D$, by the Cauchy Integral Formula.

The Cauchy Integral Formula allows us to differentiate inside the integral sign, as follows:

**Theorem 15.1.**

$$f'(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta) d\zeta}{(\zeta - z)^2}, \ f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\partial D} \frac{f(\zeta) d\zeta}{(\zeta - z)^{n+1}}.$$

*In particular, an analytic function has (complex) derivatives of all orders.*

The theorem follows from the following lemma:

**Lemma 15.2.** *Let $\gamma : [\alpha, \beta] \to \mathbf{C}$ be a piecewise differentiable arc and $\phi : Im\ \gamma \to \mathbf{C}$ be a continuous map. Then $F_n(z) = \int_\gamma \frac{\phi(\zeta) d\zeta}{(\zeta - z)^n}$ is holomorphic on $\mathbf{C} - Im\ \gamma$ and its derivative is $F_n'(z) = nF_{n+1}(z)$.*

We will only prove the continuity and differentiability of $F_1(z)$. The rest is similar and is left as an exercise.

**Continuity of $F_1(z)$:** Suppose $z_0 \in \mathbf{C} - \text{Im } \gamma$. Since $\mathbf{C} - \text{Im } \gamma$ is open, there is a $\delta' > 0$ such that $D_{\delta'}(z_0) \stackrel{def}{=} \{|z - z_0| < \delta'\} \subset \mathbf{C} - \text{Im } \gamma$. Let $z \in D_{\delta'/2}(z_0)$. We write

$$F_1(z) - F_1(z_0) = \int_\gamma \left( \frac{\phi(\zeta)}{\zeta - z} - \frac{\phi(\zeta)}{\zeta - z_0} \right) d\zeta = (z - z_0) \int_\gamma \frac{\phi(\zeta) d\zeta}{(\zeta - z)(\zeta - z_0)}.$$

We estimate that $|\zeta - z| > \frac{\delta'}{2}$ and $|\zeta - z_0| > \frac{\delta'}{2}$, and

$$|F_1(z) - F_1(z_0)| \leq |z - z_0| \frac{4}{|\delta'|^2} \int_\gamma |\phi(\zeta)||d\zeta|.$$

If we further shrink $D_{\delta'/2}(z_0)$, we can make $|F_1(z) - F_1(z_0)|$ arbitrarily small.

**Differentiability of $F_1(z)$:** We write

$$\frac{F_1(z) - F_1(z_0)}{z - z_0} = \int_\gamma \frac{\phi(\zeta) d\zeta}{(\zeta - z)(\zeta - z_0)}. \tag{1}$$

Now, use the above step with $\frac{\phi(\zeta)}{\zeta - z_0}$ instead of $\phi(\zeta)$. [Note that $\frac{\phi(\zeta)}{\zeta - z_0}$ is continuous on Im $\gamma$ since $z_0$ avoids Im $\gamma$.] By the above step, Equation 1 is continuous on $\mathbf{C} - \gamma$. Hence as

$z \to z_0$ we have

$$\int_\gamma \frac{\phi(\zeta)d\zeta}{(\zeta - z)(\zeta - z_0)} \to \int_\gamma \frac{\phi(\zeta)d\zeta}{(\zeta - z_0)^2}.$$

Therefore $F_1'$ is holomorphic and $F_1'(z) = F_2(z)$.

15.2. **Corollaries of Theorem 15.1.**

15.2.1. *Morera's Theorem.*

**Theorem 15.3.** *If $f : \Omega \to \mathbf{C}$ is continuous and $\int_\gamma f dz = 0$ for all closed curves $\gamma : [\alpha, \beta] \to \Omega$, then $f$ is analytic on $\Omega$.*

*Proof.* We already showed that $f = F'$ for some analytic function $F$. $f$ is then itself analytic by Theorem 15.1. □

15.2.2. *Liouville's Theorem.*

**Theorem 15.4.** *Any analytic function $f : \mathbf{C} \to \mathbf{C}$ which is bounded is constant.*

*Proof.* Let $D = \{|z - a| < r\}$. Suppose $|f(z)| \leq M$ for all $z \in \mathbf{C}$. If we apply the integral formula to $\partial D$, then

$$f^{(n)}(a) \leq \frac{n!}{2\pi} \frac{M \cdot 2\pi r}{r^{n+1}} = \frac{n! M}{r^n}.$$

But $r$ is arbitrary, so (taking $n = 1$) $f'(a) = 0$ for all $a$. Hence $f$ is constant. □

15.2.3. *Fundamental Theorem of Algebra.*

**Theorem 15.5.** *Any polynomial $P(z)$ of degree $> 0$ has a root.*

*Proof.* Suppose not. Then $\frac{1}{P(z)}$ is analytic on the whole plane. As $z \to \infty$, $\frac{1}{P(z)} \to 0$, so $\frac{1}{P(z)}$ is bounded. By Liouville, $\frac{1}{P(z)}$ is constant, which is a contradiction. □

## 16. Removable singularities, Taylor's theorem, zeros and poles

### 16.1. Removable singularities.

**Theorem 16.1.** *Let $a \in \Omega$. If $f : \Omega - \{a\} \to \mathbf{C}$ is analytic, then $f$ can be extended to an analytic function $\Omega \to \mathbf{C}$ iff $\lim_{z\to a}(z-a)f(z) = 0$.*

*Proof.* If $f$ is analytic on $\Omega$, then it is continuous on $\Omega$. Hence $\lim_{z\to a}(z-a)f(z) = \lim_{z\to a}(z-a) \cdot \lim_{z\to a} f(z) = 0 \cdot f(a) = 0$.

Suppose $\lim_{z\to a}(z-a)f(z) = 0$. Take a small disk $D \subset \Omega$ centered at $a$. Consider the function $g(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(\zeta)d\zeta}{\zeta - z}$. This is an analytic function on all of the interior of $D$. We claim that, away from $z = a$, $f(z) = g(z)$. Indeed, in the proof of the Cauchy Integral Formula, we used $G(\zeta) = \frac{f(\zeta)-f(z)}{\zeta - z}$. There are two points $b = z$ and $b = a$, where we need to check that $\lim_{\zeta\to b}(\zeta - b)G(\zeta) = 0$ (in order to use Cauchy's Theorem). If $b = z$, then we have $\lim_{\zeta\to z}(f(\zeta) - f(z)) = 0$ by the continuity of $f$ at $z$. If $b = a$, then we have $\lim_{\zeta\to a} \frac{f(\zeta)-f(z)}{a-z}(\zeta - a) = 0$ by the condition $\lim_{z\to a}(z-a)f(z) = 0$. □

For example, we can use the removable singularities theorem if $f(z)$ is bounded in a neighborhood of $z = a$.

### 16.2. Taylor's Theorem.

Consider the function $F(z) = \frac{f(z)-f(a)}{z-a}$. If $f$ is analytic on $\Omega$, then $F$ is analytic on $\Omega - \{a\}$. By the removable singularities theorem, $F$ extends to an analytic function on all of $\Omega$. By continuity of $F$ at $a$, we must have $F(a) = \lim_{z\to a} F(z) = f'(a)$.

We can therefore write

$$f(z) = f(a) + (z-a)f_1(z),$$

where $f_1 = F$. Recursively we let $f_1(z) = f_1(a) + (z-a)f_2(z)$, and so on, to obtain

$$f(z) = f(a) + f_1(a)(z-a) + f_2(a)(z-a)^2 + \cdots + f_{n-1}(a)(z-a)^{n-1} + (z-a)^n f_n(z).$$

We can differentiate $f(z)$ and set $z = a$ to obtain:

**Theorem 16.2** (Taylor's theorem)**.** *Suppose $f : \Omega \to \mathbf{C}$ is an analytic function. Then there is an analytic function $f_n : \Omega \to \mathbf{C}$ such that*

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(z-a)^{n-1} + (z-a)^n f_n(z).$$

*The remainder term $f_n(z)$ can be written as:*

$$f_n(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(\zeta)d\zeta}{(\zeta-a)^n(\zeta-z)},$$

*where $D \subset \Omega$ is a small disk centered at $a$.*

*Proof.* We only need to prove the expression for the remainder. We prove the statement for $n = 1$, leaving the general case as an exercise.

$$\begin{aligned} f_1(z) &= \frac{1}{2\pi i}\int_{\partial D} \frac{f_1(\zeta)d\zeta}{\zeta - z} = \frac{1}{2\pi i}\int_{\partial D} \frac{(f(\zeta)-f(a))d\zeta}{(\zeta-a)(\zeta-z)} \\ &= \frac{1}{2\pi i}\int_{\partial D} \frac{f(\zeta)d\zeta}{(\zeta-a)(\zeta-z)} - \frac{f(a)}{2\pi i}\int_{\partial D}\frac{d\zeta}{(\zeta-a)(\zeta-z)} \end{aligned}$$

Now, we can write $\frac{1}{(\zeta-a)(\zeta-z)} = \frac{1}{a-z}\left(\frac{1}{\zeta-a} - \frac{1}{\zeta-z}\right)$. Since $\int_{\partial D}\frac{d\zeta}{\zeta-a} = \int_{\partial D}\frac{d\zeta}{\zeta-z} = 2\pi i$ ($a$ and $z$ are in the same connected component of $\mathbf{C} - \partial D$), the second term in the equation vanishes. □

**Remark:** We will discuss *Taylor series* next time.

16.3. **Zeros.**

**Proposition 16.3.** *Let $f : \Omega \to \mathbf{C}$ be an analytic function, $a \in \Omega$, and $\Omega$ be connected. If $f(a) = f'(a) = \cdots = f^{(n)}(a) = 0$ for all $n$, then $f$ is identically zero.*

*Proof.* Suppose $f^{(n)}(z) = 0$ for all $n$. Then $f(z) = f_n(z)(z-a)^n$ for all $n$. Using the above expression for $f_n(z)$, we write:

$$|f_n(z)| \le \frac{1}{2\pi}\int_{\partial D}\frac{M|d\zeta|}{R^n(R-|z-a|)} = \frac{1}{2\pi}\frac{M}{R^n}\frac{2\pi R}{(R-|z-a|)} = \frac{M}{R^{n-1}(R-|z-a|)}.$$

Here $D$ is a disk of radius $R$ centered at $a$, $|f(\zeta)| \le M$ for $\zeta$ on $\partial D$, and $z \in int(D)$. This implies:

$$|f(z)| \le \left(\frac{|z-a|}{R}\right)^n \frac{MR}{R-|z-a|}.$$

As $n \to \infty$, $\left(\frac{|z-a|}{R}\right)^n \to 0$, and we're done. □

By Proposition 16.3, all the zeros of an analytic function have *finite order*, i.e., there exist an integer $h > 0$ and an analytic function $f_h$ such that $f(z) = (z-a)^h f_h(z)$ and $f_h(a) \neq 0$. We say that $z = a$ is a *zero of order $h$*.

**Corollary 16.4.**

(1) *The zeros of an analytic function $f \not\equiv 0$ are isolated.*
(2) *If $f, g$ are analytic on $\Omega$ and if $f(z) = g(z)$ on a set which has an accumulation point in $\Omega$, then $f \equiv g$. (Here, $f \equiv g$ means $f(z) = g(z)$ for all $z \in \Omega$.)*

*Proof.* Follows from observing that $f_h(a) \neq 0$ and $f_h$ is continuous $\Rightarrow f(z) \neq 0$ in a neighborhood of $a$, provided $z \neq a$. □

In other words, an analytic function $f : \Omega \to \mathbf{C}$ is uniquely determined by $f|_X$ ($f$ restricted to $X$), where $X \subset \Omega$ is any set with an accumulation point. For example, if $f, g$ agree on a nontrivial subregion of $\Omega$ or agree on a nontrivial (= does not map to a point) arc, then $f \equiv g$.

## 17. Analysis of isolated singularities

**Definition 17.1.** *An analytic function $f$ has an* isolated singularity at $a$ *if $f$ is analytic on $0 < |z-a| < \delta$ for some $\delta > 0$.*

We already discussed *removable singularities*, i.e., isolated singularities where $\lim_{z\to a}(z-a)f(z) = 0$. In such a case $f$ can be extended across $a$. (The isolated singularity was a singularity only because of lack of information.)

### 17.1. Poles.

Suppose $a \in \Omega$ and $f$ is analytic on $\Omega - \{a\}$. If $\lim_{z\to a} f(z) = \infty$, then $z = a$ is a *pole* of $f(z)$.

**Claim.** *A holomorphic function $f : \Omega - \{a\} \to \mathbf{C}$ with a pole at $z = a$ can be extended to a holomorphic function $f : \Omega \to S^2$.*

*Proof.* Change coordinates on $S^2$ from $z$-coordinates (about 0) to $w = \frac{1}{z}$-coordinates (about $\infty$). To distinguish these coordinates from the $z$-coordinates on $\Omega$, we write them as $z_2$ and $w_2$. Then $z_2 = f(z) \Leftrightarrow w_2 = g(z) = \frac{1}{f(z)}$. Since $\lim_{z\to a} f(z) = \infty$, it follows that $\lim_{z\to a} g(z) = 0$. Hence, by the removable singularities theorem, we can extend $g(z)$ holomorphically across $z = a$. □

The *order* of the pole $z = a$ is $h$ if $g(z) = (z-a)^h g_h(z)$, $g_h(a) \neq 0$. Note that, from the perspective of the claim, zeros and poles are completely analogous: one counts the order of a preimage under $f$ of $0 \in S^2$ and the other counts the order of a preimage under $f$ of $\infty \in S^2$.

If the pole $z = a$ of $f$ is of order $h$, then we can write

$$(z-a)^h f(z) = a_h + a_{h-1}(z-a) + \cdots + a_1(z-a)^{h-1} + \phi(z)(z-a)^h,$$

where we have expanded $(z-a)^h f(z)$ using Taylor's theorem. Here the $a_i$ are constants and $\phi$ is an analytic function. This implies that

$$f(z) = \frac{a_h}{(z-a)^h} + \frac{a_{h-1}}{(z-a)^{h-1}} + \cdots + \frac{a_1}{(z-a)} + \phi(z).$$

**Definition 17.2.** *A holomorphic function $f : \Omega \to S^2$ is said to be* meromorphic *on $\Omega$.*

### 17.2. Essential singularities.

In order to analyze isolated singularities, consider the following conditions:

(1) $\lim_{z\to a} |z-a|^\alpha |f(z)| = 0$. Here $\alpha \in \mathbf{R}$.
(2) $\lim_{z\to a} |z-a|^\alpha |f(z)| = \infty$.

**Proposition 17.3.** *Given an isolated singularity $z = a$ there are three possibilities:*

A. *$f \equiv 0$ and (1) holds for all $\alpha$.*
B. *There is an integer $h$ such that (1) holds for $\alpha > h$ and (2) holds for $\alpha < h$.*
C. *Neither (1) nor (2) holds for any $\alpha$.*

*Proof.* Suppose (1) holds for some $\alpha$. If (1) holds for some $\alpha$, it holds for larger $\alpha$. Therefore, take $\alpha$ to be a sufficiently large integer $m$. We may assume $(z-a)^m f(z)$ has a removable singularity and vanishes at $z=a$. Therefore, we can write $(z-a)^m f(z) = (z-a)^k g(z)$, with $g(a) \neq 0$, provided $f$ is not identically zero. Therefore, $f(z) = (z-a)^{k-m} g(z)$, and (1) holds for $\alpha > m-k$ and (2) holds for $\alpha < m-k$.

The case where (2) holds for some $\alpha$ is similar. □

In case B, $h$ is called the *algebraic order* of $f$ at $z=a$. If case C holds, then $z=a$ is called an *essential singularity.*

At first glance, essential singularities are not easy to understand or visualize – they have rather peculiar properties.

**Theorem 17.4** (Casorati-Weierstrass)**.** *An analytic function $f$ comes arbitrarily close to any complex value in every neighborhood of an essential singularity.*

*Proof.* We argue by contradiction. Suppose there exists $A$ such that $|f(z)-A| > \delta > 0$ for all $z$ in a neighborhood of $a$. Then it follows that

$$\lim_{z\to a} |z-a|^\alpha |f(z)-A| = \infty$$

for $\alpha < 0$. Therefore, $a$ is not an essential singularity for $f(z)-A$. Using Proposition 17.3, it follows that there is $\beta >> 0$ so that

$$\lim_{z\to a} |z-a|^\beta |f(z)-A| = 0.$$

Now, $|z-a|^\beta |A| \to 0$, so $|z-a|^\beta |f(z)| \to 0$ as well, a contradiction. □

**Remark:** The converse is also true. If $z=a$ is an isolated singularity, and, for any disk $|z-a| < \delta$ about $a$, $f(z)$ comes arbitrarily close to any complex value $A$, then $a$ is an essential singularity. (It is easy to verify that B cannot hold under the above conditions.)

**Example:** $f(z) = e^z$ has an essential singularity at $\infty$. (Equivalently, by changing to $w = \frac{1}{z}$ coordinates, $g(w) = e^{1/w}$ has an essential singularity at $w=0$.) Indeed, given any small open disk about $z=\infty$, i.e., $D = \{|z| > R\} \cup \{\infty\}$ for $R$ large, if we take $C >> R$, then the infinite annulus $\{C \leq \text{Im } z \leq C + 2\pi\} \subset D$ maps onto $\mathbf{C} - \{0\}$ via $f$. In this case, the essential singularity misses all but one point, namely the origin.

**Remark:** According to the "big" Picard theorem, near an essential singularity $z=a$, $f$ takes every complex value $A$, with at most one exception! (Moreover, the set $\{f(z) = A\}$ is infinite.) We will not prove this fact in this course.

## 18. Local mapping properties

Let $f$ be an analytic function on an open disk $D$. Suppose $f \not\equiv 0$.

**Theorem 18.1.** *Let $z_j$ be the zeros of $f$. If $\gamma$ is a closed curve in $D$ which does not pass through any $z_j$, then*

$$\sum_j n(\gamma, z_j) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz.$$

*Proof.* Suppose $f$ has a finite number of zeros $z_1, \dots, z_n$ on $D$. We assume that a zero is repeated as many times as its order. Then we can write

$$f(z) = (z - z_1) \cdots (z - z_n) g(z),$$

where $g(z)$ is analytic and $\neq 0$ on $D$. We compute that

$$f'(z) = \sum_j (z - z_1) \cdots \widehat{(z - z_j)} \cdots (z - z_n) g(z) + (z - z_1) \cdots (z - z_n) g'(z),$$

where a hat $\widehat{}$ indicates the term is omitted. Hence we obtain

$$\frac{f'(z)}{f(z)} = \frac{1}{z - z_1} + \cdots + \frac{1}{z - z_n} + \frac{g'(z)}{g(z)}.$$

**Remark:** This is what we would get if $\log f(z)$ was a well-defined single-valued function. If so, we could write $(\log f(z))' = \frac{f'(z)}{f(z)}$ and $\log f(z) = \log(z - z_1) + \cdots + \log(z - z_n) + \log g(z)$, so we would have the above equation.

Now,

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = \frac{1}{2\pi i} \left( \int_\gamma \frac{dz}{z - z_1} + \cdots + \int_\gamma \frac{dz}{z - z_n} + \int_\gamma \frac{g'(z)}{g(z)} dz \right) = n(\gamma, z_1) + \cdots + n(\gamma, z_n),$$

observing that $\int_\gamma \frac{g'(z)}{g(z)} dz = 0$ by Cauchy's Theorem, since $g(z) \neq 0$ on $D$

Now suppose there are infinitely many zeros of $f$ on $D$. Since $f$ is not identically zero, the zeros do not accumulate inside $D$. Take a smaller disk $D' \subset D$ which contains Im $\gamma$. Then there exist finitely many zeros of $f$ on $D'$, and the previous considerations apply. $\square$

**Interpretation:** If $f$ maps from the $z$-plane to the $w$-plane, i.e., $w = f(z)$, then we can write

$$n(f \circ \gamma, 0) = \int_{f \circ \gamma} \frac{dw}{w} = \int_\gamma \frac{f'(z)}{f(z)} dz.$$

Hence we can interpret Theorem 18.1 as follows:

$$n(f \circ \gamma, 0) = \sum_{f(z_i) = 0} n(\gamma, z_i).$$

More generally, we can write

$$n(f\circ\gamma, a) = \sum_{f(z_i)=a} n(\gamma, z_i),$$

provided $a \notin \text{Im } \gamma$.

**Theorem 18.2.** *Suppose $f(z)$ is analytic near $z_0$, $f(z_0) = w_0$, and $f(z) - w_0$ has a zero of order $n$ at $z_0$. For a sufficiently small $\varepsilon > 0$, there exists $\delta > 0$ such that for all $w \neq w_0$ in $D_\delta(w_0)$, $f(z) = w$ has $n$ distinct roots on $D_\varepsilon(z_0)$.*

**Remark:** If $f(z_0) = w_0$, then $z_0$ is a zero of $f(z) - w_0$ of order 1 iff $f'(z_0) \neq 0$. (Why?)

*Proof.* Consider a small disk $D_\varepsilon(z_0)$ centered at $z_0$ which does not contain any other zeros of $f(z) - w_0$. By taking $\varepsilon > 0$ sufficiently small, we may assume that $f'(z) \neq 0$ for $z \neq z_0$ in $D_\varepsilon(z_0)$. [If $f'(z_0) \neq 0$, then it is clear such a disk exists; if $f'(z_0) = 0$, then such a disk exists since the zeros of $f'$ are isolated.] By the above remark, all the zeros of $f(z) - A$ are of order 1, if $z \neq z_0 \in D_\varepsilon(z_0)$. Let $\gamma = \partial D_\varepsilon(z_0)$.

Now take a small disk $D_\delta(w_0)$ about $w_0 = f(z_0)$ which misses Im $f \circ \gamma$. Then for $w \in D_\delta(w_0)$

$$\sum_{f(z_i)=w} n(\gamma, z_i) = n(f\circ\gamma, w) = n(f\circ\gamma, w_0) = n,$$

and the preimages $z_i$ of $w$ are distinct by the previous paragraph. □

**Corollary 18.3** (Open Mapping Theorem)**.** *An analytic function $f \not\equiv 0$ maps open sets to open sets, i.e., is an* open mapping.

**Corollary 18.4.** *If $f$ is analytic at $z_0$ and $f'(z_0) \neq 0$, then $f$ is 1-1 near $z_0$ and its local inverse $f^{-1}$ is analytic.*

*Proof.* By Theorem 18.2, $f$ is 1-1 near $z_0$ if $f'(z_0) \neq 0$; by the previous corollary, $f$ is an open mapping. It follows from the Claim from Day 11 that the local inverse $f^{-1}$ is analytic. □

**Factorization of maps:** Suppose $f(z) - w_0$ has a zero of order $n$ at $z_0$, i.e., $f(z) - w_0 = (z - z_0)^n g(z)$ with $g(z_0) \neq 0$. Provided $|g(z) - g(z_0)| < |g(z_0)|$, i.e., $g(z)$ is contained in a disk of radius $|g(z_0)|$ about $g(z_0)$, $h(z) \overset{def}{=} \sqrt[n]{g(z)}$ exists, and we can write $f(z) - w_0 = ((z - z_0)h(z))^n$. We can then factor $w = f(z)$ into maps $z \mapsto \zeta = (z - z_0)h(z)$, followed by $\zeta \mapsto w = w_0 + \zeta^n$. The first map is locally 1-1 and the second is a standard $n$th power.

## 19. Maximum principle, Schwarz lemma, and conformal mappings

### 19.1. **Maximum Principle.**

A corollary of the Open Mapping Theorem is the following:

**Theorem 19.1** (Maximum Principle)**.** *If $f(z)$ is analytic and nonconstant on an open set $\Omega$, then $|f(z)|$ has no maximum on $\Omega$.*

*Proof.* Suppose $|f(z)|$ attains a maximum at $z_0 \in \Omega$. Then there is an open disk $D_\delta(z_0) \subset \Omega$ which maps to an open set about $f(z_0)$. In particular, $|f(z_0)|$ is not maximal. □

A reformulation of the Maximum Principle is the following:

**Theorem 19.2.** *If $f$ is defined and continuous on an closed bounded set $E$ and analytic on $int(E)$, then* $\max|f(z)|$ *occurs on the boundary of $E$. (Here $bdry(E) \overset{def}{=} E - int(E)$.)*

*Proof.* Since $E$ is compact, $\max|f(z)|$ is attained on $E$. It cannot occur on $int(E)$, so must occur on $bdry(E)$. □

### 19.2. **Schwarz Lemma.**

**Theorem 19.3** (Schwarz Lemma)**.** *Suppose $f$ is analytic on $|z| < 1$ and satisfies $|f(z)| \leq 1$ and $f(0) = 0$. Then $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$. If $|f(z)| = |z|$ for some $z \neq 0$, or if $|f'(0)| = 1$, then $f(z) = cz$ for $|c| = 1$.*

We can view the Schwarz Lemma as a "Contraction Mapping Principle", namely either the distance from $z$ to 0 is contracted under $f$ for all $z \in D$, or else $f$ is a rotation.

*Proof.* By comparison with $g(z) = z$. Take $f_1(z) = \frac{f(z)}{g(z)} = \frac{f(z)}{z}$. Since $f(0) = 0$, $f_1(z)$ is analytic and $f_1(0) = f'(0)$. On $|z| = r \leq 1$, $|f_1(z)| \leq \frac{1}{r}$, so $|f_1(z)| \leq \frac{1}{r}$ on $|z| \leq r$ by the Maximum Principle. By taking the limit $r \to 1$, we have $|f_1(z)| \leq 1$ on $|z| < 1$. Hence $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$.

If $|f_1(z)| = 1$ on $|z| < 1$, then $f_1$ is constant. Hence $f(z) = cz$ with $|c| = 1$. □

### 19.3. **Automorphisms of the open disk and the half plane.**

**Terminology:** A holomorphic map $f : \Sigma_1 \to \Sigma_2$ between two Riemann surfaces is *biholomorphism* if it has a holomorphic inverse. $\Sigma_1$ an $\Sigma_2$ are then *biholomorphic*. If $\Sigma_1 = \Sigma_2 = \Sigma$, then $f$ is often called an *automorphism*. We denote by $Aut(\Sigma)$ the group of automorphisms of $\Sigma$.

As an application of the Schwarz Lemma, we determine all automorphisms $f$ of the *open* unit disk $D$. By the results from last time, if $f$ is analytic and $f : D \to D$ is 1-1 and onto, then $f^{-1}$ is also analytic.

**Theorem 19.4.** *Let $f : D \to D$ be an automorphism satisfying $f(0) = 0$. Then $f(z) = e^{i\theta} z$ for some $\theta \in \mathbf{R}$.*

**HW 15.** *Use the Schwarz Lemma to prove Theorem 19.4.*

To understand the general case, i.e., $f(\alpha) = \beta$, we need an automorphism $g_\alpha : D \to D$ which maps $\alpha$ to 0. Consider $g_\alpha(z) = \frac{\alpha - z}{1-\overline{\alpha}z}$, where $\alpha \in D$.

**HW 16.** *Verify that $g_\alpha$ is a fractional linear transformation which takes $\overline{D}$ to itself.*

If $f : D \to D$ is an automorphism which takes $\alpha$ to $\beta$, then $g_\beta \circ f \circ g_\alpha^{-1}$ maps 0 to 0, hence is $e^{i\theta}z$. This implies that the group $Aut(D)$ of automorphisms of $D$ consists of $f(z) = g_\beta^{-1}(e^{i\theta}g_\alpha(z))$. In particular, automorphisms of the open unit disk are all fractional linear transformations.

We now consider automorphisms of the upper half plane $\mathbf{H} = \{\mathrm{Im}\ z > 0\}$. Let $g : \mathbf{C} \to \mathbf{C}$ be the fractional linear transformation $z \mapsto \frac{z-i}{z+i}$.

**HW 17.** *Prove that $g$ maps $\mathbf{H}$ biholomorphically onto $D$.*

Observe that $f$ is an automorphism of $\mathbf{H}$ iff $g \circ f \circ g^{-1}$ is an automorphism of $D$. (We say $f$ is *conjugated* by $g$.) Hence, $Aut(\mathbf{H}) = g^{-1}Aut(D)g$. Alternatively, we can use the following HW to show that $Aut(\mathbf{H}) = PSL(2, \mathbf{R})$.

**HW 18.** *The fractional linear transformations that preserve $\mathbf{H}$ are given by $PSL(2, \mathbf{R})$.*

19.4. **Conformal mappings.** Towards the end of this course, we will prove the following fundamental result:

**Theorem 19.5** (Riemann Mapping Theorem)**.** *Any simply connected (connected) region $\Omega \subsetneq \mathbf{C}$ is biholomorphic to the open unit disk $D$.*

**Remark:** If $\Omega = \mathbf{C}$, then any analytic function $f : \mathbf{C} \to D$ is bounded, and hence constant. Hence, $\mathbf{C}$ and $D$ are *not biholomorphic*!

We will define *simple connectivity* in due course, but for the time being, it roughly means that there are no holes in $\Omega$. It turns out that the only simply connected Riemann surfaces are $D$, $S^2$, and $\mathbf{C}$, and we have the following theorem:

**Theorem 19.6.**

(1) $Aut(D) \simeq Aut(\mathbf{H}) \simeq PSL(2, \mathbf{R})$.
(2) $Aut(S^2) \simeq PSL(2, \mathbf{C})$.
(3) $Aut(\mathbf{C}) \simeq \{az + b \mid a, b \in \mathbf{C}\}$.

The second and third assertions are left for HW. Also observe that $Aut(\Omega)$ in the Riemann Mapping Theorem is conjugate to (and hence isomorphic to) $Aut(D)$.

**Examples:** We now give examples of simply connected regions $\Omega$ that are biholomorphic to $D$. We emphasize that all the regions are open! By a succession of biholomorphisms, all the regions that appear here are biholomorphic to $D$.

1. $\mathbf{H} \simeq D$ by $z \mapsto \frac{z-i}{z+i}$ as before.

2. (First quadrant) $\{x > 0, y > 0\} \simeq \mathbf{H}$ by sending $z \mapsto z^2$.

3. (Quarter disk) $\{x > 0, y > 0, |z| < 1\} \simeq$ (half disk) $\{|z| < 1, y > 0\}$ via $z \mapsto z^2$.

4. (Half disk) $\{|z| < 1, y > 0\} \simeq$ (first quadrant) $\{x > 0, y > 0\}$ via $z \mapsto \frac{1+z}{1-z}$.

5. (Half disk) $\{|z| < 1, y > 0\} \simeq$ (semi-infinite strip) $\{0 < y < \pi, x < 0\}$ via $z \mapsto \log z$.

6. $\mathbf{H} \simeq$ (infinite strip) $\{0 < y < \pi\}$ via $z \mapsto \log z$.

7. (Half plane with slit) $\{x > 0\} - \{y = 0, 0 < x \leq 1\} \simeq \{x > 0\}$ by a succession of maps $z \mapsto z^2$, $z \mapsto z - 1$, $z \mapsto \sqrt{z}$, whose composition is $z \mapsto \sqrt{z^2 - 1}$.

8. $\mathbf{H} - \{e^{i\theta} | 0 < \theta < a < \pi\} \simeq \{x > 0\} - \{y = 0, 0 < x \leq b\}$ via $z \mapsto \frac{z-1}{z+1}$.

9. $\{0 < y < \pi\} - \{x = 0, 0 < y < a\} \simeq \mathbf{H} - \{e^{i\theta} | 0 < \theta < a < \pi\}$ via $z \mapsto e^z$.

## 20. Weierstrass' theorem and Taylor series

20.1. **Weierstrass' theorem.** Consider the sequence $\{f_n\}$, where $f_n$ is analytic on the open set $\Omega_n$. Suppose in addition that $\Omega_1 \subset \Omega_2 \subset \dots \subset \Omega_n \subset \dots$ and $\Omega = \cup_n \Omega_n$.

**Theorem 20.1.** *Suppose $\{f_n\}$ converges to $f$ on $\Omega$, uniformly on every compact subset of $\Omega$. Then $f$ is analytic on $\Omega$. Moreover, $f_n'$ converges uniformly to $f'$ on every compact subset of $\Omega$.*

We will often write UCOCS to mean "uniform convergence on compact sets".

**Remark:** Every compact set $E \subset \Omega$ is covered by $\{\Omega_n\}$, so must be contained in some $\Omega_n$ and hence all $\Omega_{n'}$ for $n' \geq n$.

*Proof.* Suppose $\overline{D} = \{|z-a| \leq r\} \subset \Omega$. (Here $D$ will denote the open disk.) By the Cauchy integral formula,

$$f_n(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f_n(\zeta)d\zeta}{\zeta - z},$$

for all $z \in D$. As $n \to \infty$,

$$f(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(\zeta)d\zeta}{\zeta - z},$$

so $f$ is analytic on $D$. [Since $f_n(z) \to f(z)$ uniformly on $\overline{D}$, $\int_{\partial D} \frac{f_n(\zeta)d\zeta}{\zeta-z} \to \int_{\partial D}\frac{f(\zeta)d\zeta}{\zeta-z}$ as $n \to \infty$; this is a standard property of Riemann integrals under uniform convergence.]

Similarly, $f_n'(z) = \frac{1}{2\pi i}\int_{\partial D}\frac{f_n(\zeta)d\zeta}{(\zeta-z)^2}$, so $f_n'(z) \to f'(z)$ UOCS. □

**Corollary 20.2.** *If a series $f(z) = f_1(z) + f_2(z) + \dots$, with $f_i$ analytic on $\Omega$, has UCOCS of $\Omega$, then $f$ is analytic on $\Omega$, and its derivative can be differentiated term-by-term.*

**Remark:** It suffices to prove uniform convergence on the boundary of compact sets $E$, by the maximum principle. $|f_n(z) - f_m(z)| \leq \varepsilon$ on $\partial E$ iff $|f_n(z) - f_m(z)| \leq \varepsilon$ on $E$.

**Theorem 20.3.** *If $f_n$ is analytic and nowhere zero on $\Omega$, and $f_n \to f$ UOCS, then $f(z)$ is either identically 0 or never zero on $\Omega$.*

*Proof.* Suppose $f$ is not identically zero. If $f(z_0) = 0$, then there exists $\delta > 0$ such that $f(z) \neq 0$ on $D_\delta(z_0) - \{z_0\} \subset \Omega$. Then $f_n \to f$ uniformly on $\partial D_\delta(z_0)$ and also $f_n' \to f'$ uniformly on $\partial D_\delta(z_0)$. Hence we have

$$\lim_{n\to\infty} \frac{1}{2\pi i}\int_{\partial D_\delta(z_0)} \frac{f_n'(z)}{f_n(z)}dz = \frac{1}{2\pi i}\int_{\partial D_\delta(z_0)}\frac{f'(z)}{f(z)}dz.$$

For sufficiently large $n$, $f_n(z) \neq 0$ on $\partial D$ since $f \neq 0$ on $\partial D$; therefore the left-hand side makes sense. Now, the LHS is zero, but the RHS is nonzero, a contradiction. □

20.2. **Taylor series.**

**Theorem 20.4.** *Let $f$ be an analytic function on $\Omega$ and $a \in \Omega$. Then*

$$f(z) = f(a) + f'(a)(z-a) + \cdots + \frac{f^{(n)}(a)}{n!}(z-a)^n + \dots.$$

*The power series converges in the largest open disk $D \subset \Omega$ centered at $a$.*

*Proof.* Recall that $f(z)$ has a Taylor expansion:

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(z-a)^{n-1} + (z-a)^n f_n(z),$$

with remainder term:

$$f_n(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(\zeta)d\zeta}{(\zeta-a)^n(\zeta-z)}.$$

As before, we estimate

$$|(z-a)^n f_n(z)| \leq \left(\frac{|z-a|}{R}\right)^n \frac{MR}{R-|z-a|},$$

where $R$ is the radius of $D$. On a slightly smaller disk $D_\rho(a) = \{|z-a| \leq \rho\}$, where $\rho < R$, the remainder term $|(z-a)^n f_n(z)| \to 0$ uniformly. This proves the theorem. □

One often refers to this theorem as "a holomorphic function (one that has a complex derivative) is analytic (can be written as a power series)."

## 21. Plane topology

### 21.1. **Chains and cycles.** Let $\Omega \subset \mathbf{C}$ be an open region.

**Definition 21.1.** *A* chain *is a formal sum $\gamma = a_1\gamma_1 + \dots + a_n\gamma_n$, where $a_i \in \mathbf{Z}$ and $\gamma_i : [\alpha_i, \beta_i] \to \Omega$ is piecewise differentiable. In other words, a chain is an element of the free abelian group generated by piecewise differentiable arcs in $\Omega$.*

Write $C_0(\Omega)$ for the free abelian group generated by points in $\Omega$, and $C_1(\Omega)$ for the free abelian group generated by piecewise differentiable arcs in $\Omega$.

Consider the map $\partial : C_1(\Omega) \to C_0(\Omega)$, which maps an arc $\gamma : [a, b] \to \Omega$ to $\gamma(b) - \gamma(a)$. It is extended linearly to elements of $C_1(\Omega)$, i.e., formal linear combinations of arcs. Elements of $\ker \partial$ are called *cycles* of $\Omega$.

Let $\omega$ be a differential $pdx+qdy$ or $f(z)dz$. Consider the integration map $C_1(\Omega) \xrightarrow{\int \omega} \mathbf{C}$, which sends an arc $\gamma$ to $\int_\gamma \omega$, and is extended linearly to $C_1(\Omega)$. We observe the following:

1. If $\gamma : [a, b] \to \Omega$ and $\gamma_1 = \gamma|_{[a,c]}$, $\gamma_2 = \gamma|_{[c,b]}$, where $a < c < b$, then $\int_\gamma \omega = \int_{\gamma_1} \omega + \int_{\gamma_2} \omega$.
2. If $\gamma'$ is an orientation-preserving reparametrization of $\gamma$, then $\int_\gamma \omega = \int_{\gamma'} \omega$.
3. If $-\gamma$ is the opposite arc of $\gamma$, then $\int_{-\gamma} \omega = -\int_\gamma \omega$.

Therefore, as far as integration is concerned, we may take equivalence relations $\gamma_1 + \gamma_2 \sim \gamma$, $\gamma \sim \gamma'$, and $(-\gamma) \sim -1(\gamma)$.

**Remark:** Given a cycle $\gamma = \sum a_i\gamma_i$, it is equivalent to a sum $\gamma' = \gamma_1' + \dots + \gamma_k'$, where each $\gamma_i'$ is a closed curve. First, by reversing orientations if necessary, we can write $\gamma \sim \sum \gamma_i$, where $\gamma_i$ may not be closed. Since $\partial\gamma = \sum \partial\gamma_i = 0$, given some $\gamma_i : [\alpha_i, \beta_i] \to \Omega$, there exists $\gamma_j : [\alpha_j, \beta_j] \to \Omega$ such that $\gamma_i(\beta_i) = \gamma_j(\alpha_j)$. We can then concatenate $\gamma_i$ and $\gamma_j$ to obtain a shorter chain. Repeat until all the $\gamma_i$ are closed curves.

### 21.2. **Simple connectivity.** We give three alternate definitions for a connected open set $\Omega \subset \mathbf{C}$ to be *simply connected*, and prove their equivalence.

**Definition 21.2.** *A connected open region $\Omega \subset \mathbf{C}$ is* simply connected *if one of the following equivalent conditions holds:*

1. *$\overline{\mathbf{C}} - \Omega$ is connected.*
2. *$n(\gamma, a) = 0$ for all cycles $\gamma$ in $\Omega$ and $a \notin \Omega$.*
3. *Every continuous closed curve $\gamma$ in $\Omega$ is contractible.*

Definition (3) is the usual definition of simple connectivity.

Roughly speaking, a simply connected region is a region without holes. Examples of simply connected $\Omega$ are $\mathbf{C}$, the open unit disk, the upper half plane, and $\{0 < y < 2\pi\}$.

Two continuous closed curves $\gamma_0, \gamma_1 : [a, b] \to \Omega$ are *homotopic* in $\Omega$ if there is a continuous function $\Gamma : [a, b] \times [0, 1] \to \Omega$ such that $\Gamma(s, 0) = \gamma_0(s)$ and $\Gamma(s, 1) = \gamma_1(s)$. $\Gamma$ is a *homotopy* from $\gamma_0$ to $\gamma_1$. A continuous closed curve $\gamma : [a,b] \to \Omega$ is *contractible* if $\gamma$ is homotopic to a closed curve that maps to a point in $\Omega$.

(1)$\Rightarrow$(2). Suppose $\overline{\mathbf{C}} - \Omega$ is connected. $\overline{\mathbf{C}} - \text{Im } \gamma$ is open and $n(\gamma, a)$ is constant for all $a$ in a given component of $\overline{\mathbf{C}} - \text{Im } \gamma$. Since $\overline{\mathbf{C}} - \Omega$ is connected, the component of $\overline{\mathbf{C}} - \text{Im } \gamma$ containing $\infty$ must also contain $\overline{\mathbf{C}} - \Omega$. Now $n(\gamma, a) = 0$ for all $a$ in the component of $\overline{C} - \text{Im } \gamma$ containing $\infty$.

(2)$\Rightarrow$(1). Suppose $\overline{\mathbf{C}} - \Omega$ is not connected. We can write $\overline{\mathbf{C}} - \Omega = A \sqcup B$, where $A$ and $B$ are open sets (and hence closed sets), and $\infty \in B$. $A$ and $B$ are both closed and bounded in $\overline{\mathbf{C}}$, hence compact. Hence $d(A, B) = \min_{x \in A, y \in B} d(x, y)$ exists and is positive.

**HW 19.** *Given compact subsets $A, B$ of a complete metric space $X$, prove that $d(A, B) = \min_{x \in A, y \in B} d(x, y)$ exists and is positive if $A$ and $B$ are disjoint.*

Now cover $\mathbf{C}$ with a net of squares whose side is $< \frac{d(A,B)}{\sqrt{2}}$. Let $\{Q_i\}_{i=1}^{\infty}$ be the set of (closed) squares. Observe that each $Q_i$ intersects at most one of $A$ or $B$. Then take $\gamma = \sum_{Q_i \cap A \neq \emptyset} \partial Q_i$.

We claim that if we simplify $\gamma$ by canceling pairs of edges, then $\gamma$ does not meet $A$ and $B$, hence $\gamma \subset \Omega$. Not meeting $B$ is immediate, since $Q_i \cap A \neq \emptyset$ implies $Q_i \cap B = \emptyset$. If $a \in A$ is contained in an edge of $\gamma$, then $a$ is in adjacent $Q_i$ and $Q_j$ whose common edge is subjected to a cancellation, a contradiction. [A similar argument holds in case $a$ is a corner of $Q_i$.]

Now, $n(\partial Q_i, a) = 1$ and $n(\partial Q_j, a) = 1$ if $i \neq j$, for $a \in int(Q_i)$. Hence $n(\gamma, a) = 1$ for all $a \in int(\cup Q_i)$, where the union is over all $Q_i \cap A \neq \emptyset$. We could have taken the net so that $a \in A$ is at the center of one of the squares, so we have obtained $\gamma$ on $\Omega$ and $a \in A$ so that $n(\gamma, a) = 1$, a contradiction.

## 22. The general form of Cauchy's theorem

22.1. **Simple connectivity, cont'd.** We continue the proof of the equivalence of the three definitions of simple connectivity.

(3)⇒(2). We first define $n(\gamma, a)$, $a \in \text{Im } \gamma$, for a continuous closed curve $\gamma$ as follows: Divide $\gamma$ into subarcs $\gamma_1, \dots, \gamma_n$ so that Im $\gamma_i$ is contained in a disk $\subset \Omega$ which does not intersect $a$. (This is possible by compactness.) Replace $\gamma_i$ by a line segment $\sigma_i$ with the same endpoints as $\gamma_i$. Then $\sigma = \sigma_1 + \dots + \sigma_n$, and define $n(\gamma, a) = n(\sigma, a)$.

**HW 20.** *Prove that this definition of $n(\gamma, a)$ does not depend on the choice of $\sigma_j$, and agrees with the standard definition of $n(\gamma, a)$ for piecewise smooth closed curves.*

**HW 21.** *Prove that if two closed curves $\gamma_0$ and $\gamma_1$ are homotopic through a homotopy that does not intersect $a$, then $n(\gamma_0, a) = n(\gamma_1, a)$.*

Given a piecewise differentiable closed curve $\gamma$, $n(\gamma, a) = n(pt, a) = 0$, by the above HW, if $a \notin \Omega$. Hence $n(\gamma, a) = 0$ for all $\gamma$ on $\Omega$ and $a \notin \Omega$.

(1),(2)⇒(3). We will cheat slightly and appeal to the Riemann Mapping Theorem, i.e., a simply connected, connected region $\Omega \subsetneq \mathbf{C}$ is biholomorphic to the open disk. (We will prove the Riemann Mapping Theorem later, using only definitions (1) and (2) of simple connectivity.) Any closed curve $\gamma$ on $D$ can easily be contracted to a point and any closed curve $\gamma$ on $\mathbf{C}$ can also be contracted to a point.

22.2. **General form of Cauchy's Theorem.** A cycle $\gamma$ on $\Omega$ is *homologous to zero* (or *nullhomologous*) in $\Omega$ if $n(\gamma, a) = 0$ for all $a \in \overline{\mathbf{C}} - \Omega$. Two cycles $\gamma_1$ and $\gamma_2$ are *homologous* if $\gamma_1 - \gamma_2$ is nullhomologous. We will write $[\gamma]$ to denote the equivalence class of cycles homologous to $\gamma$. In particular, $[\gamma] = 0$ means $\gamma$ is nullhomologous.

**Remark:** This is not the usual definition of a nullhomologous $\gamma$.

**Theorem 22.1.** *If $f(z)$ is analytic on $\Omega$, then $\int_\gamma f(z)dz = 0$ for all cycles $\gamma$ which are nullhomologous in $\Omega$.*

**Corollary 22.2.** *If $f(z)$ is analytic on a simply connected $\Omega$, then $\int_\gamma f(z)dz = 0$ for all cycles $\gamma$ on $\Omega$.*

*Proof of Corollary.* If $\Omega$ is simply connected, then any cycle $\gamma$ on $\Omega$ is nullhomologous by condition (2). □

Suppose $\Omega$ is simply connected and $f$ is analytic on $\Omega$. Since $\int f(z)dz$ is independent of the choice of path, $f(z)$ is the complex derivative of an analytic function $F(z)$ which is defined on all of $\Omega$.

**Corollary 22.3.** *If $f(z)$ is analytic and nowhere zero on a simply connected $\Omega$, then $\log f$ and $\sqrt[n]{f}$ admit single-valued branches on $\Omega$.*

*Proof.* Let $F(z)$ be the analytic function on $\Omega$ whose derivative is $\frac{f'(z)}{f(z)}$. This is basically $\log f(z)$, but we need to check that it satisfies $e^{F(z)} = f(z)$. First we verify that $f(z)e^{-F(z)}$ has derivative 0, hence $f(z)e^{-F(z)}$ is constant. Choosing $z_0 \in \Omega$, we set $f(z)e^{-F(z)} = f(z_0)e^{-F(z_0)}$, and hence

$$e^{F(z)-F(z_0)-\log f(z_0)} = f(z).$$

We take $F(z) - F(z_0) - \log f(z_0)$ to be $\log f$. We also take $\sqrt[n]{f} = e^{\frac{1}{n}\log f}$. $\square$

**Remark:** This corollary will become crucial in the proof of the Riemann Mapping Theorem.

We will prove the following more general result:

**Theorem 22.4.** *Let $\omega = pdx + qdy$ be a locally exact differential on $\Omega$, i.e., it is exact in some neighborhood of each point of $\Omega$. Then $\int_\gamma \omega = 0$ for all nullhomologous cycles $\gamma$ in $\Omega$.*

*Proof.*

**Step 1:** Let $[\gamma] = 0$. We may assume that the cycle $\gamma$ is a finite sum of closed curves by the Remark from last time. We now replace $\gamma$ by a piecewise linear one $\sigma$ with each piece a horizontal or vertical line segment: Subdivide $\gamma$ (which we assume WLOG to be a single closed curve) into subarcs $\gamma_i$ as above, so that each Im $\gamma_i$ lies in a small disk $\subset \Omega$. Replace $\gamma_i$ by a horizontal arc, followed by a vertical one, which we call $\sigma_i$. $\gamma_i$ and $\sigma_i$ have the same endpoints. Since $\omega$ is exact on this disk (if sufficiently small), $\int_{\gamma_i} \omega = \int_{\sigma_i} \omega$. Also one can easily verify that $[\gamma] = [\sigma]$.

**Step 2:** Extend the horizontal and vertical line segments of $\sigma$ to lines in $\mathbf{C}$. The lines cut up $\mathbf{C}$ into rectangles $R_i$ and unbounded regions. Pick $a_i \in int(R_i)$ and form

$$\sigma_0 = \sum n(\sigma, a_i)\partial R_i.$$

We claim that $\sigma$ and $\sigma_0$ become equivalent after cancelling pairs of edges. Indeed, if $a \in int(R_i)$, then $n(\sigma_0, a) = \sum n(\sigma, a_i)n(\partial R_i, a) = n(\sigma, a_i) = n(\sigma, a)$. Moreover $n(\sigma_0, a) = n(\sigma, a) = 0$ for $a$ in the unbounded regions. Suppose the reduced expression of $\sigma - \sigma_0$ contains the multiple $c\sigma_{ij}$, where $\sigma_{ij}$ is a common side of rectangles $R_i$ and $R_j$. Then $n(\sigma - \sigma_0 - c\partial R_i, a_i) = n(\sigma - \sigma_0 - c\partial R_i, a_j)$, but then the LHS is $-c$ and the RHS is 0, hence implying $c = 0$. Now $\sigma \sim \sum n(\sigma, a_i)\partial R_i$ as far as integration is concerned.

**Step 3:** If $n(\sigma, a_i) \neq 0$, then we claim that $R_i \subset \Omega$. First, $int(R_i) \subset \Omega$, since $a \in int(R_i) \Rightarrow n(\sigma, a) = n(\sigma, a_i) \neq 0$. Next, if $a$ lies on an edge of $R_i$, then either that edge is in Im $\sigma$ (and hence in $\Omega$), or else the edge is not contained in Im $\sigma$, in which case $n(\sigma, a) \neq 0$ and $a \in \Omega$. Hence, if $n(\sigma, a_i) \neq 0$, then $\int_{\partial R_i} \omega = 0$. This proves the theorem. $\square$

## 23. Multiply connected regions and residues

23.1. **Multiply connected regions.** A (connected) region $\Omega$ which is not simply connected is said to be *multiply connected*. $\Omega$ has *connectivity* $n$ if $\overline{\mathbf{C}} - \Omega$ has $n$ connected components and *infinite connectivity* if $\overline{\mathbf{C}} - \Omega$ has infinitely many connected components.

We consider the case of finite connectivity. Then there are $n$ connected components of $\overline{\mathbf{C}} - \Omega$, which we write as $A_1, \dots, A_n$, with $\infty \in A_n$.

**Lemma 23.1.** *There exist closed curves $\gamma_1, \dots, \gamma_{n-1}$ such that $n(\gamma_i, a_j) = \delta_{ij}$, if $a_j \in A_j$.*

*Proof.* Use the proof of (2)⇒(1) from Day 21. In other words, take a sufficiently fine net of squares $Q_k$ that covers $A_i$ (and $A_i$ only), and let $\gamma_i = \sum_{Q_k \cap A_i \neq \emptyset} \partial Q_k$. □

**Lemma 23.2.** *Every cycle $\gamma$ on $\Omega$ is homologous to a (unique) linear combination $c_1\gamma_1 + \dots + c_{n-1}\gamma_{n-1}$.*

*Proof.* If $c_i = n(\gamma, a_i)$ for $a_i \in A_i$, then $n(\gamma - \sum c_i\gamma_i, a_j) = 0$ for all $a_j \in A_j$, $i = 1, \dots, n-1$. $A_n$ contains $\infty$, so $n(\gamma - \sum c_i\gamma_i, a_n) = 0$ as well. Hence $[\gamma] = [\sum c_i\gamma_i]$.

Next to prove uniqueness, if $[\sum a_i\gamma_i] = [\sum b_i\gamma_i]$, then $[\sum(a_i - b_i)\gamma_i] = 0$, and evaluation on the $\gamma_j$ gives $a_i = b_i$. □

We say that $\gamma_1, \dots, \gamma_{n-1}$ form a *homology basis* for $\Omega$.

By Cauchy's Theorem, we have

$$\int_\gamma f(z)dz = c_1 \int_{\gamma_1} f(z)dz + \dots + c_{n-1} \int_{\gamma_{n-1}} f(z)dz.$$

We call $\int_{\gamma_i} f(z)dz$ the *periods* of the differential $fdz$.

**Example:** Let $\Omega$ be the annulus $\{r_1 < |z| < r_2\}$. $\Omega$ has connectivity 2, and has a homology basis consisting of one element, the circle $C$ of radius $r$, $r_1 < r < r_2$. Then any cycle $\gamma$ is homologous to $nC$ for some $n \in \mathbf{Z}$, and $\int_\gamma f(z)dz = n \int_C f(z)dz$.

23.2. **Residues.** Let $f(z)$ be an analytic function on the region $\Omega$, with the exception of isolated singularities. Suppose for the moment that there are finitely many isolated singularities $a_1, \dots, a_n \in \Omega$. We assume that they are either poles or essential singularities.

Let $C_j \subset \Omega$ be a small circle about $a_j$ which contains no other singularities in the interior. If we let $R_j = \frac{1}{2\pi i}\int_{C_j} f(z)dz$, then $f(z) - \frac{R_j}{z-a_j}$ has vanishing period about $a_j$. $R_j$ is said to be the *residue* of $f(z)$ at $z = a_j$, and is written $Res_{z=a_j} f(z)$.

**Theorem 23.3** (Residue Formula)**.** *If $f(z)$ is analytic on $\Omega$ with the exception of isolated singularities, then*

$$\frac{1}{2\pi i}\int_\gamma f(z)dz = \sum_j n(\gamma, a_j) Res_{z=a_j} f(z)$$

*for any cycle $\gamma$ on $\Omega$ which is nullhomologous on $\Omega$ and avoids the singular points.*

*Proof.* Since $[\gamma] = 0$ on $\Omega$, $n(\gamma, a) = 0$ for all $a \notin \Omega$. Hence $n(\gamma - \sum_j n(\gamma, a_j)C_j, a) = 0$ for all $a \notin \Omega' = \Omega - \{a_1, \dots, a_n\}$. In other words, $[\gamma - \sum_j n(\gamma, a_j)C_j] = 0$ in $\Omega'$. Hence we have:

$$\frac{1}{2\pi i}\int_\gamma f(z)dz = \sum_j n(\gamma, a_j) Res_{z=a_j} f(z).$$

Observe that there are only finitely many isolated singularities in the bounded connected components of $\mathbf{C} - \text{Im } \gamma$. □

23.3. **Computations of residues.** In principle, the isolated singularities can be essential singularities, but in practice we will only treat poles.

**Lemma 23.4.** *If a meromorphic function $f$ is written as $\frac{B_k}{(z-a)^k} + \dots + \frac{B_1}{z-a} + g(z)$ near $z = a$, where $g(z)$ is analytic, then $Res_{z=a} f(z) = B_1$.*

*Proof.* Observe that all the other terms besides $\frac{B_1}{z-a}$ admit antiderivatives. ($g(z)$ by Cauchy's theorem on the disk, and the others explicitly.) Hence $\frac{1}{2\pi i}\int_C f(z)dz$ only leaves $\frac{1}{2\pi i}\int_C \frac{B_1 dz}{z-a}$, where $C$ is a small circle about $z = a$. □

In particular, if $f$ has a simple pole at $z = a$, then $f(z) = \frac{B_1}{z-a} + g(z)$ and $(z-a)f(z) = B_1 + (z-a)g(z)$. Evaluating $(z-a)f(z)$ at $z = a$ gives the residue $B_1$.

**Example:** $f(z) = \frac{\cos z}{(z-1)(z-2)}$. Since $z = 1, 2$ are not zeros of $\cos z$, they are simple zeros of $f(z)$. $Res_{z=1} f(z) = -\cos 1$ and $Res_{z=2} f(z) = \cos 2$. If $\gamma$ is a closed curve which winds once around $z = 1$ and $z = 2$ (for example, if $\gamma$ is the circle $|z| = 3$, oriented counterclockwise), then $\int_\gamma f(z)dz = n(\gamma, 1)Res_{z=1} f(z) + n(\gamma, 2)Res_{z=2} f(z) = -\cos 1 + \cos 2$. If $\gamma$ winds around $z = 1$ only, say $\gamma$ is the circle $|z| = \frac{3}{2}$, oriented counterclockwise, then $\int_\gamma f(z)dz = -\cos 1$.

**Example:** $f(z) = \frac{\sin z}{z^4}$. The only possible pole is $z = 0$. Expand $\sin z$ using Taylor's theorem: $\sin z = z - \frac{z^3}{3!} + z^5 g(z)$. Then $f(z) = \frac{1}{z^3} - \frac{1}{6}\frac{1}{z} + zg(z)$, and $Res_{z=0} f(z) = -\frac{1}{6}$. Similarly, $f(z) = \frac{\sin z}{z^5}$ has $Res_{z=0} f(z) = 0$.

## 24. Residues

24.1. **Application: The argument principle.** Suppose $f$ is meromorphic on $\Omega$, with zeros $a_i$ and poles $b_i$. We will compute $n(f\circ\gamma,0)=\int_\gamma \frac{f'(z)}{f(z)}dz$, using the Residue Formula from last time. (This is a generalization of Theorem 18.1 from Day 18.)

**Theorem 24.1.** *$n(f\circ\gamma,0)=\sum_j n(\gamma,a_j)-\sum_j n(\gamma,b_j)$, where $a_j$ and $b_j$ are repeated as many times as their orders.*

*Proof.* Near a zero $z=a$ of order $h$, we have $f(z)=(z-a)^h g(z)$, where $g(z)$ is a nonzero analytic function. Therefore,

$$\frac{f'(z)}{f(z)}=\frac{h}{z-a}+\frac{g'(z)}{g(z)},$$

and $Res_{z=a}\frac{f'(z)}{f(z)}=h$, which is the order of the zero. Similarly, near a pole $z=b$ of order $k$, $f(z)=(z-a)^{-k}g(z)$, and $Res_{z=b}\frac{f'(z)}{f(z)}=-k$, which is minus the order of the pole. □

A corollary of the Argument Principle is the following theorem:

**Theorem 24.2** (Rouché's Theorem). *Suppose $[\gamma]=0$ on $\Omega$ and $n(\gamma,z)=0$ or $1$ for all $z\in\Omega-Im\ \gamma$. Also suppose that $f(z)$, $g(z)$ are analytic on $\Omega$ and satisfy $|f(z)-g(z)|<|f(z)|$ on $\gamma$. Then $f(z)$ and $g(z)$ have the same number of zeros enclosed by $\gamma$.*

*Proof.* Take $\frac{g(z)}{f(z)}$. Then on $\gamma$ we have $|\frac{g(z)}{f(z)}-1|<1$. Therefore, Im $\frac{g}{f}\circ\gamma$ is contained in the disk of radius 1 about $z=1$, and hence $n(\frac{g}{f}\circ\gamma,0)=0$. This implies that $\sum_j n(\gamma,a_j)=\sum_k n(\gamma,b_k)$, where $a_j$ are the zeros of $\frac{g}{f}$ and $b_k$ are the poles of $\frac{g}{f}$. Upon some thought, one concludes that $f$ and $g$ must have the same number of zeros enclosed by $\gamma$. [Here "$z$ is enclosed by $\gamma$" means $n(\gamma,z)\neq 0$.] □

**Example:** Consider $g(z)=z^8-5z^3+z-2$. Find the number of roots of $g(z)$ inside the unit disk $|z|\leq 1$.

Let $\gamma$ be $|z|=1$, oriented counterclockwise. Also let $f(z)=-5z^3$. Then on $\gamma$ we have $|f(z)-g(z)|=|z^8+z-2|\leq|z^3|+|z|+|2|\leq 4$, whereas $|g(z)|=5$. Hence $|f(z)-g(z)|<|g(z)|$, and the number of zeros of $g$ in $|z|<1$ is equal to the number of zeros of $f$ in $|z|<1$, which in turn is 3 (after counting multiplicities).

### 24.2. Evaluation of definite integrals.

**A.** $\int_0^{2\pi}R(\cos\theta,\sin\theta)d\theta$, where $R$ is a rational function of two variables.

**Example:** $\int_0^{2\pi}\frac{1}{a+\sin\theta}d\theta$, where $a$ is real and $>1$. We change coordinates $z=e^{i\theta}$ and integrate over the closed curve $\gamma=\{|z|=1\}$, oriented counterclockwise. Then $dz=ie^{i\theta}d\theta$, and $d\theta=-i\frac{dz}{z}$. Also we have $\frac{1}{2}(z+\frac{1}{z})=\frac{1}{2}(e^{i\theta}+e^{-i\theta})=\cos\theta$ and $\frac{1}{2i}(z-\frac{1}{z})=\sin\theta$.

We substitute

$$\int_0^{2\pi} \frac{d\theta}{a+\sin\theta} = 2\int_\gamma \frac{dz}{z^2+2iaz-1}.$$

If we write $z^2+2iaz-1 = (z-\alpha)(z-\beta)$, then $\alpha = -i(a-\sqrt{a^2-1})$ and $\beta = -i(a+\sqrt{a^2-1})$. Observe that $\alpha$ is in the unit disk, while $\beta$ is not. The residue of $\frac{1}{(z-\alpha)(z-\beta)}$ at $z=\alpha$ is $\frac{1}{\alpha-\beta}$, and

$$2\int_\gamma \frac{dz}{z^2+2iaz-1} = 2(2\pi i) Res_{z=\alpha}\frac{1}{(z-\alpha)(z-\beta)} = \frac{2\pi}{\sqrt{a^2-1}}.$$

**B.** $\int_{-\infty}^{\infty} f(x)dx$, where $f(z)$ is meromorphic, has a finite number of poles, has no poles on $\mathbf{R}$, and satisfies $|f(z)| \le \frac{B}{|z|^2}$ for $|z| >> 0$. If $f$ is a rational function, then the last condition means that the degree of the denominator is at least two larger than the degree of the numerator.

**Example:** $\int_{-\infty}^{\infty} \frac{1}{x^4+1}dx$. First consider the integral of $\frac{1}{z^4+1}$ over the closed curve $\gamma$ consisting of an arc $C_1$ from $-R$ to $R$ on $\mathbf{R}$, followed by a counterclockwise semicircle $C_2 = \{|z| = R, \mathrm{Im}\ z \ge 0\}$.

We first observe that

$$\left|\int_{C_2} \frac{1}{z^4+1}dz\right| \le \int_{C_2} \frac{1}{|z|^4}|dz| \le \frac{\pi R}{R^4} = \frac{\pi}{R^3} \to 0$$

as $R \to \infty$.

Hence,

$$\int_{-\infty}^{\infty} \frac{1}{x^4+1}dz = \int_\gamma \frac{dz}{z^4+1}.$$

The fourth roots of $-1$ are $\pm e^{\pi i/4}$, $\pm e^{3\pi i/4}$ (and are simple, in particular). Only two of them – $e^{\pi i/4}$ and $e^{3\pi i/4}$ – are contained in the region bounded by $\gamma$. Therefore,

$$\int_\gamma \frac{dz}{z^4+1} = 2\pi i(Res_{z=e^{\pi i/4}} + Res_{z=e^{3\pi i/4}}).$$

A convenient way of computing the residues is:

**HW 22.** *Prove that the residue of* $\frac{1}{f(z)}$ *at* $z=a$ *is* $\frac{1}{f'(a)}$ *if* $a$ *is a simple pole.*

Hence

$$\int_\gamma \frac{dz}{z^4+1} = 2\pi i\left(\frac{1}{4e^{3\pi i/4}} + \frac{1}{4e^{9\pi i/4}}\right).$$

[Of course, this can be further simplified to give a real expression....]

# 25. Day 25

## 25.1. **Evaluation of definite integrals, Day 2.**

**C.** (Fourier Transforms) Integrals of the form $\int_{-\infty}^{\infty} f(x)e^{iax}dx$, where $f(z)$ is meromorphic, has a finite number of poles (none of them on the $x$-axis), and $|f(z)| \leq \frac{B}{|z|}$ for $|z| >> 0$. [Observe that we only need $f(z)$ to have a zero of order at least 1 at $\infty$. Compare with B, where $f(z)$ was required to have a zero of order at least 2 at $\infty$.]

**Example:** $\int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+1}dx$, $a > 0$. Integrate on the contour $\gamma$ which is the counterclockwise boundary of the rectangle with vertices $-X_1, X_2, X_2+iY, -X_1+iY$, where $X_1, X_2, Y >> 0$. Call the edges of the rectangle $C_1, C_2, C_3, C_4$ in counterclockwise order, where $C_1$ is the edge on the $x$-axis from $-X_1$ to $X_2$. Using the bound $|\frac{1}{z^2+1}| \leq \frac{B}{|z|}$, we obtain:

$$\left|\int_{C_2} \frac{e^{iaz}}{z^2+1}dz\right| \leq \frac{B}{X_2}\int_0^Y e^{-ay}dy = \frac{B}{X_2}\left(\frac{1}{a}\right)(1-e^{-aY}),$$

$$\left|\int_{C_4} \frac{e^{iaz}}{z^2+1}dz\right| \leq \frac{B}{X_1}\left(\frac{1}{a}\right)(1-e^{-aY}),$$

$$\left|\int_{C_3} \frac{e^{iaz}}{z^2+1}dz\right| \leq \frac{B}{Y}\int_{-X_1}^{X_2} e^{-aY}|dx| = \frac{B}{Y}e^{-aY}(X_1+X_2).$$

If we take $X_1, X_2$ to be large, and then let $Y \to \infty$, then all three integrals go to zero. Hence,

$$\int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+1}dx = \int_\gamma \frac{e^{iaz}}{z^2+1}dz = 2\pi i \ Res_{z=i}\left(\frac{e^{iaz}}{z^2+1}\right) = 2\pi i \left.\frac{e^{iaz}}{z+i}\right|_{z=i} = \pi e^{-a}.$$

**Example:** $\int_{-\infty}^{\infty} \frac{\sin x}{x}dx$. In this problem, we want to compute $\int_{-\infty}^{\infty} \frac{e^{ix}}{x}dx$, which has a pole at $x = 0$. Instead, we take the Cauchy principal value $p.v. \int_{-\infty}^{\infty} \stackrel{def}{=} \lim_{\varepsilon\to 0}(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty})$. Strangely enough,

$$p.v. \int_{-\infty}^{\infty} \frac{e^{ix}}{x}dx = \left(p.v. \int_{-\infty}^{\infty} \frac{\cos x}{x}dx\right) + i\left(\int_{-\infty}^{\infty} \frac{\sin x}{x}dx\right),$$

since $\frac{\sin x}{x}$ is actually continuous at $x = 0$ and there's no need for p.v.'s.

Take $\gamma$ to be the boundary of $R \cup D$, where $R$ is the rectangle as in the previous example, and $D$ is the disk of radius $\varepsilon$ about $z = 0$. There is one pole inside $R \cup D$, which gives $2\pi i \ Res_{z=0}\frac{e^{iz}}{z} = 2\pi i$. Now,

$$2\pi i = \int_\gamma \frac{e^{iz}}{z}dz = p.v. \int_{-\infty}^{\infty} \frac{e^{iz}}{z}dz + \lim_{\varepsilon\to 0}\int_C \frac{e^{iz}}{z}dz,$$

where $C$ is the lower semicircle of $\partial D$. Writing $e^{iz} = \frac{1}{z} + g(z)$, where $g(z)$ is analytic (and hence continuous) near $z = 0$, we find that $\int_C \frac{e^{iz}}{z}dz \to \int_C \frac{1}{z}dz = \pi i$ as $\varepsilon \to 0$. We therefore obtain

$$p.v. \int_{-\infty}^{\infty} \frac{e^{iz}}{z} dz = \pi i, \ \int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi.$$

**D.** (Mellin Transforms) Integrals of the form $\int_0^\infty f(x)x^\alpha dx$, where $0 < \alpha < 1$. Here $f(z)$ is meromorphic, has a finite of poles (none of them on the $x$-axis), and $|f(z)| \leq \frac{B}{|z|^2}$ for $|z| >> 0$ and $|f(z)| \leq \frac{B}{|z|}$ for $|z|$ near 0.

We take the contour $\gamma = C_1 + C_2 + C_3 + C_4$, where $C_1 = \{|z| = R_1\}$, oriented clockwise, $C_3 = \{|z| = R_2\}$, oriented counterclockwise, $C_2$ is the line segment from $z = R_1$ to $z = R_2$, and $C_4$ is the line segment from $z = R_2$ to $z = R_1$. Here $R_1 < R_2$. [More rigorously, we're interested in the boundary of the region $\{R_1 < r < R_2, \varepsilon < \theta < 2\pi - \varepsilon\}$ (in polar coordinates) and we're letting $\varepsilon \to 0$.]

Then

$$\left|\int_{C_1} f(z)z^\alpha dz\right| \leq \frac{B}{R_1^{1-\alpha}} 2\pi R_1 = 2\pi B R_1^\alpha \to 0, \text{ as } R_1 \to 0;$$
$$\left|\int_{C_3} f(z)z^\alpha dz\right| \leq \frac{B}{R_2^{2-\alpha}} 2\pi R_2 = \frac{2\pi B}{R_2^{1-\alpha}} \to 0, \text{ as } R_2 \to \infty.$$

On the other hand, since $z^\alpha = e^{\alpha \log z}$, the values of $\log z$ on $C_2$ and $C_4$ differ by $2\pi i$ and those of $z^\alpha$ differ by $e^{2\pi i \alpha}$. Hence,

$$\int_{C_4} f(z)z^\alpha dz = -\int_{C_2} f(z)e^{2\pi i\alpha} z^\alpha dz,$$
$$(1 - e^{2\pi i \alpha}) \int_{C_2} f(z)z^\alpha dz = 2\pi i \sum Res_{z=a_j}(f(z)z^\alpha),$$
$$\int_0^\infty f(x)x^\alpha dx = 2\pi i \sum Res_{z=a_j}(f(z)z^\alpha),$$

where the sum is over all residues in the plane.

This technique can be used to compute integrals such as $\int_0^\infty \frac{x^\alpha}{x^2+1} dx$.

25.2. **Harmonic functions.** A function $u : \Omega \to \mathbf{R}$ is *harmonic* if $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. For the time being, assume $u$ has continuous partials up to second order.

**Recall:** If $f = u + iv$ and $f$ is analytic, then $u$ and $v$ are harmonic.

**Example:** The simplest harmonic functions are $u(x, y) = ax + by$.

**Example:** If $f(z) = \log z$, then $f(z) = \log r + i\theta$, where $z = re^{i\theta}$. (Suppose we are restricting attention to a domain $\Omega$ where $\theta$ is single-valued and continuous.) $u(x, y) = \log r = \log \sqrt{x^2 + y^2}$ and $v(x, y) = \theta$ are harmonic.

**Question:** Given a harmonic form $u$ on $\Omega$, is there a *harmonic conjugate* $v : \Omega \to \mathbf{R}$, such that $f = u + iv$ is analytic on $\Omega$?

Let $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$. Then we define

$$*du = \frac{\partial u}{\partial x}dy - \frac{\partial u}{\partial y}dx.$$

($*$ is called the *Hodge $*$-operator*.)

**Lemma 25.1.** *If $u$ is harmonic, then $g(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$ is analytic.*

This $g(z)$ is the (complex) derivative of the function $f(z)$ we are looking for.

*Proof.* We compute that $g(z)$ satisfies the Cauchy-Riemann equations $\frac{\partial g}{\partial x} = -i\frac{\partial g}{\partial y}$:

$$\frac{\partial g}{\partial x} = \frac{\partial^2 u}{\partial x^2} - i\frac{\partial^2 u}{\partial x \partial y},$$

$$\frac{\partial g}{\partial y} = \frac{\partial^2 u}{\partial x \partial y} + i\frac{\partial^2 u}{\partial x^2}.$$

$\square$

## 26. More on harmonic functions

26.1. **Existence of harmonic conjugate.** Let $u : \Omega \to \mathbf{R}$ be a harmonic function. We are looking for its harmonic conjugate.

**Lemma 26.1.** *$\int_\gamma *du = 0$ if $[\gamma] = 0$ (i.e., $\gamma$ is nullhomologous).*

*Proof.* Define $g(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$. Since

$$g(z)dz = \left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right)(dx + idy) = du + i * du,$$

we have

$$\int_\gamma g(z)dz = \int_\gamma du + i\int_\gamma *du.$$

The integral over $g(z)dz$ is zero by Cauchy's Theorem, if $[\gamma] = 0$. The integral over $du$ is always zero, since $du$ is exact. Hence, $\int_\gamma *du = 0$ whenever $[\gamma] = 0$. □

**Remark:** This also follows from remarking that $\omega = *du$ is a *closed differential 1-form.* A closed 1-form $\omega = pdx + qdy$ satisfies $d\omega = (\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y})dxdy = 0$. It is known (Poincaré Lemma) that closed forms are *locally exact.*

**Theorem 26.2.** *There is a single-valued harmonic conjugate $v : \Omega \to \mathbf{R}$ if $\Omega$ is simply connected. Moreover, $v$ is uniquely determined up to a constant.*

*Proof.* By the above lemma, $\int_\gamma *du = 0$ for all cycles $\gamma$ on a simply-connected $\Omega$. Hence $*u$ is exact on $\Omega$. To show uniqueness, if $f_1 = u + iv_1$ and $f_2 = u + iv_2$ are both analytic, then $f_1 - f_2 = i(v_1 - v_2)$ is also analytic. Recall that a nonconstant analytic function is an open mapping. Therefore, $v_1 - v_2$, which has image on the $y$-axis, must be constant. □

In general, the obstacles to the existence of a single-valued harmonic conjugate $v$ are the periods $\int_{\gamma_i} *du$ where $\gamma_i$ are basis elements of the homology of $\Omega$.

26.2. **Mean value property.**

**Theorem 26.3.** *Let $u : \Omega \to \mathbf{R}$ be a harmonic function and $\overline{D}_r(z_0) \subset \Omega$ be a closed disk. Then $u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})d\theta$.*

This is called the *mean value property*, since the value of $u$ at the center of a disk is the average of the values on the boundary of the disk.

*Proof.* On the disk $\overline{D}_r(z_0)$ the harmonic function $u$ has a harmonic conjugate $v$. Hence $f = u + iv$ is analytic and by the Cauchy integral formula we have:

$$f(z_0) = \frac{1}{2\pi i}\int_{\partial D_r(z_0)} \frac{f(z)dz}{z - z_0} = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})d\theta.$$

Taking the real part of the equation yields the theorem. □

**Corollary 26.4.** *$u$ does not attain its maximum or minimum in the interior of $\Omega$.*

**Theorem 26.5.** *Let $u$ be a harmonic function defined on the annulus $A = \{r_1 < |z| < r_2\}$. Then $\frac{1}{2\pi}\int_{|z|=r} u d\theta = \alpha \log r + \beta$, where $\alpha, \beta$ are constant. Here $r_1 < r < r_2$.*

In view of the mean value property, if $u$ is defined on $D = \{|z| < r_2\}$, then $\alpha = 0$ and $\beta = u(0)$.

*Proof.* The proof we give has elements of *analytic continuation*, which we will discuss in more detail later. Consider subsets $U_1 = \{0 < \theta < \pi\}$, $U_2 = \{\frac{\pi}{2} < \theta < \frac{3\pi}{2}\}$, $U_3 = \{\pi < \theta < 2\pi\}$, and $U_4 = \{\frac{3\pi}{2} < \theta < \frac{5\pi}{2}\}$ (of $A$). Since each $U_j$ is simply-connected, on each $U_j$ there exists a harmonic conjugate $v_j$ of $u$. Having picked $v_1$, pick $v_2$ so that $v_1 = v_2$ on $U_1 \cap U_2$. (Recall they initially differ by a constant. Then modify $v_2$ so it agrees with $v_1$.) Likewise, pick $v_3$ so that $v_2 = v_3$ on $U_2 \cap U_3$, and pick $v_4$.... Write $f_j = u + v_j$.

Suppose $f_4 - f_1 = iC$, where $C$ is a constant. Consider the functions $F_j = f_j - \frac{C}{2\pi}\log z$, where branches of $\log z$ are chosen so that $f_j - \frac{C}{2\pi}\log z$ agrees with $f_{j+1} - \frac{C}{2\pi}\log z$. Now, $\log z$ for $F_4$ and $\log z$ for $F_1$ differ by $2\pi i$ and $F_4 - F_1 = iC - \frac{C}{2\pi}2\pi i = 0$. Therefore, the $F_j$ glue to give a globally defined analytic function $F$. Thus,

$$\frac{1}{2\pi i}\int_{|z|=r} \frac{F(z)}{z} dz = \frac{1}{2\pi}\int_0^{2\pi} F(re^{i\theta}) d\theta$$

is constant and its real part is:

$$\frac{1}{2\pi}\int_{|z|=r} u d\theta + \frac{1}{2\pi}\int_{|z|=r} \frac{-C}{2\pi}\log r d\theta = \frac{1}{2\pi}\int_{|z|=r} u d\theta + \frac{-C}{2\pi}\log r.$$

□

26.3. **Poisson's formula.** We now give an explicit formula for which expresses the values of a harmonic function $u$ on a disk in terms of the values of $u$ on the boundary of the disk. This is analogous to the Cauchy integral formula (and is indeed a consequence of it).

**Theorem 26.6.** *Suppose $u(z)$ is harmonic on $|z| < R$ and continuous for $|z| \leq R$. Then*

$$u(a) = \frac{1}{2\pi}\int_{|z|=R} \frac{R^2 - |a|^2}{|z-a|^2} u(z) d\theta = \frac{1}{2\pi}\int_{|z|=R} Re\frac{z+a}{z-a} u(z) d\theta,$$

*if $|a| < R$.*

Sometimes we will use the center expression and at other times we will use the RH expression.

*Proof.* For simplicity, we'll assume that $R = 1$ and $u$ is harmonic on all of $|z| \leq R$. The Poisson formula is just a restatement of the mean value property.

First observe that the fractional linear transformation $S(z) = \frac{z+a}{\bar{a}z+1}$ sends the unit disk $D = \{|z| < 1\}$ to itself and 0 to $a$. (Recall our discussion of automorphisms of $D$.) Then

$u \circ S$ is a harmonic function on $\overline{D}$ and by the mean value property we have:

$$u(a) = u \circ S(0) = \frac{1}{2\pi} \int_0^{2\pi} u(S(z)) d \arg z. \tag{2}$$

We now rewrite the RH integral in terms of $w = S(z)$. Inverting, we obtain $z = S^{-1}(w) = \frac{w-a}{-\overline{a}w+1}$ and

$$d \arg z = -i \frac{dz}{z} = -i \left( \frac{1}{w-a} + \frac{\overline{a}}{-\overline{a}w+1} \right) dw = \left( \frac{w}{w-a} + \frac{\overline{a}w}{-\overline{a}w+1} \right) d\theta.$$

By using $w = \frac{1}{\overline{w}}$ on the unit circle, we can rewrite the RHS as:

$$\left( \frac{w}{w-a} + \frac{\overline{a}}{\overline{w}-\overline{a}} \right) d\theta = \frac{1-|a|^2}{|w-a|^2} d\theta,$$

or alternatively as

$$\frac{1}{2} \left( \frac{w+a}{w-a} + \frac{\overline{w}+\overline{a}}{\overline{w}-\overline{a}} \right) d\theta = Re \frac{w+a}{w-a} d\theta.$$

□

By using the RH expression in the Poisson integral formula (in Theorem 26.6), it follows that $u(z)$ is the real part of the analytic function

$$f(z) = \frac{1}{2\pi i} \int_{|z|=R} \frac{\zeta + z}{\zeta - z} u(\zeta) \frac{d\zeta}{\zeta} + iC.$$

($f$ is analytic by Lemma 15.2 from Day 15.)

## 27. Schwarz reflection principle

### 27.1. **Schwarz's Theorem.**

**Theorem 27.1.** *Given a piecewise continuous function $U(\theta)$ on $0 \leq \theta \leq 2\pi$, the Poisson integral*

$$P_U(z) = \frac{1}{2\pi}\int_0^{2\pi} Re\frac{e^{i\theta}+z}{e^{i\theta}-z}U(\theta)d\theta$$

*is harmonic for $|z| < 1$ and $\lim_{z\to e^{i\theta_0}} P_U(z) = U(\theta_0)$, provided $U$ is continuous at $\theta_0$.*

*Proof.* By Lemma 15.2, $f(z) = \frac{1}{2\pi i}\int_{|\zeta|=1}\frac{\zeta+z}{\zeta-z}U(\zeta)\frac{d\zeta}{\zeta}$ is analytic on $|z| < 1$, and hence $P_U(z)$ is harmonic for $|z| < 1$.

$P$ is a linear operator which maps piecewise continuous functions $U$ on the unit circle to harmonic functions $P_U$ on the open unit disk. It satisfies $P_{U_1+U_2} = P_{U_1} + P_{U_2}$ and $P_{cU} = cP_U$ for $c$ constant. We can easily deduce that $P_c = c$ and that $m \leq U \leq M$ implies $m \leq P_U \leq M$. (You can verify this using Equation 2.)

WLOG assume that $U(\theta_0) = 0$; otherwise we can take $U - U(\theta_0)$ instead (and still call it $U$). (Observe that we can do this because $P_c = c$.) Take a short arc $C_2 \subset \partial D$ (here $D$ is the unit disk) containing $\theta_0$ in its interior, such that $|U(\theta)| < \varepsilon$ on $C_2$. (This is possible by the continuity of $U$ at $\theta_0$.) Let $C_1$ be its complement in $\partial D$. Let $U_i$, $i = 1, 2$, equal $U$ on $C_i$ and zero elsewhere. Then $U = U_1 + U_2$.

$P_{U_1}$ is harmonic away from $C_1$; in particular it is harmonic (and continuous) in a neighborhood of $e^{i\theta_0}$. Moreover, by the center expression in Theorem 26.6, we can verify that $P_{U_1}$ is zero on $C_2$. On the other hand, $P_{U_2}$ satisfies $|P_{U_2}| < \varepsilon$ since $|U_2| < \varepsilon$. Adding up $P_{U_1}$ and $P_{U_2}$ we see that $|P_U|$ can be made arbitrarily small in a neighborhood of $e^{i\theta_0}$. $\square$

Therefore, there is a 1-1 correspondence between continuous functions on the unit circle and continuous functions on the closed unit disk that are harmonic on the open unit disk. The correspondence is given by $U \mapsto P_U$, and its inverse map is just the restriction $P_U|_{\partial D}$.

### 27.2. **Schwarz reflection principle.**

**Goal:** Extend/continue an analytic function $f : \Omega \to \mathbf{C}$ to a larger domain. The ultimate goal is to find the maximal domain on which $f$ can be defined.

**Observation:** If $f(z)$ is analytic on $\Omega$, then $\overline{f(\overline{z})}$ is analytic on $\Omega' = \{\overline{z} | z \in \Omega\}$.

If $f(z)$ is an analytic function, defined on a region $\Omega$ which is symmetric about the $x$-axis, and $f(z) = \overline{f(\overline{z})}$, then $f(z)$ is real on the $x$-axis. We have the following converse:

**Theorem 27.2.** *Let $\Omega$ be a symmetric region about the $x$-axis and let $\Omega^+ = \Omega \cap \{Im\ z > 0\}$, $\sigma = \Omega \cap \{Im\ z = 0\}$. If $f(z)$ is continuous on $\Omega^+ \cup \sigma$, analytic on $\Omega^+$, and real for all $z \in \sigma$, then $f(z)$ has an analytic extension to all of $\Omega$ such that $f(z) = \overline{f(\overline{z})}$.*

Theorem 27.2 follows from the following result for harmonic functions:

**Theorem 27.3.** *Suppose $v(z)$ is continuous on $\Omega^+ \cup \sigma$, harmonic on $\Omega^+$, and zero on $\sigma$. Then $v$ has a harmonic extension to $\Omega$ satisfying $v(z) = -v(\overline{z})$.*

*Proof.* Define $V(z)$ to be $v(z)$ for $z \in \Omega^+$, 0 for $z \in \sigma$, and $-v(\overline{z})$ for $z \in \Omega^- = \Omega \cap \{\text{Im } z < 0\}$. We want to prove that $V(z)$ is harmonic. For each $z_0 \in \sigma$, take an open disk $D_\delta(z_0) \subset \Omega$. Then define $P_V$ to be the Poisson integral of $V$ with respect to the boundary $\partial D_\delta(z_0)$. By Theorem 27.1 above, $P_V$ is harmonic on $D_\delta(z_0)$ and continuous on $\overline{D}_\delta(z_0)$. It will be shown that $V = P_V$.

On the upper half disk, $V$ and $P_V$ are both harmonic, so $V - P_V$ is harmonic. $V - P_V = 0$ on the upper semicircle, since $V(z) = v(z)$ by definition and $P_V(z) = v(z)$ by the continuity of $P_V$ (here $z$ is on the semicircle). Also, $V - P_V = 0$ on $\sigma \cap D_\delta(z_0)$, since $v(z) = 0$ by definition and $P_V(z) = \frac{1}{2\pi}\int_{|\zeta|=\delta} \frac{\delta^2 - |z|^2}{|\zeta - z|^2} V(\zeta) d\theta$, and we note that the contributions from the upper semicircle cancel those from the lower semicircle.

Summarizing, $V - P_V$ is harmonic on the upper half disk $D_\delta(z_0) \cap \Omega^+$, continuous on its closure, and zero on its boundary. Therefore, $V = P_V$ on the upper half disk. $\square$

*Proof of Theorem 27.2.* Given $f(z) = u(z) + iv(z)$ on $\Omega^+$, extend $f(z)$ to $\Omega^-$ by defining $f(z) = \overline{f(\overline{z})} = u(\overline{z}) - iv(\overline{z})$ for $z \in \Omega^-$. From above, $v(z)$ is extended to a harmonic function $V(z)$ on all of $\Omega$ as above. Since $-u(z)$ is the harmonic conjugate of $v(z)$ on $\Omega^+$, we define $U(z)$ to be a harmonic conjugate of $-V(z)$ (at least in a neighborhood $D_\delta(z_0)$ of $z_0 \in \sigma$). Adjust $U(z)$ (by adding a constant) so that $U(z) = u(z)$ on the upper half disk.

We prove that $g(z) = U(z) - U(\overline{z}) = 0$ on $D_\delta(z_0)$. Indeed, $U(z) = U(\overline{z})$ on $\sigma$, so $\frac{\partial g}{\partial x} = 0$ on $\sigma$. Also, $\frac{\partial g}{\partial y} = 2\frac{\partial u}{\partial y} = -2\frac{\partial v}{\partial x} = 0$ on $\sigma$. Therefore, the analytic function $\frac{\partial g}{\partial x} - i\frac{\partial g}{\partial y}$ vanishes on the real axis, and hence is constant. Since $g(z) = 0$ on $\sigma$, $g(z)$ is identically zero. This implies that $U(z) = U(\overline{z})$ on all of $D_\delta(z_0)$, hence proving the theorem. $\square$

I want to emphasize that the symmetry about the $x$-axis is simply a *normalization*, and that the reflection principle is applicable in far greater generality.

## 28. Normal families, Arzela-Ascoli

28.1. **Normal families.** Let $\mathcal{F}$ be a collection (called "family") of functions $f : \Omega \to \mathbf{C}$. (Much of what we discuss hold for functions $f : \Omega \to X$, where $X$ is a metric space, but we'll keep things simple.)

**Definition 28.1.** *$\mathcal{F}$ is* equicontinuous *on $E \subset \Omega$ if $\forall \varepsilon > 0$ $\exists \delta > 0$ such that $|z - z'| < \delta$, $z, z' \in E \Rightarrow |f(z) - f(z')| < \varepsilon$ for any $f \in \mathcal{F}$.*

Basically we are asking for all $f \in \mathcal{F}$ to be simultaneously uniformly continuous with the same constants.

**Definition 28.2.** *$\mathcal{F}$ is* normal *if every sequence $f_1, f_2, \dots$ in $\mathcal{F}$ has a subsequence which converges uniformly on compact subsets.*

**Remark:** The limit $f$ of a convergent sequence in $\mathcal{F}$ does not need to be in $\mathcal{F}$. (In other words, the closure $\overline{\mathcal{F}}$ must be compact.)

28.2. **Arzela-Ascoli Theorem.**

**Theorem 28.3** (Arzela-Ascoli)**.** *A family $\mathcal{F}$ of continuous functions $f : \Omega \to \mathbf{C}$ is normal iff (i) $\mathcal{F}$ is equicontinuous on each compact $E \subset \Omega$ and (ii) for any $z \in \Omega$ the set $\{f(z) | f \in \mathcal{F}\}$ is compact.*

Roughly speaking, if the functions are $f : \mathbf{R} \to \mathbf{R}$ with usual coordinates $y = f(x)$ and we're imagining the graph of $f$, then (i) controls the vertical spread in values of a single function $f|_E$ (the vertical spread is uniform for all $f \in \mathcal{F}$) and (ii) controls the vertical spread at a single point $z \in E$ as we vary $f$.

*Proof.* Suppose $\mathcal{F}$ is normal. We prove (i). If $\mathcal{F}$ is not equicontinuous on a compact subset $E$, then there are sequences $z_n, z_n' \in E$ and $f_n \in \mathcal{F}$ such that $|z_n - z_n'| \to 0$ but $|f_n(z_n) - f_n(z_n')| \geq \varepsilon$. Since $E$ is compact, there is a subsequence (we abuse notation and also call it $z_n$) such that $z_n \to z \in E$. Also, since $z_n$ and $z_n'$ are close, $z_n' \to z \in E$. (We still have $|f_n(z_n) - f_n(z_n')| \geq \varepsilon$.)

On the other hand, since $\mathcal{F}$ is normal, there is a subsequence (still called $f_n$) $f_n \in \mathcal{F}$ which converges to a continuous function $f$ on $E$. (Why is $f$ continuous?) But then

$$|f_n(z_n) - f_n(z_n')| \leq |f_n(z_n) - f(z_n)| + |f(z_n) - f(z_n')| + |f(z_n') - f_n(z_n')|,$$

and the first and third terms on the RHS $\to 0$ by uniform convergence, and the second term $\to 0$ as $z_n$ and $z_n'$ get arbitrarily close (by continuity of $f$). This contradicts $|f_n(z_n) - f_n(z_n')| \geq \varepsilon$.

Next we prove (ii). Any sequence $\{f_n\}$ has a convergent subsequence (which we still call $f_n$), so given any $z \in \Omega$, $f_n(z) \to f(z)$. Since any sequence in $\{f(z) | f \in \mathcal{F}\}$ has a convergent subsequence, the set is compact.

Now suppose (i) and (ii) hold. Given $f_1, \dots, f_n, \dots \in \mathcal{F}$, take a sequence $\{\zeta_1, \zeta_2, \dots\}$ which is everywhere dense in $\Omega$ (e.g., the set of points with rational coordinates). Take a subsequence of $\{f_n\}$ which converges at $\zeta_1$. Denote it by indices $n_{11} < n_{12} < \dots < n_{1j} < \dots$. Next, take a subsequence of it which converges at $\zeta_2$. Denote it by indices $n_{21} < n_{22} < \dots <$

$n_{2j} < \dots$. Continuing in this manner, we then take the diagonal $n_{11} < n_{22} < n_{33} < \dots$. Then $f_{n_{jj}}$ converges at all the points $\zeta_i$.

We now claim that $\{f_{n_{jj}}\}$ converges uniformly on any compact $E \subset \Omega$. Indeed, given any $z \in E$,

$$|f_{n_{ii}}(z) - f_{n_{jj}}(z)| \leq |f_{n_{ii}}(z) - f_{n_{ii}}(\zeta_k)| + |f_{n_{ii}}(\zeta_k) - f_{n_{jj}}(\zeta_k)| + |f_{n_{jj}}(\zeta_k) - f_{n_{jj}}(z)|,$$

where $\zeta_k$ is within a distance $\delta$ of $z$. Since $f_{n_{ii}}$ and $f_{n_{jj}}$ are equicontinuous, given $\varepsilon > 0$, there is $\delta > 0$ so that the first and last terms on the RHS are $< \frac{\varepsilon}{3}$. Since $f_{n_{ii}}(\zeta_k)$ converges, there is an $N$ such that $n_{ii} > N$ implies that the middle term is $< \frac{\varepsilon}{3}$. $\square$

28.3. **Montel's theorem.** We now apply the Arzela-Ascoli theorem in the setting of a family $\mathcal{F}$ of analytic functions.

**Theorem 28.4** (Montel)**.** *A family $\mathcal{F}$ of analytic functions $f : \Omega \to \mathbf{C}$ is* normal *iff functions $f \in \mathcal{F}$ are* uniformly bounded *on each compact set $E \subset \Omega$, i.e., there exists a constant $M$ such that $|f(z)| < M$ for all $z \in E$ and $f \in \mathcal{F}$.*

*Proof.* Suppose $\mathcal{F}$ is normal. Cover $E$ with a finite number of disks $D_i$ of radius $\delta$ (this is possible by compactness). On each disk $D_i$ centered at $z_i$, if $f_\alpha, f_\beta \in \mathcal{F}$ and $z \in D_i$ we have:

$$|f_\alpha(z) - f_\beta(z_i)| \leq |f_\alpha(z) - f_\alpha(z_i)| + |f_\alpha(z_i) - f_\beta(z_i)|,$$

and first term on the RHS is bounded above by equicontinuity and the second term is bounded since $\{f(z_i) | f \in \mathcal{F}\}$ is compact. Since there are only finitely many balls, we have uniform boundedness.

Now suppose $\mathcal{F}$ is uniformly bounded on $E$. It is sufficient to prove equicontinuity, in view of Arzela-Ascoli. If $D_r(z_0)$ is a disk of radius $r$ about $z_0$ and $z, z' \in E \cap D_{r/2}(z_0)$, then

$$f(z) - f(z') = \frac{1}{2\pi i} \int_{\partial D_i} \left( \frac{f(\zeta)}{\zeta - z} - \frac{f(\zeta)}{\zeta - z'} \right) d\zeta = \frac{z - z'}{2\pi i} \int_{\partial D} \frac{1}{(\zeta - z)(\zeta - z')} f(\zeta) d\zeta,$$

and

$$|f(z) - f(z')| \leq \frac{|z - z'|}{2\pi} \frac{2\pi r}{(r/2)(r/2)} M \leq \frac{4M}{r} |z - z_0|.$$

This proves equicontinuity on $D_{r/2}(z_0)$.

Now, cover $E$ with finitely many disks of radius $\frac{r}{2}$. If $z, z' \in E$ and $|z - z'| < \frac{r}{4}$, then there is some disk $D_{r/2}(z_0)$ which contains both $z$ and $z'$. Given $\varepsilon > 0$, pick $\delta = \min(\frac{r}{4}, \frac{r}{4M}\varepsilon)$. Then $|z - z'| < \delta \Rightarrow |f(z) - f(z')| < \varepsilon$. $\square$

## 29. Riemann mapping theorem

**Theorem 29.1.** *A simply connected, connected, open $\Omega \subsetneq \mathbf{C}$ is biholomorphic to the open unit disk $D$.*

**Step 1:** Let $a \in \mathbf{C} - \Omega$. Since $\Omega$ simply connected, there is a single-valued branch of $f(z) = \sqrt{z-a}$ on $\Omega$. (Recall that $f(z) = e^{\frac{1}{2}\log(z-a)}$ and $\log(z-a)$ can be defined as an antiderivative of $\frac{1}{z-a}$. This is because, according to Cauchy's Theorem, $\int_\gamma \frac{dz}{z-a} = 0$ for all closed curves $\gamma$ in a simply connected $\Omega$.) $f$ gives a biholomorphism of $\Omega$ onto its image. For simplicity take $a = 0$. Then consider $\overline{D_\delta(b)} \subset \operatorname{Im} f$. We have $\overline{D_\delta(-b)} \cap \operatorname{Im} f = \emptyset$. (Observe that $\overline{D_\delta(-b)}$ is $\overline{D_\delta(b)}$ which has been reflected across the origin. This means that they would have the same image under the map $w \mapsto w^2$.) Now take an FLT which maps $\overline{D_\delta(-b)}$ to the complement of the open unit disk. From now on we assume that $\Omega$ is a subset of $D$, and also that $0 \in \Omega$. (Recall there is an FLT which is an automorphism of $D$ and takes any interior point of $D$ to 0.)

**Step 2:** Consider a holomorphic map $f : \Omega \to D$ which satisfies:

(1) $f$ is 1-1,
(2) $f(0) = 0$,
(3) $f'(0) > 0$ (this means $f'(0)$ is real and positive).

Let $\mathcal{F}$ be the family of all such holomorphic maps. In particular, *id* is one such map, so $\mathcal{F}$ is nonempty.

If $f$ is not onto, then we find $h \in \mathcal{F}$ with larger $h'(0)$. Let $a \in D - f(\Omega)$. We take $g$ to be the composition $g_3 g_2 g_1$ of the following maps:

(1) FLT $g_1$, an automorphism of $D$ which sends $a$ to 0,
(2) $g_2(z) = \sqrt{z}$, which is single-valued on $g_1 f(\Omega)$ (since it is simply connected),
(3) FLT $g_3$, an automorphism of $D$ which sends $g_2 g_1(0)$ to 0 and satisfies $(g_3 g_2 g_1)'(0) > 0$. (The latter condition can be achieved by composition with an appropriate rotation about the origin.)

We claim that $(g \circ f)'(0) > f'(0)$. Indeed $g^{-1}$ can be viewed as a map from $D$ to itself which sends $0 \mapsto 0$ such that $(g^{-1})'(0) > 0$. By the Schwarz lemma, $(g^{-1})'(0) < 1$ since $g^{-1}$ is not a rotation. This means that $g'(0) > 1$, and hence $(g \circ f)'(0) > f'(0)$.

**Step 3:** Now consider $\mathcal{F}$. $\mathcal{F}$ is uniformly bounded (since all $f$ map to $D$), so take a sequence $f_1, \dots, f_n, \dots$ in $\mathcal{F}$ for which $f_i'(0) \to M = \sup_{f \in \mathcal{F}} f'(0)$. Then there is a subsequence of $\{f_n\}$ converging UCOCS to a holomorphic map $f : \Omega \to D$ by Montel's theorem. (This in particular implies that $M < \infty$.)

It remains to show that $f$ is 1-1. We argue using the argument principle. By uniform convergence,

$$\int_\gamma \frac{f_i'(z)}{f_i(z)} dz \to \int_\gamma \frac{f'(z)}{f(z)} dz,$$

for closed curves $\gamma$ that avoid the zeros of $f$. But these represent winding numbers – this implies that the winding numbers do not change in the limit, and that $f$ is 1-1. Hence $f \in \mathcal{F}$. By Step 2, $f$ must be onto $D$. This completes the proof of the Riemann Mapping Theorem.

**Enhancements:**

1. Let $\Omega$ and $\Omega'$ be open regions. If $f : \Omega \xrightarrow{\sim} \Omega'$ is a homeomorphism and $\{z_n\}$ is a sequence that tends to $\partial\Omega$, then $\{f(z_n)\}$ tends to $\partial\Omega'$. (Proof left as exercise.)

We define $\partial\Omega = \overline{\Omega} - int(\Omega)$ (the closure of $\Omega$ minus (the interior of) $\Omega$). We also say that $z_n$ *tends to* $\partial\Omega$ if for all $z \in \Omega$, $\exists\ D_\varepsilon(z) \subset \Omega$ and $N > 0$ such that $z_n \notin D_\varepsilon(z)$ if $n > N$. Observe that if $K \subset \Omega$ is compact, then by covering $K$ with open disks, there exists $N > 0$ such that $z_n \notin K$ if $n > N$.

2. The *Uniformization Theorem* is a generalization of the Riemann Mapping Theorem which says that a simply connected (=any closed curve can be contracted to a point), connected Riemann surface (without boundary) is biholomorphic to one of $\mathbf{C}$, $S^2$, or $D$.

3. A multiply connected region $\Omega \subset \mathbf{C}$ of connectivity $n+1$ (recall that this means that $\mathbf{C} - \Omega$ has $n+1$ connected components) is biholomorphic to the annulus $1 < |z| < \lambda$ (for some $\lambda > 1$) with $n-1$ arcs, each of which is a subarc of $|z| = \lambda_i$ for some $1 < \lambda_i < \lambda$, removed.

## 30. Analytic continuation

30.1. **Riemann mapping theorem and boundary behavior.** Recall that last time we proved that a simply connected $\Omega \subsetneq \mathbf{C}$ is biholomorphic to the open unit disk $D$. We give more enhancements:

4. If $\partial\Omega$ contains a *free one-sided analytic arc* $\gamma$, then the biholomorphism $f : \Omega \xrightarrow{\sim} D$ has an analytic extension to $\Omega \cup \gamma$, and $\gamma$ is mapped onto an arc of $\partial D$.

An *analytic arc* $\gamma : [a, b] \to \Omega$ is an arc which, in a neighborhood of each $t_0 \in [a, b]$, is given by a Taylor series

$$\gamma(t) = a_0 + a_1(t - t_0) + a_2(t - t_0)^2 + \dots,$$

with a nonzero radius of convergence.

A *free one-sided boundary arc* $\gamma$ is a regular (i.e., $\gamma'(t) \neq 0$ on $[a, b]$), simple (i.e., 1-1) arc such that there are neighborhoods $\Delta \subset \mathbf{C}$ of $t_0 \in [a, b]$ and $\Omega' \subset \mathbf{C}$ of $\gamma(t_0)$ such that $\Delta \cap \{\mathrm{Im}\ z > 0\}$ gets mapped onto $\Omega \cap \Omega'$ and $\Delta \cap \{\mathrm{Im}\ z < 0\}$ gets mapped to $\mathbf{C} - \Omega$.

*Proof.* Use the Schwarz reflection principle, after changing coordinates so that $f$ looks like a map from a subset of the upper half plane to itself. $\square$

5. If $\Omega$ is simply connected and $\partial D$ is given by a simple (continuous) closed curve, then the biholomorphism $f : \Omega \to D$ extends to a homeomorphism $f : \overline{\Omega} \to \overline{D}$.

The proof is omitted.

30.2. **Analytic continuation.** Denote by $(f, \Omega)$ an analytic function $f$ defined on an open region $\Omega$. (We assume, as usual, that $\Omega$ is connected.)

**Definition 30.1.** *$(f_1, \Omega_1)$ and $(f_2, \Omega_2)$ are* direct analytic continuations *of each other, if $\Omega_1 \cap \Omega_2 \neq \emptyset$ and $f_1 = f_2$ on $\Omega_1 \cap \Omega_2$.*

Observe that if $(f_2, \Omega_2)$ and $(g_2, \Omega_2)$ are direct analytic continuations of $(f_1, \Omega_1)$ to $\Omega_2$, then $f_2 = g_2$, since $f_2|_{\Omega_1 \cap \Omega_2} = g_2|_{\Omega_1 \cap \Omega_2}$. (Recall that if two analytic functions with the same domain agree on a set with an accumulation point, then they are identical.)

**Definition 30.2.** *If there exists a sequence $(f_1, \Omega_1), \dots, (f_n, \Omega_n)$ such that $(f_{i+1}, \Omega_{i+1})$ is a direct analytic continuation of $(f_i, \Omega_i)$, then $(f_n, \Omega_n)$ is an* analytic continuation *of $(f_1, \Omega_1)$.*

**Remark:** It is possible that $\Omega_1 = \Omega_n$ but $f_1 \neq f_n$.

**Example:** Consider $f(z) = \log z$. Define $\Omega_j = \{\frac{\pi j}{2} < \theta < \frac{\pi j}{2} + \pi\}$. Then on $\Omega_j$, we define $f_j(z) = \log|z| + i\arg(z)$, where $\frac{\pi j}{2} < \arg(z) < \frac{\pi j}{2} + \pi$. We have $\Omega_0 = \Omega_4$, but $f_4 = f_0 + 2\pi i$.

30.3. **Germs and sheaves.** Now consider a pair $(f, \zeta)$, where $\zeta \in \mathbf{C}$ and $f$ is analytic in a neighborhood of $\zeta$. We view $(f_1, \zeta_1)$ and $(f_2, \zeta_2)$ as equivalent iff $\zeta_1 = \zeta_2$ and $f_1 = f_2$ on some neighborhood of $\zeta_1 = \zeta_2$. Such an equivalence class is called a *germ* of a holomorphic function. Notice that $(f, \Omega)$ gives rise to a germ $(f, \zeta)$ at each $\zeta \in \Omega$.

If $D$ is an open set in $\mathbf{C}$, then the set of all germs of holomorphic functions $(f, \zeta)$ with $\zeta \in D$ is called the *sheaf of germs of holomorphic functions* over $D$, and will be denoted $\mathcal{F}_D$ or $\mathcal{F}$, if $D$ is understood. There is a projection map $\pi : \mathcal{F} \to D$, which sends $(f, \zeta) \mapsto \zeta$. $\pi^{-1}(\zeta)$ is called the *stalk* at $\zeta$, and is also denoted $\mathcal{F}_\zeta$.

**Theorem 30.3.** *The set $\mathcal{F}$ can be given the structure of a Hausdorff topological space such that $\pi : \mathcal{F} \to D$ becomes a local homeomorphism (i.e., for each $(f, \zeta) \in \mathcal{F}$ there is an open neighborhood $U$ whose image $\pi(U)$ is open and homeomorphic to $U$).*

*Proof.* A set $V \subset \mathcal{F}$ is open iff for every $(f, \zeta) \in V$ there exists a function element $(f, \Omega)$ with $\zeta \in \Omega$ (which restricts to $(f, \zeta)$) such that all $(f, \zeta') \in V$ for $\zeta' \in \Omega$. (Verify that this satisfies the axioms of a topology!)

Now, given $(f, \zeta) \in \mathcal{F}$, take some function element $(f, \Omega)$ which restricts to $(f, \zeta)$. (Here we're assuming that $\zeta \in \Omega$.) Then take $V$ to be the set $\{(f, \zeta') | \ \zeta' \in \Omega\}$ of restrictions of $f$ to all points in $\Omega$. It is clear that $\pi$ maps $V$ homeomorphically onto $\Omega$.

It remains to show that $\mathcal{F}$ is Hausdorff. Given $(f_1, \zeta_1)$ and $(f_2, \zeta_2)$, if $\zeta_1 \neq \zeta_2$, then that's easy. (Take open sets $\Omega_1$ and $\Omega_2$ about $\zeta_1$ and $\zeta_2$, respectively, on which $f_1$ and $f_2$ can be defined, and $(f_1, \Omega_1)$ and $(f_2, \Omega_2)$ give rise to disjoint open sets, as in the previous paragraph.) Now suppose $\zeta_1 = \zeta_2$. Let $\Omega$ be an open set containing $\zeta_1 = \zeta_2$ on which $f_1$ and $f_2$ are both defined. If $(f_1, \zeta') = (f_2, \zeta')$ for any $\zeta' \in \Omega$, then $f_1 = f_2$ on all of $\Omega$. Therefore the open sets $V_1$ and $V_2$ corresponding to $(f_1, \Omega)$ and $(f_2, \Omega)$ do not intersect. □

**Remark:** $\pi : \mathcal{F} \to D$ is not quite a *covering space*, if you know what that means. This is because not every component of $\pi^{-1}U$ (for an open set $U \subset D$) is homeomorphic to $U$. (Why?)

## 31. Analytic continuation

Last time we defined $\mathcal{F}$, the sheaf of germs of holomorphic functions, and showed that it can be given the structure of a Hausdorff topological space such that the projection $\pi : \mathcal{F} \to \mathbf{C}$ (today, the base space is $\mathbf{C}$ instead of $D$) is a local homeomorphism. A basis for the topology of $\mathcal{F}$ is the set of $U_{(f,\Omega)} = \{(f,\zeta) | \zeta \in \Omega\}$, where $f$ is an analytic function on an open set $\Omega \subset \mathbf{C}$.

31.1. **Riemann surface of a function.** Given a function element $(f,\Omega)$, take its corresponding open set $U_{(f,\Omega)}$ in $\mathcal{F}$, and the connected component of $\mathcal{F}$ which contains $U_{(f,\Omega)}$. Call this $\Sigma_{(f,\Omega)}$, or simply $\Sigma$, if $(f,\Omega)$ is understood.

**Claim.** *$\Sigma$ can be given the structure of a Riemann surface, where the holomorphic coordinate charts are given by the local homeomorphism $\pi : \Sigma \to \mathbf{C}$.*

$\Sigma$ will be called the *Riemann surface of* $(f,\Omega)$. Note that $\Sigma$ is the set of all $(g,\zeta')$ for which there is an analytic continuation from $(f,\Omega)$ to $(g,\Omega')$ with $\zeta' \in \Omega'$. (Effectively, we have pasted together such $(g,\Omega')$ to obtain $\Sigma$.)

There also is a holomorphic map (often called a *global analytic function*) $\mathbf{f} : \Sigma \to \mathbf{C}$ obtained by setting $(f,\zeta) \mapsto f(\zeta)$. (Verify that this is holomorphic!) We refer to $\Sigma$ as the *Riemann surface of* $\mathbf{f}$ and write $\Sigma = \Sigma_{\mathbf{f}}$.

**Remark:** $\Sigma_{\mathbf{f}}$ is Hausdorff since $\mathcal{F}$ is. Observe that we haven't shown that $\Sigma_{\mathbf{f}}$ is second countable. (See the definition of a Riemann surface.) The verification is not trivial but won't be done here.

**Example:** Riemann surface of $f(z) = \log z$. Above each point of $\mathbf{C} - \{0\}$, there are infinitely many sheets, corresponding to $f_i(z) = \log|z| + i\arg(z)$, where $\frac{\pi j}{2} < \arg(z) < \frac{\pi j}{2} + \pi$. As we move around the origin, we can move from one sheet to another. Observe that an alternate way of obtaining $\Sigma_f$ is to start with $\Omega_j$ given above, and glue $\Omega_j$ to $\Omega_{j+1}$ along the region where $f_j$ and $f_{j+1}$ agree.

**Example:** Riemann surface of $f(z) = \sqrt{z}$. Above each point of $\mathbf{C} - \{0\}$ there are two sheets corresponding to $re^{i\theta} \mapsto \pm\sqrt{r}e^{i\theta/2}$. Notice that as we continue one choice of $\sqrt{z}$ around the origin, we reach the other choice, and circling twice around the origin gives the original function element.

**Remark:** We haven't dealt with the branch points, e.g., $z = 0$ for $f(z) = \sqrt{z}$. There is a reasonable way to fill in the point $z = 0$ to give a genuine Riemann surface (one without any singularities). Note that the usual picture of a "Riemann surface of $f(z) = \sqrt{z}$" is rather misleading, since it exhibits what looks like a singular point at the origin.

31.2. **Analytic continuation along arcs and the monodromy theorem.** Let $\gamma : [a,b] \to \mathbf{C}$ be a continuous arc. Consider a connected component $\Sigma_{\mathbf{f}}$ of $\mathcal{F}$. An arc $\tilde{\gamma} : [a,b] \to \Sigma_{\mathbf{f}}$

is an *analytic continuation of the global analytic function* $\mathbf{f}$ *along* $\gamma$, if $\pi \circ \tilde{\gamma} = \gamma$. (This is called a *lift* of $\gamma$ to $\Sigma_{\mathbf{f}}$ in topological jargon.)

**Lemma 31.1.** *Two analytic continuations* $\tilde{\gamma}_0$ *and* $\tilde{\gamma}_1$ *of a global analytic function* $\mathbf{f}$ *along* $\gamma$ *are either identical, or* $\tilde{\gamma}_0(t) \neq \tilde{\gamma}_1(t)$ *for all* $t$.

*Proof.* It follows from the Hausdorff/local homeomorphism property that the set of $t$ for which $\tilde{\gamma}_0(t) = \tilde{\gamma}_1(t)$ is open and the set for which $\tilde{\gamma}_0(t) \neq \tilde{\gamma}_1(t)$ is also open. □

Let $\Omega$ be a region in $\mathbf{C}$. Suppose $\mathbf{f}$ is a global analytic function which can be continued along all continuous arcs $\gamma$ in $\Omega$ and starting at any $(f, \zeta) \in \Sigma_{\mathbf{f}}$. Let $\gamma_0$ and $\gamma_1$ be two continuous arcs in $\Omega$ which are homotopic in $\Omega$ relative to their endpoints, i.e., there is a continuous map $\Gamma : [a, b] \times [0, 1] \to \Omega$ so that:

(1) $\Gamma(x, 0) = \gamma_0(x)$ and $\Gamma(x, 1) = \gamma_1(x)$,
(2) $\Gamma(a, t)$ and $\Gamma(b, t)$ do not depend on $t$.

Then we have the following:

**Theorem 31.2** (Monodromy theorem)**.** *Suppose* $\gamma_0$ *and* $\gamma_1$ *are homotopic relative to their endpoints and* $\mathbf{f}$ *is as in the preceding paragraph. Given a germ of* $\mathbf{f}$ *at the common initial point of* $\gamma_0$ *and* $\gamma_1$*, their continuations along* $\gamma_0$ *and* $\gamma_1$ *lead to the same germ at the common terminal point.*

*Proof.* It suffices to prove this theorem when the homotopy $\gamma_t(x) \stackrel{def}{=} \Gamma(x, t)$ ranges over a small subinterval $t \in [t_0, t_0 + \varepsilon] \subset [0, 1]$. We can therefore subdivide $[a, b]$ into $a = x_0 < x_1 < x_2 < \cdots < b = x_k$ so that the germ of $\mathbf{f}$ at each $\gamma_{t_0}(x_i)$ is defined in a neighborhood $U_i$, and $\gamma_t(x) \in U_i$ for all $t \in [t_0, t_0 + \varepsilon]$ and $x \in [x_i, x_{i+1}]$. By using an argument similar to that of Lemma 31.1, we're done. □

**Corollary 31.3.** *If* $\mathbf{f}$ *is a global analytic function which can be continued along all arcs in a simply connected* $\Omega$*, then* $\mathbf{f}$ *is a single-valued analytic function.*

This gives another proof of the fact that $f(z) = \sqrt{z - a}$ admits a single-valued analytic branch on a simply-connected $\Omega$, if $a \notin \Omega$.

## 32. Universal covers and the little Picard theorem

### 32.1. Universal cover of $\mathbf{C} - \{0,1\}$.

**Definition 32.1.** *A* covering space *of $X$ is a topological space $\tilde{X}$ and a map $\pi : \tilde{X} \to X$ such that the following holds: there is an open cover $\{U_\alpha\}$ of $X$ such that $\pi^{-1}(U_\alpha)$ is a disjoint union of open sets, each of which is homeomorphic to $U_\alpha$ via $\pi$.*

**Definition 32.2.** *A* universal cover *$\pi : \tilde{X} \to X$ is a covering space of $X$ which is simply connected.*

**Theorem 32.3.** *Any "reasonable" space $X$ has a universal cover $\tilde{X}$. Moreover, the universal cover is unique, in the sense that given any other $\pi' : \tilde{X}' \to X$ there exists a homeomorphism $\phi : \tilde{X} \to \tilde{X}'$ such that $\pi' \circ \phi = \pi$.*

As a set, "the" universal cover $\tilde{X}$ is given as follows: Pick a basepoint $x_0 \in X$. Then $\tilde{X}$ is the quotient of the set of paths $\gamma$ with initial point $x_0$, by the equivalence relation $\sim$ which identifies $\gamma$ and $\gamma'$ if they have the same endpoints and are homotopic arcs relative to their endpoints.

We'll presently construct the universal cover of $\mathbf{C} - \{0,1\}$. Let $\Omega = \{0 < \text{Re } z < 1, |z - \frac{1}{2}| > \frac{1}{2}, \text{Im } z > 0\}$ be a region of the upper half plane $\mathbf{H}$. $\Omega$ has three boundary components $C_1 = \{\text{Re } z = 0, \text{Im } z > 0\}$, $C_2 = \{|z - \frac{1}{2}| = \frac{1}{2}, \text{Im } z > 0\}$, and $C_3 = \{\text{Re } z = 1, \text{Im } z > 0\}$ (together with points $0, 1$). Using the Riemann mapping theorem (and its enhancements), there exists a biholomorphic map $\pi : \Omega \to \mathbf{H}$, which extends to a map from $C_1$ to $\{\text{Im } z = 0, \text{Re } z < 0\}$, $C_2$ to $\{\text{Im } z = 0, 0 < \text{Re } z < 1\}$, and $C_3$ to $\{\text{Im } z = 0, 1 < \text{Re } z\}$. Using the Schwarz reflection principle, we can reflect along each of the $C_j$. For example, if we reflect across $C_1$, $\Omega$ goes to $\Omega' = \{-1 < \text{Re } z < 0, |z + \frac{1}{2}| > \frac{1}{2}, \text{Im } z > 0\}$, and $\pi$ extends to a holomorphic map on $\Omega \cup \Omega' \cup C_1$, where $\Omega'$ is mapped to the lower half plane. Continuing in this manner, we define a map $\pi : \mathbf{H} \to \mathbf{C} - \{0,1\}$. It is not hard to verify that this is a covering map. Since $\mathbf{H} \simeq D$ is simply-connected, $\pi$ is the universal covering map.

### 32.2. Little Picard Theorem.

**Theorem 32.4.** *Let $f(z)$ be a nonconstant entire function. Then $\mathbf{C}$ − Im $f$ consists of at most one point.*

*Proof.* Suppose $f : \mathbf{C} \to \mathbf{C}$ misses at least two points. By composing with some fractional linear transformation, we may assume that $f$ misses 0 and 1.

We now construct a lift $\tilde{f} : \mathbf{C} \to \mathbf{H}$ of $f$ to the universal cover, i.e., find a holomorphic map $\tilde{f}$ satisfying $\pi \circ \tilde{f} = f$. [This is called the *homotopy lifting property*, and is always satisfied if the domain is a "reasonable" simply connected topological space $X$.]

Given $z_0 \in \mathbf{C}$, there is a neighborhood $V \supset f(z_0)$ and $W \subset \mathbf{H}$ so that $W \xrightarrow{\pi} V$ is a biholomorphism. By composing with $\pi^{-1}$, we can define $\tilde{f} : U \to W \subset \mathbf{H}$, where $U$ is a sufficiently small neighborhood of $z_0$. Take some lift $\tilde{f}$ in a neighborhood of a reference point $z = 0$. Let $\gamma : [0,1] \to \mathbf{C}$ be a continuous arc in $\mathbf{C}$ with $\gamma(0) = 0$, $\gamma(1) = z$. There exist

$0 = t_0 < t_1 < \cdots < t_n = 1$ such that each $[t_{i-1}, t_i]$ is mapped into $V_i \subset \mathbf{C} - \{0, 1\}$, where $\pi^{-1}(V_i)$ is the disjoint union of biholomorphic copies of $V_i$ (by the covering space property). Suppose we have extended $\tilde{f}$ along $\gamma$ up to $t_i$. The pick the component $W$ of $\pi^{-1}(V_{i+1})$ which contains $\tilde{f}(\gamma(t_i))$. Continue the lift to $[t_i, t_{i+1}]$ by composing $f$ with $\pi^{-1} : V_{i+1} \xrightarrow{\sim} W$.

Since $\mathbf{C}$ is simply connected, the monodromy theorem tells us that the value of $\tilde{f}(z)$ does not depend on the choice of path $\gamma$. This gives a holomorphic map $\tilde{f} : \mathbf{C} \to \mathbf{H}$. Now, composing with the biholomorphism $\mathbf{H} \xrightarrow{\sim} D$, we obtain a bounded entire function. Since we know that a bounded entire function is a constant function, it follows that $\tilde{f}$ is constant. □

**Enhancements:**

1. (Montel) Let $\Omega \subset \mathbf{C}$ be an open set and $\mathcal{F}$ be a family of analytic maps $\Omega \to \mathbf{C}$. If each $f \in \mathcal{F}$ misses the same two points $a, b$, then $\mathcal{F}$ is normal.

2. (Big Picard) Suppose $f$ is holomorphic on $\Omega - \{z_0\}$ and has an essential singularity at $z_0$. If $U \subset \Omega$ is any (small) neighborhood of $z_0$, then $f$ assumes all points of $\mathbf{C}$ infinitely many times in $U - \{z_0\}$, with the possible exception of one point. [Big Picard implies Little Picard.]