VOLUME 1

HANDBOOK OF BOOLEAN ALGEBRAS

Edited by
J. Donald Monk
with *Robert Bonnet*

Sabine Koppelberg

NORTH-HOLLAND

Kurtele

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VOLUME 1

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SABINE KOPPELBERG

Freie Universität Berlin

Edited by

J. DONALD MONK

Professor of Mathematics, University of Colorado

with the cooperation of

ROBERT BONNET

Université Claude-Bernard, Lyon I

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Introduction to the Handbook

The genesis of the motion of a Boolean algebra (BA) is, of course, found in the works of George Boole; but his works are now only of historical interest – cf. HAILPERIN [1981] in the bibliography (elementary part). The notions of Boolean algebra were developed by many people in the early part of this century – Schröder, Löwenheim, etc. usually working with the concrete operations union, intersection, and complementation. But the abstract notion also appeared early, in the works of Huntington and others. Despite these early developments, the modern theory of BAs can only be considered to have started in the 1930s with works of M.H. Stone and A. Tarski. Since then there has been a steady development of the subject.

The present Handbook treats those parts of the theory of Boolean algebras of most interest to pure mathematicians: the set-theoretical abstract theory and applications and relationships to measure theory, topology, and logic. Aspects of the subject *not* treated here are discussion of axiom systems for BAs, finite Boolean algebras and switching circuits, Boolean functions, Boolean matrices, Boolean algebras with operators – including cylindric algebras and related algebraic forms of logic – and the role of BAs in ring theory and in functional analysis.

The Handbook is divided into two parts (published in three volumes). The first part (Volume 1) is a completely self-contained treatment of the fundamentals of the subject, which mathematicians in various fields may find interesting and useful. Here one will find the main results on disjointness (the Erdös–Tarski theorem), free algebras (the Gaifman–Hales, Shapirovskiĭ–Shelah, and Balcar–Franěk theorems), and the basic decidability and undecidability results for the elementary theory of BAs, as well as the systematic development of the abstract theory (ultrafilters, representation, subalgebras, ideals, topological duality, free algebras, free products, measure algebras, distributivity, interval algebras, superatomic algebras, tree algebras).

The second part of the Handbook (Volumes 2 and 3) is intended to indicate most of the frontiers of research in the subject; it consists of articles which are more or less independent of each other, although most of them assume a knowledge of at least the easier portions of Part I. The second part is arranged in four sections, with two appendices and a bibliography. Section A, Arithmetical properties of BAs, has two chapters: on distributive laws and their relationships to games on BAs, and on disjoint refinements, treating extensively this elementary notion discussed in Part I. Section B, Algebraic properties of BAs, treats subalgebras, particularly the lattice of all subalgebras and the existence of complements in this lattice; cardinal functions on Boolean spaces; the number of BAs of various sorts; endomorphisms of BAs, including the existence of endo-rigid BAs; automorphisms groups; reconstruction of BAs from their automorphism groups; embeddings and automorphisms, especially for complete rigid

BAs; rigid BAs; and homogeneous BAs. Section C is devoted to special classes of BAs: superatomic algebras, mainly thin-tall and related BAs; projective BAs; and two lengthy chapters on countable BAs, with Ketonen's theorem; and on measure algebras, giving an extensive survey of this topic which is perhaps the most important subfield of the theory of BAs for most mathematicians. Section D deals with logical questions: decidable and undecidable theories of BAs in various languages; recursive BAs; Lindenbaum–Tarski algebras; and Boolean-valued models of set theory. Two appendices, on set theory and on topology, explain some more advanced notions used in some places in the Handbook. There is a chart of topological duality. Finally, there is a comprehensive Bibliography on the aspects of the theory of Boolean algebras treated in the Handbook.

Many people contributed to the Handbook by checking manuscripts for mathematical and typographical errors; in addition to several of the authors of the Handbook, the editor is indebted to the following for help of this sort: Hajnal Andréka, Aleksander Błaszczyk, Tim Carlson, Ivo Düntsch, Francisco J. Freniche, Lutz Heindorf, Istvan Németi, Stevo Todorčević, and Petr Vojtaš. Thanks are also due to the North-Holland staff, especially Leland Pierce, for their editorial work.

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Part I

GENERAL THEORY OF BOOLEAN ALGEBRAS

Sabine KOPPELBERG

Freie Universität Berlin

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To the memory of Bernhard Görnemann
who incited my interest in mathematics.

Introduction to Part I

The history of Boolean algebras goes back to George Boole (BOOLE [1854]). Boole stated a list of algebraic identities governing the "laws of thought", i.e. of classical propositional logic. The algebraic structures satisfying Boole's identities were first considered in HUNTINGTON [1904] and called Boolean *algebras* in SHEFFER [1913].

Boole had in mind two interpretations for his identities. The first of these is the two-element Boolean algebra $2 = \{0, 1\}$, where 0 is identified with the truth-value "false", and 1 with the truth-value "true", together with the operations corresponding to the logical ones of disjunction, conjunction and negation. The second interpretation was the "algebra of classes", where the Boolean operations were interpreted by those of union, intersection and complementation of arbitrary classes. More generally, and in a setting which avoids proper classes, every algebra of sets is a Boolean algebra. Here for any set X, an algebra of sets over X is a non-empty subfamily of the power set $P(X)$ closed under the finitary set-theoretical operations of union, intersection and complementation with respect to X. Boole's observation amounted, in algebraic language, to saying that his identities held true under both interpretations. Interestingly enough, Boole's second example was a precursor to Cantor's set theory which began to emerge around 1874.

Only in 1921 (respectively 1936), was it proved that Boole's identities give in fact a complete axiomatization for both of his interpretations: the completeness theorem for propositional logic (POST [1921]) amounts to saying that every identity valid in the two-element Boolean algebra is derivable from Boole's axioms, and Stone's representation theorem (STONE [1936]) asserts that every Boolean algebra is isomorphic to an algebra of sets. The proofs of both results are closely connected.

Stone duality, a fundamental part of the theory of Boolean algebras, sets up an equivalence between Boolean algebras and Boolean spaces, i.e. totally disconnected compact Hausdorff spaces. Thus, through a growing interest of topologists in Boolean spaces, Boolean algebras have a bearing on topology. They are also important in measure theory and functional analysis by way of measure spaces and Boolean algebras of projections in Banach spaces. But the main applications of Boolean algebras are still in parts of mathematics related to logic: in switching algebra, a topic not covered by our presentation, classical propositional (respectively predicate) calculus and set theory. The latter applications are based on the fact that sentences of propositional or predicate logic can be given truth values not just in the two-element Boolean algebra $2 = \{0, 1\} = \{\text{false}, \text{true}\}$ but in an arbitrary Boolean algebra. The first success of this concept of Boolean-valued model was a new and particularly lucid algebraic proof for the completeness theorems of propositional and predicate logic found by Rasiowa and Sikorski in 1950 (RASIOWA and SIKORSKI [1963]). The major breakthrough, however, was the observation made in 1967 by Scott, Solovay and Vopěnka that Cohen's construction of generic models for independence proofs in set theory can be conceived as an instance of Boolean-valued models. Not only do complete Boolean algebras thus contribute to the understanding of models of set theory, but conversely the Boolean-valued version of Cohen's method has been applied to prove mathematical theorems on Boolean algebras by metamathematical means.

CHAPTER 1

Elementary Arithmetic

Sabine KOPPELBERG
Freie Universität Berlin

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Introduction

Boolean algebras are defined by a list of algebraic axioms; this chapter deals with arithmetical laws derivable from the axioms and with notions arising in a particularly simple way from the arithmetic of Boolean algebras: atoms, relative algebras, and disjointness.

We give, in Section 1, a somewhat lengthy list of axioms for Boolean algebras. There are many other, in particular shorter, axiom systems for Boolean algebras; a lot of work on these has been done but is outside the scope of this book. Our axiom system will, however, appear completely natural to a reader familiar with lattice theory: the axioms simply state that a Boolean algebra is a distributive and complemented lattice. We derive a certain amount of additional algebraic laws and give examples of Boolean algebras. The most sophisticated of these come from logic, functional analysis and topology: Lindenbaum–Tarski algebras, Boolean algebras of projections in Banach spaces, and regular open algebras of topological spaces.

In Section 2 we use the laws established so far to prove Stone's representation theorem, the most important fact in the structure theory of Boolean algebras: every Boolean algebra is isomorphic to an algebra of sets. This gives enough insight to dispense once and for all with proofs of further arithmetical laws since it implies that an equation holds in every Boolean algebra iff it holds in every algebra of sets. In Section 3 we prove elementary results on disjointness and two non-trivial theorems on disjoint families: the Balcar–Vojtáš theorem on disjoint refinements and the Erdös–Tarski theorem on the existence of large disjoint families, the latter one being a standard tool in combinatorial questions on Boolean algebras.

We freely introduce and use some basic algebraic notions such as subalgebras, homomorphisms, etc. as they occur naturally in the proofs. They will be studied in greater detail in Chapter 2.

1. Examples and arithmetic of Boolean algebras

This section presents a study of the arithmetic of Boolean algebras and a variety of examples, the most important ones arising in set theory, logic, and topology: algebras of sets, Lindenbaum–Tarski algebras, and regular open algebras.

1.1. Definitions and notation

1.1. Definition. A *Boolean algebra* is a structure $(A, +, \cdot, -, 0, 1)$ with two binary operations $+$ and $\cdot$, a unary operation $-$, and two distinguished elements 0 and 1 such that for all x, y and z in A,

(associativity) (B1) $x + (y + z) = (x + y) + z$, (B1′) $x \cdot (y \cdot z) = (x \cdot y) \cdot z$,

(commutativity)	(B2) $x + y = y + x$,	(B2′) $x \cdot y = y \cdot x$,
(absorption)	(B3) $x + (x \cdot y) = x$,	(B3′) $x \cdot (x + y) = x$,
(distributivity)	(B4) $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$,	(B4′) $x + (y \cdot z) = (x + y) \cdot (x + z)$,
(complementation)	(B5) $x + (-x) = 1$,	(B5′) $x \cdot (-x) = 0$.

Two standard examples of Boolean algebras, algebras of sets and Lindenbaum–Tarski algebras, arise in set theory and logic. The operations $+$, $\cdot$ and $-$ of a Boolean algebra are therefore often written as $\cup$, $\cap$, $-$ or $\vee$, $\wedge$, $\neg$ and called union, intersection, complement or disjunction, conjunction, negation or, as in lattice theory, join, meet, complement. We follow Boole's original notation which is frequently used in modern texts on axiomatic set theory and think about $+$, $\cdot$, $-$ as being spelled as sum, product, complement. It should be pointed out that the distinguished elements 0 and 1 of a Boolean algebra are not assumed to be distinct – see Example 1.5.

The roles of $+$ and $\cdot$ respectively of 0 and 1 in the above axiom system are completely symmetrical, a fact more thoroughly expressed in the duality principle 1.13. Nevertheless, to save notation we shall henceforth adopt the familiar convention that multiplication binds stronger than addition and omit parentheses around products whenever possible. We also agree that $-$ binds stronger than both $+$ and $\cdot$ and write, for example, $-x + y \cdot -z$ for $(-x) + y \cdot (-z)$, etc. By the first two couples of axioms, we tacitly omit parentheses and permute summands (respectively factors) in sums (respectively products).

The structure $(A, +, \cdot, -, 0, 1)$ is usually identified with its underlying set A. This gives rise to the following definition.

1.2. DEFINITION. A Boolean algebra $(A, +, \cdot, -, 0, 1)$ *is finite* (*countable*, *of cardinality* $\kappa, \ldots$) if its underlying set A is finite (countable, of cardinality $\kappa, \ldots$).

There is a perfectly natural notion of homomorphism between Boolean algebras which makes the class of all Boolean algebras, together with all homomorphisms between them, into a category. When dealing with different Boolean algebras $(A, +_A, \cdot_A, -_A, 0_A, 1_A)$ and $(B, +_B, \cdot_B, -_B, 0_B, 1_B)$, we drop the subscripts on $+$, $\cdot$, etc. if no confusion arises.

1.3. DEFINITION. A *homomorphism* from a Boolean algebra A into a Boolean algebra B is a map $f\colon A \to B$ such that

$$f(0) = 0\,, \qquad f(1) = 1\,,$$

and for all x, y in A,

$$f(x + y) = f(x) + f(y)\,, \qquad f(x \cdot y) = f(x) \cdot f(y)\,,$$

$$f(-x) = -f(x)\,.$$

f is an *isomorphism* from A onto B if f is a bijective homomorphism from A onto

B. A and B are *isomorphic* ($A \cong B$) if there exists an isomorphism from A onto B.

1.2. Algebras of sets

The examples given in the following list are simple but quite basic and will often be referred to. Two of these, power set algebras and interval algebras, have very special properties not shared by every Boolean algebra. For example, power set algebras are complete but interval algebras of infinite linear orders are not, as explained in the subsection on infinite operations. On the other hand, algebras of sets provide the most general example of Boolean algebras: every Boolean algebra is, by Stone's representation theorem 2.1, isomorphic to an algebra of sets, more precisely, by Theorem 7.8, to the clopen algebra of some topological space.

1.4. EXAMPLE (power set algebras). Let X be any set and $P(X)$ its power set. The structure

$$(P(X), \cup, \cap, -, \emptyset, X),$$

with $-a$ the complement $X \setminus a$ of a with respect to X, is a Boolean algebra – the axioms (B1) through (B5′) simply state elementary laws of set theory. $P(X)$ is called the *power set algebra* of X.

1.5. EXAMPLE (the trivial Boolean algebra). For X the empty set, $P(X)$ reduces to the Boolean algebra $A = (P(\emptyset), \ldots, 0_A, 1_A)$ with $0_A = 1_A$, the *trivial* or *one-element Boolean algebra*. Obviously, any two Boolean algebras with exactly one element are isomorphic.

1.6. EXAMPLE (the two-element Boolean algebra). For a singleton $X = \{x\}$, $P(X)$ reduces to $\{0, 1\}$, where $0 = \emptyset$ and $1 = X$. This algebra is called the *two-element algebra* and, following a convention in set theory, denoted by 2. Its operations are given by the table below.

x	y	$x+y$	$x \cdot y$	$-x$
0	0	0	0	1
0	1	1	0	1
1	0	1	0	0
1	1	1	1	0

If 0 and 1 are identified with the truth values "false" and "true", then the Boolean operations on 2 represent the logical operations of disjunction, conjunction and negation. It follows from 1.18 and 1.21 below that any Boolean algebra with exactly two elements is isomorphic to 2.

1.7. DEFINITION. A structure $(A, +_A, \cdot_A, -_A, 0_A, 1_A)$ is a *subalgebra* of the Boolean algebra $(B, +_B, \cdot_B, -_B, 0_B, 1_B)$ if $A \subseteq B$, $0_A = 0_B$, $1_A = 1_B$ and the operations $+_A$, $\cdot_A$, $-_A$ are the restrictions of $+_B$, $\cdot_B$, $-_B$ to A.

Again we drop the subscripts whenever possible. By identification of Boolean algebras with their underlying sets, a subalgebra of B is simply a subset A of B containing 0_B and 1_B and closed under the operations of B. A is then a Boolean algebra in its own right since the axioms are valid for arbitrary elements of B and, a fortiori, of A.

1.8. DEFINITION. A subalgebra of a power set algebra $P(X)$ is called an *algebra of subsets of X* or *an algebra of sets over X*. A Boolean algebra is an *algebra of sets* if it is an algebra of sets over X, for some set X.

Note that A being an algebra of sets over X not only requires that the elements of A are subsets of X but also that the operations of A are the set-theoretical ones inherited from $P(X)$ and that $\emptyset$ and X are contained in A.

1.9. EXAMPLE (finite–cofinite algebras). Let X by any set. A subset a of X is called *cofinite in X* or, for fixed X, simple cofinite, if $X \backslash a$ is finite. Let

$$A = \{a \subseteq X: a \text{ finite or cofinite}\} .$$

A is then an algebra of sets over X, the finite–cofinite algebra on X. To check that $a \cup b$ and $a \cap b$ are in A for a, b in A, note that $a \cup b$ is finite for a, b finite and cofinite otherwise; $a \cap b \in A$ follows from de Morgan's law,

$$a \cap b = X \backslash ((X \backslash a) \cup (X \backslash b)) ,$$

since A is closed under $-$ and $\cup$.

If X has infinite cardinality κ, then so has the finite–cofinite algebra on X, since X has exactly κ finite subsets. Thus, every infinite cardinal is the cardinality of a Boolean algebra. A non-negative integer k, however, is the cardinality of a Boolean algebra iff $k = 2^n$ for some $n \in \omega$, as follows from Corollary 2.8.

1.10. EXAMPLE (clopen algebras). Let X be a topological space. A subset of X is *clopen* if it is both closed and open. The set Clop X of clopen subsets of X is an algebra of sets over X, the clopen algebra of X.

1.11. EXAMPLE (interval algebras). Let L be a linearly ordered set with first element 0_L. Extend the linear order of L to $L \cup \{\infty\}$, where ∞ is an element not contained in L, by letting $x < \infty$ for $x \in L$. For x, $y \in L \cup \{\infty\}$, the set

$$[x, y) = \{z \in L: x \leq z < y\}$$

is the *half-open interval* of L determined by x and y. We show that the set A of finite unions of half-open intervals is an algebra of sets over L. In fact, $L = [0_L, \infty)$ and $\emptyset = [x, x)$, where $x \in L$, are in A, and A is closed under finite unions. By de Morgan's law, it suffices as in the example of finite–cofinite algebras to check that A is closed under complementation.

To see this, first consider the relative position of two non-empty intervals $[x, y)$ and $[s, t)$ in L. If $y < s$, then $[x, y)$ lies left of $[s, t)$ and $[y, s)$ is non-empty; if $t < x$, then $[s, t)$ lies left of $[x, y)$ and $[t, x)$ is non-empty. In both cases, $[x, y) \cup [s, t)$ is not an interval. In the remaining case we have $s \leq y$, $x \leq t$ and

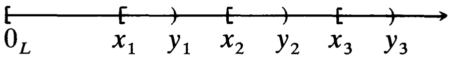
$$[x, y) \cup [s, t) = [\min(x, s), \max(y, t))$$

is a half-open interval.

Now let

$$a = [x_1, y_1) \cup \cdots \cup [x_n, y_n)$$

be an element of A where the number n of intervals is minimal for a. Then no interval $[x_i, y_i)$ is empty; hence, $x_i < y_i$. For $i \neq j$, $[x_i, y_i) \cup [x_j, y_j)$ is not an interval by minimality of n, so $[x_i, y_i)$ lies left of $[x_j, y_j)$ or vice versa. Thus, after a permutation of indices, we may assume that $x_1 < y_1 < x_2 < y_2 < \cdots < x_n < y_n$:



and conclude that

$$L \backslash a = [0_L, x_1) \cup [y_1, x_2) \cup \cdots \cup [y_n, \infty)$$

is again in A.

The Boolean algebra A is called the *interval algebra* of L and denoted by Intalg L. It can also be constructed for linear orderings without first element, by simply attaching one.

1.3. Lindenbaum–Tarski algebras

Our second major example of Boolean algebras, the Lindenbaum–Tarski algebras, exemplify the connection between Boolean algebras and logic. It will be proved in Section 9 that every Boolean algebra is isomorphic to a Lindenbaum–Tarski algebra. By their very definition, Lindenbaum–Tarski algebras are not algebras of sets.

1.12. EXAMPLE (algebras of formulas and Lindenbaum–Tarski algebras). Let L be a language for propositional or first order logic and T a theory, i.e. an arbitrary set of sentences, in L. For formulas α, β of L, define

$$\alpha \sim \beta \quad \text{iff } T \vdash \alpha \leftrightarrow \beta ,$$

i.e. iff $\alpha \leftrightarrow \beta$ is formally provable from the axioms of T in classical propositional (respectively predicate) calculus. $\sim$ is then an equivalence relation on the set of all formulas; e.g. it follows from $\alpha \sim \beta$ and $\beta \sim \gamma$ by

$$\vdash (\alpha \leftrightarrow \beta) \wedge (\beta \leftrightarrow \gamma) \rightarrow (\alpha \leftrightarrow \gamma)$$

and modus ponens that $\alpha \sim \gamma$. Let $[\alpha]$ be the equivalence class of α with respect to $\sim$ and put

$$B(T) = \{[\alpha]\colon \alpha \text{ a formula of } L\} .$$

By defining

$$[\alpha] + [\beta] = [\alpha \vee \beta] ,$$

$$[\alpha] \cdot [\beta] = [\alpha \wedge \beta] ,$$

$$-[\alpha] = [\neg \alpha] ,$$

$$1 = [\alpha_0 \rightarrow \alpha_0] ,$$

$$0 = [\alpha_0 \wedge \neg \alpha_0] ,$$

where α_0 is an arbitrary formula, we make $B(T)$ into a Boolean algebra, the *algebra of* (*equivalence classes of*) *formulas with respect to* T. In fact, the operations of $B(T)$ are well-defined since

$$\vdash (\alpha \leftrightarrow \alpha') \wedge (\beta \leftrightarrow \beta') \rightarrow ((\alpha \vee \beta) \leftrightarrow (\alpha' \vee \beta')) ,$$

etc.; the associative law (B1) holds in $B(T)$ since

$$\vdash \alpha \vee (\beta \vee \gamma) \leftrightarrow (\alpha \vee \beta) \vee \gamma ,$$

etc. For every formula α, $[\alpha] = 1$ iff $T \vdash \alpha$. Thus, T is syntactically consistent (i.e. not every formula is derivable from T) iff $B(T)$ is not the trivial Boolean algebra.

There are several naturally defined subalgebras of $B(T)$, e.g. if we are dealing with first order logic and $x_1, \ldots, x_n$ is a fixed list of individual variables, then

$$A = \{[\alpha]\colon \text{all free variables of } \alpha \text{ are among } x_1, \ldots, x_n\}$$

is a subalgebra. In particular, the subalgebra

$$\{[\alpha]\colon \alpha \text{ a sentence of } L\}$$

is called the *Lindenbaum–Tarski algebra* of T.

1.4. *The duality principle*

Before embarking on the arithmetic of Boolean algebras, we state a general principle which saves a good deal of computational work. We do not bother to formulate and prove it in great detail but think that even a reader inexperienced in logic will understand it. For every statement ϕ on Boolean algebras, let the *dual statement* $\mathrm{d}\phi$ of ϕ be obtained by systematically exchanging the symbols $+$, $\cdot$ and 0, 1 in ϕ. Obviously $\mathrm{dd}\phi = \phi$ and each of the axioms (B_i') in Definition 1.1 is the dual of (B_i). It is this self-duality of the axiom system which gives rise to the following theorem.

1.13. THEOREM (duality principle). *If a statement holds in every Boolean algebra, then so does its dual.*

PROOF. Let ϕ hold in every Boolean algebra and let $\boldsymbol{B} = (B, +, \cdot, -, 0, 1)$ be any Boolean algebra; we show that $\mathrm{d}\phi$ holds in $\boldsymbol{B}$. Consider the dual structure $\mathrm{d}\boldsymbol{B} = (B, \cdot, +, -, 1, 0)$ obtained from $\boldsymbol{B}$ by exchanging the operations $+$ and $\cdot$ and the elements 0 and 1. An arbitrary statement ψ holds in $\mathrm{d}\boldsymbol{B}$ iff $\mathrm{d}\psi$ holds in $\mathrm{dd}\boldsymbol{B} = \boldsymbol{B}$. In particular by self-duality of the axiom system in Definition 1.1, $\mathrm{d}\boldsymbol{B}$ is a Boolean algebra. So ϕ holds in $\mathrm{d}\boldsymbol{B}$ and $\mathrm{d}\phi$ holds in $\boldsymbol{B}$. □

1.5. *Arithmetic of Boolean algebras. Connection with lattices*

We now prove a number of laws governing the elementary arithmetic of Boolean algebras. They will be used in the future without specific reference. All laws are assumed to be assertions about arbitrary elements x, y, z ... of a Boolean algebra. By the duality principle, we shall prove only one of two dual statements in the following.

1.14. LEMMA. (a) (idempotence) $x + x = x$ *and* $x \cdot x = x$.
(b) $x + y = y$ *iff* $x \cdot y = x$.

PROOF. (a)

$$\begin{aligned} x + x &= x + x \cdot (x + x) && \text{by (B3}') \\ &= x && \text{by (B3)}. \end{aligned}$$

(b) If $x + y = y$, then, by (B3′),

$$x \cdot y = x \cdot (x + y) = x,$$

and the converse implication follows by duality. □

We take for granted the notion of a partial order or partially ordered set $(P, \leq)$, i.e. a set P with a binary relation $\leq$ which is reflexive, transitive, and antisymmetric (if $x \leq y$ and $y \leq x$, then $x = y$). The subsequent lemmas prove that every Boolean algebra is partially ordered, in a natural way, and its operations can be recovered from the partial order. In fact, Boolean algebras are often defined as being distributive complemented lattices, a special kind of partially ordered sets.

1.15. DEFINITION. Let $(P, \leq)$ be a partial order. For $M \subseteq P$ and $a \in P$, a is a *lower bound* (an *upper bound*) of M if $a \leq m$ ($m \leq a$) for every $m \in M$. $a \in P$ is a *least element* (*greatest element*) of P if a is a lower bound (an upper bound) of P. For $M \subseteq P$ and $a \in P$, a is a *greatest lower bound* of M, $a = \text{glb}\, M$, (a *least upper bound* of M, $a = \text{lub}\, M$) if a is a lower bound (an upper bound) of M and $a' \leq a$ holds for each lower bound a' of M ($a \leq a'$ holds for each upper bound a' of M). $(P, \leq)$ is a *lattice* if, for any x and y in P, $\text{glb}\{x, y\}$ and $\text{lub}\{x, y\}$ exist.

In the above definition, a lower bound for $M \subseteq P$ does not necessarily exist nor is it unique. It is, however, easily checked that a least element of P or a greatest lower bound of $M \subseteq P$, if they exist, are uniquely determined. Similar reasoning applies to greatest elements of P and least upper bounds of M, and thus justifies the notation $a = \text{glb}\, M$ and $a = \text{lub}\, M$.

1.16. LEMMA. *For every Boolean algebra A, the relation $\leq$ defined by*

$$x \leq y \quad \textit{iff } x + y = y$$

(iff $x \cdot y = x$, by 1.14) is a partial order on A. $(A, \leq)$ is a lattice in which $\text{glb}\{x, y\} = x \cdot y$ *and* $\text{lub}\{x, y\} = x + y$.

PROOF. The relation $\leq$ is reflexive by 1.14(a). If $x \leq y$ and $y \leq z$, then

$$\begin{aligned} x + z &= x + (y + z) && \text{by } y \leq z \\ &= (x + y) + z \\ &= y + z && \text{by } x \leq y \\ &= z && \text{by } y \leq z\,, \end{aligned}$$

so $x \leq z$ and $\leq$ is a transitive relation. It follows from $x \leq y$ and $y \leq x$ that $y = x + y = y + x = x$; thus, the relation $\leq$ is antisymmetric and $(A, \leq)$ is a partially ordered set.

$x + y$ is an upper bound of both x and y: e.g. by 1.14(a),

$$x + (x + y) = (x + x) + y = x + y\,,$$

which shows $x \leq x + y$. Let z be an arbitrary upper bound of x and y. Then

$$(x+y)+z = x+(y+z),$$
$$= x+z \qquad \text{by } y \leq z$$
$$= z \qquad \text{by } x \leq z,$$

thus $x+y \leq z$ and we have proved $x+y = \mathrm{lub}\{x, y\}$. It follows by dual considerations that $x \cdot y = \mathrm{glb}\{x, y\}$. □

The partial order constructed in the preceding lemma is sometimes referred to as the *canonical* or the *Boolean partial order* on A. For algebras of sets, the Boolean partial order simply is set-theoretical inclusion.

In the proofs of 1.14 and 1.16, we only used the axioms (B1) through (B3′), hence every structure $(A, +, \cdot)$ satisfying (B1) through (B3′) can be made into a lattice $(A, \leq)$. There is a converse to this construction: for $(A, \leq)$ a lattice, define binary operations $+$ and $\cdot$ by letting $x+y = \mathrm{lub}\{x, y\}$ and $x \cdot y = \mathrm{glb}\{x, y\}$. The structure $(A, +, \cdot)$ then clearly satisfies (B2) and (B2′) (commutativity of $+$ and $\cdot$). Axiom (B1) (associativity of $+$) holds since both $x+(y+z)$ and $(x+y)+z$ coincide with $\mathrm{lub}\{x, y, z\}$ and (B3) holds since $x \cdot y \leq x$. (B1′) and (B3′) follow by dual reasoning.

It is not difficult to see that the constructions outlined above, of lattices from algebraic structures $(A, +, \cdot)$ satisfying (B1) through (B3′) and vice versa, are converses of each other. We shall freely identify lattices $(A, \leq)$ with their associated algebraic structures.

1.17. LEMMA. *The distributive laws* (B4) *and* (B4′) *are equivalent in every lattice.*

PROOF. By duality, it is sufficient to prove (B4′) from (B4) and the lattice axioms (B1) through (B3′):

$$(x+y)\cdot(x+z) = (x+y)\cdot x + (x+y)\cdot z \qquad \text{by (B4)}$$
$$= x + (x \cdot z + y \cdot z) \qquad \text{by (B3′), (B4)}$$
$$= x + y \cdot z \qquad \text{by (B1), (B3)}. \quad \square$$

We turn to implications of (B5) and (B5′). The following laws characterize the distinguished elements 0 and 1 of a Boolean algebra A as the unique least and greatest elements of the lattice $(A, \leq)$.

1.18. LEMMA. (a) $0 \leq x \leq 1$.
(b) $x+0 = x$ *and* $x \cdot 1 = x$.
(c) $x \cdot 0 = 0$ *and* $x+1 = 1$.

PROOF. By (B5′) and 1.16, 0 is the greatest lower bound of x and $-x$; in particular, $0 \leq x$. Hence, $x+0 = x$ and $x \cdot 0 = 0$. The rest follows by duality. □

1.19. DEFINITION. Let $(L, \leq)$ be a lattice. L is *distributive* if the distributive law (B4) (hence, by 1.17, (B4′)) holds in L. If L has a least element 0 and a greatest element 1, then $y \in L$ is a *complement* of $x \in L$ if $x + y = 1$ and $x \cdot y = 0$. L is *complemented* if L has a least and a greatest element and each element of L has a complement.

Inspection of our axiom system shows that Boolean algebras can be described as being distributive complemented lattices. By the following proof, complements are unique in distributive lattices.

1.20. LEMMA. *$-x$ is the unique complement of x.*

PROOF. By (B5) and (B5′), $-x$ is a complement of x. If y and z are both complements of x, then

$$\begin{aligned} z &= z \cdot 1 \\ &= z \cdot (x + y) && \text{by } x + y = 1 \\ &= z \cdot x + z \cdot y && \text{by (B4)} \\ &= 0 + z \cdot y && \text{by } x \cdot z = 0 \\ &= z \cdot y \,, \end{aligned}$$

hence $z \leq y$. Similarly, $y \leq z$ and thus $y = z$ by antisymmetry of the Boolean partial order. □

Since 0, 1 and the operations $+$, $\cdot$, $-$ of a Boolean algebra are definable in terms of its partial order, we find (taking for granted the notion of isomorphism for partial orders) that two Boolean algebras are isomorphic as algebraic structures iff their associated partial orders are isomorphic.

1.21. LEMMA. (a) $- - x = x$,
(b) *if* $-x = -y$, *then* $x = y$,
(c) $-0 = 1$ and $-1 = 0$,
(d) (de Morgan's laws) $-(x + y) = -x \cdot -y$ *and* $-(x \cdot y) = -x + -y$.

PROOF. (a) Both x and $--x$ are complements of $-x$.
(b) If $-x = -y$, then $x = - - x = - - y = y$.
(c) By 1.18(c), $0 + 1 = 1$ and $0 \cdot 1 = 0$. So 0 and 1 are complements of each other.
(d) By distributivity and absorption,

$$\begin{aligned} (x + y) \cdot -x \cdot -y &= x \cdot -x \cdot -y + y \cdot -x \cdot -y \,, \\ &= 0 + 0 \,, \\ &= 0 \,, \end{aligned}$$

and

$$\begin{aligned}(x+y)+-x\cdot -y &= x\cdot(y+-y)+y+-x\cdot -y\\ &= x\cdot y+x\cdot -y+y+-x\cdot -y\\ &= y+x\cdot -y+-x\cdot -y\\ &= y+-y\\ &= 1\,,\end{aligned}$$

thus $-x\cdot -y$ is the complement of $x+y$. The second law follows by duality. □

1.22. LEMMA (monotonicity laws). *If* $x\leq x'$ *and* $y\leq y'$, *then* $x+y\leq x'+y'$, $x\cdot y\leq x'\cdot y'$ *and* $-x'\leq -x$.

PROOF. The first assertion follows from

$$(x+y)+(x'+y')=(x+x')+(y+y')=x'+y'\,,$$

the second one by duality, and the third one by de Morgan's law and

$$-x'+-x=-(x\cdot x')=-x\,. \quad \square$$

1.23. LEMMA. (a) $x\leq y$ *iff* $-y\leq -x$ *iff* $x\cdot -y=0$.
(b) $z\cdot x\leq y$ *iff* $x\leq -z+y$.

PROOF. (a) By 1.22, $x\leq y$ implies $-y\leq -x$. Conversely, $-y\leq -x$ implies $x=--x\leq --y=y$. If $x\leq y$, then $x=x\cdot y$ and $x\cdot -y=x\cdot y\cdot -y=0$. Conversely, $x\cdot -y=0$ implies that

$$\begin{aligned}x &= x\cdot(y+-y)\\ &= x\cdot y+x\cdot -y\\ &= x\cdot y\end{aligned}$$

and $x\leq y$.

(b) If $z\cdot x\leq y$, then by monotonicity of $+$,

$$x=z\cdot x+-z\cdot x\leq y+-z\,.$$

If $x\leq -z+y$, then by monotonicity of $\cdot$,

$$z\cdot x\leq z\cdot(-z+y)=z\cdot y\leq y. \quad \square$$

1.6. Connection with Boolean rings

There is a close connection between Boolean algebras and particular commutative rings with unit, the so-called Boolean rings. For an outline of this, we consider one more operation on Boolean algebras definable in terms of $+$, $\cdot$ and $-$, symmetric difference. Two other less important operations are defined in Exercise 1.

1.24. DEFINITION. For any elements x, y of a Boolean algebra A, the *symmetric difference* of x and y is

$$x \triangle y = x \cdot -y + y \cdot -x .$$

Thus, for an algebra A of sets, $x \triangle y$ as defined above is the set-theoretic symmetric difference $(x \setminus y) \cup (y \setminus x)$ of x and y.

1.25. LEMMA. (a) $x = y$ *iff* $x \triangle y = 0$.
(b) $x \triangle (y \triangle z) = (x \triangle y) \triangle z$.
(c) $x \cdot (y \triangle z) = (x \cdot y) \triangle (x \cdot z)$.

PROOF. (a)

$$\begin{aligned} x = y \quad & \text{iff } x \leq y \text{ and } y \leq x \\ & \text{iff } x \cdot -y = 0 \text{ and } y \cdot -x = 0 \quad \text{by 1.23(a)} \\ & \text{iff } x \triangle y = 0 . \end{aligned}$$

(b) By de Morgan's laws and distributivity,

$$\begin{aligned} -(y \triangle z) &= (-y + z) \cdot (-z + y) \\ &= y \cdot z + -y \cdot -z . \end{aligned}$$

Evaluation of $x \triangle (y \triangle z)$ gives, using the same laws,

$$\begin{aligned} x \triangle (y \triangle z) &= x \cdot (y \cdot z + -y \cdot -z) + -x \cdot (y \cdot -z + -y \cdot z) \\ &= x \cdot y \cdot z + x \cdot -y \cdot -z + -x \cdot y \cdot -z + -x \cdot -y \cdot z . \end{aligned}$$

By symmetry, evaluation of $(x \triangle y) \triangle z$ gives the same result.
(c)

$$\begin{aligned} x \cdot (y \triangle z) &= x \cdot (y \cdot -z + z \cdot -y) \\ &= x \cdot y \cdot -z + x \cdot z \cdot -y \end{aligned}$$

and

$$\begin{aligned} (x \cdot y) \triangle (x \cdot z) &= x \cdot y \cdot (-x + -z) + x \cdot z \cdot (-x + -y) \\ &= x \cdot y \cdot -z + x \cdot z \cdot -y . \quad \square \end{aligned}$$

1.26. DEFINITION. An element x of a ring R is *idempotent* if $x \cdot x = x$. R is a *Boolean ring* if R is a ring with unit and every element of R is idempotent.

1.27. PROPOSITION. *The following assignments give a one-to-one correspondence between Boolean algebras and Boolean rings: for a Boolean algebra* $\boldsymbol{B} = (B, +, \cdot, -, 0, 1)$, *let*

$$r\boldsymbol{B} = (B, \triangle, \cdot, 0, 1)\ ;$$

for a Boolean ring $\boldsymbol{R} = (R, \oplus, \cdot, 0, 1)$, *let*

$$b\boldsymbol{R} = (R, +, \cdot, -, 0, 1)\ ,$$

where for x *and* y *in* R, $x + y = x \oplus y \oplus (x \cdot y)$ *and* $-x = x \oplus 1$.

PROOF. $r\boldsymbol{B}$ is a Boolean ring; this follows immediately from Lemma 1.25 and the fact that symmetric difference is commutative and has 0 as a neutral element.

To prove that $b\boldsymbol{R}$ is a Boolean algebra we first list some consequences of $\boldsymbol{R}$ being Boolean. As usual in ring theory, we omit parentheses around products whenever possible.

(1) $\qquad x \cdot y \oplus y \cdot x = 0\ ,$

since

$$\begin{aligned} x \oplus y &= (x \oplus y) \cdot (x \oplus y) \\ &= x \cdot x \oplus x \cdot y \oplus y \cdot x \oplus y \cdot y \\ &= x \oplus x \cdot y \oplus y \cdot x \oplus y\ . \end{aligned}$$

Letting $x = y$ in (1), we find

(2) $\qquad x \oplus x = 0\ ;$

then (1) and (2) imply that

(3) $\qquad x \cdot y = y \cdot x\ ,$

i.e. that a Boolean ring is commutative. The Boolean algebra axioms for $b\boldsymbol{R}$ are proved by straightforward computation, using the ring axioms for $\boldsymbol{R}$ and the additional laws (2) and (3).

It follows from the elementary facts proved above and easy computations that $br\boldsymbol{B} = \boldsymbol{B}$ and $rb\boldsymbol{R} = \boldsymbol{R}$. $\square$

By this equivalence between Boolean algebras and Boolean rings, the theory of Boolean algebras could be subsumed under that of rings, and there is in fact some literature where this is done. But since the intuitive idea of a Boolean algebra is rather that of being an algebra of sets – and Stone's representation theorem 2.1 proves this intuition to be correct – or a lattice, we will never use the ring-theoretic approach.

The construction of a Boolean algebra from a Boolean ring can be somewhat generalized to other rings. For an arbitrary ring R, let $\operatorname{Idp} R$ be the set of all idempotent elements of R. Then if $(R, \oplus, \cdot, 0, 1)$ is a commutative ring with unit, $(\operatorname{Idp} R, +, \cdot, -, 0, 1)$ with $+$ and $-$ defined as in 1.27 is a Boolean algebra. For a ring $(R, \oplus, \cdot, 0, 1)$ with unit but not necessarily commutative, $(B, +, \cdot, -, 0, 1)$ is a Boolean algebra where

$$B = \{x \in \operatorname{Idp} R: x \cdot r = r \cdot x \text{ for every } r \in R\}$$

and $+$, $-$ are defined as in 1.27. Similar examples of Boolean algebras are relevant to functional analysis and described below in the subsection on Boolean algebras of projections.

1.7. *Infinite operations*

The Boolean operations $+$, $\cdot$, and $-$ reflect the finitary set-theoretical operations of union, intersection, and complement. We now introduce two operations Σ (sum) and Π (product) defined for some, but not necessarily all, subsets of a Boolean algebra; they reflect several properties of unions and intersections of arbitrary families of sets. The formal definition 1.28 of sums and products is motivated by the fact that, for M a family of subsets of X, $\bigcup M$ and $\bigcap M$ are the least upper bound and the greatest lower bound of M in the partial order $(P(X), \subseteq)$. Also if $x_1, \ldots, x_n$ are finitely many elements of a lattice L and $n \geq 1$, then an easy induction shows that $x_1 + \cdots + x_n$ and $x_1 \cdot \cdots \cdot x_n$ are the least upper bound and the greatest lower bound of $\{x_1, \ldots, x_n\}$ in L.

1.28. DEFINITION. Let A be a Boolean algebra. For $M \subseteq A$, ΣM (ΠM) is the least upper bound (the greatest lower bound) of M in the partial order $(A, \leq)$ if it exists. We write $\Sigma_{i \in I} m_i$ ($\Pi_{i \in I} m_i$) if $M = \{m_i: i \in I\}$ and ΣM (ΠM) exists. A is *complete* if ΣM and ΠM exist for every $M \subseteq A$. Let κ be an infinite cardinal. A is *κ-complete* if ΣM and ΠM exist for each $M \subseteq A$ of cardinality less than κ. A is σ-complete if ΣM and ΠM exist for each countable $M \subseteq A$.

A word of caution is in order here. If A is an algebra of sets and $\bigcup M \in A$ for some subset M of A, then clearly $\bigcup M = \Sigma M$. It is however possible that ΣM exists but does not coincide with $\bigcup M$: e.g. let A be the interval algebra of the real line $\boldsymbol{R}$ and

$$M = \{[x, \infty): x \in \boldsymbol{R}, 0 < x\} ;$$

then $\bigcup M$ is the open interval $(0, \infty)$ and ΣM is the half-open interval $[0, \infty)$. Thus, when dealing with a subalgebra A of a Boolean algebra B, it is usually necessary to distinguish, for $M \subseteq A$, the least upper bound $\Sigma^A M$ of M in A and the least upper bound $\Sigma^B M$ of M in B; similarly for $\Pi^A M$ and $\Pi^B M$.

1.29. DEFINITION. A subalgebra A of a Boolean algebra B is a *regular subalgebra* of B if for each $M \subseteq A$ such that $\Sigma^A M$ ($\Pi^A M$) exists, also $\Sigma^B M$ ($\Pi^B M$) exists and $\Sigma^A M = \Sigma^B M$ ($\Pi^A M = \Pi^B M$). An algebra A of sets is a *complete algebra of sets* if $\bigcup M \in A$ and $\bigcap M \in A$ for each $M \subseteq A$. A is a *κ-algebra* (a *σ-algebra*) *of sets* if $\bigcup M \in A$ and $\bigcap M \in A$ for each $M \subseteq A$ of size less than κ (each countable $M \subseteq A$).

Thus, a complete algebra of sets is, by definition, an algebra of sets in which sums and products of arbitrary subsets not only exist but are formed set-theoretically. It could also be defined, in a more abstract way, as being a regular subalgebra of a power set algebra which is complete in its own right. Every power set algebra is a complete algebra of sets; Exercise 4 shows that, conversely, a complete algebra of sets is isomorphic to a power set algebra. The regular open algebra of the reals considered in 1.37 is complete and, by Stone's theorem 2.1, isomorphic to an algebra of sets. It is, however, not isomorphic to any σ-algebra of sets; cf. Exercise 3 of Section 2. For any infinite regular cardinal κ and any set X, a simple example of a κ-algebra of sets over X is given by

$$A = \{a \subseteq X: |a| < \kappa \text{ or } |X \setminus a| < \kappa\} .$$

1.30. EXAMPLE (Borel algebras). Let X be a topological space and $\boldsymbol{O}$ its topology, the family of open subsets of X. Then

$$\text{Bor } X = \bigcap \{A \subseteq P(X): A \text{ a } \sigma\text{-algebra of sets over } X \text{ and } \boldsymbol{O} \subseteq A\}$$

is a σ-algebra of sets over X containing all open sets, in fact the smallest σ-algebra of sets over X containing all open sets. It is called the *Borel algebra* of X; the sets in Bor X are the *Borel subsets* of X.

1.31. EXAMPLE (Baire algebras). Let X be a topological space. A subset a of X is said to have the *Baire property* if $a \triangle u$ is meager, for some open subset u of X. Elementary topological reasoning shows that the set

$$\text{Bai } X = \{a \subseteq X: a \text{ has the Baire property}\}$$

is a σ-algebra of sets over X, the *Baire algebra* of X; see Exercise 6 for the relevant definitions. Since every open subset of X has the Baire property, Bor X is a subalgebra of Bai X.

1.32. EXAMPLE (the algebra of measurable sets). The set Leb $\boldsymbol{R}$ of all Lebesgue-measurable subsets of $\boldsymbol{R}$ is a σ-algebra of sets over $\boldsymbol{R}$. Since every open subset of $\boldsymbol{R}$ is Lebesgue-measurable, Bor $\boldsymbol{R}$ is a subalgebra of Leb $\boldsymbol{R}$. More generally, σ-algebras of sets arise in measure theory through the concept of a measurable space, i.e. a pair (X, A), where X is a set and A a σ-algebra of sets over X.

The algebras considered in Examples 1.9 and 1.11 are, in all interesting cases, not even σ-complete. E.g. let A be the finite–cofinite algebra over an infinite set X and let $Y \subseteq X$ be such that both Y and $X \setminus Y$ are infinite; then $M = \{\{y\}: y \in Y\}$

has no least upper bound in A. More generally, if κ is regular and infinite and X has cardinality at least κ, then the algebra $\{a \subseteq X: |a| < \kappa \text{ or } |X \setminus a| < \kappa\}$ is κ-complete but not κ^+-complete. For L an infinite linear order, the interval algebra A of L as defined in 1.11 is not σ-complete since without loss of generality there is a strictly ascending sequence $(x_n)_{n\in\omega}$ in L. Then

$$M = \{[x_{2n}, x_{2n+1}): n \in \omega\}$$

has no least upper bound in A. For suppose

$$a = [y_1, z_1) \cup \cdots \cup [y_k, z_k)$$

is an upper bound. For some $s \in \{1, \ldots, k\}$, the set N consisting of those n such that $[x_{2n}, x_{2n+1}) \subseteq [y_s, z_s)$ is infinite, and there is some n such that both n and $n+1$ are in N. Then $y_s < x_{2n+1} < x_{2n+2} < z_s$ and $a \setminus [x_{2n+1}, x_{2n+2})$ is an upper bound of M in A strictly smaller than a.

1.33. LEMMA. *Let A be a Boolean algebra and suppose the least upper bounds of the subsets M, N, $M_1, \ldots, M_k$, M_i ($i \in I$) exist. Then the right-hand sides in* (a), (b), (c) *exist and*

(a) (de Morgan's law) $-\Sigma M = \Pi\{-m: m \in M\}$,

(b) (distributivity) for $a \in A$, $a \cdot \Sigma M = \Sigma\{a \cdot m: m \in M\}$,

(c) (distributivity) $\Sigma M \cdot \Sigma N = \Sigma\{m \cdot n: m \in M, n \in N\}$ *and*

$$\Sigma M_1 \cdot \cdots \cdot \Sigma M_k = \Sigma\{m_1 \cdot \ldots \cdot m_k: m_i \in M_i \text{ for } 1 \le i \le k\},$$

(d) (associativity) $\Sigma_{i\in I}(\Sigma M_i) = \Sigma(\bigcup_{i\in I} M_i)$ *in the sense that, if one of these sums exists, then so does the other one, and they coincide.*

PROOF. (d) holds because the subsets $\{\Sigma M_i; i \in I\}$ and $\bigcup_{i\in I} M_i$ of A have the same set of upper bounds.

(a) We show that, for $s = \Sigma M$, $-s$ is the greatest lower bound of the set $\{-m: m \in M\}$. Now for $m \in M$, $m \le s$ and hence $-s \le -m$ by 1.22. If x is another lower bound of $\{-m: m \in M\}$ in A, then for $m \in M$, $x \le -m$ and, again by 1.22, $m \le -x$, hence $s \le -x$ and $x \le -s$.

(b) Again let $s = \Sigma M$; we show that $a \cdot s$ is the least upper bound of the set $\{a \cdot m: m \in M\}$. By $a \cdot m \le a \cdot s$ for $m \in M$, $a \cdot s$ is an upper bound; let t be another one. For $m \in M$, $a \cdot m \le t$, so $m \le -a + t$ by 1.23(b). Then $s \le -a + t$ and, again by 1.23(b), $a \cdot s \le t$.

(c) follows by iterated application of (b), making use of (d). □

By duality, we obtain under assumptions dual to those of 1.33,

$$-\Pi M = \Sigma\{-m: m \in M\},$$

$$a + \Pi M = \Pi\{a + m: m \in M\},$$

$$\Pi M_1 + \cdots + \Pi M_k = \Pi\{m_1 + \cdots + m_k: m_i \in M_i \text{ for } 1 \le i \le k\},$$

$$\prod_{i\in I}\left(\Pi M_i\right) = \Pi\left(\bigcup_{i\in I} M_i\right).$$

de Morgan's laws show that in the definition of complete and κ-complete Boolean algebras, one only needs to insist that for all M (respectively for all M of power less than κ), ΣM exists – or that, for all M (of power less than κ), ΠM exists.

The algebraic laws in 1.33 and Stone's theorem 2.1 might suggest that the infinite operations Σ and Π obey exactly the same laws as the set-theoretical ones of union and intersection. It will, however, be explained in Section 14 that distributive laws more general than 1.33(c) do hold in complete algebras of sets but not in arbitrary complete Boolean algebras.

1.8. Boolean algebras of projections

We sketch an example of Boolean algebras which arises in functional analysis. The reader is advised to consult the book by DUNFORD and SCHWARTZ [1957–71] for proofs and more detailed definitions.

For V any vector space over a field K, the linear maps of V into itself form a ring $\mathrm{End}_K V$ under the operations $+$ of pointwise addition and $\circ$ of composition, the endomorphism ring of V. The zero O_E (respectively the unit 1_E) of $\mathrm{End}_K V$ is the zero map (respectively the identity map) id_X. With the additional operation of scalar multiplication by elements of K, $\mathrm{End}_K V$ is also a K-algebra. It is, of course, non-commutative if V has dimension at least 2. The idempotent elements of $\mathrm{End}_K V$, i.e. the linear transformations e of V satisfying $e \circ e = e$, are called *projections*. More precisely, e is said to be a *projection* onto the range of e, i.e. the linear subspace $\mathrm{im}\, e$ or V.

1.34. DEFINITION. A *Boolean algebra of projections in V* is a set A of idempotent elements in $\mathrm{End}_K V$ such that $O_E \in A$, $1_E \in A$, the elements of A are pairwise commuting and A is closed under the operations defined by

$$e \vee f = e + f - e \circ f\,,$$
$$e \wedge f = e \circ f\,,$$
$$\neg e = 1_E - e\,.$$

A straightforward ring-theoretical computation shows, as in 1.27 and the remarks following it, that a Boolean algebra of projections as defined above is really a Boolean algebra. Its Boolean partial order is given by

(4) $\quad e \leq f \quad \text{iff}\ \mathrm{im}\, e \subseteq \mathrm{im}\, f\,.$

Historically, Boolean algebras of projections arose from the following examples. If V is a finite-dimensional unitary vector space over the complex numbers and $f\colon V \to V$ is a self-adjoint linear transformation, then the spectral theorem says that V has an orthonormal base $\{v_1, \ldots, v_n\}$ consisting of eigenvectors of f, and the eigenvalue λ_i of v_i is a real number. Now if e_i denotes the orthogonal projection of V onto the subspace spanned by v_i, then

$$f = \sum_{1 \le i \le n} \lambda_i e_i \,,$$

where the e_i can be considered as elements of the finite Boolean algebra

$$A = \left\{ \sum_{i \in M} e_i \colon M \subseteq \{1, \ldots, n\} \right\}$$

of projections in V.

This example has a less trivial generalization to infinite-dimensional spaces: for V a Banach space over the reals $\boldsymbol{R}$ or the complex numbers $\boldsymbol{C}$, the bounded linear transformations of V into itself form a Banach algebra B, a subalgebra of $\mathrm{End}_K V$, and a Boolean algebra of projections is then naturally assumed to be a subset of B. The finite-dimensional spectral theorem generalizes to the situation where V is a Hilbert space over the complex numbers and $f\colon V \to V$ a bounded self-adjoint transformation; it gives a representation

$$f = \int \lambda E(\mathrm{d}\lambda)$$

of f as the integral of the function $\mathrm{id}_{\boldsymbol{R}}$ with respect to a spectral measure E. Here a spectral measure is a homomorphism from the Boolean algebra Bor $\boldsymbol{R}$ of Example 1.30 into a Boolean algebra A of projections in V which satisfies an additional requirement of countable completeness, similar to the definition 1.35 below. In the finite-dimensional case considered above, E would be the map defined by

$$E(u) = \left\{ \sum e_i \colon 1 \le i \le n,\ \lambda_i \in u \right\}$$

for each Borel set u in $\boldsymbol{R}$.

It is natural to require, for A a Boolean algebra of projections in a Banach space V, that the infinite operations of A, whenever defined, respect the topological structure of V. In view of (4), we define:

1.35. DEFINITION. A Boolean algebra of projections in a Banach space is a *complete* (σ*-complete*) *algebra of projections* if for each $M \subseteq A$ (each countable $M \subseteq A$), $\Sigma^A M$ and $\Pi^A M$ exist and

$$\mathrm{im}\left(\Pi^A M\right) = \bigcap \{\mathrm{im}\, e \colon e \in M\} \,,$$

$$\mathrm{im}\left(\Sigma^A M\right) = \text{the topological closure of the linear subspace generated by } \bigcup \{\mathrm{im}\, e \colon e \in M\} \,.$$

It can be shown that A is a complete Boolean algebra of projections iff, for each subset M of A, $\Pi^A M$ ($\Sigma^A M$) is the limit, in the strong operator topology, of the net $\{\Pi^A M' \colon M' \subseteq M \text{ finite}\}$ (of the net $\{\Sigma^A M' \colon M' \subseteq M \text{ finite}\}$, respectively).

Let us consider two simple examples of Boolean algebras of projections. For the Hilbert space

$$H=\left\{(x_n)_{n\in\omega}: x_n \in \boldsymbol{R},\ \sum_{n\in\omega} x_n^2<\infty\right\}$$

and $u \subseteq \omega$, define $e_u: H \to H$ by

$$e_u(x) = x \cdot \chi_u$$

(pointwise multiplication), where $\chi_u: \omega \to \{0, 1\}$ is the characteristic function of u. Then e_u is a projection of H.

$$A = \{e_u: u \subseteq \omega\}$$

is a Boolean algebra of projections in H which is isomorphic to $P(\omega)$ and hence complete as a Boolean algebra. It is not difficult to check that A is also a complete Boolean algebra in the sense of Definition 1.35.

On the other hand, let I be an infinite set, V the Banach space of all bounded functions from I into $\boldsymbol{R}$ with the sup-norm; again for $u \subseteq I$, define $e_u: V \to V$ by $e_u(x) = x \cdot \chi_u$. Then

$$A = \{e_u: u \subseteq I\}\ ,$$

being isomorphic to $P(I)$, is complete as a Boolean algebra but not complete in the sense of 1.35. To see this, let $u \subseteq I$ be the union of a strictly increasing sequence $(u(n))_{n\in\omega}$ of subsets of u. If x denotes the map from I to $\boldsymbol{R}$ with constant value 1, then $e_u(x) = \chi_u$ is not the limit of the sequence $e_{u(n)}(x) = \chi_{u(n)}$ since $\|e_u(x) - e_{u(n)}(x)\| = 1$ for every n; thus $(e_{u(n)})_{n\in\omega}$ does not converge to e_u in the strong operator topology.

1.9. *Regular open algebras*

We give a standard example of a complete Boolean algebra, the regular open algebra of a topological space. For X a topological space and $a \subseteq X$, we denote by int a the interior and by cl a the closure of a in X.

1.36. DEFINITION. Let X be a topological space. For $a \subseteq X$, $\mathrm{r}a = \operatorname{int}\operatorname{cl} a$ is the *regularization* of a. $u \subseteq X$ is *regular open* if $\mathrm{r}u = u$.

$$\mathrm{RO}(X) = \{u \subseteq X: u \text{ regular open}\}$$

is the *regular open algebra* of X.

The name regular open *algebra* is justified by the following theorem where, for notational convenience, a Boolean algebra is identified with its associated partial order.

1.37. THEOREM. RO(X) *is a complete Boolean algebra under set-theoretical inclusion. The distinguished elements and the operations of* RO(X) *are given by*

$$0 = \emptyset\,, \qquad 1 = X\,,$$
$$u + v = \mathrm{r}(u \cup v)\,, \qquad u \cdot v = u \cap v\,, \qquad -u = \mathrm{int}(X \setminus u)\,,$$
$$\sum M = \mathrm{r}\Big(\bigcup M\Big)\,, \qquad \prod M = \mathrm{r}\Big(\bigcap M\Big)\,.$$

PROOF. We establish six elementary facts on regular open sets. Let u, v, w range over open subsets of X.

(5) $u \subseteq \mathrm{r}u$

since $u \subseteq \mathrm{cl}\, u$ and $u = \mathrm{int}\, u \subseteq \mathrm{int\,cl}\, u \mathrm{r} u$.

(6) If u and v are regular open, then so is $u \cap v$,

since $u \cap v \subseteq \mathrm{r}(u \cap v)$ by (5), while $\mathrm{r}(u \cap v) \subseteq \mathrm{r}u = u$ and $\mathrm{r}(u \cap v) \subseteq \mathrm{r}v = v$ imply that $\mathrm{r}(u \cap v) \subseteq u \cap v$.

(7) $\mathrm{r}a$ is regular open for each $a \subseteq X$:

we have $\mathrm{r}a \subseteq \mathrm{cl}\, a$, $\mathrm{cl\,r}a \subseteq \mathrm{cl}\, a$, so $\mathrm{rr}a = \mathrm{int\,cl\,r}a \subseteq \mathrm{int\,cl}\, a = \mathrm{r}a$. But $\mathrm{r}a \subseteq \mathrm{rr}a$ holds by (5).

(8) $\mathrm{r}u$ is the least regular open set including u,

since $u \subseteq \mathrm{r}u \in \mathrm{RO}(X)$ by (5) and (7), and $u \subseteq v \in \mathrm{RO}(X)$ implies $\mathrm{r}u \subseteq \mathrm{r}v = v$.

(9) $u \cap \mathrm{r}a \subseteq \mathrm{r}(u \cap a)$ for each $a \subseteq X$:

openness of u implies $u \cap \mathrm{cl}\, a \subseteq \mathrm{cl}(u \cap a)$, hence $u \cap \mathrm{r}a = \mathrm{int}\, u \cap \mathrm{int\,cl}\, a = \mathrm{int}(u \cap \mathrm{cl}\, a) \subseteq \mathrm{int\,cl}(u \cap a) = \mathrm{r}(u \cap a)$.

(10) If $u \cap v = \emptyset$, then $\mathrm{r}u \cap \mathrm{r}v = \emptyset$

since $u \subseteq X \setminus v$, $\mathrm{cl}\, u \subseteq \mathrm{cl}(X \setminus v) = X \setminus v$, $\mathrm{cl}\, u \cap v = \emptyset$ and $\mathrm{r}u \cap v = \emptyset$. By the same reasoning, $\mathrm{r}u \cap \mathrm{r}v = \emptyset$.

Now for a proof of the theorem, note that $(\mathrm{RO}(X), \subseteq)$ is a partial order with a least and a greatest element, since both the empty set and X are regular open. For any subset M of RO(X), $\bigcup M$ is the least set including each $m \in M$ and it is open, so by (8) $\mathrm{r}(\bigcup M)$ is the least upper bound of M in RO(X). To show that $w = \mathrm{r}(\bigcap M)$ is the greatest lower bound of M, observe that $w \in \mathrm{RO}(X)$ by (7). Since $\bigcap M \subseteq m$ for $m \in M$, it follows that $w \subseteq \mathrm{r}m = m$; so w is a lower bound of M. If v is another lower bound of M in RO(X), then $v \subseteq \bigcap M$ and $v = \mathrm{r}v \subseteq \mathrm{r}(\bigcap \mathrm{M}) = \mathrm{w}$.

We have thus proved that $(\mathrm{RO}(X), \subseteq)$ is a lattice in which every subset has both a least upper bound and a greatest lower bound. In particular,

$(\mathrm{RO}(X), +, \cdot)$ satisfies the lattice axioms (B1) through (B3′) since by (6), $u \cdot v = u \cap v$ for $u, v \in \mathrm{RO}(X)$. Also $u \cdot \mathrm{int}(X \backslash u) = \emptyset$ for $u \in \mathrm{RO}(X)$ since u and $\mathrm{int}(X \backslash u)$ are disjoint sets and $u + \mathrm{int}(X \backslash u) = 1$ since $u \cup \mathrm{int}(X \backslash u)$ is a dense subset of X.

We are left with the proof of the distributive laws, and by Lemma 1.17 we need only check (B4). So let u, v, w be regular open. By the lattice axioms, we have $u \subseteq u + v$, so $w \cdot u \subseteq w \cdot (u + v)$. Similarly, $w \cdot v \subseteq w \cdot (u + v)$ which gives $w \cdot u + w \cdot v \subseteq w \cdot (u + v)$. Conversely,

$$\begin{aligned} w \cdot (u + v) &= w \cap \mathrm{r}(u \cup v) \\ &\subseteq \mathrm{r}(w \cap (u \cup v)) \qquad \text{by (9)} \\ &= \mathrm{r}((w \cap u) \cup (w \cap v)) \\ &= w \cdot u + w \cdot v\,. \quad \square \end{aligned}$$

Even a very special case of this last theorem gives, up to isomorphism, all complete Boolean algebras: every partial order $(P, \leq)$ can be topologized in a trivial way by taking the sets $\{q \in P \colon q \leq p\}$, where $p \in P$, as the base of a topology. It will be proved in Section 4 that every complete Boolean algebra B is isomorphic to $\mathrm{RO}(P)$ for some partial order P. This connection between partial orders and complete Boolean algebras is, in axiomatic set theory, responsible for the equivalence between forcing with partially ordered sets and forcing with complete Boolean algebras.

Exercises

1. Define, in a Boolean algebra A, the binary operations $\mid$ (Sheffer stroke) and $\uparrow$ (Peirce arrow) by

$$x \mid y = -x \cdot -y\,, \qquad x \uparrow y = -x + -y\,.$$

Prove that 0, 1, and the Boolean operations are definable, in terms of equations, both by $\mid$ and by $\uparrow$.

2. Using the notation of Proposition 1.27, check that $b\boldsymbol{R}$ is a Boolean algebra and that $br\boldsymbol{B} = \boldsymbol{B}$, $rb\boldsymbol{R} = \boldsymbol{R}$.

3. Let A be the finite–cofinite algebra on X and B the power set algebra of X. Then, for any subset M of A, $\Sigma^A M$ exists iff $\bigcup M \in A$, and in this case $\Sigma^A M = \bigcup M = \Sigma^B M$.

4. Every complete algebra of sets, as defined in 1.29, is isomorphic to a power set algebra.

Hint. If A is a complete algebra of sets over X, define an equivalence relation $\sim$ on X by

$$x \sim y \quad \text{iff for each } a \in A,\ x \in a \text{ iff } y \in a\,.$$

Then $A \cong P(Y)$, where Y is the set of equivalence classes of $\sim$.

5. Let T be a first order theory in the language L and $\phi(xx_1 \ldots x_n)$ an L-formula. Let, in the algebra $B(T)$ of (equivalence classes of) formulas,

$$M_\phi = \{[\phi(tx_1 \ldots x_n)]: t \text{ a term of } L\},$$

where $\phi(tx_1 \ldots x_n)$ arises from $\phi(xx_1 \ldots x_n)$ by substitution of t for x, after renaming bound variables in ϕ if necessary. Prove that

$$\sum M_\phi = [\exists x\, \phi(xx_1 \ldots x_n)], \qquad \prod M_\phi = [\forall x\, \phi(xx_1 \ldots x_n)].$$

6. In a topological space X, call a subset a of X *nowhere dense* if $\operatorname{int} \operatorname{cl} a = \emptyset$ and *meager* if it is the union of countably many nowhere dense sets. Prove that the Baire algebra of X, as defined in 1.31, is a σ-algebra of sets over X.

7. In a topological space X, call $a \subseteq X$ *regular closed* if $a = \operatorname{cl} \operatorname{int} a$. The set of all regular closed subsets of X is a complete Boolean algebra, under inclusion, which is isomorphic to $\mathrm{RO}(X)$.

2. Atoms, ultrafilters and Stone's theorem

The principal result of this section is the following theorem.

2.1. THEOREM (Stone's representation theorem, set-theoretical version). *Every Boolean algebra is isomorphic to an algebra of sets.*

We first give a simple proof under the additional hypothesis that the algebra involved has enough atoms – see Corollary 2.7; in particular this gives a complete description of finite Boolean algebras in 2.8 and 2.9. In the general case, the notion of an atom has to be replaced by that of an ultrafilter, and the Boolean prime ideal theorem 2.16 guarantees that every Boolean algebra has sufficiently many ultrafilters. It does so, however, only at the expense of using the axiom of choice in a significant way; see the discussion following Corollary 2.17.

As a consequence of Stone's theorem, we find an elementary method for checking validity of an equation in all Boolean algebras (Proposition 2.19). We finally consider the Rasiowa–Sikorski lemma 2.21, an extension of the Boolean prime ideal theorem which is useful in predicate logic and axiomatic set theory.

2.1. Atoms

2.2. NOTATION. For x and y in a Boolean algebra A, $x \not\leq y$ if $x \leq y$ does not hold. $x < y$ (or $y > x$, x is *strictly smaller* than y) if $x \leq y$ but $x \neq y$.

$$A^+ = \{x \in A: 0 < x\}$$

($= A \setminus \{0\}$, by 1.18) is the set of *positive* elements of A.

2.3. DEFINITION. Let A be a Boolean algebra. $a \in A$ is an *atom* of A if $0 < a$ but there is no x in A satisfying $0 < x < a$. At A is the set of atoms of A. A is *atomless* if it has no atoms and *atomic* if for each positive element x of A, there is some atom a such that $a \leq x$.

For example, a power set algebra $P(X)$ and the finite–cofinite algebra on X (Example 1.9) are atomic, the atoms being the singletons $\{x\}$, where $x \in X$. Also each finite Boolean algebra is atomic since if $x > 0$ in a Boolean algebra A and there is no atom below x, then there exists a strictly decreasing infinite sequence $x_0 = x > x_1 > x_2 > \cdots$ in A^+. The interval algebra of the real line (Example 1.11) is atomless and so is the regular open algebra of the reals in their usual topology (Theorem 1.37). To see the latter, note that for $s < t$ in $\boldsymbol{R}$, the open interval (s, t) of $\boldsymbol{R}$ is regular open. Now if a is a positive element of $\mathrm{RO}(\boldsymbol{R})$, then there are $u < s < t < v$ in $\boldsymbol{R}$ such that $(u, v) \subseteq a$ which shows that, in $\mathrm{RO}(\boldsymbol{R})$, $0 < (s, t) < a$ and thus that a is not an atom. Similarly for each dense linear order L, Intalg L and $\mathrm{RO}(L)$ are atomless if L is given the order topology. If L is a linear order isomorphic to a copy of ω followed by a copy of $\boldsymbol{R}$, then the interval algebra of L is neither atomic nor atomless.

2.4. LEMMA. *The following are equivalent, for every element a of A:*
(a) *a is an atom of A,*
(b) *for every x in A, $a \leq x$ or $a \leq -x$ but not both,*
(c) *$a > 0$ and, for all x and y in A, $a \leq x + y$ iff $a \leq x$ or $a \leq y$.*

PROOF. (a) implies (b): let a be an atom and $x \in A$. If both $a \leq x$ and $a \leq -x$, then $a \leq x \cdot -x = 0$, a contradiction. If $a \not\leq x$ then $0 < a \cdot -x \leq a$ by 1.23; since a is an atom $a \cdot -x = a$ and thus $a \leq -x$.

(b) implies (c): trivially both $a \leq x$ and $a \leq y$ imply $a \leq x + y$. For the converse, assume $a \leq x + y$ but $a \not\leq x$. Then $a \leq -x$ by (b) and $a \leq -x \cdot (x + y) = -x \cdot y \leq y$. Also $a > 0$ since otherwise, both $a \leq x$ and $a \leq -x$ hold for every x in A.

(c) implies (a): assume that $b \in A$ and $0 \leq b < a$ with the aim of showing that $b = 0$. Now $a = a \cdot b + a \cdot -b = b + a \cdot -b$, and since $a \not\leq b$, we obtain $a \leq a \cdot -b$ by (c). So $a \leq -b$ and, by 1.23, $0 = a \cdot b = b$. □

A homomorphism between two Boolean algebras was defined in 1.3 to be a map respecting all Boolean operations.

2.5. DEFINITION. A homomorphism $f\colon A \to B$ between the Boolean algebras A and B is a *monomorphism* or an *embedding* of A into B if it is one-to-one. It is an *epimorphism* if it is onto.

The image $f[A]$ of A under a homomorphism $f\colon A \to B$ is a subalgebra of B; if f is an embedding, then A is isomorphic to $f[A]$ via f.

2.6. PROPOSITION. *For every Boolean algebra A, the map from A into the power set algebra $P(\mathrm{At}\ A)$ defined by*

$$f(x) = \{a \in \mathrm{At}\, A\colon a \leq x\}$$

is a homomorphism. It is an embedding if A is atomic and an epimorphism if A is complete.

Proof. Obviously, $f(0) = \emptyset$ and $f(1) = \mathrm{At}\, A$. By 2.4(b),

$$\begin{aligned} f(-x) &= \{a \in \mathrm{At}\, A\colon a \leq -x\} \\ &= \mathrm{At}\, A \setminus \{a \in \mathrm{At}\, A\colon a \leq x\} \\ &= \mathrm{At}\, A \setminus f(x)\,, \end{aligned}$$

and $f(x + y) = f(x) \cup f(y)$ follows similarly from 2.4(c). Also, $f(x \cdot y) = f(x) \cap f(y)$ since

$$a \leq x \cdot y \quad \text{iff } a \leq x \text{ and } a \leq y$$

holds for every a in A. Thus, f is a homomorphism.

Now let A be atomic and $x \neq y$ in A; without loss of generality, $x \not\leq y$. But then $x \cdot -y$ is non-zero by 1.23, and there is an atom a of A such that $a \leq x \cdot -y$. It follows that $a \in f(x)$ and $a \not\in f(y)$, so $f(x) \neq f(y)$ and f is one-to-one.

Let A be complete and $Y \subseteq \mathrm{At}\, A$. We prove that $Y = f(s)$, where $s = \Sigma\, Y$, which shows that f is onto: if $a \in Y$, then $a \leq s$ and $a \in f(s)$. Conversely, consider $a \in \mathrm{At}\, A \setminus Y$. Then for every $y \in Y$, a and y are distinct atoms of A, hence $a \not\leq y$, $a \leq -y$ by 2.4(b), and $a \cdot y = 0$ by 1.23. It follows from the distributive law 1.33(b) that $a \cdot s = 0$, so $a \not\in f(s)$. □

The last proposition not only gives a weak version of Stone's theorem, but also describes the complete atomic and the finite Boolean algebras.

2.7. Corollary. *Every atomic Boolean algebra is isomorphic to an algebra of sets. Every complete and atomic Boolean algebra is isomorphic to a power set algebra.* □

2.8. Corollary. *The finite Boolean algebras are, up to isomorphism, exactly the power set algebras of finite sets.*

Proof. If A is a finite Boolean algebra then $\mathrm{At}\, A$ is finite and A is both complete and atomic. By Proposition 2.6, A is isomorphic to $P(\mathrm{At}\, A)$. □

In particular, a natural number is the cardinality of a Boolean algebra iff it is a power of 2.

2.9. Corollary. *Two finite Boolean algebras are isomorphic iff they have the same cardinality.*

PROOF. If A and B have both cardinality k, then by $A \cong P(\mathrm{At}\, A)$ and $B \cong P(\mathrm{At}\, B)$ we have $k = 2^n$, where $n = |\mathrm{At}\, A| = |\mathrm{At}\, B|$. Now any bijection between $\mathrm{At}\, A$ and $\mathrm{At}\, B$ gives rise to an isomorphism between $P(\mathrm{At}\, A)$ and $P(\mathrm{At}\, B)$, hence between A and B. □

2.2. *Ultrafilters and Stone's theorem*

Ultrafilters, the main tool in the proof of Stone's theorem, arise naturally from embeddings of Boolean algebras into power set algebras. If $e: A \to P(X)$ is such an embedding or, more generally, a homomorphism, then for any point x of X, the subset

$$p = \{a \in A: x \in e(a)\}$$

of A has the following properties:

$$\begin{aligned} 1 &\in p\,, \qquad 0 \notin p\,, \\ a \cdot b \in p &\quad \text{iff } a \in p \text{ and } b \in p\,, \\ a + b \in p &\quad \text{iff } a \in p \text{ or } b \in p\,, \\ -a \in p &\quad \text{iff } a \notin p\,. \end{aligned}$$

The subsets of A with these properties are exactly the ultrafilters of A defined below, as follows from Proposition 2.15. Thus, if e embeds A into $P(X)$, then the points of X give rise to ultrafilters of A. Conversely, Stone's theorem is proved by taking the ultrafilters of A as the points of a set $\mathrm{Ult}\, A$; Corollary 2.16 then says that $\mathrm{Ult}\, A$ is large enough to embed A into $P(\mathrm{Ult}\, A)$.

2.10. DEFINITION. A *filter* in a Boolean algebra A is a subset p of A such that

$1 \in p$,

if $x \in p$, $y \in A$ and $x \leq y$, then $y \in p$,

if $x \in p$ and $y \in p$, then $x \cdot y \in p$.

For example, for each $a \in A$, the set $\{x \in A; a \leq x\}$ is a filter in A; for $a = 1$ it reduces to $\{1\}$ and for $a = 0$ it coincides with A, giving rise to the following definition.

2.11. DEFINITION. A filter p of A is a *principal filter* if $p = \{x \in X: a \leq x\}$ for some $a \in A$; p is then the *principal filter generated by a*. p is the *trivial filter* if $p = \{1\}$; it is a *proper filter* if $0 \notin p$, i.e. if $p \neq A$.

2.12. LEMMA AND DEFINITION. Let E be a subset of A. Then the set

$$\{x \in A: e_1 \cdot \cdots \cdot e_n \leq x \text{ for some } n \in \omega \text{ and } e_1, \ldots, e_n \in E\}$$

is a filter of A, *the filter generated by* E *in* A. E has the *finite intersection property* if, for all $n \in \omega$ and $e_1, \ldots, e_n \in E$, $e_1 \cdot \cdots \cdot e_n > 0$.

2.13. LEMMA. *The filter generated by E in A is the least filter of A including E. It is proper iff E has the finite intersection property.* □

Filters can also be characterized as being subsets p of A such that $1 \in p$ and

(1) for all $x, y \in A$, $x \cdot y \in p$ iff $x \in p$ and $y \in p$:

if p is a filter and $x \cdot y \in p$, then $x \in p$ and $y \in p$ since $x \cdot y \leq x$ and $x \cdot y \leq y$; so p satisfies (1). Conversely, if p satisfies (1) and $x \in p$, $y \in A$ and $x \leq y$, then $x \cdot y = x \in p$ implies $y \in p$, so p is a filter. For more special filters we get stronger properties similar to the properties of atoms listed in Lemma 2.4:

2.14. DEFINITION. A filter p of A is an *ultrafilter* if, for each $x \in A$, $x \in p$ or $-x \in p$ but not both. p is a *prime filter* if it is proper and, for $x, y \in A$, $x + y \in p$ implies that $x \in p$ or $y \in p$. p is a *maximal filter* if it is proper and there is no proper filter of A having p as a proper subset.

A typical example of an ultrafilter is provided, for A an algebra of sets over X and x any point of X, by the set $\{a \in A: x \in a\}$. The principal filter generated by an element a of an arbitrary Boolean algebra A is, by Lemma 2.4, an ultrafilter iff it is prime iff a is an atom of A; distinct atoms of A generate distinct ultrafilters. Thus ultrafilters can be considered as generalizations of atoms.

2.15. PROPOSITION AND DEFINITION. For every filter, the properties of being maximal, an ultrafilter and prime are equivalent. An arbitrary subset p of a Boolean algebra A is an ultrafilter iff its characteristic function $\chi_p: A \to 2$ is a homomorphism from A into the two-element Boolean algebra, the *characteristic homomorphism*.

PROOF. For any filter of A, each of the three properties implies properness.

Let p be maximal; we prove that p is an ultrafilter. Let $x \in A$. Since p is proper, x and $-x$ cannot both be in p. Suppose $x \notin p$. By maximality of p, the filter generated by $p \cup \{x\}$ contains 0, hence by 2.13, $a \cdot x = 0$ for some $a \in p$. Thus, $a \leq -x$ and $-x \in p$.

Every ultrafilter is prime: assuming $x \notin p$ and $y \notin p$, we have to show that $x + y \notin p$. This follows from $-x \in p$, $-y \in p$ and $-x \cdot -y = -(x + y) \in p$.

Primeness implies maximality: we assume $x \notin p$ and prove that $p \cup \{x\}$ generates the improper filter. Since $1 = x + -x$ is in p but x is not, $-x \in p$. So the filter generated by $p \cup \{x\}$ contains $x \cdot -x = 0$.

An arbitrary subset p of A is, by (1), a proper filter iff $\chi_p(1) = 1$, $\chi_p(0) = 0$ and χ_p preserves the operation $\cdot$. It is, by the first part of our proposition, an ultrafilter iff χ_p also preserves the operations $+$ and $-$. □

Because of the equivalence between ultrafilters, prime filters and the prime ideals defined in Section 5 and also for historical reasons, the following existence theorem for ultrafilters, or rather its consequence that every non-trivial Boolean algebra has an ultrafilter, is called the Boolean prime *ideal* theorem (BPI).

2.16. PROPOSITION (Boolean prime ideal theorem). *A subset of a Boolean algebra is included in an ultrafilter iff it has the finite intersection property.*

PROOF. If $E \subseteq A$ is included in an ultrafilter p of A, then by properness of p, p and hence E have the finite intersection property.

Conversely, assume $E \subseteq A$ has the finite intersection property, so the filter p_0 generated by E is proper by 2.13. The set P of all proper filters of A including p_0 is non-empty and partially ordered by inclusion, moreover each non-empty chain C in P has $\bigcup C$ as an upper bound in P. By Zorn's lemma, let p be a maximal element of P. Then p is a maximal filter and includes E; by 2.15, it is an ultrafilter. □

2.17. COROLLARY. *An element a of a Boolean algebra is contained in an ultrafilter iff $a > 0$.*

PROOF. The set $\{a\}$ has the finite intersection property iff $a > 0$. □

The proof given above of the Boolean prime ideal theorem uses Zorn's lemma, an equivalent of the axiom of choice, and it is in fact shown in FEFERMAN [1965] that BPI is not derivable from the axiom system ZF of Zermelo–Fraenkel set theory without the axiom of choice. On the other hand, HALPERN and LEVY [1971] show that in ZF, BPI is strictly weaker than the axiom of choice. Like the axiom of choice, BPI has several interesting equivalences. For example, it is shown in KELLEY [1950] that the full axiom of choice is equivalent in ZF to Tychonoff's theorem (the product space of a family of compact topological spaces is compact) for arbitrary spaces, but the restriction of Tychonoff's theorem to Hausdorff spaces is equivalent to BPI (cf. ŁOS and RYLL-NARDZEWSKI [1954]; RUBIN and SCOTT [1954]).

With the machinery of ultrafilters at our hands, we are ready to embed any Boolean algebra into a power set algebra.

2.18. DEFINITION. For A a Boolean algebra,

$$\text{Ult } A = \{p \subseteq A\colon p \text{ an ultrafilter of } A\}$$

is the set of ultrafilters of A. The map $s\colon A \to P(\text{Ult } A)$ defined by

$$s(x) = \{p \in \text{Ult } A\colon x \in p\}$$

is the *Stone map* of A.

Proof of Stone's representation theorem 2.1. It follows as in the proof of 2.6 that the Stone map s is a homomorphism from A into $P(\text{Ult } A)$. For example, $s(0) = \emptyset$ since every ultrafilter is proper,

$$\begin{aligned} s(x + y) &= \{p: x + y \in p\} \\ &= \{p: x \in p\} \cup \{p: y \in p\} \\ &= s(x) \cup s(y), \end{aligned}$$

since ultrafilters are prime, etc. We prove that s is one-to-one: let $x \neq y$ in A; without loss of generality, $x \not\leq y$. Thus, $x \cdot -y > 0$; by 2.17 let p be an ultrafilter containing $x \cdot -y$. Then $x \in p$ and $y \notin p$ which gives $p \in s(x) \setminus s(y)$. □

The set Ult A of ultrafilters of A and the Stone embedding s of A will be analyzed more thoroughly in Section 7: Theorem 7.8, the topological version of Stone's theorem, states that Ult A can be topologized in such a way that s is an isomorphism from A onto the clopen algebra of Ult A.

2.3. Arithmetic revisited

Stone's theorem makes Boolean algebras look quite simple but, apart from giving an intuitive picture of how they look like, it does by no means trivialize every problem. For example, none of the combinatorial proofs in the following section would be really easier for algebras of sets than for arbitrary Boolean algebras. However, there is one aspect of their theory which *is* simplified by Stone's theorem, or rather the method of ultrafilters developed for its proof – elementary arithmetic.

Let us understand by an *equation* an expression of the form

$$t(x_1 \ldots x_n) = t'(x_1 \ldots x_n),$$

where t and t' are terms built up from the variables $x_1, \ldots, x_n$, the constants 0 and 1 and the Boolean operation symbols $+$, $\cdot$, and $-$. We say that the above equation *holds* in a Boolean algebra A if the values of t and t' coincide for every assignment of elements $a_1, \ldots, a_n$ of A to $x_1, \ldots, x_n$.

2.19. PROPOSITION. *For every equation e, the following are equivalent*:
(a) *e holds in every Boolean algebra*,
(b) *e holds in every power set algebra*,
(c) *e holds in some non-trivial Boolean algebra*,
(d) *e holds in the two-element Boolean algebra.*

PROOF. Obviously, (a) implies (b) and (b) implies (c). Also (c) implies (d) since the two-element algebra $2 = \{0, 1\}$ is, in a canonical way, a subalgebra of every non-trivial Boolean algebra A; thus an equation holding in A holds, a fortiori, in 2.

Finally assume that (a) fails for an equation e of the form $t(x_1 \ldots x_n) = t'(x_1 \ldots x_n)$; we prove that (d) fails. There is a Boolean algebra A and $a_1, \ldots, a_n$ in A such that $b = t(a_1 \ldots a_n)$ and $b' = t'(a_1 \ldots a_n)$ are distinct, say $b \not\leq b'$. But then, as in the proof of Stone's theorem, there is an ultrafilter p of A such that $b \in p$ and $b' \notin p$. Let $f\colon A \to 2$ be the characteristic function of p. Then

$$1 = f(b) = f(t(a_1 \ldots a_n)) = t(f(a_1) \ldots f(a_n))$$

since f is a homomorphism from A into 2. Similarly,

$$0 = t'(f(a_1) \ldots f(a_n))\,;$$

thus e does not hold in the two-element Boolean algebra. □

The validity of a particular equation may thus be decided either by (b) above or, using the table in Example 1.6, i.e. the well-known truth-table method, by (d). We shall, in the rest of this text, not prove elementary equations holding in every Boolean algebra but leave it to the reader to convince himself by either method.

Also, inequalities of the form $t(x_1 \ldots x_n) \leq t'(x_1 \ldots x_n)$ are decidable in this fashion since $t \leq t'$ is equivalent to the equation $t \cdot t' = t$. Our method is, however, limited to equations and inequalities involving only finitary operations. For example, (a) and (b) in the above proposition are not necessarily equivalent for equations involving the infinite operations Σ and Π since the Stone map from a Boolean algebra A into the power set algebra of Ult A generally fails to preserve the infinite sums and products which happen to exist in A. See the following subsection and Exercise 1 for a discussion of this phenomenon. In fact, in Section 14 infinite distributive laws will be considered which trivially hold in every power set algebra but can fail badly in atomless complete algebras.

2.4. *The Rasiowa–Sikorski lemma*

The Boolean prime ideal theorem 2.16, or rather its corollary 2.17, gives an easy proof for the completeness theorem of propositional logic; see Exercise 5. In fact these two theorems are equivalent in ZF set theory without the axiom of choice (RASIOWA and SIKORSKI [1963]). The proof works on the grounds that the Boolean structure of an algebra of formulas, as defined in 1.12, reflects the propositional connectives of disjunction, conjunction and negation. In predicate logic, the quantifiers are reflected by, possibly infinite, sums and products (see Exercise 5 of Section 1), thus a similar proof for the completeness theorem of predicate logic would require ultrafilters which preserve some infinite sums and products. Existence of these ultrafilters is guaranteed by the Rasiowa–Sikorski lemma 2.21 below, and the completeness theorem of predicate logic for countable languages follows without difficulty (Exercise 6). The Rasiowa–Sikorski lemma is also used in set theory to ensure existence of generic filters over countable structures.

2.20. DEFINITION. Let $f: A \to B$ be a homomorphism of Boolean algebras and $M \subseteq A$ such that $\Sigma^A M (\Pi^A M)$ exists. *f preserves* $\Sigma^A M$ (respectively $\Pi^A M$) if $\Sigma^B f[M]$ exists ($\Pi^B f[M]$ exists) and

$$f\left(\Sigma^A M\right) = \Sigma^B f[M]$$

(respectively $f(\Pi^A M) = \Pi^B f[M]$). Let p be an ultrafilter of A and $M \subseteq A$ such that ΣM (ΠM) exists. *p preserves* ΣM if $\Sigma M \in p$ implies that $m \in p$ for some $m \in M$. *p preserves* ΠM if $M \subseteq p$ implies that $\Pi M \in p$.

Thus, an ultrafilter p of A preserves ΣM (ΠM) iff its characteristic homomorphism $\chi_p: A \to 2$ does. And by the infinitary version 1.33 of de Morgan's laws, p preserves ΣM (respectively ΠM) iff it preserves $\Pi \{-m: m \in M\}$ (respectively $\Sigma \{-m: m \in M\}$).

2.21. THEOREM (Rasiowa–Sikorski lemma). *Let, in a non-trivial Boolean algebra A, S and P be at most countable families of subsets of A such that ΣM exists for each M in S and ΠN exists for each N in P. Then there is an ultrafilter of A preserving ΣM for each M in S and ΠN for each N in P.*

PROOF. By the remarks preceding this theorem, it suffices to find an ultrafilter preserving ΣM for each M in S. If S is empty, then any ultrafilter of A will do – note that A has at least one ultrafilter by the Boolean prime ideal theorem, being non-trivial. Thus, let

$$S = \{M_n : n \in \omega\}$$

by an enumeration of S. We construct by induction a decreasing sequence

$$1 = a_0 \geq a_1 \geq a_2 \geq \cdots$$

in $A^+ = A \setminus \{0\}$ such that, for each $n \in \omega$,

$$(2_n) \qquad a_{n+1} \cdot \Sigma M_n = 0 \quad \text{or} \quad a_{n+1} \leq m \text{ for some } m \in M_n .$$

Let $a_0 = 1$. After $a_n > 0$ has been constructed, we find a_{n+1} as follows. If $a_n \cdot \Sigma M_n = 0$, we let $a_{n+1} = a_n$. Otherwise by the distributive law 1.33(b),

$$0 < a_n \cdot \Sigma M_n = \Sigma \{a_n \cdot m: m \in M_n\} ;$$

hence, there is some $m \in M_n$ such that $0 < a_n \cdot m$ and we let $a_{n+1} = a_n \cdot m$. So (2_n) holds.

The set $E = \{a_n : n \in \omega\}$ is a decreasing chain in A^+ and thus has the finite intersection property; let, by the Boolean prime ideal theorem, p be an ultrafilter including E. Then p preserves ΣM for each M in S. To see this, assume $M = M_n$ and $\Sigma M_n \in p$. Since $a_{n+1} \in p$ and p is a proper filter, $a_{n+1} \cdot \Sigma M_n > 0$. So by (2_n), $a_{n+1} \leq m$ for some $m \in M$, which implies $m \in p$. $\square$

One might naturally try to remove the countability restriction on the sets S and P in the Rasiowa–Sikorski lemma. This is partially done in the following statement; the "countable chain condition" referred to is defined in Section 3.

MARTIN'S AXIOM. Let, in a non-trivial Boolean algebra A satisfying the countable chain condition, S be a family of subsets of A such that $|S| < 2^\omega$ and ΣM exists for each M in S. Then there is an ultrafilter of A preserving ΣM for each M in S.

Martin's axiom is consistent with, but not provable from, the axioms of ZFC set theory. It is a consequence of the continuum hypothesis CH ($2^\omega = \omega_1$) since under CH it simply reduces to the Rasiowa–Sikorski lemma proved above in ZFC. Several consequences of CH can be also proved from Martin's axiom; cf. MARTIN and SOLOVAY [1970] for a discussion of these topics. The analogue of Martin's axiom to families S of cardinality at least 2^ω, however, contradicts the axioms of ZFC, see Exercise 3 in Section 3. And Exercise 7 in Section 4 shows that there is a Boolean algebra A not satisfying the countable chain condition and a family S of subsets of A such that $|S| = \omega_1$ and no ultrafilter of A preserves ΣM for each M in S.

Exercises

1. In a Boolean algebra A, let $M \subseteq A$ such that ΣM exists. The Stone homomorphism $s: A \to P(\text{Ult } A)$ preserves ΣM iff $\Sigma M = \Sigma M_0$ for some finite subset M_0 of M.

Similarly, let M and N be subsets of A such that $\bigcap s[M] \subseteq \bigcup s[N]$. Then there are finite subsets M_0 of M and N_0 of N such that $\bigcap s[M_0] \subseteq \bigcup s[N_0]$.

2. Let X be a compact Hausdorff space. Then for every ultrafilter p of $\text{RO}(X)$, there is a unique point of X lying in $\bigcap \{\text{cl } u: u \in p\}$.

3. Let, in the regular open algebra $\text{RO}([0, 1])$ of the real unit interval, S be the set

$$S = \{(a, b): a, b \text{ rational and } 0 \le a < b \le 1\}.$$

Using Exercise 2, show that no ultrafilter of $\text{RO}([0, 1])$ preserves $\Sigma S'$ for every subset S' of S. Conclude that $\text{RO}([0, 1])$ is not isomorphic to any σ-algebra of sets.

4. Consider the Lindenbaum–Tarski algebra

$$A = \{[\alpha]: \alpha \text{ a sentence of } L\}$$

of a fixed theory T in a first order language L, as defined in Example 1.12. A *completion* of T is an L-theory T^* such that $T \subseteq T^*$ and T^* is maximally consistent. Prove that for each completion T^* of T, the set $\{[\alpha]: a \in T^*\}$ is an ultrafilter of A, and that this assignment gives a one-to-one correspondence between completions of T and ultrafilters of A.

5. (Completeness theorem for propositional logic). Let L be a language for propositional logic with V as its set of propositional variables and T a consistent theory in L. Prove that T has a model, i.e. there is an assignment $h: V \to 2 = \{\text{false}, \text{true}\}$ under which every formula in T is true.

Hint. Fix an ultrafilter p of the algebra $B(T)$ and put $h(v) = 1$ iff $[v] \in p$. Then for every formula α in L, α is true under h iff $[\alpha] \in p$.

6. (Completeness theorem for countable languages in first order logic). Let L be a countable language for first order logic and T a consistent theory in L. Then T has a countable model.

Hint. Fix, by the Rasiowa–Sikorski lemma, an ultrafilter p of $B(T)$ preserving ΣM_ϕ for each L-formula $\phi(xx_1 \dots x_n)$; the set $M_\phi = \{\phi(tx_1 \dots x_n)\colon t \text{ a term in } L\}$ has been considered in Exercise 5 of Section 1.

(a) Prove that

$$t \equiv t' \quad \text{iff } [t = t'] \in p$$

defines an equivalence relation $\equiv$ on the set of L-terms; let A be the set of equivalence classes $t/{\equiv}$.

(b) Prove that there is a unique L-structure $\boldsymbol{A}$ with A as its underlying set such that, for each atomic L-formula $\alpha(x_1 \dots x_n)$ and arbitrary terms $t_1, \dots, t_n$,

$$(*) \qquad \boldsymbol{A} \models \alpha[t_1/{\equiv}, \dots, t_n/{\equiv}] \quad \text{iff } [\alpha(t_1 \dots t_n)] \in p\,.$$

(c) Show that $(*)$ holds for every L-formula α. It follows that $\boldsymbol{A}$ is a model of T.

3. Relativization and disjointness

We describe a simple construction of new Boolean algebras from old ones, the relative algebras. Relativization is a useful device since certain aspects of Boolean algebras, e.g. cardinal invariants, are sometimes easier to investigate for suitably chosen relative algebras of a Boolean algebra than for the algebra itself.

The main topic of this section is the study of disjoint families and partitions of unity in a Boolean algebra, in particular of the cardinality functions of cellularity (respectively saturation) connected with these notions. It is not difficult to show that if X is a partition of unity in a sufficiently complete Boolean algebra A, then A is isomorphic to the cartesian product of the relative algebras $A \upharpoonright x$, $x \in X$; see Section 6. Thus, questions about A can often be reduced to questions about some relative algebras of A.

The method of relativization is applied in proving the first of two non-trivial combinatorial results on pairwise disjoint families, the Erdös–Tarski theorem on attainment of cellularity and the Balcar–Vojtáš theorem on disjoint refinements. They will be used in Section 13 where they play a prominent role in the proof of the Balcar–Franěk theorem.

3.1. *Relative algebras and pairwise disjoint families*

3.1. LEMMA AND DEFINITION. Let A be a Boolean algebra and $a \in A$. Then the subset

$$A \restriction a = \{x \in A : x \leq a\}$$

of A is, with the partial order inherited from A, a Boolean algebra, the *relative algebra* or *factor algebra* of A with respect to a.

PROOF. The subset $A \restriction a$ of A is closed under the operations $+$ and $\cdot$ of A, hence a distributive lattice. It has 0 as its least and a as its greatest element, and each x in $A \restriction a$ has $a \cdot -x$ as a complement. Thus, $(A \restriction a, \leq)$ is a distributive complemented lattice, i.e. a Boolean algebra. □

For example, for any set X and any $a \subseteq X$, $P(X) \restriction a$ is the power set algebra of a. If X is a topological space and $a \subseteq X$ is clopen, then $(\text{Clop } X) \restriction a$ is the clopen algebra of a where a has the topology induced by X.

$A \restriction a$ is not a subalgebra of A for $a \neq 1$, since its unit element and complementation are not those inherited from A. It is, however, a homomorphic image of A: one easily checks that the "projection" map

$$p_a : A \to A \restriction a\,, \qquad p_a(x) = a \cdot x$$

is a homomorphism which is onto since $p_a(x) = x$ for $x \in A \restriction a$.

The cartesian product set $A \times B$ of two Boolean algebras A and B can be made into a Boolean algebra by defining all operations componentwise; we shall study cartesian products of (arbitrarily many) Boolean algebras in greater detail in Section 6. The following lemma explains the name of "factor algebras" for relative algebras.

3.2. LEMMA. *For each a in A,*

$$A \cong (A \restriction a) \times (A \restriction -a)\,.$$

PROOF. Define functions $g: A \to (A \restriction a) \times (A \restriction -a)$ and $h: (A \restriction a) \times (A \restriction -a) \to A$ by

$$g(x) = (x \cdot a, x \cdot -a)\,, \qquad h(y, z) = y + z\,.$$

Then g is a homomorphism since $g(x) = (p_a(x), p_{-a}(x))$ and it is bijective since g and h are inverses of each other. □

3.3. DEFINITION. Let A be a Boolean algebra, x and y in A and $X \subseteq A$. x and y are *disjoint* if $x \cdot y = 0$. X is a *pairwise disjoint family* if $0 < x$ for $x \in X$ and any two distinct elements of X are disjoint.

In algebras of sets, this notion of disjointness coincides with the usual one known from set theory. It is of course an abuse of notation to call a sub*set* of A a pairwise disjoint *family*; we will also call an indexed family $(x_i)_{i\in I}$ in A a pairwise disjoint family if $x_i > 0$ and $x_i \cdot x_j = 0$ for $i \neq j$.

3.4. PROPOSITION. *In every infinite Boolean algebra, there are an infinite pairwise disjoint family, a strictly increasing infinite sequence and a strictly decreasing infinite sequence.*

PROOF. Let A be infinite; we first construct a strictly decreasing sequence $(a_n)_{n\in\omega}$ in A such that, for every $n \in \omega$, $A \restriction a_n$ is infinite. Let $a_0 = 1$. Assume that a_n has been constructed such that $A \restriction a_n$ is infinite; pick $a \in A \restriction a_n$ such that $0 < a < a_n$. If $A \restriction a$ is infinite, we let $a_{n+1} = a$; otherwise by Lemma 3.2, $A \restriction (a_n \cdot -a)$ must be infinite and we let $a_{n+1} = a_n \cdot -a$.

The rest of the proposition follows immediately: if the sequence $(a_n)_{n\in\omega}$ is strictly decreasing, then $(-a_n)_{n\in\omega}$ is strictly increasing. Also, $\{a_n \cdot -a_{n+1}: n \in \omega\}$ is an infinite pairwise disjoint family since $m < n$ implies $a_n \leq a_{m+1}$ and

$$(a_m \cdot -a_{m+1}) \cdot (a_n \cdot -a_{n+1}) \leq a_n \cdot -a_{m+1} = 0 . \quad \square$$

We have seen in Corollary 2.8 that the cardinalities of finite Boolean algebras are exactly the powers of 2 and in the remark following 1.9 that every infinite cardinal is the cardinality of some Boolean algebra. The next corollary, a simple consequence of Proposition 3.4, shows that there are restrictions on the cardinality of σ-complete infinite algebras. The possible cardinalities of infinite σ-complete and also of κ-complete ($\kappa > \omega_1$) or complete algebras will be determined in Section 12: they are exactly those cardinals κ satisfying $\kappa = \kappa^\omega$.

3.5. COROLLARY. *An infinite σ-complete Boolean algebra has cardinality at least 2^ω.*

PROOF. Let A be infinite and σ-complete; by Proposition 3.4, fix a pairwise disjoint family $\{d_m: m \in \omega\}$. We show that the function $f: P(\omega) \to A$ defined by

$$f(M) = \sum_{m\in M} d_m$$

is one-to-one. If M and N are distinct subsets of ω, then there is, without loss of generality, some $m \in M \setminus N$. Then $d_m \leq f(M)$, hence

$$d_m \cdot f(M) = d_m > 0 ,$$

whereas, by 1.33(b),

$$d_m \cdot f(N) = \sum_{n\in N} d_m \cdot d_n = 0 . \quad \square$$

It is a consequence of Zorn's lemma that every pairwise disjoint family can be extended to a maximal one.

3.6. LEMMA. *A pairwise disjoint family X is maximal iff $\Sigma X = 1$.*

PROOF. For the non-trivial direction of the equivalence, assume that ΣX does not exist or is strictly smaller than 1. Thus, there is an upper bound b of X satisfying $b < 1$, and the pairwise disjoint family $X \cup \{-b\}$ shows that X is not maximal. □

In a power set algebra $P(M)$, the maximal pairwise disjoint families are exactly the partitions of M. This motivates the following definition.

3.7. DEFINITION. A subset of a Boolean algebra A is a *partition* (*a partition of* A, *a partition of* 1, *a partition of unity*) if it is a maximal pairwise disjoint family.

3.2. Attainment of cellularity: the Erdös–Tarski theorem

Proposition 3.4 implies that every countably infinite Boolean algebra has a countably infinite pairwise disjoint family. Is it possible to find, in an arbitrary infinite Boolean algebra A, a pairwise disjoint family of cardinality $|A|$? A strong counterexample to this question is provided in Section 9 by free Boolean algebras: they can have any prescribed infinite cardinality but have only countable pairwise disjoint families.

We describe, in this subsection, the possible sizes of pairwise disjoint families in a Boolean algebra by the cardinal invariant of cellularity, the first of several cardinal functions on Boolean algebras to be studied later. In particular the countable chain condition is a very strong assumption on a Boolean algebra. It is relevant to set theory where it arises through the subjects of forcing and Martin's axiom; cf. the discussion following the Rasiowa–Sikorski lemma in Section 2.

3.8. DEFINITION. Let A be a Boolean algebra, κ a cardinal. Then

$$\mathrm{c}A = \sup\{|X|: X \text{ a pairwise disjoint family in } A\}$$

and

$$\operatorname{sat} A = \min\{\mu: \mu \text{ a cardinal, } |X| < \mu \text{ for each pairwise disjoint family } X \text{ in } A\}$$

are the *cellularity* and the *saturation* of A. cA *is attained* if $\mathrm{c}A = |X|$ for some pairwise disjoint family X in A.

A satisfies the *κ-chain condition* if $|X| < \kappa$ for each pairwise disjoint family X in A, i.e. if $\operatorname{sat} A \leq \kappa$. A satisfies the *countable chain condition* if each pairwise disjoint family in A is at most countable.

The notion of cellularity is also considered in topology: for a topological space X, define the cellularity of X by

$$\mathrm{c}X = \sup\{|U|: U \text{ a family of pairwise disjoint non-empty open subsets of } X\}.$$

For example, $\mathrm{c}X \leq \omega$ if X is separable, i.e. if it has a countable dense subset D. For suppose U is a family of non-empty pairwise disjoint open subsets of X. Then for $u \in U$, choose a point d_u in $U \cap D$; the d_u are pairwise distinct, and thus $|U| \leq |D| \leq \omega$.

The definitions of cellularity for Boolean algebras and topological spaces are connected by

$$\mathrm{c}X = \mathrm{c}\,\mathrm{RO}(X);$$

this follows from the fact that, for any two disjoint open subsets u and v of X, their regularizations $\operatorname{int}\operatorname{cl} u$ and $\operatorname{int}\operatorname{cl} v$ are non-empty and disjoint by (10) in the proof of Theorem 1.37. As an example, consider the algebra $\mathrm{RO}(X)$ for an infinite separable Hausdorff space X. It satisfies the countable chain condition as shown above but has cardinality at least 2^ω by Corollary 3.5; in fact $|\mathrm{RO}(X)| = 2^\omega$ since each regular open subset of X is determined by its intersection with a (countable) dense subset of X.

Pairwise disjoint families are sometimes called antichains, and the κ-chain condition could be more properly called the κ-antichain condition. In fact, a Boolean algebra with only countable pairwise disjoint families can have uncountable chains, i.e. subsets which are linearly ordered under the Boolean partial ordering; see Exercise 2.

Let us make some simple remarks on attainment of cellularity. For a finite (or, more generally, an atomic) Boolean algebra A, $\mathrm{c}A$ is attained since the set $\mathrm{At}\,A$ of all atoms of A is a pairwise disjoint family in A and $\mathrm{c}A = |\mathrm{At}\,A|$. If $\mathrm{c}A = \omega$, then A is infinite and $\mathrm{c}A$ is attained by Proposition 3.4; also $\mathrm{c}A$ is trivially attained if it is a successor cardinal. For κ a weakly inaccessible, i.e. a regular uncountable limit cardinal, there is a Boolean algebra A with $\mathrm{c}A = \kappa$ not attained, cf. Example 11.14. The remaining case of a singular cardinal is handled in the following theorem.

3.9. NOTATION. For A a Boolean algebra and $a \in A$,

$$\mathrm{c}_A a = \mathrm{c}(A \upharpoonright a).$$

If A is understood, we write $\mathrm{c}a$ for $\mathrm{c}_A a$.

3.10. THEOREM (Erdös–Tarski). *For every Boolean algebra A, $\mathrm{c}A$ is attained if singular.*

PROOF. We shall use the following fact several times: if $a \in A$ and $\mathrm{c}a > \mu$ for some cardinal μ, then there is a pairwise disjoint family X of size μ in $A \upharpoonright a$. This holds since $\mu < \mathrm{c}a$ implies that there is some pairwise disjoint family Y in $A \upharpoonright a$ satisfying $|Y| \geq \mu$; let then X be any subset of Y of cardinality μ.

Let $\lambda = \mathrm{c}A$, κ its cofinality and $(\lambda_\alpha)_{\alpha<\kappa}$ a strictly increasing sequence of cardinals such that $\lambda = \sup_{\alpha<\kappa} \lambda_\alpha$. The proof is broken up into three cases.

Case 1. There is some $b \in A$ such that, for each x satisfying $0 < x \leq b$, $cx = \lambda$. In this case, a pairwise disjoint family Z of size λ is easily constructed: by singularity of λ, we have $\kappa < \lambda = cb$; let $\{b_\alpha : \alpha < \kappa\}$ be a pairwise disjoint family in $A \restriction b$. Now $\lambda_\alpha < \lambda = cb_\alpha$ for $\alpha < \kappa$, so let Z_α be a pairwise disjoint family of size λ_α in $A \restriction b_\alpha$ and put $Z = \bigcup_{\alpha<\kappa} Z_\alpha$. This finishes Case 1.

Now assume that Case 1 fails; we define two sets S and X relevant to the rest of the proof. Let

$$S = \{a \in A^+ : ca < \lambda\}$$

and let X be maximal among the pairwise disjoint families included in S. By failure of Case 1, there is for each $b \in A^+$ some $s \in S$ such that $s \leq b$. Thus, X is, in the terminology of Definition 3.7, a partition of unity.

Case 2. $\sup_{x \in X} cx = \lambda$. Again we find a pairwise disjoint family Z of size λ. Since $cx < \lambda$ for $x \in X$ but $\sup_{x\in X} cx = \lambda$, choose by induction pairwise distinct elements x_α of X for $\alpha < \kappa$ such that $\lambda_\alpha < cx_\alpha$. For each α, pick a pairwise disjoint family Z_α of size λ_α in $A \restriction x_\alpha$ and again let $Z = \bigcup_{\alpha<\kappa} Z_\alpha$.

Case 3. $\sup_{x\in X} cx < \lambda$. We prove that, in this case, the partition X has cardinality λ. For otherwise let

$$\mu = \sup_{x \in X} cx\,, \qquad \mu' = \max(|X|, \mu)^+\,;$$

thus, $\mu' < \lambda = cA$ since λ is a limit cardinal. Let Y be a pairwise disjoint family in A of size μ' and consider, for $x \in X$, the set

$$Y_x = \{y \in Y : x \cdot y > 0\}\,.$$

Since $\Sigma X = 1$ by Lemma 3.6, we find that $Y = \bigcup_{x\in X} Y_x$. Now $\{x \cdot y : y \in Y_x\}$ is a pairwise disjoint family in $A \restriction x$, so $|Y_x| \leq cx \leq \mu$. It follows that $|Y| \leq \mu \cdot |X| < \mu'$, a contradiction. □

Saturation, the other cardinal function defined in 3.8, is connected with cellularity as follows. Clearly

$$\text{sat}\, A = \sup\{|X|^+ : X \text{ a pairwise disjoint family in } A\}\,.$$

Thus, sat $A = (cA)^+$ if cA is attained (in particular if $cA = n \in \omega$, then sat $A = n + 1$), and sat $A = cA$ otherwise. Considering all finite cardinals to be regular, we obtain a reformulation of the Erdös–Tarski theorem in terms of saturation.

3.11. COROLLARY. *For every Boolean algebra A,* sat A *is a regular cardinal.* □

3.3. Disjoint refinements: the Balcar–Vojtáš theorem

It is sometimes useful to "disjoint" a family $(a_i)_{i\in I}$ consisting of arbitrary elements of a Boolean algebra, i.e. to replace the a_i by pairwise disjoint elements b_i such that $b_i \leq a_i$ and the sequence $(b_i)_{i\in I}$ shares some properties with $(a_i)_{i\in I}$.

In sufficiently complete algebras, Lemma 3.12 describes a standard procedure to do this. Since the proof requires a well-ordering of the index set I anyway, we assume that I is a cardinal κ.

3.12. LEMMA. *Let A be a κ-complete Boolean algebra. Then for every family $(a_\alpha)_{\alpha<\kappa}$ in A there is a family $(b_\alpha)_{\alpha<\kappa}$ consisting of pairwise disjoint elements such that $b_\alpha \leq a_\alpha$ and*

$$\sum_{\alpha<\kappa} a_\alpha = \sum_{\alpha<\kappa} b_\alpha$$

if one of these sums exists.

PROOF. Let

$$b_\alpha = a_\alpha \cdot -\sum_{\beta<\alpha} a_\beta .$$

Then $b_\alpha \leq a_\alpha$ and, for $\beta < \alpha$, $b_\alpha \cdot b_\beta = 0$ since $b_\beta \leq a_\beta$ and $b_\alpha \leq -a_\beta$. It follows by induction that $\Sigma_{\beta<\alpha} b_\beta = \Sigma_{\beta<\alpha} a_\beta$ for each $\alpha < \kappa$. So $\{b_\alpha: \alpha < \kappa\}$ and $\{a_\alpha: \alpha < \kappa\}$ have the same set of upper bounds in A, which proves the last assertion of the lemma. □

The construction in the preceding proof does not ensure that $(b_\alpha)_{\alpha<\kappa}$ is a pairwise disjoint family in the sense of Definition 3.3 and the remark following it, since some of the b_α may be zero. In fact, if A satisfies the countable chain condition and κ is uncountable, only countably many b_α can be positive. For a less trivial result, we have to assume a bit more on cellularity in A.

3.13. DEFINITION. Let $(a_\alpha)_{\alpha<\kappa}$ be any sequence in A^+. Then $(b_\alpha)_{\alpha<\kappa}$ is a *disjoint refinement* of $(a_\alpha)_{\alpha<\kappa}$ if $0 < b_\alpha \leq a_\alpha$ for $\alpha < \kappa$ and $b_\beta \cdot b_\alpha = 0$ for $\beta < \alpha < \kappa$.

3.14. THEOREM (Balcar–Vojtáš). *Assume κ is an infinite cardinal and $\kappa^+ \leq \mathrm{c}x$ for each $x \in A^+$. Then each sequence of type κ in A^+ has a disjoint refinement.*

PROOF. Let $(a_\alpha)_{\alpha<\kappa}$ be a sequence of positive elements of A. For each pairwise disjoint family X in A and $a \in A$, write

$$X(a) = \{x \in X: x \cdot a > 0\} .$$

It suffices to construct a pairwise disjoint family X such that $|X(a_\alpha)| \geq \kappa$ for $\alpha < \kappa$; given X with this property, inductively choose

$$x_\alpha \in X(a_\alpha) \setminus \{x_\beta: \beta < \alpha\}$$

and put $b_\alpha = a_\alpha \cdot x_\alpha$.

To find X, we shall construct by induction X_α for $\alpha < \kappa$ such that

$(X_\alpha)_{\alpha<\kappa}$ is an increasing chain of pairwise disjoint families in A;

for $\alpha, \gamma < \kappa$, either $X_\alpha(a_\gamma) = \emptyset$ or $|X_\alpha(a_\gamma)| = \kappa^+$;

$|X_{\alpha+1}(a_\alpha)| = \kappa^+$;

and then let $X = \bigcup_{\alpha<\kappa} X_\alpha$.

Put $X_0 = \emptyset$ and $X_\lambda = \bigcup_{\alpha<\lambda} X_\alpha$ for a limit ordinal $\lambda < \kappa$. If X_α has been constructed, let $X_{\alpha+1} = X_\alpha$ if $|X_\alpha(a_\alpha)| = \kappa^+$. Otherwise $X_\alpha(a_\alpha) = \emptyset$ and hence $X_\alpha \subseteq A \restriction (-a_\alpha)$. Then by $c(a_\alpha) \geq \kappa^+$, $A \restriction a_\alpha$ has a pairwise disjoint family Y of size κ^+; note that $X_\alpha \cup Y$ is a pairwise disjoint family. Define

$$X_{\alpha+1} = (X_\alpha \cup Y) \setminus \bigcup \{Y(a_\gamma): \gamma < \kappa \text{ and } |Y(a_\gamma)| \leq \kappa\} . \quad \square$$

Disjoint refinements can be applied to questions on ultrafilters and to topology; these two topics are closely related since for every Boolean algebra A, the set Ult A of all ultrafilters of A is, in a natural way, a topological space – cf. Section 7. We give here an application to ultrafilters which generalizes the following argument. If $cx \geq \omega$ for every $x \in A^+$, then $A \restriction x$ is infinite for $x > 0$ and A has no atoms; consequently, no ultrafilter of A is principal. Hence, no ultrafilter of A is, in the sense of Definition 2.12, generated by finitely many elements.

3.15. PROPOSITION. *Assume A is a κ^+-complete Boolean algebra and $\kappa^+ \leq cx$ for each x in A^+. Then no ultrafilter of A is generated by less than κ^+ elements.*

PROOF. Assume for contradiction that p is an ultrafilter of A generated by the set

$$E = \{e_\alpha : \alpha < \lambda\} ,$$

where $\lambda = |E| \leq \kappa$. We may assume that λ is minimal for p, i.e. that no set of size less than λ generates p and, since the set of all finite products of elements of E also has cardinality λ and generates p, that E is closed under finite products. Thus, for each $e \in E$,

(1) $\quad |\{f \in E: f \leq e\}| = \lambda ,$

for the set on the left-hand side of (1) generates p.

Using (1), we choose by induction elements f_α and g_α of E for $\alpha < \lambda$ such that

$$f_\alpha, g_\alpha \in \{e \in E: e \leq e_\alpha\} \setminus \bigcup_{\beta<\alpha} \{f_\beta, g_\beta\} ,$$

$$f_\alpha \neq g_\alpha .$$

By the Balcar–Vojtáš theorem and $|E| \leq \kappa$, there is a disjoint refinement $(d(e))_{e\in E}$ of E. Put

$$a = \sum_{\alpha<\lambda} d(f_\alpha), \qquad b = \sum_{\alpha<\lambda} d(g_\alpha).$$

Then $a \cdot b = 0$ since the $d(e_\alpha)$, $d(g_\alpha)$ are pairwise disjoint. We prove that $a \in p$: otherwise, $-a \in p$. The set E generates p and is closed under finite products, thus there is some $\alpha < \lambda$ such that $e_\alpha \leq -a$ which gives $e_\alpha \cdot a = 0$. Now $d(f_\alpha) \leq f_\alpha \leq e_\alpha$; also $d(f_\alpha) \leq a$ by definition of a. This implies $d(f_\alpha) = 0$, a contradiction. The same reasoning shows that $b \in p$. This, however, is impossible since $a \cdot b = 0$ and p is a proper filter. □

Exercises

1. $P(\omega)$ is embeddable into every infinite σ-complete Boolean algebra.

Hint. For A infinite and σ-complete, find an embedding $g: P(\omega) \to A$ by slightly modifying the map $f: P(\omega) \to A$ in the proof of 3.5.

2. A subset X of a Boolean algebra A is called a *chain* (respectively a well-ordered chain) in A if X is, under the partial order inherited from A, a linear order (respectively a well-ordering).

(a) If A satisfies the countable chain condition, as defined in 3.8, then each well ordered chain in A is countable. The converse holds if A is σ-complete.

(b) In the finite–cofinite algebra A over an infinite set, every chain is countable but A does not necessarily satisfy the countable chain condition.

(c) Find an interval algebra which has an uncountable chain but satisfies the countable chain condition.

3. Conclude from Exercise 3 in Section 2 that even for Boolean algebras satisfying the countable chain condition, the hypothesis $|S| < 2^\omega$ cannot be removed from the formulation of Martin's axiom in Section 2 without contradicting ZFC.

4. In the interval algebra of the reals, the subset

$$\{(a, b): a \text{ and } b \text{ rational}, a < b\}$$

has no disjoint refinement. Thus, the assumption that $cx \geq \kappa^+$ for each $x \in A^+$ cannot be removed from the Balcar–Vojtáš theorem.

5. A map $\mu: A \to [0, 1]$ from a Boolean algebra A into the real unit interval is called a finitely additive *measure* if $\mu(1) = 1$ and, for every finite set $\{a_i: i \in I\}$ of pairwise disjoint elements, $\mu(\Sigma_{i\in I} a_i) = \Sigma_{i\in I} \mu(a_i)$ (and it is a σ-additive measure if, in addition, $\mu(\Sigma_{i\in I} a_i) = \Sigma_{i\in I} \mu(a_i)$ holds for every countable set $\{a_i: i \in I\}$ of pairwise disjoint elements for which $\Sigma_{i\in I} a_i$ happens to exist). μ is *strictly positive* if $a > 0$ implies $\mu(a) > 0$.

Show that each algebra admitting a strictly positive finitely additive measure satisfies the countable chain condition.

CHAPTER 2

Algebraic Theory

Sabine KOPPELBERG

Freie Universität Berlin

Contents

HANDBOOK OF BOOLEAN ALGEBRAS
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Introduction

This chapter describes, more thoroughly than the preceding one, three basic methods for constructing new Boolean algebras from old ones: subalgebras, quotient algebras (i.e. homomorphic images), and cartesian products. These notions are of general importance when dealing with algebraic structures, for the following reason.

Consider a class $\boldsymbol{V}$ of algebraic structures of a fixed similarity type L. $\boldsymbol{V}$ is said to be a *variety* if there is a set E of L-equations defining $\boldsymbol{V}$, i.e. such that

$$\boldsymbol{V} = \{A\colon A \text{ an } L\text{-algebra such that every equation in } E \text{ holds in } A\}\,.$$

For example, the class $\boldsymbol{BA}$ of all Boolean algebras is a variety, being defined by the equations (B1) through (B5′) in Section 1.

For every class $\boldsymbol{K}$ of L-algebras there is a smallest variety $V(\boldsymbol{K})$ including $\boldsymbol{K}$, the variety generated by $\boldsymbol{K}$: simply let $E(\boldsymbol{K})$ consist of all L-equations holding in every member of $\boldsymbol{K}$, and let then $V(\boldsymbol{K})$ be the variety defined by $E(\boldsymbol{K})$. It is a consequence of Proposition 2.19 that there are only two varieties of Boolean algebras. For let $\boldsymbol{K}$ be any class of Boolean algebras. If all members of $\boldsymbol{K}$ are trivial, i.e. one-element algebras, then every equation holds in every member of $\boldsymbol{K}$ and $V(\boldsymbol{K})$ is the class consisting of all trivial Boolean algebras. Otherwise by 2.19, $E(\boldsymbol{K})$ consists of all equations holding in the two-element Boolean algebra 2 and, again by 2.19, $V(\boldsymbol{K}) = \boldsymbol{BA}$.

Now Birkhoff's theorem, one of the fundamentals of universal algebra, characterizes the varieties as being those classes of L-algebras which are closed under the operations of taking subalgebras, homomorphic images, and products. More precisely, it says that, for any class $\boldsymbol{K}$ of L-algebras, $V(\boldsymbol{K}) = \mathrm{HSP}(\boldsymbol{K})$ where

$$\mathrm{HSP}(\boldsymbol{K}) = \{A\colon A \text{ a homomorphic image of a subalgebra of a product of members of } \boldsymbol{K}\}\,;$$

here the inclusion $\mathrm{HSP}(\boldsymbol{K}) \subseteq V(\boldsymbol{K})$ is easily verified. In the special case of a class $\boldsymbol{K}$ of Boolean algebras, the non-trivial inclusion $V(\boldsymbol{K}) \subseteq \mathrm{HSP}(\boldsymbol{K})$ can be derived from Stone's theorem as follows. First note that the assertion is obvious if all members of $\boldsymbol{K}$ are trivial. So assume $\boldsymbol{K}$ has a non-trivial member A; let B be an arbitrary Boolean algebra with the aim of showing that $B \in \mathrm{HSP}(\{A\}) \subseteq \mathrm{HSP}(\boldsymbol{K})$. By Stone's theorem, B is isomorphic to a subalgebra of some power set algebra $P(X)$. But $P(X)$ is isomorphic to the cartesian product $\Pi_{x\in X} C_x$, where $C_x = 2$ for $x \in X$, via the map assigning to each subset of X its characteristic function; moreover, 2 is a subalgebra of A. Thus, B is isomorphic to a subalgebra of $\Pi_{x\in X} A_x$, where $A_x = A$, which proves that $B \in \mathrm{HSP}(\{A\})$.

It actually turns out that, in the variety of Boolean algebras, formation of cartesian products is less important than that of subalgebras and homomorphic images; in fact, the direct factors of a Boolean algebra are easily described, in Section 6, as being its relative algebras and thus are well known. The most basic and frequently used topics of this chapter are then the process of generating subalgebras from subsets of a Boolean algebra and finding normal forms for their elements, as well as Sikorski's extension theory for homomorphisms. We also introduce another important construction of Boolean algebras: the completion of a Boolean algebra and, more generally, of an arbitrary partial order.

4. Subalgebras, denseness, and incomparability

One of the most frequently applied constructions of Boolean algebras is, for an arbitrary subset X of an algebra B, the formation of the subalgebra $\langle X \rangle$ of B generated by X. This subalgebra is easily definable by abstract reasoning: it is the intersection of all subalgebras of B including X. The normal form theorem 4.4 describes the elements of $\langle X \rangle$: they are exactly those elements of B representable in a certain normal form over X. The normal form has a somewhat messy notation but is in many situations the only tool to get detailed information on $\langle X \rangle$. Normal forms are essential in Section 5 for a characterization of those maps from X into an arbitrary Boolean algebra B' which can be extended to a homomorphism from $\langle X \rangle$ into B'.

We further study the notions of denseness, irredundance, and incomparability for subsets of a Boolean algebra; they are related to subalgebras by McKenzie's result 4.23 that maximally irredundant subsets generate dense subalgebras and Shelah's theorem 4.25 that every Boolean algebra has a "large" irredundant subset consisting of pairwise incomparable elements. The most important one of these notions is denseness. For example, a Boolean algebra B is called a completion of an algebra A if B is complete and A is a dense subalgebra of B. We shall see that every Boolean algebra has a unique completion, up to isomorphism, and that the embedding into the completion preserves arbitrary sums and products. The process of completion can be generalized to arbitrary partially ordered sets. Again the completion of a partial order is a complete Boolean algebra; we follow KUNEN [1980] in describing axiomatically the relationship between a partial order and its completion. The passage from a partial order to the completion is reflected, in axiomatic set theory, by the connection of forcing with partiallly ordered sets and forcing with complete Boolean algebras.

4.1. Normal forms

A subalgebra of a Boolean algebra B was defined, in Section 1, to be a subset A of B containing 0 and 1 and closed under the operations $+$, $\cdot$, and $-$. It suffices, however, to insist on non-emptiness and closure under $+$ and $-$ (or, dually, under and $-$) since $1 = x + -x$, $0 = -1$, and $x \cdot y = -(-x + -y)$. We define, for reference in later sections:

4.1. DEFINITION. A subalgebra A of a Boolean algebra B is a *κ-complete subalgebra* (*a σ-complete subalgebra, a complete subalgebra*) of B if for each subset M of A of size less than κ (each countable subset M of A, each subset M of A) such that $\Sigma^B M$ exists, also $\Sigma^A M$ exists and $\Sigma^A M = \Sigma^B M$.

Thus, A is a κ-algebra of sets as defined in 1.29 if it is a κ-complete subalgebra of a power set algebra; similarly for σ-algebras of sets and complete algebras of sets.

The intersection $\bigcap \mathbf{S}$ of a non-empty family $\mathbf{S}$ of subalgebras of B is again a subalgebra of B, and so is the union $\bigcup \mathbf{S}$ of $\mathbf{S}$ if $\mathbf{S}$ is a chain under inclusion, i.e. if $S \subseteq T$ or $T \subseteq S$ holds for all $S, T \in \mathbf{S}$. More generally, if $\mathbf{S}$ is a non-empty directed family of subalgebras, i.e. if for arbitrary $S, T \in \mathbf{S}$ there is some $R \in \mathbf{S}$ satisfying $S \cup T \subseteq R$, then $\bigcup \mathbf{S}$ is a subalgebra of B.

4.2. DEFINITION AND LEMMA. Let X be a subset of a Boolean algebra B. Then

$$\langle X \rangle = \bigcap \{A \subseteq B: X \subseteq A,\ A \text{ a subalgebra of } B\}$$

is the *subalgebra generated by X* in B; it is the least subalgebra of B including X. The elements of $\langle X \rangle$ are said to be *generated by X*. X is a *set of generators* for a Boolean algebra A if $X \subseteq A$ and $\langle X \rangle = A$. We write $\langle x_i: i \in I \rangle$ for $\langle \{x_i: i \in I\} \rangle$.

Formation of the generated subalgebra is, by its very definition, an order-preserving operation: if $X \subseteq X' \subseteq B$, then $\langle X \rangle \subseteq \langle X' \rangle$. There is a standard procedure to prove, for X and A subsets of B, that $\langle X \rangle = A$: one checks that A is a subalgebra of B, $X \subseteq A$, and every subalgebra of B including X also includes A. This reasoning shows, for example, that in a power set algebra $P(Y)$, the singletons $\{y\}$, $y \in Y$, generate the finite–cofinite algebra on Y. The interval algebra of a linear order L (Example 1.11) is generated by the half-open intervals $[0_L, x)$, where $x \in L$.

We turn to a closer analysis of what the elements of $\langle X \rangle$ look like. Here the main result is the normal form theorem 4.4, in particular Step 2 of the proof and its notation. They will be frequently used in the sequel.

4.3. DEFINITION. Let B be a Boolean algebra. For $x \in B$ and ε one of the integers $+1$ or -1, define the element εx of B by

$$(+1)x = x\,, \qquad (-1)x = -x\,.$$

For $X \subseteq B$, an *elementary product over X* is a finite product with factors of the form εx, $\varepsilon \in \{+1, -1\}$, $x \in X$. An element of B is in (additive) *normal form over X* if it is a finite sum of pairwise disjoint elementary products over X.

In view of the connection between Boolean algebras and propositional calculus, the normal form defined above is also called the disjunctive normal form. Plainly, each element of B representable in normal form over X is generated by X; the converse is stated in the following proposition.

4.4. PROPOSITION (normal form theorem). *The subalgebra generated by $X \subseteq B$ contains exactly the elements of B representable in normal form over X.*

PROOF.

Step 1. We first prove that an element of B is generated by X iff it is generated by a finite subset of X, i.e. that

(1) $$\langle X \rangle = \bigcup \{\langle Y \rangle : Y \text{ a finite subset of } X\}.$$

To see this, denote by A the right-hand side of (1). Then $A \subseteq \langle X \rangle$ by monotonicity. Also by monotonicity, $\langle Y \rangle \cup \langle Z \rangle \subseteq \langle Y \cup Z \rangle$ for arbitrary subsets Y, Z of X, so the subalgebras $\langle Y \rangle$, Y a finite subset of X, constitute a directed family and A is a subalgebra of B. For every $x \in X$, $x \in \langle x \rangle \subseteq A$. Hence, $X \subseteq A$ and $\langle X \rangle \subseteq A$, which proves (1).

Step 2. We are left with the proof that if a finite subset X of B generates $b \in B$, then b is representable in normal form over X. Let

$$X = \{x_1, \ldots, x_n\}$$

and

$$E = {}^{\{1,\ldots,n\}}\{+1, -1\} = \{e : e \text{ a function from } \{1, \ldots, n\} \text{ into } \{+1, -1\}\}.$$

For $e \in E$, let p_e be the elementary product over X defined by

$$p_e = e(1)x_1 \cdot \cdots \cdot e(n)x_n .$$

We then have

(2) $$p_e \cdot p_{e'} = 0 \quad \text{for } e \neq e',$$

since if, for example, $e(i) = +1$ and $e'(i) = -1$ fo some i, then $p_e \leq x_i$ and $p_{e'} \leq -x_i$. Moreover,

(3) $$\sum_{e \in E} p_e = 1,$$

as follows by evaluating the right-hand side of

$$1 = (x_1 + -x_1) \cdot \cdots \cdot (x_n + -x_n)$$

by distributivity. Now for $M \subseteq E$, the sum

$$s_M = \sum_{e \in M} p_e$$

is in normal form over X; let $A = \{s_M : M \subseteq E\}$. We show that $\langle X \rangle \subseteq A$, thus proving our theorem. A is non-empty and closed under $+$ and $-$ since, by (2) and (3),

(4) $$s_M + s_{M'} = s_{M \cup M'},$$

(5) $$-s_M = s_{E\setminus M}\,;$$

so it is a subalgebra of B. Every x_i is in A, since by

$$x_i = x_i \cdot \prod_{j\neq i} (x_j + -x_j)$$

and distributivity, $x_i = s_M$ where $M = \{e \in E: e(i) = +1\}$. This proves $\langle X\rangle \subseteq A$. □

4.5. COROLLARY. *If $X \subseteq B$ has cardinality $n < \omega$, then the subalgebra generated by X contains exactly the elements*

$$\sum_{e\in M} \prod_{x\in X} e(x)x\,,$$

where $M \subseteq {}^X\{+1, -1\}$, and thus has cardinality at most 2^{2^n}. So each finitely generated Boolean algebra is finite. If $\kappa = |X|$ is infinite, then $\langle X\rangle$ has cardinality κ.

PROOF. The first assertion follows from Step 2 in the preceding proof. If $\kappa = |X|$ is infinite, then $\kappa \leq |\langle X\rangle|$ by $X \subseteq \langle X\rangle$ and $|\langle X\rangle| \leq \omega\cdot\kappa = \kappa$ by (1), the first assertion and since X has exactly κ finite subsets. □

For every $n \in \omega$, the bound 2^{2^n} in Corollary 4.5 is actually attained for suitably chosen sets X, as shown by Exercise 1. Normal forms for more special situations are easily obtained from the general result 4.4.

4.6. DEFINITION. Let A by a subalgebra of B. For $n \in \omega$ and $x_1, \ldots, x_n$ in B,

$$A(x_1 \ldots x_n) = \langle A \cup \{x_1, \ldots, x_n\}\rangle$$

is the *finite extension* of A by $x_1, \ldots, x_n$. For $x \in B$,

$$A(x) = \langle A \cup \{x\}\rangle$$

is the *simple extension* of A by x.

4.7. COROLLARY. *For A a subalgebra of B and $x \in B$,*

$$\begin{aligned} A(x) &= \{a\cdot x + a'\cdot -x\colon a, a' \in A\} \\ &= \{a_1\cdot x + a_2\cdot -x + a_3\colon a_1, a_2, a_3 \in A \textit{ pairwise disjoint}\}\,. \end{aligned}$$

PROOF. The first assertion is an immediate consequence of the normal form theorem. For the second one, note that

$$a\cdot x + a'\cdot -x = (a\cdot -a')\cdot x + (a'\cdot -a)\cdot -x + a\cdot a'\,,$$

where $a \cdot -a'$, $a' \cdot -a$ and $a \cdot a'$ are disjoint, and that, conversely,

$$a_1 \cdot x + a_2 \cdot -x + a_3 = (a_1 + a_3) \cdot x + (a_2 + a_3) \cdot -x . \quad \square$$

If $A_1, \ldots, A_m$ are subalgebras of B, then the elements of $\langle A_1 \cup \cdots \cup A_m \rangle$ are, by the normal form theorem, exactly the finite sums of products $a_1 \cdot \cdots \cdot a_m$, where $a_i \in A_i$. In the special case where $m = 2$, $A_1 = A$, and $A_2 = \langle x_1, \ldots, x_n \rangle$, $\langle A_1 \cup A_2 \rangle$ coincides with $A(x_1, \ldots, x_n)$. Using the notation of Step 2 in 4.4, in particular $p_e = e(1)x_1 \cdot \cdots \cdot e(n)x_n$, and collecting, for $e \in E$, summands of the form $a \cdot p_e$, where $a \in A$, we obtain:

$$A(x_1 \ldots x_n) = \left\{ \sum_{e \in E} a_e \cdot p_e \colon a_e \in A \text{ for } e \in E \right\} .$$

4.2. *The completion of a partial order*

We begin this subsection by defining, for a subset of a Boolean algebra, the notion of denseness and list a few simple equivalences. Denseness and the associated cardinal function of density are a strong tool for studying complete Boolean algebras. For example, it is a consequence of Theorem 4.14 that two complete Boolean algebras are isomorphic if they have dense subsets which are, with the partial orderings induced by the Boolean ones, order-isomorphic.

4.8. DEFINITION. Let B be a Boolean algebra. A subset X of $B^+ = B \setminus \{0\}$ is *dense* in B if for every $b \in B^+$ there is some $x \in X$ such that $0 < x \leq b$. A subalgebra A of B is a *dense subalgebra* if A^+ is dense in B.

$$\pi B = \min\{|X| \colon X \subseteq B \text{ dense in } B\}$$

is the *density* of B.

For example, in the regular open algebra of the reals, the set of all open intervals with rational endpoints is dense, and hence $\pi(\mathrm{RO}(\boldsymbol{R})) = \omega$. A Boolean algebra A is, by definition, atomic iff the atoms of A form a dense subset of A, and then $\pi A = |\mathrm{At}\, A|$. The notation πB comes, via an equivalence between Boolean algebras and particular topological spaces described in Section 7, from the cardinal invariant called π-weight (pseudo-weight) in topology.

4.9. LEMMA. *The following are equivalent, for* $X \subseteq B^+$:

(a) *X is dense in B,*

(b) *for every $b \in B$, there is a pairwise disjoint family $M \subseteq X$ such that $b = \Sigma\, M$,*

(c) *for every $b \in B$, there exists $M \subseteq X$ such that $b = \Sigma\, M$,*

(d) *for every $b \in B$, $b = \Sigma\, \{x \in X \colon x \leq b\}$.*

PROOF. Only the direction from (a) to (b) is non-trivial. So assume $b \in B$ and, by Zorn's lemma, let M be maximal with respect to the properties that $M \subseteq X \cap (B \upharpoonright b)$ and M is a pairwise disjoint family. If $b \neq \Sigma M$, then there is an upper bound b' of M strictly smaller than b. By the denseness of X, pick $x \in X$ such that $0 < x \leq b \cdot -b'$. The pairwise disjoint family $M \cup \{x\}$ then contradicts maximality of M. □

It follows immediately from this lemma that $|B| \leq 2^{|X|}$ if $X \subseteq B$ is dense in B, and hence that $|B| \leq 2^{\pi B}$.

We shall now assign in a canonical way to every partial order $(P, \leq)$ a complete Boolean algebra, its completion. This process may be illustrated by the following standard example, a partial order used in axiomatic set theory to construct models which collapse cardinals and violate the general continuum hypothesis.

4.10. EXAMPLE (partial functions). Let I and J be arbitrary sets, λ a regular infinite cardinal and

$$\mathrm{Fn}(I, J, \lambda) = \{p\colon p \text{ a function from a subset of } I \text{ into } J,\ |\mathrm{dom}\ p| < \lambda\}\ ;$$

for $p, q \in \mathrm{Fn}(I, J, \lambda)$, let

$$q \leq p \quad \text{iff } p \subseteq q \text{ (i.e. iff dom } p \subseteq \text{dom } q \text{ and } p(i) = q(i) \text{ for } i \in \text{dom } p)\,.$$

4.11. DEFINITION AND LEMMA. Let P be a partial order. Two elements p and q of P are *compatible* if there is some $r \in P$ such that $r \leq p$ and $r \leq q$, otherwise *incompatible*. For $p \in P$, let

$$u_p = \{q \in P\colon q \leq p\}\,.$$

The set $\{u_p\colon p \in P\}$ is the base of a topology of P, the *partial order topology*. A subset u of P is open in this topology iff $p \in u$ and $q \leq p$ imply $q \in u$.

For example, in the partial order $\mathrm{Fn}(I, J, \lambda)$ of 4.10, p and q are compatible iff the relation $p \cup q$ is a partial function from I into J, i.e. iff $p(i) = q(i)$ for every i in dom $p \cap$ dom q.

4.12. DEFINITION. Let P be a partial order. A *completion* of P is a pair (e, B) such that B is a complete Boolean algebra, e a mapping from P into B^+ and

(6) e is order preserving: if $q \leq p$ in P, then $e(q) \leq e(p)$ in B,

(7) e preserves incompatibility: if p and q are incompatible in P, then $e(p) \cdot e(q) = 0$ in B,

(8) $e[P]$ is dense in B.

Note that it follows from (6) and from $e(r) > 0$, for $r \in P$, that the converse of

(7) holds true: if $e(p) \cdot e(q) = 0$, then p and q are incompatible. If X is a dense subset of a complete Boolean algebra B, then X, equipped with the partial order inherited by B, has (id_X, B) as a completion; (7) holds since $p \cdot q > 0$ implies, by denseness of X, that $0 < r \leq p \cdot q$ for some $r \in X$, thus p and q are compatible in X.

4.13. THEOREM. *Every partial order P has* $\mathrm{RO}(P)$ *as a completion.*

PROOF. Equip P with the partial order topology of 4.11 and define B to be the regular open algebra of P; let $e: P \to B$ be the mapping defined by

$$e(p) = \operatorname{int} \operatorname{cl} u_p .$$

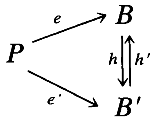
We show that the pair (e, B) is a completion of P.

For every $p \in P$, the regularization $e(p)$ of u_p, in the sense of Definition 1.36, is regular open by (7) in Section 1. Since u_p is a non-empty open subset of P, $u_p \subseteq e(p)$ by (5) of Section 1, so $e(p) \in B^+$. The following argument shows that $e[P]$ is dense in B: let b be a non-empty regular open subset of P. Since $\{u_p: p \in P\}$ is a base for the topology of P, there is some p such that $u_p \subseteq b$. By (8) in Section 1, $e(p)$ is the least regular open subset of P including u_p, so $e(p) \subseteq b$.

Monotonicity of e is trivial since $q \leq p$ implies $u_q \subseteq u_p$ and thus $e(q) \subseteq e(p)$. Finally, suppose that p and q are incompatible. Then u_p and u_q are disjoint and so are their regularization $e(p)$ and $e(q)$, by (10) in Section 1. □

4.14. THEOREM. *Any two completions of a partial order P are isomorphic over P.*

PROOF. Let (e, B) and (e', B') be completions of P; we prove that there is a unique isomorphism $h: B \to B'$ such that $h \circ e = e'$.



First suppose that h exists. Then by the denseness of $e[P]$ and Lemma 4.9,

$$b = \sum \{e(p): p \in P, e(p) \leq b\}$$

for every $b \in B$, and since an isomorphism preserves arbitrary sums,

$$\begin{aligned} h(b) &= \sum \{h(e(p)): p \in P, e(p) \leq b\} \\ &= \sum \{e'(p): p \in P, e(p) \leq b\} . \end{aligned}$$

This shows uniqueness of h and suggests its definition for the existence proof: since B and B' are complete, we can define mappings $h: B \to B'$ and $h': B' \to B$ by

$$h(b) = \sum \{e'(p)\colon p \in P, e(p) \le b\},$$

$$h'(b') = \sum \{e(p)\colon p \in P, e'(p) \le b'\}.$$

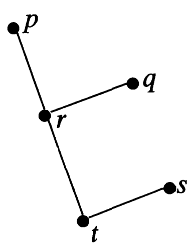
h and h' are order preserving and will be isomorphisms of the partially ordered sets $(B, \le)$ and $(B', \le)$, hence of the Boolean algebras B and B', if we can show that $h' \circ h = \mathrm{id}_B$ and $h \circ h' = \mathrm{id}_{B'}$. The proof of this relies on

(9) for $p \in P$ and $b \in B$, $e(p) \le b$ iff $e'(p) \le h(b)$.

To prove the non-trivial direction of (9), assume that $e'(p) \le h(b)$ but $e(p) \not\le b$. By the denseness of $e[P]$ in B, pick $q \in P$ such that $e(q) \le e(p) \cdot -b$. Then p and q are compatible by (7); let $r \in P$ satisfy $r \le p$ and $r \le q$. So

$$e'(r) \le e'(p) \le h(b) = \sum \{e'(s)\colon s \in P, e(s) \le b\};$$

by the distributive law 1.33(b), there is some $s \in P$ such that $e(s) \le b$ and $0 < e'(r) \cdot e'(s)$. Thus, r and s are compatible; let $t \in P$ satisfy $t \le r$ and $t \le s$. It follows that $e(t) \le e(q) \le -b$ and $e(t) \cdot b = 0$; on the other hand, $e(t) \le e(s) \le b$, a contradiction.



By (9) and the denseness of $e[P]$,

$$\begin{aligned} h'(h(b)) &= \sum \{e(p)\colon e'(p) \le h(b)\} \\ &= \sum \{e(p)\colon e(p) \le b\} \\ &= b; \end{aligned}$$

and $h \circ h' = \mathrm{id}_{B'}$ follows similarly.

We finally prove that $h \circ e = e'$: by definition of h, we have $e'(p) \le h(e(p))$ for $p \in P$. Similarly, $e(p) \le h'(e'(p))$ and thus $h(e(p) \le h(h'(e'(p))) = e'(p)$. □

The uniqueness theorem implies an assertion mentioned at the beginning of this subsection: let B and B' be complete Boolean algebras with dense subsets X (respectively X') which are, with the partial orderings inherited from B (respectively B'), isomorphic. Then (id_X, B) is a completion of X and $(\mathrm{id}_{X'}, B')$ is a completion of X', as remarked above; thus B and B' are isomorphic.

We sketch, for readers with some background in forcing, the fundamental connection between forcing with partial orders and Boolean-valued models of set theory. Let P be a partial order and (e, B) its completion as constructed in 4.13. In the theory of forcing, one defines a proper class $V^{(P)}$ depending on P, a formal language, the forcing language, for set theory which has the elements of $V^{(P)}$ as constants, and a binary relation $\Vdash$ between elements of P and sentences of the forcing language; $p \Vdash \phi$ reads "p forces ϕ". For each sentence ϕ of the forcing language, the subset $\|\phi\|$ of P, defined by

$$\|\phi\| = \{p \in P: p \Vdash \phi\},$$

is regular open, hence an element of B, and the assignment of $\|\phi\|$ to ϕ gives a Boolean valuation of the sentences of the forcing language. Conversely, the forcing relation can be recovered from the Boolean valuation $\| \ldots \|$ by

$$p \Vdash \phi \quad \text{iff } e(p) \leq \|\phi\| \text{ in } B .$$

It is shown in the theory of forcing that each axiom of ZFC is forced by every $p \in P$. Thus, $\|\phi\| = 1$ for each axiom ϕ of ZFC, i.e. $(V^{(P)}, \| \ldots \|)$ is a Boolean-valued model of ZFC with underlying class $V^{(P)}$.

By the existence and uniqueness theorems 4.13 and 4.14, we may speak about *the completion* of a partial order. There is a desirable property of completions which does not hold for arbitrary partially ordered sets: one would like e to be an isomorphism, rather than only an order preserving map, from P onto the dense subset $e[P]$ of B.

4.15. DEFINITION. A partial order P is *separative* if for all p and q in P such that $q \not\leq p$, there is some $r \leq q$ incompatible with p.

The partial order $\mathrm{Fn}(I, J, \lambda)$ of partial functions defined in 4.10 is separative if J has at least two elements. For assume $q \not\leq p$; then the function q does not extend p. Thus, either $p(i) \neq q(i)$ for some $i \in \mathrm{dom}\, p$, and $r = q$ is incompatible with p. Otherwise, there is some $i \in \mathrm{dom}\, p \setminus \mathrm{dom}\, q$, and $r = q \cup \{(i, j)\}$ is incompatible with p, where $j \in J \setminus \{p(i)\}$. Also for a Boolean algebra A, every dense subset X of A with the partial order inherited from A is separative, for if $q \not\leq p$ in X, then $q \cdot -p$ is non-zero in A; by denseness of X in A, there is some $r \in X$ such that $0 < r \leq q \cdot -p$. Then $r \leq q$ and r is incompatible with p in X since $r \cdot p = 0$.

4.16. PROPOSITION. *Let P be a partial order and $(e, B = \mathrm{RO}(P))$ the completion of P constructed in* 4.13. *The following are equivalent*:

(a) *P is separative*,

(b) *$e(p) = \mathrm{int}\,\mathrm{cl}\, u_p$ coincides with u_p, for every $p \in P$*,

(c) *e is an isomorphism from P onto the partial order $e[P] \subseteq B$.*

PROOF. (a) implies (b): first note that for every subset a of P we have, in the partial order topology,

$$\operatorname{int} a = \{x \in P\colon u_x \subseteq a\}$$

$$\operatorname{cl} a = \{x \in P\colon u_x \cap a \neq \emptyset\}\,,$$

since u_x is the least neighborhood of x in P. For $p \in P$,

$$\operatorname{cl} u_p = \{y \in P\colon u_p \cap u_y \neq \emptyset\} = \{y \in P\colon y \text{ and } p \text{ are compatible}\}\,,$$

$$\operatorname{int} \operatorname{cl} u_p = \{x \in P\colon \text{every } y \leq x \text{ is compatible with } p\}\,.$$

Hence, if P is separative, then $q \not\in u_p$ implies $q \not\in \operatorname{int} \operatorname{cl} u_p$, i.e. $\operatorname{int} \operatorname{cl} u_p \subseteq u_p$; also since u_p is open, $u_p \subseteq \operatorname{int} \operatorname{cl} u_p$ by (5) of Section 1.

(b) implies (c): if $q \not\leq p$, then u_q is not included in u_p and $e(q) \not\leq e(p)$ by (b). This shows that e is one-to-one and e^{-1} is order preserving; thus (c) holds.

(c) implies (a): since, by the remark preceding our proposition, dense subsets of Boolean algebras are separative. □

4.3. The completion of a Boolean algebra

For a Boolean algebra A, the completion of the partial order $(A, \leq)$ in the sense of Definition 4.12 is the two-element Boolean algebra, since for each partial order P with a least element, the regular open algebra of P reduces to $\{\emptyset, P\}$. Any reasonable definition of completion for Boolean algebras would, of course, require the completion of A to be a complete Boolean algebra B having A as a subalgebra and being, in a way, minimal over A. Now by Stone's representation theorem 2.1, A is embedded into the complete algebra $P(\text{Ult}\, A)$ via the Stone homomorphism. This embedding has a serious drawback: it does not preserve any non-trivial infinite sums and products, as shown by Exercise 1 in Section 2. The following proposition describes a better behaved class of embeddings. Recall from Definition 1.29 that a subalgebra A of a Boolean algebra B is a regular subalgebra if $\Sigma^A M = \Sigma^B M$ for each $M \subseteq A$ such that $\Sigma^A M$ exists.

4.17. PROPOSITION. *Every dense subalgebra of a Boolean algebra B is regular in B.*

PROOF. Suppose A is a dense subalgebra of B, $M \subseteq A$ and $a = \Sigma^A M$ exists. Then a is an upper bound of M in A, hence in B. Let b be any upper bound of M in B; we claim that $a \leq b$. Otherwise, pick by denseness of A some $a' \in A^+$ such that $a' \leq a \cdot -b$. Then $a \cdot -a'$ is an upper bound of M in A strictly smaller than a, a contradiction. □

This proposition partly motivates the following definition.

4.18. DEFINITION AND NOTATION. Let A be a Boolean algebra. A *completion* of A is a complete Boolean algebra B having A as a dense subalgebra. We write $B = \bar{A}$ if B is a completion of A.

4.19. THEOREM. *Every Boolean algebra has a unique completion, up to isomorphism.*

PROOF. Uniqueness is an immediate consequence of the remark following Theorem 4.14: if A is a dense subalgebra of the complete algebras B and B', then $A^+ = A\setminus\{0\}$ is, with the partial order inherited from A, a dense subset of both B and B', hence there is an isomorphism $h: B \to B'$ such that $h \restriction A = \mathrm{id}_A$.

To prove existence of a completion, it is enough to find a complete Boolean algebra B and a monomorphism f from A onto a dense subalgebra of B; A can then be identified with the isomorphic algebra $f[A]$. To define B, note that A^+ is a dense subset of A, hence a separative partial order by the remark preceding Proposition 4.16. Let (e, B) be the completion of A^+ constructed in the proof of Theorem 4.13, i.e. $B = \mathrm{RO}(A^+)$ and $e(p) = \mathrm{int\,cl}(u_p)$ $(= u_p$, by 4.16); then extend $e: A^+ \to B$ to a function $f: A \to B$ by defining $f(0) = 0$.

By separativity of A^+, e is one-to-one and so is f since $f[A] \subseteq B^+$ and $f(0) = 0$. We prove that f is a Boolean homomorphism. Clearly, $f(0_A) = 0_B$ and $f(1_A) = 1_B$. If $p, q \in A$ are such that $p \cdot q > 0$, then

$$f(p \cdot q) = u_{p \cdot q} = u_p \cap u_q = f(p) \cdot f(q)\,;$$

if $p \cdot q = 0$, then both $f(p \cdot q)$ and $f(p) \cdot f(q) = f(p) \cap f(q)$ are empty. In particular,

$$f(p) \cdot f(-p) = f(0) = 0\,.$$

Also, $f(p) + f(-p) = 1$: otherwise $p \neq 0$, $-p \neq 0$ and by denseness of $e[A^+]$ in B, there is some $q \in A^+$ such that $e(q) \cdot e(p) = 0 = e(q) \cdot e(-p)$. This means that $q \cdot p = 0 = q \cdot -p$, contradicting $q > 0$. Thus, f preserves $\cdot$ and $-$ and is a homomorphism.

Finally, $f[A]$ is a subalgebra of B since f is a homomorphism; it is a dense subalgebra since $e[A^+]$ is dense in B. □

Again we speak about *the* completion $\bar{A}$ of A, having in mind any complete Boolean algebra with A as a dense subalgebra. For example, the power set algebra $P(X)$ of X is the completion of the finite–cofinite algebra on X. There are constructions of $\bar{A}$ different from that given in 4.19, e.g. we introduce in Section 7 a topology in the set Ult A of ultrafilters of A and prove that $\bar{A} \cong \mathrm{RO}(\mathrm{Ult}\, A)$. The construction above has, however, the advantage of not using any form of the axiom of choice. For another characterization of $\bar{A}$ over A, see Exercise 2 of Section 5.

4.20. PROPOSITION. *A Boolean algebra is complete iff it is isomorphic to the regular open algebra of some topological space.*

PROOF. The regular open algebra of a topological space is complete by Theorem 1.37. Conversely, let B be complete and endow the partial order B^+ with its partial order topology. The proof of Theorem 4.19 shows that $\mathrm{RO}(B^+)$ is a completion of B; trivially B is another one. Thus, $B \cong \mathrm{RO}(B^+)$ by uniqueness of completions. □

4.4. *Irredundance and pairwise incomparable families*

We prove here three combinatorial theorems involving the notion of denseness and the process of generating subalgebras from subsets with, possibly, additional properties.

4.21. DEFINITION. A subset X of a Boolean algebra is *irredundant* if no element x of X is generated by $X \setminus \{x\}$.

The subsequent results show that a Boolean algebra need not have an irredundant set of generators; it has, however, large subalgebras in the sense of denseness which are irredundantly generated. This follows from 4.23 since, by Zorn's lemma, each Boolean algebra has a maximally irredundant subset.

4.22. PROPOSITION. *No infinite σ-complete Boolean algebra has an irredundant set of generators.*

PROOF. Assume for contradiction that A is infinite, σ-complete and generated by an irredundant subset X. Thus, X is infinite; let x_n, $n \in \omega$, be distinct elements of X and put $Y = X \setminus \{x_n: n \in \omega\}$. A is the union of the countable chain $(A_n)_{n\in\omega}$ of subalgebras where $A_n = \langle Y \cup \{x_k: k < n\}\rangle$, and by irredundance of X, each A_n is a proper subalgebra of A. This situation, however, is impossible as shown by the following argument.

Call $a \in A$ *small* if $A \restriction a \subseteq A_n$ for some $n \in \omega$, and *large* otherwise. If a, $a' \in A$ are such that $a + a'$ is large, then a or a' is large. Otherwise, there is $n \in \omega$ such that both $A \restriction a$ and $A \restriction a'$ are included in A_n, and then $A \restriction (a + a') \subseteq A_n$, contradicting largeness of $a + a'$, since for $x \in A \restriction (a + a')$, $x = x \cdot a + x \cdot a'$ where $x \cdot a$ and $x \cdot a'$ are in A_n.

For $a \in A$, define the *height* of a by

$$h(a) = \min\{n \in \omega: a \in A_n\} .$$

We claim that if $a \in A$ is large and $n \in \omega$, then there is $b \leq a$ such that $h(b) > n$ and $a \cdot -b$ is large. To see this, pick $a' \leq a$ such that $h(a') > \max(n, h(a))$; this is possible since a is large. Then $h(a \cdot -a') > h(a)$ (otherwise, $a' = a \cdot -(a \cdot -a')$ is in $A_{h(a)}$) and at least one of the elements a' and $a \cdot -a'$ must be large, as shown above. So let $b = a'$ if $a \cdot -a'$ is large and $b = a \cdot -a'$ if a' is large.

Using this claim and the fact that the unit element 1 of A is large, we inductively choose pairwise disjoint non-zero elements d_n, $n \in \omega$, of A such that $h(d_n) > n$ and $-(d_0 + \cdots + d_{n-1})$ is large. Now let $\{N_k: k \in \omega\}$ be a partition of ω into infinite subsets; by σ-completeness, put

$$s_k = \sum \{d_n: n \in N_k\}$$

and pick, for $k \in \omega$, an element $l(k)$ of N_k such that $k \leq l(k)$ and $h(s_k) \leq l(k)$. Define

$$x = \sum \{d_{l(k)}: k \in \omega\} .$$

By disjointness of the d_n, the distributive law 1.33(c) and $N_k \cap \{l(j): j \in \omega\} = \{l(k)\}$, we obtain, for every $k \in \omega$,

$$s_k \cdot x = d_{l(k)} .$$

Assume $x \in A_k$. By the above choice of $l(k)$, both x and s_k are in $A_{l(k)}$. But then $d_{l(k)} \in A_{l(k)}$, a contradiction. So $x \notin \bigcup_{k\in\omega} A_k$ and $A \neq \bigcup_{k\in\omega} A_k$. □

4.23. PROPOSITION (McKenzie). *Every maximally irredundant subset of a Boolean algebra generates a dense subalgebra.*

PROOF. Let X be maximally irredundant in a Boolean algebra A and D the subalgebra of A generated by X. Suppose $a \in A^+$; we want to find some $d \in D^+$ such that $d \leq a$. We may assume that $a \notin D$. By maximality of X and $a \notin D = \langle X \rangle$, there is an $x \in X$ such that $x \in \langle Y \cup \{a\} \rangle$ where $Y = X \setminus \{x\}$. By Corollary 4.7, write

$$x = y \cdot a + z \cdot -a ,$$

where $y, z \in \langle Y \rangle$. Then $x \cdot -a = z \cdot -a$, thus the symmetric difference $d = x \triangle z$, an element of D, satisfies $d \leq a$. Since $z \in \langle Y \rangle$ but $x \notin \langle Y \rangle$ by irredundance of X, $x \neq z$; so d is non-zero. □

4.24. DEFINITION. Two elements x and y of a Boolean algebra A are *comparable* if $x \leq y$ or $y \leq x$, *incomparable* otherwise. A subset X of A is a *pairwise incomparable family* if any two distinct elements of X are incomparable.

A Boolean algebra of infinite cardinality κ does not necessarily have a pairwise incomparable family of size κ or a chain, i.e. a set of pairwise comparable elements, of size κ. For example, in SHELAH [1981] a Boolean algebra is constructed under CH which has cardinality ω_1 but no uncountable pairwise incomparable family or uncountable chain. At least, by the following result and the consequence $|A| \leq 2^{\pi A}$ of 4.9, a Boolean algebra A of cardinality κ has an incomparable family of size λ, for some cardinal λ satisfying $2^\lambda \geq \kappa$.

In the following proof, call $(a_i)_{i\in I}$ an irredundant family in A if $a_i \notin \langle a_j: j \in I \setminus \{i\} \rangle$ for every $i \in I$.

4.25. THEOREM (Shelah). *Every infinite Boolean algebra A has an irredundant pairwise incomparable family of size πA.*

PROOF. We start out with two preliminary remarks.

Claim 1. If $f: A \to B$ is a homomorphism of Boolean algebras, $(a_\alpha)_{\alpha<\kappa}$ a sequence in A such that $(f(a_\alpha))_{\alpha<\kappa}$ is an irredundant family in B, then $(a_\alpha)_{\alpha<\kappa}$ is an irredundant family in A.

For otherwise, some a_α is generated by $\{a_\beta: \beta \neq \alpha\}$. Writing a_α in normal form over $\{a_\beta: \beta \neq \alpha\}$ shows that $f(a_\alpha)$ can be written in normal form over $\{f(a_\beta): \beta \neq \alpha\}$ and that $f(a_\alpha)$ is generated by $\{f(a_\beta): \beta \neq \alpha\}$, a contradiction.

Claim 2. Let, in a Boolean algebra B, $(b_\alpha)_{\alpha<\kappa}$ be a sequence of positive elements such that, for $\alpha < \kappa$, no positive element of $B \upharpoonright b_\alpha$ is generated by $\{b_\beta: \beta < \alpha\}$. Then $(b_\alpha)_{\alpha<\kappa}$ is an irredundant family.

Otherwise there are a finite subset X of κ and some $\alpha \in \kappa$ such that $\alpha \not\in X$ but $b_\alpha \in \langle b_\xi: \xi \in X\rangle$; we may assume that $|X|$ is minimal for this situation. Since b_α is not generated by $\{b_\beta: \beta < \alpha\}$, X is non-empty, say

$$X = \{\alpha(1), \ldots, \alpha(n)\}, \quad \alpha(1) < \cdots < \alpha(n).$$

By assumption of Claim 2, $\alpha < \alpha(n)$. Let

$$A = \langle b_{\alpha(i)}: 1 \leq i < n\rangle, \qquad u = b_{\alpha(n)};$$

thus $b_\alpha \in A(u)$ and by Corollary 4.7, there are pairwise disjoint a, a', a'' in A such that

$$b_\alpha = a \cdot u + a' \cdot -u + a''.$$

Now $a \cdot b_\alpha = a \cdot u \leq u = b_{\alpha(n)}$; this implies $a \cdot b_\alpha = 0$ since $a \cdot b_\alpha$ is generated by $\{b_\beta: \beta < \alpha(n)\}$; it follows that $a \cdot u = 0$. Similarly, $a' \cdot -u \leq b_\alpha$, thus $a' \cdot -b_\alpha \leq u$, $a' \cdot -b_\alpha = 0$ and $a' \leq b_\alpha$ which gives $a' \cdot -u = a' \cdot b_\alpha = a'$. So

$$b_\alpha = 0 + a' + a''$$

is generated by $\{b_{\alpha(i)}: 1 \leq i < n\}$ which contradicts the minimal choice of X and finishes Claim 2.

We now prove the theorem by induction on the cardinal invariant πA of A. Note that A is infinite iff πA is infinite, so let $\kappa = \pi A$ be infinite and suppose the theorem holds for every infinite Boolean algebra C with $\pi C < \kappa$. Define, for $a \in A$, $\pi(a) = \pi(A \upharpoonright a)$.

Case 1. There is some $a \in A$ such that $\pi(b) = \kappa$ for all $b \in A$ satisfying $0 < b \leq a$. Then a is not an atom of A, so choose disjoint non-zero elements a_1 and a_2 in $A \upharpoonright a$. For $i = 1, 2$, define inductively a sequence $(a_{i\alpha})_{\alpha<\kappa}$ in $(A \upharpoonright a_i)^+$ such that no positive element of $A \upharpoonright a_{i\alpha}$ is generated, in $A \upharpoonright a_i$, by $\{a_{i\beta}: \beta < \alpha\}$. This is possible since by $\pi(a_i) = \kappa$, the subalgebra generated in $A \upharpoonright a_i$ by $\{a_{i\beta}: \beta < \alpha\}$ cannot be dense in $A \upharpoonright a_i$. Define, for $\alpha < \kappa$,

$$c_\alpha = a_{1\alpha} + a_2 \cdot -a_{2\alpha};$$

we show that $(c_\alpha)_{\alpha<\kappa}$ is an irredundant family of pairwise incomparable elements. Obviously, $\beta < \alpha$ implies that $a_{1\beta} \not\leq a_{1\alpha}$ and hence $c_\beta \not\leq c_\alpha$; similarly for $\beta < \alpha$, $a_{2\beta} \not\leq a_{2\alpha}$, $-a_{2\alpha} \not\leq -a_{2\beta}$ and $c_\alpha \not\leq c_\beta$. So the c_α are pairwise incomparable. To show irredundance, consider the projection homomorphism $p: A \to A \restriction a_1$ defined by $p(x) = x \cdot a_1$. By Claim 2, $(a_{1\alpha})_{\alpha<\kappa}$ is an irredundant family in $A \restriction a_1$; so by Claim 1 and $p(c_\alpha) = a_{1\alpha}$, $(c_\alpha)_{\alpha<\kappa}$ is an irredundant family in A.

If Case 1 fails, then

$$D = \{a \in A^+ : \pi(a) < \kappa\}$$

is dense in A. By Lemma 4.9, there is a partition X of unity included in D.

Case 2. $|X| = \kappa$. Then X is an irredundant pairwise incomparable family in A, and we are finished.

Case 3. $|X| < \kappa$. Then for $x \in X$, choose a dense subset D_x in $A \restriction x$ of size $\pi(x) < \kappa$. $\bigcup_{x\in X} D_x$ is dense in A and thus has cardinality at least $\pi A = \kappa$. So κ is singular; let $\kappa = \sup_{\alpha<\mathrm{cf}\,\kappa} \kappa_\alpha$ where each κ_α is an infinite cardinal less than κ. Since $|X| < \kappa$, $|D_x| = \pi(x)$ for $x \in X$ and

$$\kappa \le \left|\bigcup_{x\in X} D_x\right| = \sup_{x\in X} \pi(x) ,$$

we can choose a sequence $(x_\alpha)_{\alpha<\mathrm{cf}\,\kappa}$ of pairwise distinct elements of X such that $\pi(x_\alpha) > \kappa_\alpha$. By induction hypothesis and $\kappa_\alpha < \pi(x_\alpha) < \kappa$, each $A \restriction x_\alpha$ has an irredundant pairwise incomparable family M_α of size κ_α. Then $\bigcup_{\alpha<\mathrm{cf}\,\kappa} M_\alpha$ is an irredundant pairwise incomparable family in A of size κ. □

Exercises

1. Let $n \in \omega$, A the power set algebra of ${}^n2 = {}^{\{0,\ldots,n-1\}}\{0, 1\}$; so $|A| = 2^{2^n}$. Find a subset of A of size n which generates A.

2. Show that a Boolean algebra A is atomic (respectively atomless) iff its completion $\bar{A}$ is atomic (respectively atomless). Moreover, $\pi A = \pi \bar{A}$.

3. Prove that no dense proper subalgebra of a Boolean algebra can be complete.

4. Show that $\mathrm{RO}(\boldsymbol{R})$ is (isomorphic to) a completion of Intalg $\boldsymbol{R}$.

5. Let $(P, \le)$ be a partial order. Call $X \subseteq P$ an *antichain* of P if the elements of X are pairwise incompatible and define

$$\mathrm{c}_{\mathrm{po}}P = \sup\{|X| : X \subseteq P \text{ an antichain}\} ,$$

the *cellularity* of $(P, \le)$. Prove that, if P is equipped with its partial order topology,

$$\begin{aligned} \mathrm{c}_{\mathrm{po}}P &= \text{the cellularity of the topological space } P \\ &= \text{the cellularity of the Boolean algebra } \mathrm{RO}(P) . \end{aligned}$$

6. Let J be a discrete topological space; endow ${}^IJ$ with the product topology and the partial order $\mathrm{Fn}(I, J, \omega)$ with the partial order topology. Then $\mathrm{RO}({}^JI) \cong \mathrm{RO}(\mathrm{Fn}(I, J, \omega))$.

7. Let P be the separative partial order $\mathrm{Fn}(I, J, \omega)$, where $|I| = \omega$, $|J| = \omega_1$ and let (e, B) be the completion of P. Prove that

(a) B does not satisfy the countable chain condition.

(b) For each $j \in J$, the set

$$M_j = \{e(p) \colon p \in P,\ j \in \operatorname{ran} p\}$$

is dense in B, hence $\Sigma\, M_j = 1$.

(c) There is no ultrafilter x of B preserving $\Sigma\, M_j$ for each $j \in J$, for otherwise, $\bigcup \{p \in P \colon e(p) \in x\}$ is a function mapping I onto J.

Thus, in Martin's axiom, the assumption of the countable chain condition cannot be dispensed with.

8. Let (X, d) be a metric space. For $a \subseteq X$ non-empty, the *diameter* of a is defined by

$$\operatorname{diam}(a) = \sup\{d(x, y) \colon x, y \in a\}\,.$$

(a) For $u \subseteq X$ open and $\mathrm{r}u = \operatorname{int}\operatorname{cl} u$ its regularization (cf. Definition 1.36), show that $\operatorname{diam}(u) = \operatorname{diam}(\mathrm{r}u)$.

(b) For $1 \leq n < \omega$, fix a pairwise disjoint family P_n in $\mathrm{RO}(X)$ maximal with respect to the property that $\operatorname{diam}(u) \leq 1/n$ for $u \in P_n$. Then each P_n is a partition of unity in $\mathrm{RO}(X)$ and $\bigcup_{1 \leq n < \omega} P_n$ is dense in $\mathrm{RO}(X)$.

(c) Conclude that the cellularity of $\mathrm{RO}(X)$ equals $\sup\{|P_n| \colon 1 \leq n < \omega\}$, and is actually attained.

5. Homomorphisms, ideals, and quotients

In this section we deal with several important notions, constructions and results on homomorphisms. The first major topic is Sikorski's theory on extensions of maps between Boolean algebras to homomorphisms and extensions of homomorphisms defined on subalgebras to larger algebras; Sikorski's extension criterion 5.5 is one of the bread-and-butter theorems on Boolean algebras. Two special cases of Sikorski's extension criterion imply Vaught's isomorphism theorem 5.15 for countable Boolean algebras and a characterization of complete Boolean algebras via Sikorski's extension theorem 5.9. Both results are fundamental for the theory of countable (respectively complete) Boolean algebras.

Our second topic is the formation of the quotient of a Boolean algebra modulo an ideal. This is a frequently applied and quite general construction since every homomorphic image of a Boolean algebra A is isomorphic to a quotient of A. As a particularly enlightening example, we study the quotient of the power set algebra $P(\omega)$ of the natural numbers modulo the ideal of finite sets.

5.1. Homomorphic extensions

A homomorphism between Boolean algebras A and B was defined, in 1.3, to be a map $g \colon A \to B$ preserving the Boolean operations $+$, $\cdot$, $-$ and the distinguished elements 0 and 1. It suffices, however, to assume that g preserves the operations $+$ and $-$ or, dually, $\cdot$ and $-$, since the remaining operations and constants can be recovered from these. Even preservation of $+$, 0, 1 and disjointness is sufficient (and hence preservation of $+$, $\cdot$, 0 and 1), for it implies $g(x)+g(-x)=1$ and $g(x)\cdot g(-x)=0$, i.e. $g(-x)=-g(x)$. Every homomorphism preserves the symmetric difference $x \triangle y = x\cdot -y + y\cdot -x$ and also the Boolean partial order, for it follows from $x \leq y$ that $x = x\cdot y$, $g(x)=g(x)\cdot g(y)$ and $g(x)\leq g(y)$.

In the class of complete (κ-complete, σ-complete) algebras, there is a stronger and more natural notion of homomorphism:

5.1. DEFINITION. A Boolean homomorphism is *complete* (*κ-complete*, *σ-complete*) if it preserves, in the sense of 2.20, ΣM for every $M \subseteq A$ (for every $M \subseteq A$ of cardinality less than κ, for every countable $M \subseteq A$) for wich $\Sigma^A M$ happens to exist.

In Definition 2.5, a one-to-one (onto, bijective) homomorphism was called a monomorphism (an epimorphism, an isomorphism). Let us list, for further reference, two more notions.

5.2. DEFINITION. Let A be a Boolean algebra. An *endomorphism* of A is a homomorphism from A into itself. An *automorphism* of A is an isomorphism from A onto itself.

5.3. LEMMA. *A homomorphism g: $A \to B$ is a monomorphism iff, for every x in A, $g(x)=0$ implies $x=0$.*

PROOF. This follows from the equivalence of $y=z$ and $0=y\triangle z$ in A (see Lemma 1.25) (respectively $g(y)=g(z)$ and $0=g(y)\triangle g(z)=g(y\triangle z)$ in B), since g preserves symmetric differences. □

In the rest of this subsection, we deal with the question in which circumstances a map f defined on a set X of generators of A can be extended to a homomorphism g: $A\to B$. Such an extension, if it exists, is uniquely determined.

5.4. LEMMA. *If g and g' are homomorphisms from A to B coinciding on a set of generators of A, then $g=g'$.*

PROOF. This is an immediate consequence of the normal form theorem 4.4. Here is a less computational proof: the set

$$A_0 = \{x \in A\colon g(x)=g'(x)\}$$

is a subalgebra of A. If it includes a set of generators of A, then $A_0 = A$ and $g=g'$. □

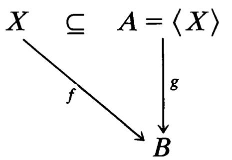
In the following proof, we use the fact that, loosely speaking, the union of a non-empty directed family of homomorphisms is a homomorphism. More precisely, assume that for $i \in I$, g_i: $A_i \to B$ is a homomorphism from a subalgebra A_i of A into B and that for arbitrary $i, j \in I$ there is $k \in I$ such that $A_i \cup A_j \subseteq A_k$ and g_k extends both g_i and g_j. Then $\bigcup_{i\in I} A_i$ is a subalgebra of A and $\bigcup_{i\in I} g_i$: $\bigcup_{i\in I} A_i \to B$ is a homomorphism. In particular, the union of any non-empty chain of homomorphisms from subalgebras of A into B is a homomorphism from a subalgebra of A into B.

With the normal form theorem 4.4 and its notation at our hands, we are in a position to prove the fundamental result of this subsection.

5.5. THEOREM (Sikorski's extension criterion). *Assume X generates A and f maps X into a Boolean algebra B. The following condition is necessary and sufficient for f to extend to a homomorphism from A into B:*

for $n \in \omega$, $x_1, \ldots, x_n \in X$ and $\varepsilon_1, \ldots \varepsilon_n \in \{+1, -1\}$,

(1) *if $\varepsilon_1 x_1 \cdot \cdots \cdot \varepsilon_n x_n = 0$ in A, then $\varepsilon_1 f(x_1) \cdot \cdots \cdot \varepsilon_n f(x_n) = 0$ in B.*



PROOF. Necessity of the condition is obvious. For sufficiency, it is enough to consider the case where X is finite. For suppose the theorem holds true for finite sets of generators. Then for each finite subset Y of X, there is by 5.4 a unique homomorphism

$$g_Y: \langle Y \rangle \to B$$

extending $f \restriction Y$. Now if Z is a finite subset of X including Y, then also the restriction of g_Z to $\langle Y \rangle$ is a homomorphism extending $f \restriction Y$, so g_Z extends g_Y. Thus, $\{g_Y$: Y a finite subset of $X\}$ is a directed set of homomorphisms, and its union is a homomorphism from A into B extending f.

So assume $X = \{x_1, \ldots, x_n\}$ is finite. We define, as in the proof of the normal form theorem 4.4, the set

$$E = {}^{\{1,\ldots,n\}}\{+1, -1\}$$

of all functions from $\{1, \ldots, n\}$ to $\{+1, -1\}$ and, in A, the elementary products

$$p_e = e(1)x_1 \cdot \cdots \cdot e(n)x_n$$

for $e \in E$, and elements written in normal form:

$$s_M = \sum_{e\in M} p_e$$

for $M \subseteq E$. Similarly, we define in B

$$q_e = e(1)f(x_1) \cdot \cdots \cdot e(n)f(x_n)$$

for $e \in E$ and

$$t_M = \sum_{e \in M} q_e$$

for $M \subseteq E$. By the normal form theorem, $A = \langle X \rangle = \{s_M : M \subseteq E\}$; we will see that the map $g: A \to B$, given by

$$g(s_M) = t_M$$

for $M \subseteq E$, works for the theorem.

First, g is well-defined, i.e. $s_M = s_{M'}$ implies $t_M = t_{M'}$, for $M, M' \subseteq E$. To prove this, note that by assertions (4) and (5) in the proof of the normal form theorem, the map assigning s_M to M is a homomorphism from $P(E)$ into $\langle x_1, \ldots, x_n \rangle$. Thus, if $s_M = s_{M'}$, then $0 = s_M \triangle s_{M'} = s_{M \triangle M'}$, i.e. $p_e = 0$ for each $e \in M \triangle M'$. By condition (1) above, also $q_e = 0$ for $e \in M \triangle M'$, and this shows $t_M = t_{M'}$.

g is a homomorphism since, by (4) in 4.4,

$$g(s_M + s_{M'}) = g(s_{M \cup M'}) = t_{M \cup M'} = t_M + t_{M'} = g(s_M) + g(s_{M'})\ ;$$

similarly, g preserves complements by (5) in 4.4.

Finally, letting $M = \{e \in E: e(i) = +1\}$, we have $s_M = x_i$ and $g(x_i) = t_M = f(x_i)$; thus g is an extension of f. □

The same proof gives the following more general but somewhat messy formulation which is also referred to as Sikorski's extension criterion.

5.6. PROPOSITION. *Let $r \subseteq A \times B$ be a relation between elements of two Boolean algebras A and B. Then r is a function from* dom $r \subseteq A$ *into B and extends to a homomorphism from* $\langle \text{dom}\, r \rangle \subseteq A$ *onto* $\langle \text{ran}\, r \rangle \subseteq B$ *iff*:

for $n \in \omega$, $(x_1, y_1), \ldots, (x_n, y_n) \in r$ and $\varepsilon_1, \ldots, \varepsilon_n \in \{+1, -1\}$,

(2) *if $\varepsilon_1 x_1 \cdot \cdots \cdot \varepsilon_n x_n = 0$ in A, then $\varepsilon_1 y_1 \cdot \cdots \cdot \varepsilon_n y_n = 0$ in B.*

r extends to an isomorphism from $\langle \text{dom}\, r \rangle \subseteq A$ *onto* $\langle \text{ran}\, r \rangle \subseteq B$ *iff*:

for $n \in \omega$, $(x_1, y_1), \ldots, (x_n, y_n) \in r$ and $\varepsilon_1, \ldots, \varepsilon_n \in \{+1, -1\}$,

(2′) *$\varepsilon_1 x_1 \cdot \cdots \cdot \varepsilon_n x_n = 0$ in A iff $\varepsilon_1 y_1 \cdot \cdots \cdot \varepsilon_n y_n = 0$ in B.*

PROOF. We only prove that r is a function; the rest follows from 5.3 and 5.5. Assume (x, y_1) and (x, y_2) are in r with the aim of showing $y_1 = y_2$. With $n = 2$,

$x_1 = x_2 = x$ and $\varepsilon_1 = +1$, $\varepsilon_2 = -1$, we see that $\varepsilon_1 x_1 \cdot \varepsilon_2 x_2 = x \cdot -x = 0$. Thus, by (2), $\varepsilon_1 y_1 \cdot \varepsilon_2 y_2 = y_1 \cdot -y_2 = 0$ and $y_1 \leq y_2$. Similarly, $y_2 \leq y_1$ and thus $y_1 = y_2$. □

Let us state two useful special cases of Sikorski's extension criterion.

5.7. COROLLARY. *Assume that, for $i \in I$, f_i: $A_i \to B$ is a homomorphism from a subalgebra A_i of A into B and that $\bigcup_{i \in I} A_i$ generates A. There exists a homomorphism from A into B extending each f_i iff*:

for $n \in \omega$, distinct $i(1), \ldots, i(n) \in I$ and $a_{i(k)} \in A_{i(k)}$,

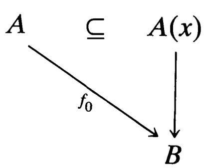
(3) *if $a_{i(1)} \cdot \cdots \cdot a_{i(n)} = 0$ in A, then $f_{i(1)}(a_{i(1)}) \cdot \cdots \cdot f_{i(n)}(a_{i(n)}) = 0$ in B.*

PROOF. In the preceding proposition, put $r = \bigcup_{i \in I} f_i$. Since, for every $i \in I$, A_i is a subalgebra and f_i is a homomorphism, the elementary products considered in (2) can be reduced to the form considered in (3). □

Recall from Section 4 that, for A a subalgebra of a Boolean algebra A' and $x \in A'$, $A(x)$ is the subalgebra of A' generated by $A \cup \{x\}$, the simple extension of A by x.

5.8. COROLLARY. *Let f_0: $A \to B$ be a homomorphism of Boolean algebras, $A(x)$ a simple extension of A and y an element of B. There exists a homomorphism g: $A(x) \to B$ extending f_0 and mapping x to y iff for all a and a' in A*:

(4) *if $a \leq x \leq a'$ in A, then $f_0(a) \leq y \leq f_0(a')$ in B.*



PROOF. Let, in Proposition 5.6, $r = f_0 \cup \{(x, y)\}$. Existence of g is equivalent, by (2), to the assertions

$$\text{if } c \cdot x = 0, \text{ then } f_0(c) \cdot y = 0,$$

and

$$\text{if } d \cdot -x = 0, \text{ then } f_0(d) \cdot -y = 0,$$

for c, d in A. But $c \cdot x = 0$ iff $x \leq -c$ and, since f_0 is a homomorphism, $f_0(c) \cdot y = 0$ iff $y \leq f_0(-c)$. Similar reasoning applies to the second assertion. □

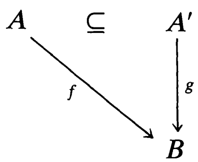
5.2. *Sikorski's extension theorem*

As a consequence of Corollary 5.8, we shall now prove Sikorski's extension theorem, a powerful principle characteristic of complete Boolean algebras. The core of its proof is the fact that, given a homomorphism $f_0: A \to B$ into a complete Boolean algebra B and a simple extension $A(x)$ of A as in 5.8, there is always a homomorphic extension g of f_0 to $A(x)$. For, by completeness of B, we can consider

$$s = \textstyle\sum^B \{f_0(a): a \in A, a \le x\}, \qquad t = \prod^B \{f_0(a'): a' \in A, x \le a'\}.$$

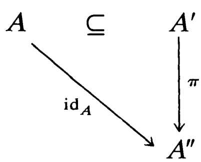
Now $s \le t$ since $f_0(a) \le f_0(a')$ for $a \le x \le a'$ $(a, a' \in A)$, and every $y \in B$ such that $s \le y \le t$ satisfies (4) in 5.8, guaranteeing existence of g. Sikorski's extension theorem slightly generalizes this situation.

5.9. THEOREM (Sikorski's extension theorem). *Let A be a subalgebra of A' and f a homomorphism from A into a complete Boolean algebra B. Then f can be extended to a homomorphism of A' into B.*



PROOF. Let P be the set of those homomorphisms g from a subalgebra of A' into B such that dom g includes A and g extends f. P is non-empty and partially ordered by inclusion; moreover each non-empty chain C in P has $\bigcup C$ as an upper bound. Thus, by Zorn's lemma, P has a maximal element g. The subalgebra dom g of A' coincides with A', i.e. g is as required. Otherwise let $x \in A' \setminus \text{dom } g$; by completeness of B and the remark preceding this theorem, g has a homomorphic extension $g': (\text{dom } g)(x) \to B$, contradicting maximality of g. □

5.10. COROLLARY. *Let A be a subalgebra of A'. Then there is an epimorphism π from A' onto a Boolean algebra A'' having A as a dense subalgebra such that $\pi \restriction A = \text{id}_A$.*



PROOF. By Sikorski's extension theorem, the identity map id_A from A into the completion $\bar{A}$ of A has a homomorphic extension $\pi: A' \to \bar{A}$, and the subalgebra $A'' = \text{ran } \pi$ of $\bar{A}$ has A as a dense subalgebra since $A \subseteq A''$ and A is dense in $\bar{A}$. □

A homomorphic extension g of f as given by Sikorski's extension theorem is, in general, neither uniquely determined nor can we expect g to be one-to-one if f is. Injectivity of g is sometimes ensured by the following simple fact.

5.11. LEMMA. *Let A be a dense subalgebra of A', $f\colon A \to B$ a monomorphism and $g\colon A' \to B$ a homomorphic extension of f. Then also g is a monomorphism.*

PROOF. By Lemma 5.3, we have to show that $g(y) > 0$ in B if $y > 0$ in A'. Choose, by denseness of A in A', $x \in A$ such that $0 < x \leq y$. Then $0 < f(x) = g(x) \leq g(y)$ since f is one-to-one. □

Let us consider two consequences of the preceding results. First, assume that f is the inclusion map from A into its completion $\bar{A} = B$. Then every A' having A as a dense subalgebra embeds into $\bar{A}$ over A; thus the extensions A' of A with A dense in A' can be described as being the subalgebras of $\bar{A}$ lying between A and $\bar{A}$. Second, assume A' is the completion $\bar{A}$ of A and B is an arbitrary complete algebra into which A is embedded via f; then $\bar{A}$ is embeddable into B via an extension of f. For this reason, $\bar{A}$ is sometimes called the minimal completion of A.

We finish this subsection with a category-theoretic description of complete algebras.

5.12. DEFINITION. A Boolean algebra B is *injective* if every homomorphism from an arbitrary algebra A into B can be extended to every extension of A.

B is a *retract* of an algebra C if there are homomorphisms $e\colon B \to C$ and $r\colon C \to B$ such that $r \circ e = \mathrm{id}_B$.

$$B \underset{e}{\overset{r}{\leftrightarrows}} C$$

If B is a retract of C via e and r, then e is one-to-one and r is onto; thus, B is, up to isomorphism, both a subalgebra and a homomorphic image of C. If $e\colon B \to C$ happens to be the inclusion map, then the epimorphism $r\colon C \to B$ satisfies $r \restriction B = \mathrm{id}_B$ and is called a *retraction* of C onto the subalgebra B.

5.13. THEOREM. *The following are equivalent, for every Boolean algebra B:*

(a) *B is complete,*

(b) *B is injective,*

(c) *B is a retract of a complete Boolean algebra.*

PROOF. The implication from (a) to (b) is asserted by Sikorski's extension theorem 5.9.

(b) implies (c): fix any complete Boolean algebra C having B as a subalgebra (e.g. let C be the completion of B). By injectivity of B, the identity map $\mathrm{id}_B\colon B \to B$ has an extension $g\colon C \to B$, so B is a retract of C via id_B and g.

(c) implies (a): suppose B is a retract of a complete algebra C via $e\colon B \to C$ and $r\colon C \to B$. Let M be a subset of B; we prove completeness of B by showing that

$$b = r\left(\Sigma^C e[M]\right)$$

is the least upper bound of M in B. For every $m \in M$, $e(m) \leq \Sigma^C e[M]$ and thus $m = r(e(m)) \leq b$; so b is an upper bound of M in B. If b' is another one, then $\Sigma^C e[M] \leq e(b')$ and hence $b \leq r(e(b')) = b'$. □

A category-theoretic dual of injectivity, the notion of projectivity, is studied and characterized in the chapter by KOPPELBERG [Ch. 20 in this Handbook]. The characterization, however, is much more complicated than that of injectivity given in the preceding theorem.

5.3. Vaught's isomorphism theorem

We turn to Vaught's isomorphism theorem, another application of Sikorski's extension criterion. Virtually every isomorphism theorem for countable Boolean algebras can be recovered from it, easy examples being provided by Corollary 5.16 and Proposition 6.6. The theorem itself and the notions involved are basic for the deep study of countable Boolean algebras undertaken in KETONEN [1978]; see the chapter by PIERCE [Ch. 20 in this Handbook] on countable Boolean algebras.

5.14. DEFINITION. A binary relation R between Boolean algebras is a *Vaught relation* if it satisfies the following conditions:

(5) R is symmetric, i.e. ARB implies BRA,

(6) if ARB and A is trivial (i.e. $|A| = 1$), then B is trivial,

(7) (back and forth property) if ARB and $a \in A$, then there is $b \in B$ such that $(A \upharpoonright a)R(B \upharpoonright b)$ and $(A \upharpoonright -a)R(B \upharpoonright -b)$.

For example, being isomorphic is a Vaught relation. We have seen in Lemma 3.2 that, for every element a of a Boolean algebra A, A is isomorphic to the product of the relative algebras $A \upharpoonright a$ and $A \upharpoonright -a$, and it will be proved in Section 6 that every product decomposition of A arises in this way. Thus, existence of a Vaught relation R such that ARB essentially means that A and B decompose into factors in a similar way.

5.15. THEOREM (Vaught's isomorphism theorem). *Assume A and B are at most countable Boolean algebras and ARB for some Vaught relation R. Then A is isomorphic to B.*

PROOF. We first set up some notation motivated by the proof of Sikorski's extension criterion 5.5 and its modification 5.6. Following the convention from set theory that each $n \in \omega$ is the set $\{0, \ldots, n-1\}$, we define

$$E_n = {}^n\{1, -1\} .$$

Then if $(a_i)_{i\in\omega}$ and $(b_i)_{i\in\omega}$ are arbitrary sequences in A (respectively B), define, for $e \in E_n$, elementary products in A and B by

$$p_e = \prod_{i\in n} e(i)a_i , \qquad q_e = \prod_{i\in n} e(i)b_i .$$

We aim to construct $a_i \in A$ and $b_i \in B$ in such a way that

$$A = \{a_i : i \in \omega\} , \qquad B = \{b_i : i \in \omega\}$$

and, for $n \in \omega$,

$(8)_n$ $\quad (A \restriction p_e)R(B \restriction q_e)$ for each $e \in E_n$.

Then putting $A_n = \langle a_i : i < n\rangle$ and $B_n = \langle b_i : i < n\rangle$, we see that $A = \bigcup_{n\in\omega} A_n$ since $A = \{a_i : i \in \omega\}$; similarly, $B = \bigcup_{n\in\omega} B_n$. It follows from $(8)_n$, (6) and 5.6 that there is a unique isomorphism f_n from A_n onto B_n mapping a_i onto b_i for $i < n$. Clearly, $(f_n)_{n\in\omega}$ is a chain of homomorphisms from subalgebras of A into B, and $f = \bigcup_{n\in\omega} f_n$ is an isomorphism from A onto B.

To construct the elements a_i and b_i, let

$$A = \{a_0, a_2, a_4, \ldots\} , \qquad B = \{b_1, b_3, b_5, \ldots\}$$

be enumerations of A and B, possibly with repetition; we will construct subsequently $b_0 \in B$, $a_1 \in A$, $b_2 \in B$, $a_3 \in A$, etc.

Suppose n is even (so a_n is already defined) and $(a_i)_{i<n}$, $(b_i)_{i<n}$ have been constructed such that $(8)_n$ holds. For every $e \in E_n$, we have $(A \restriction p_e)R(B \restriction q_e)$ and $p_e \cdot a_n \in A \restriction p_e$; so by the back and forth property (7) of R, pick $x_e \in B \restriction q_e$ such that

$$(A \restriction p_e \cdot a_n)R(B \restriction x_e) , \qquad (A \restriction p_e \cdot -a_n)R(B \restriction q_e \cdot -x_e)$$

and define

$$b_n = \sum \{x_e : e \in E_n\} .$$

Then $(8)_{n+1}$ holds again. For consider $\varepsilon \in E_{n+1}$ and put $e = \varepsilon \restriction n$. If $\varepsilon(n) = +1$, then $p_\varepsilon = p_e \cdot a_n$ and $q_\varepsilon = q_e \cdot b_n = x_e$; if $\varepsilon(n) = -1$, then $p_\varepsilon = p_e \cdot -a_n$ and $q_\varepsilon = q_e \cdot -b_n = q_e \cdot -x_e$. In both cases, $(A \restriction p_\varepsilon)R(B \restriction q_\varepsilon)$ holds by the above choice of x_e.

If n is odd, then b_n is already defined and we find a_n by interchanging the roles of A and B and using symmetry of R. □

For a reader acquainted with model theory, it is not very hard to see that for any two Boolean algebras A and B, there is a Vaught relation R satisfying ARB iff A and B are elementarily equivalent in the logic $L_{\infty\omega}$ – in fact, being $L_{\infty\omega}$-

equivalent is a Vaught relation. Vaught's result can then be regarded as a consequence of Scott's theorem that any two countable structures elementarily equivalent in $L_{\infty\omega}$ are isomorphic.

The following application of Vaught's theorem characterizes, for example, the interval algebra of the rationals as being the unique countably infinite and atomless Boolean algebra, up to isomorphism.

5.16. Corollary. *Any two countably infinite atomless Boolean algebras are isomorphic.*

Proof. Define the relation R by

$$ARB \quad \text{iff } A, B \text{ are both trivial or both infinite and atomless}.$$

Clearly, R is a Vaught relation. □

5.4. *Ideals and quotients*

We describe the fundamental connection between homomorphisms, congruence relations, and quotients of Boolean algebras. There is, of course, nothing particular about Boolean algebras here, since the same connection exists for arbitrary universal algebras, as may be known to the reader. In view of the fact that Boolean algebras can be conceived as particular rings (cf. Section 1) and that the congruence relations of a ring R are in one-to-one correspondence with certain subsets of R, the ideals of R, also the congruence relations on Boolean algebras are determined by ideals; we shall thus define the quotient of a Boolean algebra modulo an ideal.

5.17. Definition. Let A be a Boolean algebra. A *congruence relation* on A is an equivalence relation $\sim$ on A such that, for all x, x', y, y' in A, $x \sim x'$ and $y \sim y'$ imply $-x \sim -x'$ and $x + y \sim x' + y'$.

Thus, a congruence relation is an equivalence relation respecting the Boolean operations $+$ and $-$. It also respect the operation $\cdot$ and all operations definable in terms of equations from $+$ and $-$.

Each homomorphism $f\colon A \to B$ of Boolean algebras induces the congruence relation on A defined by

$$x \sim x' \quad \text{iff } f(x) = f(x').$$

There is a converse to this process:

5.18. Definition and Lemma. Let $\sim$ be a congruence relation on A. For $x \in A$, let

$$\pi(x) = \{x' \in A\colon x \sim x'\}$$

be the equivalence class of x with respect to $\sim$, and let

$$A/\sim = \{\pi(x): x \in A\}$$

be the set of equivalence classes of $\sim$. There is a unique Boolean algebra structure on $A/\sim$ which makes

$$\pi: A \to A/\sim$$

an epimorphism of Boolean algebras. $A/\sim$ is the *quotient algebra* of A with respect to $\sim$; π is the *canonical homomorphism* from A onto $A/\sim$. The congruence relation induced by π on A coincides with $\sim$.

In Section 2, a subset F of a Boolean algebra A was called a filter if $1 \in F$, F is closed under finite products and $x \in F$, $x \leq y$ imply $y \in F$. Ideals of A are defined dually.

5.19. DEFINITION AND LEMMA. A subset I of a Boolean algebra A is an *ideal of A* if

$$0 \in I,$$

$$\text{if } x \in I,\ y \in A \text{ and } y \leq x, \text{ then } y \in I,$$

$$\text{if } x \in I \text{ and } y \in I, \text{ then } x + y \in I.$$

I is a *complete ideal* (*a σ-complete ideal, a κ-complete ideal*) if $\Sigma\, M \in I$ for each subset M of I (each countable subset M of I, each subset M of I of size less than κ) such that $\Sigma\, M$ exists. For every filter F of A,

$$-F = \{-x: x \in F\}$$

is an ideal of A, the *ideal dual to F*. For every ideal I of A,

$$-I = \{-x: x \in I\}$$

is a filter of A, the *filter dual to I*.

Thus, taking the dual of a filter sets up an order-preserving one-to-one correspondence between filters and ideals of a Boolean algebra.

The elements of an ideal I of A may be intuitively thought of as being "small" in A. For example, for any set X, the set of finite subsets of X is an ideal in the power set algebra $P(X)$, its dual filter being the set of cofinite subsets of X. Similarly, the subsets of X of cardinality less than κ form a κ-complete ideal in $P(X)$ if κ is regular. The sets of Lebesgue measure zero form a σ-complete ideal in the σ-algebra of Lebesgue-measurable subsets of the reals as considered in

Example 1.32, and so do the meager Borel sets in the Borel algebra of an arbitrary topological space (cf. Example 1.30).

As analogues of 2.11 through 2.13, we have the following definitions.

5.20. DEFINITION AND LEMMA. For every subset E of A, the set

$$\{x \in A: x \leq e_1 + \cdots + e_n \text{ for some } n \in \omega \text{ and } e_1, \ldots, e_n \in E\}$$

is an ideal of A, the *ideal generated by* E; it is the least ideal of A including E. Let I be an ideal of A and F its dual filter. I is *proper* if $1 \notin I$, i.e. if F is a proper filter. I is *trivial* if $I = \{0\}$, i.e. if F is the trivial filter. I is *principal* if I is the ideal $\{x \in A: x \leq a\}$ generated by some $a \in A$, i.e. if F is the principal filter generated by $-a$. I is *prime* if it is proper and $x \cdot y \in I$ implies that $x \in I$ or $y \in I$; i.e. if F is a prime filter.

Passing to the dual filter F of an ideal I and applying Proposition 2.15 shows that I is prime iff it is a maximal proper ideal. Moreover, by the very definition of ultrafilters, $I = A \backslash F$ iff F is an ultrafilter.

5.21. DEFINITION AND LEMMA. If $f: A \to B$ is a homomorphism of Boolean algebras, then

$$f^{-1}(0) = \{x \in A: f(x) = 0\}$$

is an ideal of A, the *kernel* of f, and

$$f^{-1}(1) = \{x \in A: f(x) = 1\}$$

is a filter of A, the *dual kernel* of f.

It is, of course, a matter of taste or technical convenience whether to deal, in a particular situation, with a filter or its dual ideal. For instance, the following construction of congruence relations is mostly described via ideals but the computational details are more suggestive in terms of filters.

5.22. LEMMA AND DEFINITION. Let I be an ideal of a Boolean algebra A and F its dual filter. Then the relation $\equiv$ on A, defined by

$$x \equiv y \quad \text{iff } x \,\triangle\, y \in I,$$

is a congruence relation on A. For all x and y in A,

$$\begin{aligned} x \equiv y \quad &\text{iff } x + i = y + i \text{ for some } i \in I \\ &\text{iff } x \cdot f = y \cdot f \text{ for some } f \in F. \end{aligned}$$

The quotient algebra $A/{\equiv}$ is denoted by A/I or A/F and called the *quotient* *algebra* of A by I (respectively F). The canonical epimorphism $\pi: A \to A/I$ has I as its kernel and F as its dual kernel.

PROOF. We first derive the above equivalences of $x \equiv y$. For x and y in A, the symmetric difference $x \triangle y$ is the least element a of A satisfying $x + a = y + a$; thus $x \triangle y \in I$ iff there is $i \in I$ such that $x + i = y + i$. Also, $x + i = y + i$, where $i \in I$, implies that $x \cdot f = y \cdot f$, where $f = -i \in F$; dually $x \cdot f = y \cdot f$ for some $f \in F$ gives $x + i = y + i$, where $i = -f \in I$.

Using these equivalences, we prove that $\equiv$ is a congruence relation on A. For example, for transitivity of $\equiv$, suppose $x \equiv y$ and $y \equiv z$. Choose f and g in F such that $x \cdot f = y \cdot f$ and $y \cdot g = z \cdot g$; then $f \cdot g$ is again in F and

$$x \cdot f \cdot g = y \cdot f \cdot g = z \cdot f \cdot g ,$$

which shows that $x \equiv z$. Also the relation $\equiv$ respects the operation $+$ ($-$ is handled similarly). For suppose $x \equiv x'$ and $y \equiv y'$, say $x \cdot f = x' \cdot f$ and $y \cdot g = y' \cdot g$ where $f, g \in F$. Then

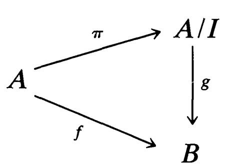
$$(x + y) \cdot f \cdot g = x \cdot f \cdot g + y \cdot f \cdot g = x' \cdot f \cdot g + y' \cdot f \cdot g = (x' + y') \cdot f \cdot g ,$$

hence $x + y \equiv x' + y'$.

The canonical homomorphism $\pi: A \to A/I$ maps an element x of A onto $0_{A/I} = \pi(0)$ iff $x \equiv 0$, i.e. iff $x = x \triangle 0 \in I$. Thus, I is the kernel of π, and it follows by passing to complements that F is the dual kernel of π. □

It similarly makes sense to form quotients in the class of κ-complete algebras, ideals and homomorphisms: let A be a κ-complete Boolean algebra and I an ideal of A. Then I is κ-complete iff the associated canonical epimorphism $\pi: A \to A/I$ is, and in this case, also A/I is κ-complete.

5.23. PROPOSITION (homomorphism theorem). *Let $f: A \to B$ be an epimorphism of Boolean algebras with kernel I. Then there is a unique isomorphism g: $A/I \to B$ such that $g \circ \pi = f$.*



□

The homomorphism theorem can often be used when checking isomorphism of two Boolean algebras. For example, let $a \in A$ be arbitrary and $I = \{x \in A: x \leq -a\}$ the principal ideal of A generated by $-a$. Then $A/I \cong A \restriction a$ since the projection epimorphism $p_a: A \to A \restriction a$ defined by $p_a(x) = x \cdot a$ has I as its kernel.

5.24. EXAMPLE. A filter F is an ultrafilter iff A/F is the two-element Boolean algebra. For let $\pi: A \to A/F$ be canonical. If $A/F = 2 = \{0, 1\}$, then $F = \pi^{-1}(1)$ is an ultrafilter. Conversely, if F is an ultrafilter, then by Proposition 2.15 the characteristic function $\chi_F: A \to 2$ of F is an epimorphism with dual kernel F, and the homomorphism theorem guarantees that $2 \cong A/F$.

5.25. EXAMPLE. Suppose $p_1, \ldots, p_n$ are distinct ultrafilters of A and F is the filter $p_1 \cap \cdots \cap p_n$. Then A/F is isomorphic to the power set of $\{1, \ldots, n\}$. Moreover, for every $M \subseteq \{1, \ldots, n\}$, there is an $a \in A$ such that, for $i \in I$, $a \in p_i$ iff $i \in M$.

This is proved by letting $X = \{1, \ldots, n\}$ and considering the map

$$f: A \to P(X)$$

defined by

$$f(a) = \{i \in X: a \in p_i\} .$$

Then, similar to the proof of Stone's theorem 2.1, f is a homomorphism; it has F as its dual kernel. Thus, $A/F \cong P(X)$ will follow from the homomorphism theorem if we can prove that f is onto.

So let $M \subseteq X$ with the aim of finding $a \in A$ such that $a \in p_i$ iff $i \in M$. For any two distinct elements i and j of X, p_i and p_j are distinct maximal filters of A, therefore fix $a_{ij} \in p_i \setminus p_j$. For $i \in M$, let

$$a_i = \prod \{a_{ij}: j \in X \setminus M\} ;$$

so $a_i \in p_i$ but $a_i \notin p_j$ for $j \in X \setminus M$. Then

$$a = \sum \{a_i: i \in M\}$$

works for our claim.

5.5. *The algebra $P(\omega)/fin$*

Most order-theoretic or combinatorial properties of Boolean algebras are not inherited by quotients. This is illustrated below by the algebra $P(\omega)/fin$, where *fin* is the ideal of finite sets in the power set algebra $P(\omega)$ of the natural numbers. The algebra is fairly well understood and can even be characterized up to isomorphism under the continuum hypothesis $2^\omega = \omega_1$; it is not quite so well understood under Martin's axiom. Some of its properties are independent from Zermelo–Fraenkel set theory ZFC. Results on $P(\omega)/fin$ are also relevant to topology since, via Stone's duality theory presented in Sections 7 and 8, studying the algebra $P(\omega)/fin$ is equivalent to studying the topological space $\omega^* = \beta\omega \setminus \omega$; here ω is given the discrete topology and $\beta\omega$ is its Stone–Čech compactification.

The algebra $P(\omega)$ is complete, atomic, and satisfies the countable chain condition; so its cellularity, as defined in Section 3, is ω. We prove below that none of these properties holds for $P(\omega)/fin$. There is, however, a property generalizing σ-completeness which carries over to quotients.

5.26. DEFINITION. A Boolean algebra A has the *countable separation property* if, for any two at most countable subsets X and Y of A satisfying $x \cdot y = 0$ for all $x \in X$ and $y \in Y$, there is an element a of A separating X and Y, i.e. such that $x \leq a$ and $y \leq -a$ for all $x \in X$ and $y \in Y$.

Clearly, every σ-complete Boolean algebra has the countable separation property – if X and Y are given as stated in 5.26, simply let $a = \Sigma X$.

5.27. LEMMA. (a) *A has the countable separation property iff for any two at most countable subsets X and Y of A such that $X \cup Y$ is a pairwise disjoint family, there is some a of A separating X and Y.*

(b) *A has the countable separation property iff for any two at most countable subsets X and Z of A satisfying $x \leq z$ for all $x \in X$ and $z \in Z$ there is an element a of A such that $x \leq a \leq z$ for all $x \in X$ and $z \in Z$.*

(c) If *A has the countable separation property, then so has every quotient of A.*

PROOF. (a) This holds because an element of A separates $\{x_n : n \in \omega\}$ and $\{y_n : n \in \omega\}$ iff it separates the pairwise disjoint families $\{x_n \cdot -\Sigma_{i<n} x_i : n \in \omega\}$ and $\{y_n \cdot -\Sigma_{i<n} y_i : n \in \omega\}$.

(b) Consider X, $Y \subseteq A$ and let $Z = \{-y : y \in Y\}$. Then $x \cdot y = 0$ for all $x \in X$ and $y \in Y$ iff $x \leq z$ for all $x \in X$ and $z \in Z$. Also, $a \in A$ satisfies $x \leq a$ and $y \leq -a$ for all $x \in X$ and $y \in Y$ iff it satisfies $x \leq a \leq z$ for all $x \in X$ and $z \in Z$.

(c) Let $f: A \to B$ be an epimorphism. Note that for each at most countable pairwise disjoint family $(b_n)_{n \in \omega}$ in B, we can choose pairwise disjoint preimages $(a_n)_{n \in \omega}$ in A by picking $c_n \in A$ such that $f(c_n) = b_n$ and letting $a_n = c_n \cdot -\Sigma_{i<n} c_i$.

Suppose that $X \cup Y$ is an at most countable pairwise disjoint family in B; by the preceding remark, there is a pairwise disjoint family $M \cup N$ in A such that f maps M onto X and N onto Y. Now if a separates M and N in A, then $f(a)$ separates X and Y in B. □

5.28. EXAMPLE. Let $B = P(\omega)/fin$. Then

(a) B has the countable separation property,

(b) B is atomless,

(c) $c(B) = 2^\omega$ is attained,

(d) each infinite partition of unity in B is uncountable,

(e) if C is a finite or countable chain in $B \setminus \{1\}$, then there is $b \in B$ such that $c \leq b < 1$ for every $c \in C$.

PROOF. (a) follows from Lemma 5.27(c) and the remark preceding it, since $P(\omega)$ is σ-complete. For the remaining assertions, let $\pi: P(\omega) \to B$ be canonical. Note that for a, $b \in P(\omega)$:

$$\pi(a)=0 \quad \text{iff } a \text{ is finite},$$

$$\pi(a)=1 \quad \text{iff } a \text{ is cofinite},$$

$$\pi(a)=\pi(b) \quad \text{iff } a \mathbin{\triangle} b \text{ is finite},$$

$$\pi(a)\leq \pi(b) \quad \text{iff } a\backslash b \text{ is finite},$$

$$\pi(a)\cdot \pi(b)=0 \quad \text{iff } a \cap b \text{ is finite}.$$

To prove (b) assume that in B, $c=\pi(a)>0$, where $a \subseteq \omega$. Thus, a is infinite; let $a' \subseteq a$ such that both a' and $a\backslash a'$ are infinite. Then $0<\pi(a')<c$ which shows that c is not an atom of B.

For a proof of (c), call a and a' in $P(\omega)$ *almost disjoint* if a and a' are infinite but $a \cap a'$ is finite; this amounts to saying that $\pi(a)$ and $\pi(a')$ are non-zero and disjoint in B. (c) follows if we can construct in $P(\omega)$ a family D of pairwise almost disjoint sets such that $|D|=2^{\omega}$. It is easier to construct such a family D not in $P(\omega)$ but in the power set of the (countable) set of rationals: for each real number x, choose a strictly increasing sequence $(r_{xn})_{n\in\omega}$ of rationals converging to x and let

$$d_x=\{r_{xn}: n\in\omega\},$$

$$D=\{d_x: x\in \boldsymbol{R}\}.$$

Then each d_x is infinite but, for $x\neq y$ in $\boldsymbol{R}$, $d_x \cap d_y$ is finite – otherwise for any infinite subset a of $d_x \cap d_y$,

$$x=\sup d_x=\sup a=\sup d_y=y,$$

a contradiction.

(d) is proved by showing that no countably infinite family D consisting of pairwise almost disjoint subsets of ω can be maximally almost disjoint. For assume $D=\{a_n: n\in\omega\}$ and pick $x_n\in a_n\backslash(a_0\cup\cdots\cup a_{n-1})$; this is possible since a_n is infinite and $a_n\cap a_k$ is finite for $k<n$. Then $a=\{x_n: n\in\omega\}$ is infinite and $a\cap a_n$, being included in $\{x_0,\ldots,x_n\}$, is finite for every n; so $D\cup\{a\}$ is an almost disjoint family strictly larger than D.

(e) We shall derive (e) from (d); it is in fact easily seen that (d) and (e) are equivalent in every infinite Boolean algebra. Our assertion is trivial if C is empty or has a greatest element. Otherwise, replacing C by a cofinal subchain, we may assume that $C=\{c_n: n\in\omega\}$, where $0=c_0<c_1<c_2<\cdots$. Defining $d_n=c_{n+1}\cdot -c_n$, we obtain a countably infinite pairwise disjoint family $D=\{d_n: n\in\omega\}$ in B. By (d), D is not a partition of unity in B, so by Lemma 3.6 it has an upper bound b strictly smaller than 1. b is also an upper bound of C, since $c_{n+1}=d_0+\cdots+d_n$ for every n. □

The algebra $P(\omega)/fin$ has cardinality 2^{ω}, since $|P(\omega)|=2^{\omega}$ and $|fin|=\omega$. The following argument shows that it is not σ-complete: by Lemma 3.4 (or by 5.28(c)), there is a countably infinite pairwise disjoint family X in $P(\omega)/fin$. If ΣX exists in $P(\omega)/fin$, then $X \cup \{-\Sigma X\}$ is a countably infinite partition of unity in $P(\omega)/fin$, contradicting 5.28(d).

The properties of $P(\omega)/fin$ listed in 5.28 have several interesting consequences; it will actually turn out that, under the continuum hypothesis, they determine $P(\omega)/fin$ up to isomorphism. For the rest of this subsection, let us call a Boolean algebra B *ω_1-universal* if every Boolean algebra of cardinality at most ω_1 is embeddable into B. B has the *strong countable separation property* if it is infinite and for any non-empty and at most countable subsets X and Y satisfying $x_1 + \cdots + x_n < y_1 \cdot \cdots \cdot y_m$ for $n, m \in \omega$, $x_1, \ldots, x_n \in X$ and $y_1, \ldots, y_m \in Y$, there is some $b \in B$ such that $x_1 + \cdots + x_n < b < y_1 \cdot \cdots \cdot y_m$ for all $x_1, \ldots, x_n \in X$ and $y_1, \ldots, y_m \in Y$. It is a matter of routine to check that B has the strong countable separation property iff it satisfies the conditions (a), (b), and (e) of Example 5.28; by equivalence of (d) and (e) in 5.28, B has the strong countable separation property iff it is infinite and atomless, satisfies the countable separation property and has no countably infinite partition of unity. In particular, $P(\omega)/fin$ has the strong countable separation property. Let us note that if B has the strong countable separation property, then for each non-zero element b of B, the cellularity $c(B \upharpoonright b)$ is at least ω_1: it is at least ω since B is atomless; by the strong countable separation property, B satisfies (d) in 5.28 and thus $c(B \upharpoonright b) \geq \omega_1$.

5.29. PROPOSITION. *Every algebra satisfying the strong countable separation property is ω_1-universal.*

PROOF. This follows immediately, by a transfinite construction of length ω_1, from the subsequent claim.

Claim. Assume that B has the strong countable separation property. Then every embedding $f: A \to B$ from an at most countable Boolean algebra A into B extends to every simple extension of A.

For suppose $A(x)$ is a simple extension of A. Consider the sets

$$I = \{i \in A : i \leq x\}, \qquad J = \{j \in A : j \leq -x\},$$

$$K = \{i + j : i \in I, j \in J\}, \qquad U = A \setminus K.$$

It suffices to find $y \in B$ such that $f(i) \leq y$ for $i \in I$, $f(j) \leq -y$ for $j \in J$, and $f(u) \not\leq y$, $f(u) \not\leq -y$ for $u \in U$, for then as in Corollary 5.8 there is a unique monomorphism $f': A(x) \to B$ extending f and mapping x onto y.

For $u \in U$, construct an element c_u of B as follows. I, J and hence K are ideals of A. Thus, for each $k \in K$, $u \cdot -k > 0$ and $f(u) \cdot -f(k) > 0$. Since $f[K]$ is a countable subset of B closed under finite sums, the strong countable separation property gives $c_u \in B$ such that $0 < c_u \leq f(u) \cdot -f(k)$ holds for *all* $k \in K$, i.e. $0 < c_u \leq f(u)$ and c_u is disjoint from each $f(k)$. By the Balcar–Vojtáš theorem 3.14 and $c(B \upharpoonright b) \geq \omega_1$ for $b \in B^+$, we may assume that $(c_u)_{u \in U}$ is a pairwise disjoint family, otherwise replacing the c_u by smaller positive elements.

For $u \in U$, write c_u as the sum of two disjoint non-zero elements d_u and e_u; this is possible since B is atomless. The countable separation property gives $y \in B$ separating the sets $f[I] \cup \{d_u : u \in U\}$ and $f[J] \cup \{e_u : u \in U\}$. Then $f(i) \leq y$ for $i \in I$ and $f(j) \leq -y$ for $j \in J$. For $u \in U$, $0 < e_u \leq -y$ and $e_u \leq c_u \leq f(u)$, thus $f(u) \not\leq y$. Similarly, $0 < d_u \leq y$ and $d_u \leq c_u \leq f(u)$; thus $f(u) \not\leq -y$. □

5.30. COROLLARY. *Assume $2^\omega = \omega_1$. Then every algebra of cardinality ω_1 with the strong countable separation property is isomorphic to $P(\omega)/fin$.*

PROOF. $P(\omega)/fin$ has the strong countable separation property and cardinality $2^\omega = \omega_1$. Thus, the Corollary follows from the Claim in the proof of 5.29 and a transfinite construction of length ω_1. □

We finally sketch another characterization of $P(\omega)/fin$, assuming some definitions and results from model theory. The first order theory of infinite atomless Boolean algebras is complete, since it has, by 5.16, exactly one countable model (up to isomorphism). Thus, under the continuum hypothesis $2^\omega = \omega_1$, it has exactly one ω_1-saturated model of size ω_1, say B. It is a straightforward consequence of ω_1-saturatedness that B satisfies the strong countable separation property (in fact, a bit more model theory shows that, for atomless algebras, the strong countable separation property is equivalent to ω_1-saturatedness). But then by 5.30, B is isomorphic to $P(\omega)/fin$. Hence, if $2^\omega = \omega_1$, then $P(\omega)/fin$ is, up to isomorphism, the unique ω_1-saturated atomless Boolean algebra of size ω_1.

5.6. The number of ultrafilters, filters, and subalgebras

We prove a theorem comparing the number of ultrafilters, filters, and subalgebras of a Boolean algebra with its cardinality. Let, in this subsection, Filt A be the set of all filters and Sub A the set of all subalgebras of the algebra A.

5.31. THEOREM. *For every infinite Boolean algebra A,*

$$|A| \leq |\mathrm{Ult}\ A| \leq |\mathrm{Filt}\ A| \leq |\mathrm{Sub}\ A| \leq 2^{|A|}.$$

Here the main assertion is the inequality $|A| \leq |\mathrm{Ult}\ A|$. The inequality $|\mathrm{Ult}\ A| \leq 2^{|A|}$ is trivial; we will see in Section 9 that the upper bound $2^{|A|}$ for $|\mathrm{Ult}\ A|$ is attained if A is a free Boolean algebra. The lower bound $|A|$ for $|\mathrm{Ult}\ A|$ is, for example, attained if A is the finite–cofinite algebra on an infinite set X, for then $|A| = |X|$ and the ultrafilters of A are the principal filters generated by the atoms and the filter of all cofinite subsets of X. More generally, it will be proved in Section 17 that $|\mathrm{Ult}\ A| = |A|$ if A is infinite and superatomic. In Theorem 5.31, $|\mathrm{Ult}\ A|$ can of course be replaced by the number of maximal ideals of A and $|\mathrm{Filt}\ A|$ by the number of ideals of A. In Section 10, we shall prove a much deeper theorem by Shelah on the number of filters (respectively ideals) of an infinite algebra A: if $\kappa = |\mathrm{Filt}\ A|$, then $\kappa^\omega = \kappa$.

We begin with two lemmas, the first one being of independent interest in the topological duality theory of Section 8. Note that, if p is an ultrafilter of A and B is a subalgebra of A, then $p \cap B$ is an ultrafilter of B.

5.32. LEMMA. *Let B be a proper subalgebra of A. Then there are distinct ultrafilters p and q of A such that $p \cap B = q \cap B$.*

PROOF. Choose $a \in A \backslash B$ and define two filters of B by

$$F = \{b \in B: a \leq b\}, \qquad F' = \{b \in B: -a \leq b\}.$$

Now $F \cup F'$ has the finite intersection property as defined in 2.12, for otherwise there are $b \in F$ and $c \in F'$ such that $b \cdot c = 0$. Then $b \leq -c \leq a$ and $a = b \in B$, a contradiction. Thus, by the Boolean prime ideal theorem 2.16, let r be an ultrafilter of B including $F \cup F'$.

Also, $r \cup \{a\}$ has the finite intersection property. Otherwise, $c \cdot a = 0$ for some $c \in r$; then $-c \in F \subseteq r$, which contradicts $c \in r$. Similarly, $r \cup \{-a\}$ has the finite intersection property. So there are ultrafilters p and q of A such that $r \cup \{a\} \subseteq p$ and $r \cup \{-a\} \subseteq q$. Clearly, $p \neq q$ but $p \cap B = r = q \cap B$. □

Recall from Definition 5.19 that, for a filter F of A, $-F = \{-x: x \in F\}$ is the ideal of A dual to F.

5.33. LEMMA. *For every filter F of A, $F \cup -F$ is a subalgebra of A. Moreover, if F and G are distinct proper non-maximal filters of A, then $F \cup -F \neq G \cup -G$.*

PROOF. The first assertion is verified either by direct computation or by noting that $F \cup -F$ is the preimage, under the canonical map, of the subalgebra $\{0, 1\}$ of A/F.

Let F and G be distinct, proper and non-maximal and fix $a \in F \backslash G$. If $a \not\in -G$, then $a \in F \cup -F$ but $a \not\in G \cup -G$, and we are finished. So assume $-a \in G$. Since F is not an ultrafilter, pick $b \in A$ such that neither b nor $-b$ is in F. Now $b + -a$ and $-b + -a$ cannot both be in F since otherwise their product $-a$ is in F, contradicting $a \in F$ and properness of F. Without loss of generality assume $b + -a \not\in F$. Then $b + -a$ is in $G \cup -G$ but not in $F \cup -F$, as desired. □

Proof of Theorem 5.31. It is obvious that $|\mathrm{Ult}\, A| \leq |\mathrm{Filt}\, A|$ and $|\mathrm{Sub}\, A| \leq 2^{|A|}$. To prove $|A| \leq |\mathrm{Ult}\, A|$, choose for any two distinct ultrafilters p and q of A an element a_{pq} of $p \backslash q$; this is possible since p, being maximal, cannot be included in q. Let B be the subalgebra of A generated by $\{a_{pq}: p \neq q \text{ in } \mathrm{Ult}\, A\}$, so $|B| \leq |\mathrm{Ult}\, A|$. But $B = A$ by Lemma 5.32 since, for any distinct ultrafilters p and q of A, $p \cap B \neq q \cap B$ is exemplified by $a_{pq} \in B \cap (p \backslash q)$.

For a proof of $|\mathrm{Filt}\, A| \leq |\mathrm{Sub}\, A|$, let Pr be the set of all proper non-maximal filters of A. We show that

$$|\mathrm{Ult}\, A| \leq |Pr| = |\mathrm{Filt}\, A|;$$

then $|\mathrm{Filt}\, A| \leq |\mathrm{Sub}\, A|$ follows since Lemma 5.33 gives a one-to-one map from Pr into $\mathrm{Sub}\, A$. Fix $p^* \in \mathrm{Ult}\, A$. The map f from $\mathrm{Ult}\, A \backslash \{p^*\}$ into Pr given by $f(q) = p^* \cap q$ is one-to-one by Example 5.25; also $|\mathrm{Ult}\, A \backslash \{p^*\}| = |\mathrm{Ult}\, A|$ since $\mathrm{Ult}\, A$ is infinite. Hence, $|\mathrm{Ult}\, A| \leq |Pr|$. Finally, since a filter of A is either improper or maximal or in Pr,

$$|\text{Filt } A| = 1 + |\text{Ult } A| + |Pr| = |Pr| . \quad \square$$

The inequalities of Theorem 5.31 are strictly limited to infinite algebras. For a finite algebra with n atoms, $|\text{Ult } A| = n$ and $|A| = 2^n = |\text{Filt } A|$ since every filter of A is principal. Every subalgebra of A is determined by its atoms and these give rise to a partition of the set of atoms of A, so $|\text{Sub } A|$ is the number of partitions of an n-element set.

Exercises

1. Assume X and X' are dense subsets of the Boolean algebras A and A' and that $f: X \to X'$ is an isomorphism of the partial orders X, X' (with respect to the orderings induced by A, A'). Using Sikorski's extension criterion, show that f extends to an isomorphism from $\langle X \rangle$ onto $\langle X' \rangle$. Derive from this the uniqueness theorem 4.14 for completions of partial orders.

2. Let B be complete and $e: A \to B$ a complete monomorphism; by Sikorski's extension theorem, there is a monomorphism $f: \bar{A} \to B$ extending e.

(a) f is uniquely determined.

(b) f is complete.

(c) Assume that B is completely generated by $e[A]$, i.e. that every complete regular subalgebra of B including $e[A]$ coincides with B. Then f is an isomorphism from A onto B.

Thus, A can be characterized as the unique complete algebra C such that A is a regular subalgebra of C, C is completely generated by A, and every complete embedding of A into a complete algebra B extends to a unique complete embedding of C into B.

3. (for model theorists) Prove that two Boolean algebras A and B are elementarily equivalent in the logic $L_{\infty\omega}$ iff ARB holds for some Vaught relation R. This gives another explanation of Vaught's theorem 5.15.

4. Let A and A' be countable infinite atomic Boolean algebras, I (respectively I') the ideals generated by their atoms; assume that A/I and A'/I' are infinite and atomless. Then A and A' are isomorphic.

5. Let A be a Boolean algebra and $I \subseteq A$. Then I is an ideal of A as defined in 5.19 iff I is an ideal of the Boolean ring associated with A, in Proposition 1.27.

6. Let I be an ideal of a Boolean algebra A, $\pi: A \to A/I$ canonical. Then the assignment $K \to \pi^{-1}[K]$ is an order preserving bijection between the ideals of A/I and the ideals of A including I.

7. An element c of a lattice L is said to be *compact* if, for every $M \subseteq L$, $c \leq \Sigma\, M$ implies that $c \leq \Sigma\, M'$ for some finite subset M' of M. Show that a lattice L is isomorphic to the lattice of ideals (under inclusion) of some Boolean algebra iff it satisfies the following conditions:

(a) L is complete; in particular it has a least element 0_L and a greatest element 1_L.

(b) L is distributive.

(c) L is algebraic, i.e. each element of L is the sum of compact elements.

(d) The set of all compact elements contains 1_L and is a complemented sublattice of L.

8. Let $f: A \to B$ be an epimorphism with kernel I, $g: A \to C$ a homomorphism with kernel J. There is a (unique) homomorphism $h: B \to C$ such that $h \circ f = g$ iff $I \subseteq J$. h is one-to-one iff $I = J$.

9. Prove that every Boolean algebra Q is the quotient of an atomic algebra A modulo the ideal generated by the atoms of A.

Hint. Embed Q into some power set $P(Y)$ and let A be a suitably chosen subalgebra of $P(X)$ where $X = Y \times \omega$.

10. Let $\mu: A \to [0, 1]$ be a finitely additive measure as defined in Exercise 5 of Section 3 and $N = \{a \in A: \mu(a) = 0\}$. Then N is an ideal of A and there is a unique finitely additive measure $\nu: A/N \to [0, 1]$ such that $\nu \circ \pi = \mu$ ($\pi: A \to A/N$ canonical). ν is strictly positive and hence A/N satisfies the countable chain condition. State and prove a similar assertion for σ-additive measures on σ-complete algebras.

11. In a simple extension $A(x)$ of A (cf. Definition 4.6), define the ideal $I(x)$ by

$$I(x) = \{a \in A: a \leq x\} .$$

Let $A(x)$, $A(y)$ be simple extensions of A. Then there is an isomorphism $h: A(x) \to A(y)$ mapping x to y and extending id_A iff $I(x) = I(y)$ and $I(-x) = I(-y)$.

12. Let I, J be ideals of A such that $i \cdot j = 0$ for $i \in I$, $j \in J$. Prove that there is a simple extension $A(x)$ of A such that (in the notation of Exercise 11) $I(x) = I$ and $I(-x) = J$.

Hint. Let $A(x) = B/M$, where B is the product algebra $A \times A$, M a suitably chosen ideal of B, $\pi: B \to B/M$ canonical, A is identified with its image under $\pi \circ e: A \to B/M$, $e(a) = (a, a)$, and $x = \pi(u)$, where u is the element $(1, 0)$ of B.

13. (for model theorists) (a) Let $a_1, \ldots, a_n$ be finitely many elements in an atomless Boolean algebra A. Show that the *elementary type*

$$\{\phi(x_1, \ldots, x_n): \phi \text{ a formula in the language of Boolean algebras and } A \models \phi[a_1 \cdots a_n]\}$$

realized by the sequence $(a_1, \ldots, a_n)$ in A is determined by the set $\{e \in E: p_e = 0\}$, where for $e \in E = {}^{\{1,\ldots,n\}}\{+1, -1\}$, p_e is the elementary product $e(1)a_1 \cdot \cdots \cdot e(n)a_n$. Conclude that the theory of infinite atomless Boolean algebras admits elimination of quantifiers.

(b) Prove that every Boolean algebra with the strong countable separation property is ω_1-saturated, in the sense of model theory.

6. Products

Cartesian products provide, together with subalgebras and quotients, the third major construction of new Boolean algebras from old ones. Unlike the situation for groups or other algebraic structures, however, the decompositions of a Boolean algebra into a product of finitely many factors are easily described – they correspond, in a one-one manner, to the finite partitions of unity. Similarly the product decompositions into infinitely many factors correspond, in sufficiently complete algebras to infinite partitions of unity.

In view of the ease of decomposing Boolean algebras into factors, there might be some hope to solve problems on isomorphism of factors like that whether two Boolean algebras are isomorphic if each of them is isomorphic to a factor of the other one. Our main result here is the counterexample, given by Hanf, of an algebra A such that $A \cong A \times 2 \times 2$ but $A \not\cong A \times 2$.

6.1. Product decompositions and partitions

6.1. DEFINITION AND LEMMA. Let $(A_i)_{i \in I}$ be a family of Boolean algebras. Then the cartesian product

$$\prod_{i \in I} A_i = \{a\colon a = (a_i)_{i \in I},\ a_i \in A_i \text{ for } i \in I\}$$

of the sets A_i is a Boolean algebra, the *product algebra* of the A_i, under the componentwise operations

$$(a+b)_i = a_i + b_i\,, \qquad (a \cdot b)_i = (a_i \cdot b_i)\,, \qquad (-a)_i = -a_i\,,$$

$$0 = (0_i)_{i \in I}\,, \qquad 1 = (1_i)_{i \in I}\,.$$

For $i \in I$, the projection map

$$\mathrm{pr}_i\colon \prod_{i \in I} A_i \to A_i$$

defined by $\mathrm{pr}_i(a) = a_i$ for $a = (a_i)_{i \in I}$, is an epimorphism.

Other notation concerning products should be self-explanatory. For example, we write $A_1 \times \cdots \times A_n$ for $\Pi_{i \in I} A_i$ if $I = \{1, \ldots, n\}$, ${}^I A$ if $A_i = A$ for all $i \in I$, and A^n (instead of ${}^n A$) if $I = n = \{0, \ldots, n-1\}$ and $A_i = A$ for $i \in n$.

Every power ${}^I 2$ of the two-element Boolean algebra 2 is isomorphic to the power set algebra $P(I)$, an isomorphism being given by the map $f\colon {}^I 2 \to P(I)$, where $f(a) = \{i \in I\colon a_i = 1\}$.

It is easily seen that for every subset M of a product algebra $P = \Pi_{i \in I} A_i$,

$$\Sigma^P M = \left(\Sigma^{A_i} \mathrm{pr}_i[M]\right)_{i \in I}$$

in the sense that the left-hand side exists iff the right-hand one does; a dual statement holds for the greatest lower bound $\Pi^P M$ of M in P. Hence $\Pi_{i \in I} A_i$ is complete (κ-complete, σ-complete) iff each A_i is.

Also the following subalgebras of $\Pi_{i \in I} A_i$ are sometimes considered, for example in the topological duality theory presented in Section 8.

6.2. DEFINITION. Let κ be an infinite cardinal. The *κ-weak product* of the A_i is the subalgebra

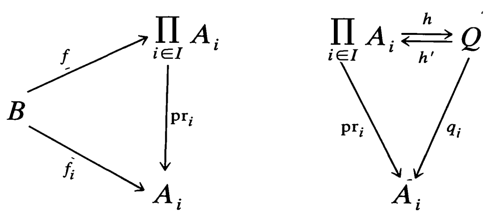
$$\prod_{i\in I}^{<\kappa} A_i = \left\{ a \in \prod_{i\in I} A_i \colon |\{i \in I\colon a_i \neq 0\}| < \kappa \text{ or } |\{i \in I\colon a_i \neq 1\}| < \kappa \right\}$$

of $\Pi_{i\in I} A_i$. For $\kappa = \omega$, it is called the *weak product* of the A_i and denoted by $\Pi^{\mathrm{w}}_{i\in I} A_i$.

The following proposition describes the cartesian product of a family of Boolean algebras by a universal property: in the category of all Boolean algebras and Boolean homomorphisms, the algebra $\Pi_{i\in I} A_i$, together with the projection maps pr_i, is a product of the family $(A_i)_{i\in I}$ as defined in category theory. We shall investigate in Section 11 a similar but less obvious construction of Boolean algebras, namely the free product, which is, formally speaking, the category-theoretic dual of the product construction.

6.3. PROPOSITION. *The pair $(\Pi_{i\in I} A_i, (\mathrm{pr}_i)_{i\in I})$ has the following universal property. Given a Boolean algebra B and a family $(f_i)_{i\in I}$ of homomorphisms $f_i\colon B \to A_i$, there is a unique homomorphism $f\colon B \to \Pi_{i\in I} A_i$ such that $\mathrm{pr}_i \circ f = f_i$ for every $i \in I$.*

Conversely, assume Q is a Boolean algebra and $q_i\colon Q \to A_i$, for $i \in I$, a homomorphism such that, given a Boolean algebra B and homomorphisms $g_i\colon B \to A_i$, there is a unique homomorphism $g\colon B \to Q$ such that $q_i \circ g = g_i$ for $i \in I$. Then there is a unique isomorphism $h\colon \Pi_{i\in I} A_i \to Q$ such that $q_i \circ h = \mathrm{pr}_i$ for $i \in I$.



PROOF. A standard argument of category theory: for the first assertion, define $f(b) = (f_i(b))_{i\in I}$ for $b \in B$. To prove the second one consider, by the universal property of $(Q, (q_i)_{i\in I})$, the unique homomorphism $h\colon \Pi_{i\in I} A_i \to Q$ such that $q_i \circ h = \mathrm{pr}_i$; similarly, by the universal property of $(\Pi_{i\in I} A_i, (\mathrm{pr}_i)_{i\in I})$, there is a unique homomorphism $h'\colon Q \to \Pi_{i\in I} A_i$ such that $\mathrm{pr}_i \circ h' = q_i$. Then $h' \circ h$ is the identity map on $\Pi_{i\in I} A_i$ and $h \circ h'$ is the identity map on Q. □

The principal result of this subsection is the simple but basic connection between product decompositions and partitions of unity. Unlike the definition given in Section 3, let us call here a family $(a_i)_{i\in I}$ in a Boolean algebra A a partition (of unity) if $a_i \cdot a_j = 0$ for $i \neq j$ and $\Sigma_{i\in I} a_i = 1$ – i.e. we drop the requirement that $a_i > 0$.

6.4. PROPOSITION. *For every partition $(a_i)_{i\in I}$ of a Boolean algebra A, the map*

$$f: A \to \prod_{i \in I} (A \restriction a_i)\,, \qquad f(x) = (x \cdot a_i)_{i \in I}$$

is a monomorphism. It is onto iff $\Sigma^A_{i \in I} c_i$ exists for each family $(c_i)_{i \in I} \in \Pi_{i \in I} (A \restriction a_i)$, in particular if A is $|I|^+$-complete.

Conversely, for every isomorphism $f: A \to \Pi_{i \in I} A_i$ from A onto a product algebra, there is a partition $(a_i)_{i \in I}$ of A such that $A_i \cong A \restriction a_i$ for $i \in I$.

PROOF. In the first assertion, f is a homomorphism since, for every $i \in I$, $\mathrm{pr}_i \circ f$ is the canonical projection homomorphism from A onto $A \restriction a_i$ considered in Section 3. f is one-to-one since if $x \neq 0$ in A, then $x \cdot a_i \neq 0$ for some $i \in I$. Moreover, if A is sufficiently complete and $c = (c_i)_{i \in I} \in \Pi_{i \in I} (A \restriction a_i)$, put $x = \Sigma^A_{i \in I} c_i$. Then $f(x) = c$ by disjointness of the a_i.

In the second assertion, let for $i \in I$ e^i be the element e of $\Pi_{i \in I} A_i$ satisfying $e_i = 1$ and $e_j = 0$ for $i \neq j$; the family $(e^i)_{i \in I}$ is a partition of unity in the algebra $\Pi_{i \in I} A_i$. Letting $a_i = f^{-1}(e^i)$, we see that $(a_i)_{i \in I}$ is a partition of unity in A. Clearly, the restriction of f to $A \restriction a_i$ is an isomorphism from $A \restriction a_i$ onto $(\Pi_{i \in I} A_i) \restriction e^i \cong A_i$. □

By the preceding proposition, the product decompositions of a complete algebra are in one-to-one correspondence with partitions of unity, a fact repeatedly used in the structure theorems on complete Boolean algebras in Section 13. The product decompositions of an arbitrary Boolean algebra A into finitely many factors are given by

$$A \cong A \restriction a_1 \times \cdots \times A \restriction a_n\,,$$

where the a_i are pairwise disjoint and $a_1 + \cdots + a_n = 1$; the factors of A in arbitrary product decompositions are, up to isomorphism, the relative algebras $A \restriction a$ of A.

6.2. *Hanf's example*

Let us consider the following problems on products of Boolean algebras.

(A) If $A \cong A \times B \times C$, does it follow that $A \cong A \times B$?

That is, are two Boolean algebras (A and $A \times B$ in Question (A)) isomorphic if each of them is a factor of the other one? The answer is positive for σ-complete algebras, and the reader can immediately proceed to the proof, given in Section 12, if he wishes. Letting $B = C$ or even $B = C = 2$ in (A) gives the successively weaker questions:

(B) If $A \cong A \times B^2$, does it follow that $A \cong A \times B$?

(C) If $A \cong A \times 2 \times 2$, does it follow that $A \cong A \times 2$?

Also the following questions might be asked:

(D) If $A^2 \cong B^2$, does it follow that $A \cong B$?

(E) If $A^3 \cong A$, does it follow that $A^2 \cong A$?

These questions were posed by Tarski for Boolean algebras and other algebraic systems; Question (A) for Boolean algebras, independently, by Sikorski. The special case of (E) for countable algebras is called "Tarski's cube problem". (A) and (D) are considered in KAPLANSKY [1968] as test problems whose solvability, in a class $\boldsymbol{K}$ of abelian groups, should indicate to what extent the structure of groups in $\boldsymbol{K}$ is understood.

The aim of this subsection is a counterexample, due to Hanf, to Question (C) – a fortiori to (B) and (A) – which also settles (D) and (E).

6.5. EXAMPLE (Hanf). For every $n \geq 2$, there is a Boolean algebra A of cardinality 2^ω such that $A^2 \cong A$ and $A \times 2^n \cong A$ but $A \times 2^k \not\cong A$ for $1 \leq k < n$.

To construct such an algebra, fix a countably infinite set X. Let us say, for subsets m and p of X, that m *splits* p if both m and $X \setminus m$ have non-empty intersection with p.

Fix a partition P of X such that each element of P has size n and let A consist of those subsets of X which split only finitely many elements of P. Then A is an atomic subalgebra of $P(X)$, the atoms of A being the singletons $\{x\}$ where $x \in X$.

To see that $A \times 2^n \cong A$, pick $p \in P$. Then $A \cong A \upharpoonright p \times A \upharpoonright -p$, $A \upharpoonright p \cong 2^n$ and $A \upharpoonright -p \cong A$. More generally, $A \upharpoonright a \cong A$ for every $a \in A$ which is the union of infinitely many elements of P, since $A \upharpoonright a$ consists of those subsets of a which split only finitely many elements of P included in a. This argument also shows that $A^2 \cong A$.

Now let $1 \leq k < n$ and assume, for contradiction, that $A \times 2^k \cong A$. Call a partition of unity (i.e. a maximal pairwise disjoint family) R in an arbitrary Boolean algebra B *complete* if $\Sigma^B M$ exists for every subset M of R. Clearly $A \times 2^k$ has a complete partition R such that exactly one element of R is the sum of exactly k atoms of $A \times 2^k$ and the other elements of R are the sum of exactly n atoms. By $A \times 2^k \cong A$, A has a complete partition with the same property, say Q; let q^* be the unique element of Q of cardinality k. Call an element q of Q *bad* if $q \neq q^*$ and q splits some element of P, i.e. if $q \in Q \setminus (\{q^*\} \cup P)$.

Claim 1. Q has infinitely many bad elements.

Otherwise, all but finitely many elements of $Q \setminus \{q^*\}$, say $q_1, \ldots, q_r$, are in P. Then the set

$$M = q^* \cup q_1 \cup \cdots \cup q_r$$

has the property that $X \setminus M$ and hence M is a union of elements of P, contradicting the fact that M has size $r \cdot n + k$, a number not divided by n.

Claim 2. There are pairwise distinct elements, $p_m \in P$, $q_m \in Q$ for $m \in \omega$ such that

(1) q_m splits p_m,

(2) $p_i \cap q_j = \emptyset$ for $i \neq j$.

To prove Claim 2, assume p_i and q_i have been constructed for $i < m$. Then each of the sets

$$s = q_0 \cup \cdots \cup q_{m-1}, \qquad P' = \{p \in P\colon p \cap s \neq \emptyset\}, \qquad u = \bigcup P'$$

is finite. By Claim 1, there is a bad element q_m of Q not intersecting u; since q_m is bad, there is some $p_m \in P$ split by q_m. Note that $p_m \not\in P'$ – otherwise q_m, intersecting p_m, would intersect u. So $p_m \cap u = \emptyset$. Now (2) holds for all $i, j \leq m$ since for every $i < m$, $p_i \cup q_i$ is contained in u: clearly, $q_i \subseteq s \subseteq u$ and, by $p_i \cap q_i \neq \emptyset$, $p_i \in P'$ and $p_i \subseteq u$.

With the elements p_m and q_m of Claim 2 at hand, we see that $a = \bigcup_{m \in \omega} q_m$ is an element of A, Q being a complete partition of A. But $a \cap p_m = p_m \cap q_m$ by (2), so a splits every p_m and is not in A. This contradiction shows that $A \times 2^k \cong A$.

Additional interesting properties of the algebra A of Hanf's example are easily derived:

For $0 \leq k < l < n$, $A \times 2^k \not\cong A \times 2^l$ – otherwise $A \times 2^k \times 2^{n-l} \cong A \times 2^l \times 2^{n-l} \cong A \times 2^n \cong A$, contradicting $1 \leq n + k - l < n$.

Since $A^2 \cong A$, the algebra $B = A \times 2$ satisfies $B^{n+1} \cong A^{n+\times}2^{n+1} \cong A \times 2^n \times 2 \cong A \times 2 \cong B$; similarly, for $2 \leq k \leq n$, $B^k \cong A \times 2^k \not\cong B$. This in particular gives, for $n = 2$, a counterexample to (E).

If $n = 2$ and $B = A \times 2$, then $B \not\cong A$ but $B^2 \cong A \times (A \times 2 \times 2) \cong A^2$, a counterexample to (D).

Denoting the algebra A constructed for $n \in \omega$ in 6.5 by A_n, we find that $A_2 \times 2 \not\cong A_2$ and $A_3 \times 2 \not\cong A_3$ but $A_2 \times A_3 \times 2 \cong (A_2 \times 2^2) \times A_3 \times 2 \cong A_2 \times (A_3 \times 2^3) \cong A_2 \times A_3$.

Thus the questions (A) through (E) are answered in the negative, but by uncountable algebras. In fact, by the following proposition, there is no countable counterexample to (C). Note here that if $A \times 2^n \cong A$, where $1 \leq n < \omega$, then A must have infinitely many atoms, for $|\mathrm{At}\, A| = k < \omega$ would imply that $|\mathrm{At}(A \times 2^n)| = k + n$.

6.6. PROPOSITION (Vaught). *If A is a countable Boolean algebra with infinitely many atoms, then $A \times 2 \cong A$.*

PROOF. It suffices, by Vaught's theorem 5.15, to check that the relation R defined by

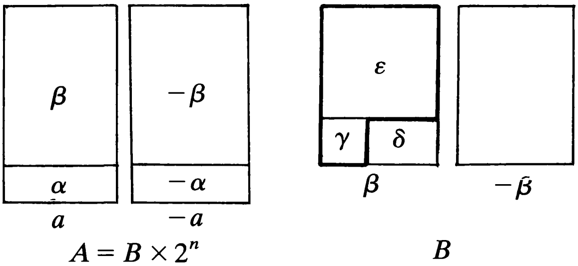
$$\begin{aligned} ARB \quad &\text{iff } A, B \text{ are finite and isomorphic} \\ &\text{or } A, B \text{ are infinite and } |\mathrm{At}\, A| = |\mathrm{At}\, B| < \omega \\ &\text{or } A, B \text{ are infinite and } |\mathrm{At}\, A| = |\mathrm{At}\, B| = \omega \text{ and,} \\ &\quad \text{for some } n \in \omega, (A \cong B \times 2^n \text{ or } B \cong A \times 2^n) \end{aligned}$$

is a Vaught relation as defined in 5.14. The least trivial part of this is the following: assume that ARB holds by the third clause in the definition of R, $A = B \times 2^n$ for simplicity, and $a \in A$. So a is an ordered pair (β, α), where $\beta \in B$ and $\alpha \in 2^n$, and $-a = (-\beta, -\alpha)$. We may assume that $B \restriction \beta$ has infinitely many atoms – otherwise interchange the roles of a and $-a$. Pick n distinct atoms in $B \restriction \beta$, say $x_1, \ldots, x_n$, and put

$$\gamma = x_1 + \cdots + x_m, \qquad \delta = x_{m+1} + \cdots + x_n,$$

where m is the number of atoms in $2^n \restriction \alpha$. Finally, let

$$\varepsilon = \beta \cdot -(\gamma + \delta), \qquad b = \varepsilon + \gamma .$$



Then $(A \restriction a)R(B \restriction b)$ since $A \restriction a$ is the product of $B \restriction b$ and a finite power of 2. Also, since $2^n \restriction -\alpha \cong 2^{n-m} \cong B \restriction \delta$ and $-b = \delta + -\beta$ where $\delta \cdot -\beta = 0$, we obtain:

$$A \restriction -a \cong (B \restriction -\beta) \times (2^n \restriction -\alpha) \cong (B \restriction -\beta) \times (B \restriction \delta) \cong B \restriction -b$$

and $(A \restriction -a)R(B \restriction -b)$. □

A solution of questions (A), (B), (D) and (E) for countable algebras turns out to be much more difficult. HANF [1957] provides a countable counterexample to (B) and hence to (A), a particularly intuitive version of the proof being given in the book by HALMOS [1963]. If A and B are countable, $A \times B^2 \cong A$ but $A \times B \ncong A$, then $C = A \times B$ is a countable algebra such that $A \ncong C$ but $A^2 \cong C^2$, which settles (D) for countable algebras. The remaining question (E), Tarski's cube problem, was answered in the negative much later by Ketonen. The solution follows from a difficult structure theorem on countable Boolean algebras in KETONEN [1978]; cf. the survey chapter by PIERCE [Ch. 21 in this Handbook] on countable Boolean algebras.

Exercises

1. For arbitrary non-trivial Boolean algebras A and B, there is a monomorphism $e: A \to A \times B$ such that $\mathrm{pr}_A \circ e = \mathrm{id}_A$, where pr_A is the projection onto the first coordinate. Hence, A is a retract of $A \times B$, as defined in 5.12.

2. Let L be a linear order with a smallest element 0_L and with a strictly increasing cofinal sequence $(x_n)_{n\in\omega}$, where $x_0 = 0_L$. Then Intalg L is isomorphic to the weak product of the algebras $\mathrm{Intalg}[x_n, x_{n+1})$, $n \in \omega$.

3. (Hanf) Let A be an interval algebra with infinitely many atoms. Then $A \times 2 \cong A$, regardless of the cardinality of A.

4. Let $(A_i)_{i\in I}$ be a family of Boolean algebras and D an ultrafilter of $P(I)$. Then

$$F = \left\{ (a_i)_{i \in I} \in \prod_{i \in I} A_i \colon \{ i \in I \colon a_i = 1 \} \in D \right\}$$

is a filter of $\Pi_{i \in I} A_i$; the quotient algebra $\Pi_{i \in I} A_i / F$ is called the *ultraproduct* of $(A_i)_{i \in I}$ with respect to D and denoted by $\Pi_{i \in I} A_i / D$. For $a \in \Pi_{i \in I} A_i$, the equivalence class of a with respect to F is also denoted by a/D.

(a) Let $a = (a_i)_{i \in I} \in \Pi_{i \in I} A_i$. Then

$$a/D \text{ is an atom of } \prod_{i \in I} A_i/D \quad \text{iff } \{ i \in I \colon a_i \text{ is an atom of } A_i \} \in D \, .$$

(b) $\Pi_{i \in I} A_i / D$ is atomic iff $\{ i \in I \colon A_i \text{ is atomic} \} \in D$.

(c) Assume there are countably many elements X_n, $n \in \omega$, of D such that $\bigcap_{n \in \omega} X_n = \emptyset$. Prove that $B = \Pi_{i \in I} A_i / D$ satisfies (a) (the countable separation property) and (e) of 5.28. (These assertions should be familiar to readers acquainted with ultraproducts in model theory.)

5. The following construction gives once more the essential step in the proof of Stone's theorem, i.e. that every Boolean algebra is embeddable into an atomic one. Let A be an arbitrary Boolean algebra, I the set of all finite subalgebras of A. For $a \in A$, let

$$X_a = \{ C \in I \colon a \in C \} \, .$$

(a) There exists an ultrafilter D of $P(I)$ containing every X_a.

(b) Let, for $i = C \in I$, $A_i = C$. Fix an ultrafilter D with $\{ X_a \colon a \in A \} \subseteq D$ and define

$$e \colon A \to \prod_{i \in I} A_i / D$$

by letting $e(a) = (a_i)_{i \in I} / D$, where $a_i = a$ if $a \in i$ and $a_i = 1$ otherwise. Then e is a monomorphism embedding A into the atomic Boolean algebra $\Pi_{i \in I} A_i / D$.

CHAPTER 3

Topological Duality

Sabine KOPPELBERG

Freie Universität Berlin

Contents

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Introduction

Stone's topological duality theory lies at the beginning, and is in fact the core, of the modern structure theory of Boolean algebras. It establishes an essentially one-to-one correspondence between Boolean algebras and zero-dimensional compact Hausdorff spaces, the so-called Boolean spaces (cf. Section 7); this is extended, in Section 8, to a correspondence of homomorphisms between Boolean algebras and continuous maps between Boolean spaces. Consequently, algebraic questions on Boolean algebras translate into topological ones on Boolean spaces, and vice versa.

As a rule, assertions and definitions concerning Boolean algebras (respectively Boolean spaces) are best understood when considered both in their algebraic and their topological formulation. Let us consider just one example: we show in Section 8 that, for each infinite set X, the power set algebra $P(X)$ corresponds, under Stone duality, to the Stone–Čech compactification βX of the discrete space X, and the quotient algebra $P(X)/fin$ (fin the ideal of finite subsets of X) corresponds to the Stone–Čech remainder $X^* = \beta X \setminus X$. For X the set ω of natural numbers, the quotient algebra $P(\omega)/fin$ is characterized up to isomorphism by algebraic properties, in Corollary 5.30, under the continuum hypothesis CH. This gives a topological description of the space ω^* under CH which is well known in topology as Parovičenko's characterization; the standard method for its proof, however, is the algebraic one outlined in 5.30.

It is often a matter of taste whether to attack a specific problem on Boolean algebras algebraically or topologically, but in some situations, one approach seems to be definitely superior to the other one. For example, complete Boolean algebras are, as a rule, easier handled algebraically; on the other hand, free Boolean algebras and free products of Boolean algebras are better visualized topologically, their dual spaces being generalized Cantor spaces (respectively products of Boolean spaces).

We will assume a working knowledge of set-theoretic topology; the book by ENGELKING [1977] is an excellent reference.

7. Boolean algebras and Boolean spaces

This section sets up the fundamental duality between Boolean algebras and particular topological spaces, the Boolean spaces. Here the dual algebra of a Boolean space X is Clop X, the algebra of clopen subsets of X, and the dual space of a Boolean algebra A is the set Ult A of ultrafilters of A, equipped with the so-called Stone topology. The point about these assignments is that every Boolean algebra A is isomorphic to Clop Ult A – more precisely the Stone map $s: A \to P(\text{Ult } A)$, as defined in 2.18, is an isomorphism from A onto Clop Ult A. In particular, we obtain a stronger topological version of Stone's theorem 2.1: every Boolean algebra is isomorphic to the clopen algebra of some topological space. Dually, each Boolean space X is homeomorphic to Ult Clop X.

Stone's duality theorems are proved after establishing some basic properties of Boolean spaces. We then give a, rather incomplete, survey of algebraic properties and notions on Boolean algebras and their topological counterparts for Boolean spaces. For example, a Boolean algebra A is countable iff its dual space is metrizable; it is atomic iff the isolated points form a dense subset of the dual space, etc. The most important one of these dualities says that the ideals of a Boolean algebra correspond to the open subsets of its dual space. Our examples also show how topological facts on Boolean spaces are sometimes very easily recovered from algebraic ones on Boolean algebras and vice versa. For example, topological considerations prove once more Theorem 2.8 that every finite Boolean algebra is isomorphic to a power set algebra, and the consequence 5.16 of Vaught's theorem implies that the Cantor discontinuum is homeomorphic to the product space ${}^{\omega}2$, where $2 = \{0, 1\}$ has the discrete topology.

7.1. *Boolean spaces*

The algebra Clop X of clopen subsets of an arbitrary topological space X is one of the standard examples for an algebra of sets. For a connected space X, however, this algebra reduces to the two-element algebra $\{\emptyset, X\}$. We consider in this subsection a particular class of topological spaces, the Boolean spaces, for which Clop X is large in the sense that it is a base for the topology of X, and prove some general facts on Boolean spaces.

7.1. DEFINITION. Let X be a topological space. X is *zero-dimensional* if Clop X is a base for the topology of X. X is a *Boolean space* if it is Hausdorff, compact and zero-dimensional.

Zero-dimensionality of a space X is equivalent to the existence of a base for X consisting of clopen sets or of a subbase for X consisting of clopen sets. For example, the space of irrational numbers with the topology inherited from the reals is zero-dimensional, having the intervals (a, b) with a and b rational as a clopen base.

7.2. EXAMPLES. (a) Every finite discrete space is Boolean. In particular, we denote by *2* the Boolean space with underlying set $2 = \{0, 1\}$ and the discrete topology.

(b) The product space of any family of Boolean spaces is Boolean. In particular, so is the product space ${}^{I}2$ for any index set I. It is called, for infinite I, a (generalized) *Cantor space* of weight $|I|$.

(c) Every closed subspace of a Boolean space is Boolean.

For a proof of (b), let $X = \prod_{i \in I} X_i$ be a product of Boolean spaces X_i. Then X is Hausdorff and, by Tychonoff's theorem, compact. For fixed $i \in I$ and $b_i \in$ Clop X_i, define a subset $s(b_i)$ of X by

$$s(b_i) = \{x \in X: x(i) \in b_i\} .$$

Then $X \setminus s(b_i) = s(X_i \setminus b_i)$, and by the very definition of the product topology on X, both $s(b_i)$ and $s(X_i \backslash b_i)$ are open, hence clopen. X is zero-dimensional since the sets $s(b_i)$ form a subbase for the product topology. The term "generalized Cantor space of weight $|I|$" will be justified by Example 7.24 and Proposition 7.23 below.

To prove (c), let X be Boolean and Y a closed subspace of X. Then Y is Hausdorff and compact; it is zero-dimensional since the intersections of the clopen subsets of X with Y constitute a clopen base of Y.

As a consequence of (b) and (c) in 7.2, every closed subspace of a Cantor space ${}^I2$ is Boolean. We shall conclude from Proposition 7.11 that, conversely, every Boolean space is homeomorphic to a closed subspace of a Cantor space.

7.3. EXAMPLE (*the classical Cantor space*). The following subspace C of the reals, the *classical Cantor space* or *Cantor discontinuum* is a standard example of set-theoretical topology. Let

$$C = \bigcap_{n \in \omega} C_n ,$$

where each C_n is a compact subset of the reals, constructed as follows. Let $C_0 = [0, 1]$. Suppose C_n has been defined and is the union of a family F_n consisting of 2^n pairwise disjoint closed intervals of length $1/3^n$. Then C_{n+1} arises from C_n by replacing each interval I in F_n, say $I = [a, b]$, by the two closed subintervals $[a, a + (b - a)/3]$ and $[b - (b - a)/3, b]$. The space C is Hausdorff; it is also compact, being a closed and bounded subspace of $\boldsymbol{R}$. To check that it is zero-dimensional, let $x \in C$ and suppose (r, s) is an open interval of $\boldsymbol{R}$ containing x; we will find a clopen subset u of C such that $x \in u \subseteq (r, s)$. Choose n so large that the unique interval I of F_n containing x is included in (r, s); this is possible since the intervals in F_n have length $1/3^n$. Then I is clopen in C_n, $u = I \cap C$ is clopen in C and $x \in u \subseteq (r, s)$.

Recall from topology that a space X is said to be connected if Clop $X = \{\emptyset, X\}$, i.e. if X is not the union of two non-empty disjoint closed subsets; Boolean spaces, on the other hand, are highly disconnected. In view of the following theorem, they are sometimes defined to be the compact hereditarily disconnected Hausdorff spaces.

7.4. DEFINITION. A topological space X is *hereditarily disconnected* (*totally disconnected*) if no subspace of X with at least two points is connected.

7.5. THEOREM. *A compact Hausdorff space is zero-dimensional* (*hence Boolean*) *iff it is hereditarily disconnected.*

PROOF. Let X be compact and Hausdorff. If X is zero-dimensional and Y a subspace of X with two distinct points y and y', then there is a clopen set a such that $y \in a$ and $y' \notin a$. So $Y \cap a$ is a proper non-empty clopen subset of Y and Y is not connected. Thus, X is hereditarily disconnected.

Conversely, suppose X is hereditarily disconnected. Let $x \in X$ and u an open neighbourhood of x; we shall find a clopen subset f of X such that $x \in f \subseteq u$, thus proving that Clop X is a base. To this end, define

$$F = \{f \in \text{Clop } X: x \in f\}, \quad q = \bigcap F .$$

It suffices to show that $q \subseteq u$. For then, since X is compact, u is open and each $f \in F$ is closed, $\bigcap F' \subseteq u$ for some finite subset F' of F, and we let $f = \bigcap F'$. We will, in fact, show that $q = \{x\}$. Otherwise q has at least two points and is not connected. So

$$q = q_1 \cup q_2 ,$$

where q_1, q_2 are non-empty, disjoint and closed in q. Now q is a closed subset of X, so each q_i is closed in X. By compactness and hence normality of X, choose disjoint open sets u_1, u_2 satisfying $q_i \subseteq u_i$. Then $q \subseteq u_1 \cup u_2$, and it follows as above by a compactness argument that $f \subseteq u_1 \cup u_2$ for some $f \in F$. Both $u_1 \cap f$ and $u_2 \cap f$ are clopen in f, hence in X. Since $x \in f$, assume that $x \in u_1 \cap f$. Then $u_1 \cap f \in F$ and $q \subseteq u_1 \cap f$. This implies $q_2 \subseteq q \subseteq u_1$, a contradiction to $q_2 \subseteq u_2$, disjointness of the u_i and non-emptiness of q_i. □

We collect some useful properties of Boolean spaces for further reference.

7.6. LEMMA. *Let X be a Boolean space.*

(a) *If $B \subseteq \text{Clop } X$ is a base for X which is closed under finite unions, then $B = \text{Clop } X$.*

(b) If *Y is a closed subspace of X, then*

$$\text{Clop } Y = \{a \cap Y: a \in \text{Clop } X\} .$$

(c) If *y and z are disjoint closed subsets of X, there is a clopen subset a of X separating y and z, i.e. such that $y \subseteq a$ and $z \subseteq X \setminus a$.*

PROOF. (a) Let a be clopen in X. Then a is the union of a subfamily B' of B; since a is compact, $a = \bigcup B''$ for some finite subset B'' of B'. So $a \in B$.

(b) The set $\{a \cap Y: a \in \text{Clop } X\}$ is a clopen base of the Boolean space Y, and it is closed under finite unions. By part (a), it coincides with Clop Y.

(c) $Y = y \cup z$ is a closed subspace of X in which y and z are clopen. By part (b), $y = a \cap Y$ for some clopen subset a of X. □

Let us point out a principle on compact Hausdorff spaces which is frequently applied when dealing with Boolean spaces. If $f: X \to Y$ is a continuous map between compact Hausdorff spaces, then f is closed, i.e. images of closed subsets of X are closed in Y. Hence, $f[X]$ carries the quotient topology induced by X and f. In particular if f is one-to-one, then it is a homeomorphism from X onto the closed subspace $f[X]$ of Y.

7.2. The topological version of Stone's theorem

We are ready to show that every Boolean algebra is isomorphic to the clopen algebra of a Boolean space. To this end, recall from Section 2 that Ult A is the set of all ultrafilters of a Boolean algebra A and that the Stone map

$$s\colon A \to P(\text{Ult } A)$$

defined by

$$s(a) = \{p \in \text{Ult } A\colon a \in p\}$$

is a monomorphism of Boolean algebras. In particular, $s[A]$ is a family of subsets of Ult A which is closed under finite intersections, hence the base of a unique topology.

7.7. DEFINITION. For a Boolean algebra A, the *Stone topology* is the unique topology on Ult A having $s[A]$ as a base. Ult A, equipped with the Stone topology, is the *Stone space* or the *dual space* or the space of ultrafilters of A.

7.8. THEOREM (Stone's representation theorem, topological version). *Every Boolean algebra is isomorphic to the clopen algebra of a Boolean space. More precisely, the dual space* Ult A *of a Boolean algebra A is Boolean and the Stone map of A is an isomorphism from A onto* Clop Ult A.

PROOF. Let A be a Boolean algebra and $X = \text{Ult } A$ its dual space. X is zero-dimensional since, by $X \setminus s(a) = s(-a)$, each of the base sets $s(a)$ is clopen. Also, X is Hausdorff, for suppose p and q are distinct ultrafilters of A; by maximality of p, pick $a \in p \setminus q$. Then $s(a)$ and $s(-a)$ are disjoint neighbourhoods of p and q.

To prove that X is compact, let U be an open cover of X. It suffices to consider the case that each element of U is a basic set; so let $U = \{s(a)\colon a \in A'\}$, where $A' \subseteq A$. Suppose no finite subset of U covers X. Then for $n \in \omega$ and $a_1, \ldots, a_n \in A'$,

$$s(a_1) \cup \cdots \cup s(a_n) \neq X = s(1),$$

hence $a_1 + \cdots + a_n \neq 1$ and $-a_1 \cdot \cdots \cdot -a_n \neq 0$. It follows that the set $-A' = \{-a\colon a \in A'\}$ has the finite intersection property; by the Boolean prime ideal theorem 2.16, let p be an ultrafilter of A including $-A'$. Then for each $a \in A'$ we have $-a \in p$, $a \notin p$ and $p \notin s(a)$ – contradicting our assumption that U covers X.

So X is a Boolean space. Since the Stone map s is a monomorphism from A into Clop X, we are left with showing that Clop $X = s[A]$. But this follows from Lemma 7.6(a) by considering the base $B = s[A]$ of X. □

7.9. DEFINITION AND LEMMA. Let X be a Boolean space. Then Clop X is the *dual algebra* of X. For each $x \in X$, the set

$$t(x) = \{a \in \text{Clop } X: x \in a\}$$

is an ultrafilter of Clop X. This defines the map

$$t: X \to \text{Ult Clop } X .$$

Theorem 7.8 can now be stated as saying that the bidual of a Boolean algebra A, i.e. the dual algebra of the dual space of A, is isomorphic to A. There is a topological dual of this fact: every Boolean space is homeomorphic to its bidual.

7.10. THEOREM. *Every Boolean space is homeomorphic to the Stone space of a Boolean algebra. More precisely, for a Boolean space X, the map $t: X \to$ Ult Clop X in 7.9 is a homeomorphism from X onto* Ult Clop X.

PROOF. It is sufficient to prove that t is continuous and bijective, since both X and Ult Clop X are compact Hausdorff spaces. Let $A = \text{Clop } X$. Continuity of t follows since preimages of the basic sets $s(a)$, $a \in A$, are open: for $a \in A$ and $x \in X$, $x \in t^{-1}[s(a)]$ iff $t(x) \in s(a)$ iff $a \in t(x)$ iff $x \in a$, so $t^{-1}[s(a)] = a$ is clopen.

t is one-to-one, for let x and y be distinct points of X. Since X is Boolean, choose $a \in \text{Clop } X$ such that $x \in a$ and $y \not\in a$. Then $a \in t(x) \setminus t(y)$.

To prove that t is onto, let p be an ultrafilter of $A = \text{Clop } X$. Now X is compact and p is a family of closed subsets of X with the finite intersection property, so pick $x \in X$ such that $x \in a$ for each $a \in p$. Then $p \subseteq t(x)$ and, by maximality of the ultrafilter p, $p = t(x)$. □

It is essential for the following proposition that the set $2 = \{0, 1\}$ carries the structure of both a Boolean algebra and a Boolean space. Thus, given a Boolean space X, we may consider ${}^X2$ as a power of the algebra 2; considering 2 as a Boolean space, we may ask which of the maps $x \in {}^X2$ are continuous. Conversely, given a Boolean algebra A, we consider ${}^A2$ as a generalized Cantor space and then, considering 2 as a Boolean algebra, ask which of the maps $x \in {}^A2$ are Boolean homomorphisms.

7.11. PROPOSITION. *For every Boolean space X, the set $\{x \in {}^X2$: x continuous$\}$ is a subalgebra of the product algebra ${}^X2$ isomorphic to* Clop X. *For every Boolean algebra A, the set $\{x \in {}^A2$: x a homomorphism$\}$ is a closed subspace of the Cantor space ${}^A2$ homeomorphic to* Ult A.

PROOF. In the first assertion, note that a map $x: X \to 2$ is continuous iff the preimage of the clopen subset $\{1\}$ of 2 under x is clopen in X, i.e. iff x is the characteristic function of some clopen subset of X. The map f: Clop $X \to {}^X2$ which assigns to each clopen subset of X its characteristic function is a monomorphism (in fact, it is the restriction of the natural isomorphism from $P(X)$ onto ${}^X2$

considered in Section 6) which maps Clop X onto the set of continuous functions in ${}^{X}2$.

For the second assertion, consider the map g: Ult $A \to {}^{A}2$ assigning to every $p \in$ Ult A its characteristic homomorphism. Clearly, g is one-to-one and by Proposition 2.15, ran g is the set of those x: $A \to 2$ which are homomorphisms. g is continuous since for any $a \in A$, the preimage of $\{x \in {}^{A}2\colon x(a) = 1\}$ under g is the clopen subset $s(a)$ of Ult A. □

By Theorem 7.10, every Boolean space is homeomorphic to the dual space of some Boolean algebra. Thus, the second part of the preceding proposition implies that every Boolean space is homeomorphic to a closed subspace of some generalized Cantor space. See Exercise 5 for a purely topological proof of this.

We conclude this subsection by characterizing the Stone monomorphism s: $A \to P$(Ult A) of a Boolean algebra A among all embeddings of A into power set algebras.

7.12. DEFINITION. A *representation* of a Boolean algebra A is a monomorphism from A into a power set algebra. A representation e: $A \to P(X)$ is *reduced* if $e[A]$ separates the points of X, i.e. if for any two distinct points x and y in X, there is an $a \in A$ such that $x \in e(a)$ and $y \notin e(a)$. It is *perfect* if for every ultrafilter p of A there is a point x of X such that for all $a \in A$, $a \in p$ iff $x \in e(a)$.

7.13. PROPOSITION. *The Stone monomorphism s: $A \to P$(Ult A) is a reduced and perfect representation of A. Conversely assume that e: $A \to P(X)$ is a reduced and perfect representation of A. Then there is a unique bijection ϕ: $X \to$ Ult A such that, for $a \in A$, $e(a) = \phi^{-1}[s(a)]$. With the topology having $e[A]$ as a base, X is a Boolean space*, Clop $X = e[A]$ *and ϕ is a homeomorphism.*

PROOF. An arbitrary representation e: $A \to P(X)$ induces a map

(1) $\quad \phi\colon X \to \text{Ult}\, A\,, \quad \phi(x) = \{a \in A\colon x \in e(a)\}\,.$

Clearly, e is reduced iff ϕ is one-to-one and perfect iff ϕ is onto.

Letting X be Ult A and e the Stone map s, we find that ϕ defined by (1) is the identity map on Ult A; thus s is reduced and perfect.

Now assume that e: $A \to P(X)$ is a reduced and perfect representation. Then ϕ as defined in (1) is bijective and satisfies $e(a) = \phi^{-1}[s(a)]$. If ϕ' is another map with this property, then for $x \in X$ and $a \in A$,

$$\phi(x) \in s(a) \quad \text{iff } x \in e(a) \quad \text{iff } \phi'(x) \in s(a)\,.$$

That is to say, the points $\phi(x)$ and $\phi'(x)$ of Ult A are contained in the same basic sets $s(a)$ of Ult A, hence $\phi(x) = \phi'(x)$ for all $x \in X$ and $\phi = \phi'$. Finally, ϕ transfers the base $e[A]$ of X onto the base $s[A]$ of Ult A and thus is a homeomorphism. □

7.3. *Dual properties of A and* Ult *A*

We compare here algebraic properties of a Boolean algebra and topological ones of its dual space. The following examples are fairly simple but will give the reader some idea of the interplay between algebraic and topological reasoning.

7.14. EXAMPLE (finite Boolean algebras). For an arbitrary Boolean algebra A, the following properties are equivalent:

A is finite,

Ult A is a finite Boolean space,

Ult A is a finite discrete space,

Clop Ult $A = P(\text{Ult } A)$,

the Stone map of A is an isomorphism from A onto $P(X)$, for some finite set X.

So we recover the fact that finite Boolean algebras are isomorphic to power sets (Corollary 2.8). It may be worth noticing that A is the trivial Boolean algebra iff the space Ult A is empty and A is the two-element algebra iff Ult A is a one-point space.

7.15. EXAMPLE. We have seen in Example 5.25 that if $p_1, \ldots, p_n$ are distinct ultrafilters of a Boolean algebra A and M is an arbitrary subset of $\{1, \ldots, n\}$, then there is $a \in A$ such that $a \in p_i$ iff $i \in M$. For a topological proof of this, choose for $1 \leq i \leq n$ a clopen neighbourhod u_i of p_i in Ult A such that $u_i \cap \{p_1, \ldots, p_n\} = \{p_i\}$; this is possible since Ult A is Hausdorff and zero-dimensional. Since Clop Ult $A = s[A]$ (s the Stone map), let $a \in A$ be such that $s(a) = \bigcup_{i \in M} u_i$. This element a works for our claim.

7.16. EXAMPLE (sums and products preserved by the Stone map). The Stone monomorphism s, considered as a map from A into $P(\text{Ult } A)$, does in general not preserve infinite sums and products. More precisely, if M is a subset of A, $\Sigma M (= \Sigma^A M)$ exists and

$$s\left(\Sigma^A M\right) = \Sigma^{P(\text{Ult } A)} s[M] = \bigcup s[M], \tag{2}$$

then $\Sigma M = \Sigma M'$ for some finite subset M' of M. For the left-hand side of (2) is a clopen, hence compact, subset of Ult A and is thus covered by finitely many elements of $s[M]$, say

$$s\left(\Sigma M\right) = s(m_1) \cup \cdots \cup s(m_n),$$

so $\Sigma M = m_1 + \cdots + m_n$.

7.17. PROPOSITION. *The regular open algebra* RO(Ult A) *of* Ult A *is a completion of* A.

PROOF. RO(Ult A) is complete, thus by Definition 4.18 we have to show that A is (isomorphic to) a dense subalgebra of RO(Ult A). Now the Stone map s is an isomorphism from A onto Clop Ult A, a subalgebra of RO(Ult A). Also Clop Ult A, being a base for the topology of Ult A, is dense in RO(Ult A). □

The Stone space of a Boolean algebra A is determined by A, and conversely the topological version of Stone's theorem says that it determines A up to isomorphism. Thus, every algebraic notion on Boolean algebras can be translated into a topological dual on Boolean spaces, and vice versa. We begin with this translation process in the following proposition.

7.18. PROPOSITION. *The atoms of a Boolean algebra A correspond to the isolated points of* Ult A. *Hence A is atomless iff* Ult A *has no isolated points and A is atomic iff the isolated points of* Ult A *constitute a dense subset of* Ult A.

PROOF. A bijection f between the set At A of atoms of A and the set *Is* of isolated points of Ult A is obtained by letting, for $a \in \text{At } A$,

$$f(a) = x \quad \text{iff } s(a) = \{x\} .$$

This holds since, first, $a > 0$ in A iff the clopen subset $s(a)$ of Ult A is non-empty and, second, a is an atom of A iff $s(a)$ is not the union of two disjoint non-empty clopen subsets, i.e. iff $s(a)$ is a singleton $\{x\}$, where x is isolated. □

The last assertion of Proposition 7.18 may be reformulated as saying that A is atomic iff Ult A is a compactification of a discrete space. We shall have a closer look at such compactifications in the following section.

7.19. EXAMPLE (one-point compactification of discrete spaces). For every cardinal κ, there is a Boolean space X of cardinality κ: if κ is finite, let X be a discrete space with κ points. Otherwise, let $X = \text{Ult } A$, where A is the finite–cofinite algebra on a set I of size κ. Then X has cardinality κ since Ult $A = \{p\} \cup \{p_i: i \in I\}$, where

$$p = \{a \in A: a \text{ cofinite}\} , \qquad p_i = \{a \in A: i \in a\} .$$

A is atomic, the atoms being the singletons $\{i\}$, $i \in I$; the p_i are the isolated points of Ult A, and Ult A is the one-point compactification of its discrete subspace $\{p_i: i \in I\}$.

7.20. DEFINITION. A topological space X is *extremally disconnected* if the closure of every open subset of X is open. It is *basically disconnected* if the closure of every countable union of clopen subsets is open.

7.21. PROPOSITION. *Let A be a Boolean algebra. For* $M \subseteq A$, $\Sigma^A M$ exists iff the closure of $\bigcup s[M]$ *in* Ult A *is open. In particular, A is complete iff* Ult A *is extremally disconnected and σ-complete iff* Ult A *is basically disconnected.*

PROOF. First note that M has a least upper bound in A iff its image $s[M]$ under the Stone map has a least upper bound in the isomorphic algebra $s[A] =$ Clop Ult A. Let c be the closure of $\bigcup s[M]$ in Ult A; thus c is the smallest closed subset of Ult A including $s(m)$ for each $m \in M$.

If c happens to be open, then it is clearly the least upper bound of $s[M]$ in $s[A]$. Conversely, suppose the least upper bound b of $s[M]$ in $s[A]$ exists. Then $\bigcup s[M] \subseteq b$, hence $c = \mathrm{cl}(\bigcup s[M]) \subseteq b$. We claim that $c = b$. Otherwise $b \setminus c$ is open and non-empty, so there exists a non-empty clopen subset d of $b \setminus c$. Then $b \setminus d$ is an upper bound of $s[M]$ in $s[A]$ strictly smaller than b, a contradiction.

The second assertion of the theorem follows since a subset u of Ult A is open iff $u = \bigcup s[M]$ for some subset M of A. □

Proposition 7.21 has an obvious generalization to κ-complete algebras: a Boolean algebra A is κ-complete iff, in Ult A, the closure of every union of less than κ clopen sets is open.

7.22. DEFINITION. The *weight* of a topological space X, denoted by wX, is the least possible cardinal of some base of X.

7.23. PROPOSITION. *For every infinite Boolean algebra* A, $w(\mathrm{Ult}\ A) = |A|$. *In particular*, Ult A *is metrizable iff* A *is at most countable.*

PROOF. $s[A]$ is a base for Ult A and thus $w(\mathrm{Ult}\ A) \leq |A|$. Conversely, assume B is any base for Ult A with the aim of proving $|A| \leq |B|$. A, and hence Clop Ult A, is infinite, so B must be infinite. For each $a \in A$, pick $B_a \subseteq B$ such that $s(a) = \bigcup B_a$; since $s(a)$ is compact, we may assume B_a to be finite. Assigning B_a to $a \in A$ gives a one-to-one map from A into the set of finite subsets of B; so $|A| \leq |B|$ since B is infinite.

For the second assertion, apply Urysohn's theorem that a compact Hausdorff space is metrizable iff it has a countable base. □

Combining Proposition 7.23 with Theorem 5.31, we obtain $w(X) \leq |X|$ for every infinite Boolean space X. For X finite, $w(X) = |X|$ holds since X is discrete. More generally, it is a well-known fact of topology that $w(X) \leq |X|$ for any compact Hausdorff space.

7.24. EXAMPLE (the classical Cantor space, again). The classical Cantor space $C \subseteq R$ defined in Example 7.3 is homeomorphic to the product space ${}^{\omega}2$. This is known from topology but can easily be recovered from duality arguments. Both C and ${}^{\omega}2$ are Boolean spaces with countable bases and without isolated points, so their dual algebras, being countably infinite and atomless, are isomorphic by Carollary 5.16. It follows that the dual spaces of the algebras are homeomorphic and

$$C \cong \mathrm{Ult\ Clop}\ C \cong \mathrm{Ult\ Clop}\ {}^{\omega}2 \cong {}^{\omega}2\,.$$

Ideals of a Boolean algebra correspond, by duality, to open subsets of the Stone space:

7.25. THEOREM. *The assignments*

$$I \mapsto o(I) = \bigcup s[I] \qquad \textit{(the open subset of } \mathrm{Ult}\, A \textit{ dual to } I\textit{)},$$

$$u \mapsto i(u) = \{a \in A\colon s(a) \subseteq u\} \quad \textit{(the ideal of } A \textit{ dual to } u\textit{)},$$

are order-preserving bijections between the ideals of a Boolean algebra A and the open subsets of Ult A. *For each ideal I of A,*

$$\mathrm{Ult}(A/I) \cong \mathrm{Ult}\, A \backslash o(I)\,.$$

PROOF. Let Id be the set of ideals of A and $\boldsymbol{O}$ the set of open subsets of Ult A. Plainly, the above assignments are order-preserving maps

$$o\colon Id \to \boldsymbol{O}\,, \qquad i\colon \boldsymbol{O} \to Id\,;$$

we prove that they are inverses of each other.

Obviously, $o(i(u)) \subseteq u$ for every $u \in \boldsymbol{O}$ and $I \subseteq i(o(I))$ for every $I \in Id$. Conversely, let $x \in u$. Choose $a \in A$ such that $x \in s(a) \subseteq u$. Then $a \in i(u)$ and $x \in o(i(u))$ which shows $u \subseteq o(i(u))$. To prove $i(o(I)) \subseteq I$, let $a \in i(o(I))$. Then $s(a) \subseteq \bigcup s[I]$ and it follows by compactness of $s(a)$ that $a \leq a_1 + \cdots + a_n$ for some $n \in \omega$ and $a_1, \ldots, a_n \in I$; so $a \in I$ holds.

Finally, assume $u = o(I)$. The homomorphism $f\colon A \to \mathrm{Clop}(\mathrm{Ult}\, A \backslash u)$ defined by

$$f(a) = s(a) \cap (\mathrm{Ult}\, A \backslash u)$$

is onto, by Lemma 7.6(b), and has the ideal I as its kernel. So by the homomorphism theorem 5.23, $A/I \cong \mathrm{Clop}(\mathrm{Ult}\, A\backslash u)$ and $\mathrm{Ult}(A/I) \cong \mathrm{Ult}\, A\backslash u$. □

Theorem 7.25 says, in a somewhat more abstract formulation, that the lattice of ideals of A is isomorphic to the lattice of open subsets of Ult A. In view of the order-preserving bijection $I \mapsto -I = \{-a\colon a \in I\}$ between ideals and filters of A and the order-reversing bijection $u \mapsto \mathrm{Ult}\, A\backslash u$ between open and closed subsets of Ult A, there is a one-to-one order-reversing correspondence between filters of A and closed subsets of Ult A. The closed set corresponding to a filter F of A is $\bigcap s[F]$ and the filter corresponding to a closed subset c of Ult A is $\{a \in A\colon c \subseteq s(a)\}$. In particular, an ultrafilter p of A corresponds to the closed subset $\{p\}$ of Ult A.

The last assertion of Theorem 7.25 gives, as a particularly simple special case:

7.26. EXAMPLE. Let A be a Boolean algebra and $a \in A$. Then the dual space of the relative algebra $A \restriction a$ is homeomorphic to the clopen subspace $s(a)$ of Ult A.

To see this, let I be the principal ideal of A generated by $-a$. Then $A \restriction a \cong A/I$ by the remark following 5.23, $o(I) = s(-a)$ in the notation of 7.25, and hence

$$\mathrm{Ult}(A \restriction a) \cong \mathrm{Ult}\, A\backslash s(-a) = s(a)\,.$$

Exercises

1. Verify that each well-ordered set with a greatest element is a Boolean space, in its order topology.

2. A subset P of an arbitrary field K is called (the set of positive elements of) an *ordering* of K if
(a) $0 \notin P$,
(b) if $x, y \in P$, then $x + y \in P$ and $x \cdot y \in P$,
(c) for every $x \in K$, either $x = 0$ or $x \in P$ or $-x \in P$.
Prove that the (possibly empty) set X of all orderings of K is a Boolean space if endowed with the following topology: for $a \in K$, let $X_a = \{P \in X: a \in P\}$ and take $S = \{X_a: a \in K\}$ as the subbase of a topology.

Hint. Identify X with the subset $\{\chi(P): P \in X\}$ of the Cantor space ${}^K 2$, where $\chi(P): K \to 2$ is the characteristic function of P.

3. Let U be an open cover of a Boolean space X. Then there are $u_1, \ldots, u_n \in U$ and clopen subsets c_k of u_k $(1 \leq k \leq n)$ such that $\{c_1, \ldots, c_n\}$ is a clopen partition of X.

4. Let X be a Boolean space and $U \subseteq X$ open. Then U is an F_σ-set (i.e. a union of countably many closed sets) iff it is a union of countably many clopen sets.

5. Prove, without using Stone's duality theory, that every Boolean space is homeomorphic to a closed subspace of a generalized Cantor space.

Hint. If X is Boolean, consider the map $f: X \to {}^A 2$, where $A = \text{Clop}\, X$ and $f(x) = (\chi_a(x))_{a \in A}$, χ_a the characteristic function of a.

6. A subset M of a Boolean algebra A generates A iff the subset $s[M]$ of Clop Ult A (s is the Stone isomorphism) separates points in Ult A.

7. Let $D \subseteq \text{Ult}\, A$. Then the homomorphism $e: A \to P(D)$ defined by $e(a) = \{p \in D: a \in p\}$ is one-to-one iff D is dense in the topological space Ult A. Consequently, $\min\{|D|: D$ a dense subset of Ult $A\}$ is the least cardinal κ such that A embeds into $P(X)$, for some set X of size κ.

8. Homomorphisms and continuous maps

In this section, the duality of Section 7 between Boolean algebras and Boolean spaces is extended to a duality between homomorphisms of Boolean algebras and continuous maps of Boolean spaces. In terms of category theory, there are contravariant functors from the category ***BA*** of Boolean algebras and homomorphisms into the category ***BS*** of Boolean spaces and continuous maps, and vice versa. The categories ***BA*** and ***BS*** are dually equivalent; commutative diagrams in ***BA*** translate into commutative diagrams in ***BS*** and vice versa. Together with Theorem 7.8 (on the canonical isomorphism of a Boolean algebra A with its bidual Clop Ult A) and Theorem 7.10 (on the canonical homeomorphism of a Boolean space X with its bidual Ult Clop X), Theorem 8.2, which expresses these facts, constitutes the core of Stone's topological duality theory.

After setting up the basic facts, we dualize the algebraic constructions of forming subalgebras and products of Boolean algebras: their topological duals consist in taking the quotient of a Boolean space by a Boolean equivalence relation (respectively the Stone–Čech compactification of a disjoint union of Boolean spaces). More generally, for any family $(A_i)_{i\in I}$ of Boolean algebra, we analyze the Stone spaces of the "intermediate algebras", i.e. the algebras lying between the weak product and the full cartesian product of the family $(A_i)_{i\in I}$.

The last subsection gives the sheaf representation of a Boolean algebra B over a subalgebra A. This representation is an excellent tool for visualizing how B lies over A. Sheaf representations have been successfully used to obtain decidability results for structures representable by sections of a sheaf over a Boolean space; see the chapter by WEESE [Ch. 33 in this Handbook] on decidable theories of Boolean algebras.

8.1. Duality of homomorphisms and continuous maps

The crucial idea for dualizing homomorphisms and continuous maps is that, for any homomorphism $f\colon A \to B$ of Boolean algebras, the preimage $f^{-1}[y]$ of an ultrafilter y of B is an ultrafilter of A. Dually, for any continuous map $\phi\colon X \to Y$ of Boolean spaces, the preimage $\phi^{-1}[b]$ of a clopen subset b of Y is clopen in X. In the formulation of Theorem 8.2, we let

$$s_A\colon A \to \text{Clop Ult } A$$

denote the Stone map and

$$t_X\colon X \to \text{Ult Clop } X$$

the canonical homeomorphism of Theorem 7.10.

8.1. DEFINITION. For every homomorphism $f\colon A \to B$ of Boolean algebras, the *dual of f* is the map

$$f^d\colon \text{Ult } B \to \text{Ult } A$$

defined by

$$f^d(y) = f^{-1}[y]\,.$$

For every continuous map $\phi\colon X \to Y$ of Boolean spaces, the *dual of ϕ* is the map

$$\phi^d\colon \text{Clop } Y \to \text{Clop } X$$

defined by

$$\phi^d(b) = \phi^{-1}[b]\,.$$

8.2. THEOREM (Stone duality for homomorphisms and continuous maps). *Let $f\colon A \to B$, $g\colon B \to C$ be homomorphisms of Boolean algebras and $\phi\colon X \to Y$, $\psi\colon Y \to Z$ continuous maps of Boolean spaces.*

(a) *$f^d\colon \mathrm{Ult}\, B \to \mathrm{Ult}\, A$ is continuous and $\phi^d\colon \mathrm{Clop}\, Y \to \mathrm{Clop}\, X$ is a homomorphism.*

(b) $(\mathrm{id}_A)^d = \mathrm{id}_{\mathrm{Ult}\, A}$ *and* $(\mathrm{id}_X)^d = \mathrm{id}_{\mathrm{Clop}\, X}$.

(c) $(g \circ f)^d = f^d \circ g^d$ *and* $(\psi \circ \phi)^d = \phi^d \circ \psi^d$.

(d) $f^{dd} \circ s_A = s_B \circ f$ *and* $\phi^{dd} \circ t_X = t_Y \circ \phi$.

$$\begin{array}{ccc} A & \xrightarrow{s_A} & \mathrm{Clop\,Ult}\, A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f^{dd}} \\ B & \xrightarrow[s_B]{} & \mathrm{Clop\,Ult}\, B \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{t_X} & \mathrm{Ult\,Clop}\, X \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \phi^{dd}} \\ Y & \xrightarrow[t_y]{} & \mathrm{Ult\,Clop}\, Y \end{array}$$

(e) *f is one-to-one (respectively onto) iff f^d is onto (respectively one-to-one); ϕ is one-to-one (respectively onto) iff ϕ^d is onto (respectively one-to-one).*

PROOF. It is convenient to have the following notation. For arbitrary sets X and Y and an arbitrary map $h\colon X \to Y$, define $h^*\colon P(Y) \to P(X)$ by letting $h^*(b) = h^{-1}[b]$, the preimage of b under h. Then h^* is a complete homomorphism between the power set algebras $P(Y)$ and $P(X)$; it is one-to-one (respectively onto) iff h is onto (respectively one-to-one).

(a) f^d is continuous since for every basic set $s_A(a)$ of Ult A, where $a \in A$,

(1) $$(f^d)^{-1}[s_A(a)] = s_B(f(a))$$

is a basic set of Ult B. Also, ϕ^d is a homomorphism, being he restriction of ϕ^* to the subalgebra Clop Y of $P(Y)$. Note that, dual to (1),

(2) $$(\phi^d)^{-1}[t_X(x)] = t_Y(\phi(x))$$

holds for every $x \in X$.

(b) and (c) are straightforward.

(d) For every $a \in A$, (1) implies that

$$f^{dd}(s_A(a)) = (f^d)^{-1}[s_A(a)] = s_B(f(a))\,.$$

The second assertion follows similarly from (2).

(e) It follows from the above remarks on the map h^* that f^d is one-to-one if f is onto and ϕ^d is one-to-one if ϕ is onto. Now assume that f is one-to-one; without loss of generality, $f = \mathrm{id}_A$, where A is a subalgebra of B. Then $f^d(y) = y \cap A$ for y in Ult B; f^d is onto since by the Boolean prime ideal theorem 2.16, every ultrafilter of A, having the finite intersection property, can be extended to an ultrafilter of B. If ϕ is one-to-one, we may again assume that $\phi = \mathrm{id}_X$, where X is a closed subspace of Y. Then $\phi^d(b) = b \cap X$ for each b in Clop Y, and ϕ^d is onto by part (b) of Lemma 7.6.

The remaining assertions follow from part (d). For example, if f^d is onto, then f^{dd} is one-to-one and so $f=(s_B)^{-1}\circ f^{dd}\circ s_A$ is one-to-one. □

Similar to the identification of a Boolean algebra or a Boolean space with their biduals and in view of 8.2(d), the bidual f^{dd} of a homomorphism f is sometimes identified with f and the bidual ϕ^{dd} of a continuous map ϕ with ϕ – i.e. the canonical isomorphisms $s_A: A\to \mathrm{Clop\,Ult}\,A$ and homeomorphisms $t_X: X\to \mathrm{Ult\,Clop}\,X$ are thought of as being identity maps.

It might be worth noticing that part (e) of 8.2 gives another proof for Lemma 5.32: let A be a proper subalgebra of B. Then the monomorphism $\mathrm{id}_A: A\to B$ is not onto and its dual map ϕ, defined by $\phi(p)=p\cap A$, is not one-to-one.

8.2. Subalgebras and Boolean equivalence relations

The last part of Theorem 8.2 says that embeddings of Boolean spaces correspond to epimorphisms of Boolean algebras or, by the homomorphism theorem 5.23, to quotients of Boolean algebras. Similarly, embeddings of Boolean algebras correspond to continuous onto maps of Boolean spaces. Inside of a fixed Boolean space X, this means that the subalgebras of Clop X correspond to those equivalence relations on X giving rise to a Boolean quotient space. We sketch a direct proof of this fact.

8.3. DEFINITION. Let $\sim$ be an equivalence relation on a Boolean space X. A subset M of X is *closed under* $\sim$ if $x\in M$ and $x'\in X$, $x\sim x'$ imply that $x'\in M$. $\sim$ is a *Boolean equivalence relation* if the subalgebra

$$B_\sim=\{b\subseteq X\colon b\in \mathrm{Clop}\,X,\ b \text{ closed under } \sim\}$$

of Clop X separates the equivalence classes of $\sim$, i.e. if distinct equivalence classes of $\sim$ are included in disjoint elements of $B_\sim$.

8.4. LEMMA. *An equivalence relation is Boolean iff its quotient space is Boolean.*

PROOF. Let $\sim$ be an equivalence relation on a Boolean space X. X, and hence its quotient space $X/{\sim}$, are compact, but $X/{\sim}$ is not necessarily Hausdorff. Now a compact space Y is Boolean iff Clop Y separates points; the non-trivial part of this follows from Theorem 7.5. Also, a subset W of $X/{\sim}$ is clopen in $X/{\sim}$ iff $\bigcup W$ is clopen in X, i.e. iff $\bigcup W$ is in $B_\sim$. Thus, $X/{\sim}$ is a Boolean space iff $B_\sim$ separates equivalence classes, i.e. iff $\sim$ is a Boolean equivalence relation. □

8.5. PROPOSITION. *Let X be a Boolean space. There is an order-reversing one-to-one correspondence between subalgebras of* Clop X *and Boolean equivalence relations on X such that, for B* a subalgebra of Clop X *and $\sim$ the corresponding equivalence relation,*

$$\mathrm{Ult}\,B\cong X/{\sim}\,.$$

PROOF. Each Boolean equivalence relation $\sim$ on X determines the subalgebra $B_\sim$ of $\mathrm{Clop}\,X$ defined in 8.3. Conversely, if B is a subalgebra of $\mathrm{Clop}\,X$, define an equivalence relation $\sim_B$ on X by

$$x \sim_B y \quad \text{iff for all } b \in B,\ x \in b \text{ iff } y \in b\,.$$

Then each $b \in B$ is closed under $\sim_B$, and $\sim_B$ is a Boolean equivalence relation. These assignments of subalgebras to Boolean equivalence relations and vice versa are clearly order-reversing. We leave it to the reader to check that they are converses of each other.

Suppose, finally, that B is a subalgebra of $\mathrm{Clop}\,X$. Then the map f: $\mathrm{Clop}(X/{\sim_B}) \to \mathrm{Clop}\,X$ defined by $f(u) = \bigcup u$ is a monomorphism, and its range consists of those clopen subsets of X which are closed under $\sim_B$. Thus, $\mathrm{ran}\,f = B_{\sim_B} = B$, $B \cong \mathrm{Clop}(X/{\sim_B})$ and $\mathrm{Ult}\,B \cong X/{\sim_B}$ by duality. □

As a consequence of Example 7.19 (every cardinal is the cardinality of a Boolean space), each set can be given a topology which makes it into a Boolean space. There is an amusing extension of this fact.

8.6. EXAMPLE. Let X be an arbitrary set and $\sim$ an arbitrary equivalence relation on X. There is a topology on X which makes X into a Boolean space and $\sim$ into a Boolean equivalence relation.

PROOF. The plan of our proof is as follows. We will define a subalgebra A of $P(X)$ such that

(3) $\quad A$ is reduced and perfect over X,

which means that $\mathrm{id}_A\colon A \to P(X)$ is a reduced and perfect representation of A in the sense of Definition 7.12. By Proposition 7.13, we equip X with the unique topology such that X is Boolean and $A = \mathrm{Clop}\,X$. We then define a subalgebra B of A such that

(4) $\quad \sim$ is the Boolean equivalence relation corresponding to B.

To begin with, let P be the set of equivalence classes of $\sim$. For each $p \in P$, choose a reduced and perfect algebra of sets A_p over p. For example, fix, by Example 7.19, a topology on p which makes p into a Boolean space and let $A_p = \mathrm{Clop}(p)$.

If P is finite, define, for any subset a of X:

$$a \in A \quad \text{iff } a \cap p \in A_p \text{ for every } p \in P\,,$$

and for $b \in A$:

$$b \in B \quad \text{iff } b \text{ is a union of elements of } P\,.$$

We omit the simple proofs of (3) and (4).

Now let P be infinite. Fix $p^* \in P$ and some $x^* \in p^*$. Let A consist of those subsets a of X whch satisfy each of the following conditions:

(5) $a \cap p \in A_p$ for every $p \in P$,

(6) if $x^* \in a$, then a includes all but finitely many $p \in P$,

(7) if $x^* \not\in a$, then $X \backslash a$ includes all but finitely many $p \in P$.

It is easily seen that A is reduced over X. To show perfectness, let g be an ultrafilter of A; we have to find $x \in X$ such that $g = \{a \in A: x \in a\}$. If g happens to be the ultrafilter

$$g^* = \{a \in A: x^* \in a\},$$

x^* will do. Otherwise, pick α in A such that $\alpha \in g \backslash g^*$. Then by $x^* \not\in \alpha$ and (7), $\alpha \subseteq \bigcup Q$ for some finite subset Q of P, and

$$\alpha = \bigcup_{q \in Q} (\alpha \cap q).$$

Each q in Q distinct from p^* is in A, hence so is $\alpha \cap q$. Also, if p^* happens to be in Q, then $\alpha \cap p^*$ is in A, by

$$\alpha \cap p^* = \alpha \backslash \bigcup \{\alpha \cap q: q \in Q, q \neq p^*\}.$$

Since g is a prime filter of A, $\alpha \cap p \in g$ for some $p \in Q$. So $f = g \cap A_p$ is an ultrafilter of A_p; since A_p is perfect over p, it is determined by some x in p. This point x also determines g, which finishes the proof of (3).

For every subset b of X, define

$$b \in B \quad \text{iff } b \in A \text{ and } b \text{ is a union of elements of } P.$$

Then B is a subalgebra of A satisfying (4) – the only non-trivial assertion is that B separates equivalence classes of $\sim$. But if $p \neq q$ in P, we may assume that $p \neq p^*$. So $b = p$ is in B and $p \subseteq b$, $q \subseteq X \backslash b$.

8.3. Product algebras and compactifications

The dualities found in the previous subsections show that homomorphic images, i.e. quotient algebras, of a Boolean algebra A correspond to closed subspaces of Ult A and subalgebras of A to (Boolean) quotient spaces. We now describe the dual spaces of product algebras – they are the Stone–Čech compactifications of disjoint unions of Boolean spaces. More generally, the subalgebras of a product algebra $\Pi_{i \in I} A_i$ which include the weak product $\Pi^{\mathrm{w}}_{i \in I} A_i$ correspond to zero-dimensional compactifications of a disjoint union of Boolean spaces.

Let us recall from topology the following definition. Suppose that, for every i in an index set I, X_i is a topological space and that the sets X_i, $i \in I$, are pairwise disjoint. Then the *disjoint union space* of the spaces X_i is the set

$$U = \bigcup_{i \in I} X_i ,$$

with the topology in which a subset u of U is open iff $u \cap X_i$ is open in X_i for every $i \in I$. Replacing the X_i by pairwise disjoint copies, one can also define the disjoint union space for an arbitrary family $(X_i)_{i \in I}$ of topological spaces.

Separation properties, like being Hausdorff, zero-dimensional or Tychonoff, are inherited by $\bigcup_{i \in I} X_i$ if they hold for each X_i. Clearly, a subset a of $\bigcup_{i \in I} X_i$ is clopen iff $a \cap X_i$ is clopen in X_i for every $i \in I$; in particular, each X_i is a clopen subspace of $\bigcup_{i \in I} X_i$. For finite I, compactness (respectively Booleanness) of each X_i implies compactness (respectively Booleanness) of $\bigcup_{i \in I} X_i$.

8.7. PROPOSITION. *For every finite product $A_1 \times \cdots \times A_n$ of Boolean algebras,*

$$\mathrm{Ult}(A_1 \times \cdots \times A_n) \cong \mathrm{Ult}\, A_1 \cup \cdots \cup \mathrm{Ult}\, A_n .$$

PROOF. Denote by X_i the Stone space of A_i and by U the disjoint union space of the X_i. The above description of clopen subsets of U implies that

$$\begin{aligned} \mathrm{Clop}\, U &\cong \mathrm{Clop}\, X_1 \times \cdots \times \mathrm{Clop}\, X_n \\ &\cong A_1 \times \cdots \times A_n . \end{aligned}$$

Now U is Boolean, so $U \cong \mathrm{Ult}(A_1 \times \cdots \times A_n)$. □

In a disjoint union space $U = \bigcup_{i \in I} X_i$, the family $(X_i)_{i \in I}$ is an open cover with no proper subcover. Hence, U fails to be compact if infinitely many of the spaces X_i are non-empty. An analogue of Proposition 8.7 for infinite index sets I is, however, obtained by replacing the disjoint union $\bigcup_{i \in I} \mathrm{Ult}\, A_i$ by a suitable compactification.

Here a *compactification* of a topological space U is a pair (γ, X) such that X is a compact Hausdorff space and $\gamma: U \to X$ is a homeomorphism from U onto a dense subspace of X. By abuse of notation, one often writes $(\gamma, \gamma U)$ or simply γU for the pair (γ, X) and calls the space γU a compactification of U if the embedding γ is understood. It is a standard fact of topology that a space U has a compactification iff it is a Tychonoff space.

In the following results, recall from Section 6 the definition of the weak product of a family of Boolean algebras: it is the subalgebra

$$\prod_{i \in I}^{\mathrm{w}} A_i = \left\{ a \in \prod_{i \in I} A_i \colon \{i \in I \colon a_i \neq 0\} \text{ finite or } \{i \in I \colon a_i \neq 1\} \text{finite} \right\}$$

of the full product $\prod_{i \in I} A_i$. Let us call, in this subsection, every subalgebra of $\prod_{i \in I} A_i$ including $\prod^{\mathrm{w}}_{i \in I} A_i$ an *intermediate algebra*.

8.8. PROPOSITION. *Let $(A_i)_{i\in I}$ be any family of Boolean algebras. The Stone spaces of intermediate algebras are exactly the zero-dimensional compactifications of $\bigcup_{i\in I}$ Ult A_i.*

PROOF. For $i \in I$, let $X_i = \text{Ult } A_i$ and let e^i be the element of $\Pi^{\text{w}}_{i\in I} A_i$ satisfying $e^i(i) = 1$ and $e^i(j) = 0$ for $j \neq i$.

First assume that B is an intermediate algebra and let

$$s\colon B \to \text{Clop Ult } B$$

be its Stone isomorphism. For $i \in I$, the element e^i is in B, the sets $s(e^i)$ are pairwise disjoint and their union is dense in Ult B since $\Sigma^B_{i\in I} e^i = 1$. Each relative algebra $B \restriction e^i$ is isomorphic to A_i; by Example 7.26 there is a homeomorphism f_i from $X_i = \text{Ult } A_i$ onto $s(e^i)$. So $\gamma = \bigcup_{i\in I} f_i$ is a homeomorphism from $\bigcup_{i\in I} X_i$ onto a dense subset of Ult B, and Ult B is a zero-dimensional compactification of $\bigcup_{i\in I} X_i$.

Conversely, let γU be a zero-dimensional compactification of $U = \bigcup_{i\in I} X_i$; we may assume that $\bigcup_{i\in I} X_i$ is a dense subspace of γU, embedded by the identity map. For every $i \in I$, let

$$s_i\colon A_i \to \text{Clop } X_i$$

be the Stone isomorphism of A_i. Then each X_i is an open and compact, hence clopen, subspace of γU. We prove that the homomorphism

$$f\colon \text{Clop } \gamma U \to \prod_{i\in I} A_i\,,$$

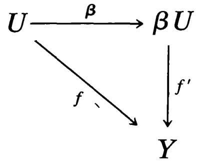
defined by

$$f(a) = (s_i^{-1}(a \cap X_i))_{i\in I}\,,$$

is an isomorphism from Clop γU onto an intermediate algebra. f is one-to-one by Lemma 5.3, since every non-empty clopen subset of γU intersects some X_i, by denseness of $\bigcup_{i\in I} X_i$ in γU. Thus, the subalgebra $B = f[\text{Clop } \gamma U]$ of $\Pi_{i\in I} A_i$ is isomorphic to Clop γU. It includes $\Pi^{\text{w}}_{i\in I} A_i$, for if $x = (a_i)_{i\in I} \in \Pi^{\text{w}}_{i\in I} A_i$ is such that $a_i = 0$ for almost every i, then $x = f(a)$, where $a = \bigcup_{i\in I} s_i(a_i)$. Consequently, B is an intermediate algebra isomorphic to Clop γU, and γU is homeomorphic to the Stone space of B. □

It is not difficult to show that the above assignments between intermediate algebras and zero-dimensional compactifications of $\bigcup_{i\in I}$ Ult A_i are essentially inverses of each other, up to homeomorphism of compactifications over $\bigcup_{i\in I}$ Ult A_i. Let us characterize the compactifications corresponding to the greatest and the least intermediate algebra, i.e. of the full and the weak product.

A compactification (γ, X) of a Tychonoff space U is called a *Stone–Čech compactification* if for each continuous map f from U into a compact Hausdorff space Y, there is a continuous map $f'\colon X \to Y$ such that $f' \circ \gamma = f$. Every Tychonoff space U has a Stone–Čech compactification; it is determined uniquely up to homeomorphism over U and denoted by $(\beta, \beta U)$.



It should be pointed out that the Stone–Čech compactification of a zero-dimensional Tychonoff space is not necessarily zero-dimensional; see Section 6.2 in ENGELKING [1977] for a thorough discussion of disconnectedness properties and their preservation in Stone–Čech compactifications.

8.9. THEOREM. *For any family* $(A_i)_{i\in I}$ *of Boolean algebras*,

$$\mathrm{Ult}\Big(\prod_{i\in I} A_i\Big) \cong \beta\Big(\bigcup_{i\in I} \mathrm{Ult}\, A_i\Big).$$

PROOF. Define the element e^i of $\Pi_{i\in I} A_i$ as in the proof of Proposition 8.8, let

$$X = \mathrm{Ult}\Big(\prod_{i\in I} A_i\Big),$$

$s: \Pi_{i\in I} A_i \to \mathrm{Clop}\, X$ the Stone isomorphism and $X_i = s(e^i)$. The proof of 8.8 shows that

$$U = \bigcup_{i\in I} X_i$$

is a dense open subset of X homeomorphic to $\bigcup_{i\in I} \mathrm{Ult}\, A_i$, so it suffices to prove that (id_U, X) is a Stone–Čech compactification of U.

Claim 1. Assume a and b are disjoint subsets of U which are closed in U. Then a and b are separated by a clopen subset of X.

To show this, pick for each $i \in I$ by Booleanness of X_i and Lemma 7.6(c) a clopen subset c_i of X_i which separates $a \cap X_i$ and $b \cap X_i$. For example, by $s(e^i) = X_i$ let $c_i = s(b_i)$, where $b_i \in \Pi_{i\in I} A_i$ and $b_i \leq e^i$. Then a and b are separated by $c = s(x)$, where x is the element $(\mathrm{pr}_i(b_i))_{i\in I}$ of $\Pi_{i\in I} A_i$.

To check the universal property of the Stone–Čech compactification for X, suppose f is a continuous map from U into a compact Hausdorff space Y.

Claim 2. For every point p of X, the subset

$$M_p = \bigcap \{\mathrm{cl}\, f[c \cap U]: c \text{ a clopen neighbourhood of } p \text{ in } X\}$$

of Y contains exactly one point (cl denotes closure in the space Y).

For, by denseness of U in X, $c \cap U$ is non-empty for every neighbourhood c of p; so $\mathrm{cl}\, f[c \cap U]$ is a non-empty closed subset of Y. Thus, M_p is non-empty since Y is compact. Assume for contradiction that y and y' are distinct points in M_p.

Using regularity of Y, choose open subsets u and u' of Y such that

$$y \in u, \quad y' \in u', \quad \mathrm{cl}\, u \cap \mathrm{cl}\, u' = \emptyset .$$

By Claim 1, there is a clopen subset c of X such that $f^{-1}[\mathrm{cl}\, u] \subseteq c$ and $f^{-1}[\mathrm{cl}\, u'] \subseteq X \setminus c$. We may assume that $p \in c$. Then $y' \in M_p \subseteq \mathrm{cl}(f[c \cap U])$ and $y' \in u'$, so $u' \cap f[c \cap U] \neq \emptyset$ since u' is open. It follows that $f^{-1}[u'] \cap (c \cap U) \neq \emptyset$, contradicting $f^{-1}[u'] \subseteq X \setminus c$.

In view of Claim 2, we define f': $X \to Y$ by

$$f'(p) = \text{the unique point of } M_p .$$

f' extends f since $f(p) \in M_p$ for each $p \in U$. Also, f' is continuous, for assume $p \in X$ and v is an open neighbourhood of $f'(p)$ in Y. Thus, $M_p \subseteq v$ and by compactness of Y, $\mathrm{cl}\, f[c \cap U] \subseteq v$ for some clopen neighbourhood c of p. But then f' maps c into v. □

For every locally compact but non-compact Hausdorff space U, the *one-point compactification* αU of U is defined, in topology, as follows. The underlying set of αU is $U \cup \{p^*\}$, where p^* is a point not contained in U and a subset u of αU is defined to be open iff either $p^* \not\in u$ and u is open in U or $p^* \in U$ and $U \setminus u$ is a compact subspace of U. It is easily verified that every compactification γU of U for which $\gamma U \setminus U$ consists of exactly one point is homeomorphic to αU over U, and that the one-point compactification of a locally compact zero-dimensional Hausdorff space is Boolean. In particular, if X_i is a Boolean space for $i \in I$, then $\bigcup_{i \in I} X_i$ is locally compact and $\alpha(\bigcup_{i \in I} X_i)$ is Boolean.

8.10. PROPOSITION. *If I is infinite and A_i is a non-trivial Boolean algebra for $i \in I$, then*

$$\mathrm{Ult}\Big(\prod_{i \in I}^{\mathrm{w}} A_i\Big) \cong \alpha\Big(\bigcup_{i \in I} \mathrm{Ult}\, A_i\Big) .$$

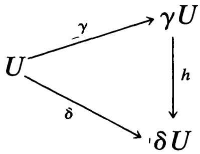
PROOF. The proof of 8.8 and its notation show that $\bigcup_{i \in I} \mathrm{Ult}\, A_i$ is homeomorphic to the dense open subspace $U = \bigcup_{i \in I} s(e^i)$ of $X = \mathrm{Ult}(\prod_{i \in I}^{\mathrm{w}} A_i)$. Now

$$p^* = \Big\{ b \in \prod_{i \in I}^{\mathrm{w}} A_i \colon b_i = 1 \text{ for all but finitely many } i \in I \Big\}$$

is an ultrafilter of $\prod_{i \in I}^{\mathrm{w}} A_i$ not contained in U. Moreover, $X = U \cup \{p^*\}$, for let $p \in X \setminus \{p^*\}$. Then there is some $b \in p$ such that $b \leq \Sigma_{i \in J} e^i$ for some finite subset J of I. So $e^i \in p$ for some $i \in J$ and $p \in s(e^i) \subseteq U$.

Thus, X is a compactification of U such that $X \setminus U$ consists of exactly one point, and $X \cong \alpha U$. □

The following sketch gives some additional information on the correspondence between intermediate algebras and compactifications of $\bigcup_{i \in I} \mathrm{Ult}\, A_i$; detailed proofs and more general results can be found in the book by DWINGER [1971]. For an arbitrary Tychonoff space U, a quasi-order (i.e. a reflexive and transitive relation) on the compactifications of U is defined, in text books on topology, as follows: we say that $(\delta, \delta U) \leq (\gamma, \gamma U)$ if there is a continuous map $h: \gamma U \to \delta U$ satisfying $h \circ \gamma = \delta$.



Call two compactifications $(\gamma, \gamma U)$ and $(\delta, \delta U)$ equivalent if there is a homeomorphism $h: \gamma U \to \delta U$ such that $h \circ \gamma = \delta$, i.e. if $(\gamma, \gamma U) \leq (\delta, \delta U)$ and $(\delta, \delta U) \leq (\gamma, \gamma U)$. The above quasi-order on the compactifications of U induces a partial order on the equivalence classes; it is called the partial order of compactifications of U. In this partial order, βU is the greatest compactification of U; if U is locally compact but non-compact, then αU is the smallest one. For $(A_i)_{i \in I}$ a family of Boolean algebras and $U = \bigcup_{i \in I} \operatorname{Ult} A_i$, it turns out that the partial order of zero-dimensional compactifications of U is isomorphic to the partial order of intermediate algebras under inclusion. This reflects, of course, the fact that subalgebras of a Boolean algebra correspond to continuous images of its Stone space.

Let us finally consider the simple but interesting special case that, for every i in an infinite set I, A_i is the two-element Boolean algebra. Then $\operatorname{Ult} A_i$ is a one-point space and $\bigcup_{i \in I} \operatorname{Ult} A_i$ is a discrete space of cardinality $|I|$; we identify it with the discrete space with underlying set I. The dual space of $\Pi^{\mathrm{w}}_{i \in I} A_i$ is then αI, the dual space of $\Pi_{i \in I} A_i = {}^{I}2 \cong P(I)$ is βI, and the intermediate algebras correspond to the zero-dimensional compactifications of the discrete space I. Identifying I with its image under $\beta: I \to \beta I$, i.e. with the set of isolated points of βI, we find by Theorem 7.25 that

$$\operatorname{Ult}(P(I)/\mathit{fin}) \cong \beta I \setminus I\,,$$

where *fin* is the ideal of finite subsets of I.

8.4. *The sheaf representation of a Boolean algebra over a subalgebra*

The notion of a sheaf of Boolean algebras is extremely useful in visualizing pairs (A, B) of Boolean algebras where A is a subalgebra of B. Sheaves of Boolean algebras are a topological generalization of the following discrete situation. Let X be an arbitrary set and for $p \in X$, let B_p be a Boolean algebra. Then the product $B = \Pi_{p \in X} B_p$ is a Boolean algebra with $A = {}^{X}2$ as a subalgebra. The elements of B are the choice functions, i.e. those functions f from X into $S = \bigcup_{p \in X} B_p$ satisfying $f(p) \in B_p$ for all $p \in X$. In a sheaf of Boolean algebras, both X and S carry a topology and we naturally restrict our attention to the continuous choice functions, obtaining a subalgebra of the full product.

We first define the somewhat more general notion of a sheaf of sets.

8.11. DEFINITION. A sequence

$$\mathcal{S} = (S, \pi, X, (B_p)_{p\in X})$$

is a *sheaf* (of sets) if each of the following holds:

(a) S and X are topological spaces,

(b) $(B_p)_{p\in X}$ is a family of pairwise disjoint non-empty sets and $S = \bigcup_{p\in X} B_p$,

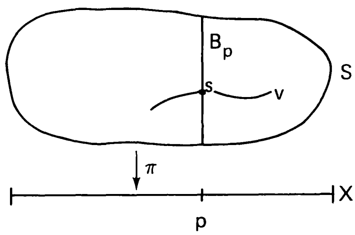
(c) $\pi: S \to X$ is the map satisfying

$$\pi(s) = p \quad \text{iff } s \in B_p ;$$

it is continuous, open and a local homeomorphism – i.e. each $s \in S$ has a neighbourhood v such that $\pi \restriction v$ is a homeomorphism from v onto $\pi[v]$; v is then a *canonical neighbourhood* of s

(d) if $u \subseteq X$ is open and $f, g \in \Pi_{p\in u} B_p$ are continuous (being functions from u into S), then the set $\{p \in u: f(p) = g(p)\}$ is open.

S is the *sheaf space*, X the *base space* and π the *projection map* of $\mathcal{S}$. The sets B_p are the *stalks* of $\mathcal{S}$. $\mathcal{S}$ is called a *Hausdorff sheaf* if its sheaf space S is Hausdorff.



8.12. DEFINITION. Let $\mathcal{S} = (S, \pi, X, (B_p)_{p\in X})$ be a sheaf. For $u \subseteq X$ and $f, g \in \Pi_{p\in u} B_p$, define

$$\|f = g\| = \{p \in u: f(p) = g(p)\} .$$

For every open subset u of X,

$$\Gamma_u(\mathcal{S}) = \left\{ f \in \prod_{p\in u} B_p : f \text{ continuous} \right\}$$

is the set of (*local*) *sections* of $\mathcal{S}$ over u.

$$\Gamma(\mathcal{S}) = \Gamma_X(\mathcal{S})$$

is the set of global sections of $\mathcal{S}$.

So if both S and X are discrete spaces and S is the disjoint union of the sets B_p, $p \in X$, then $\Gamma(\mathcal{S})$ is simply the cartesian product $\Pi_{p\in X} B_p$.

We cannot conclude from axioms (a) through (d) in 8.11 that there are any global sections, but this will be of no concern to us since in the special case 8.16

below, the set $\Gamma(\mathcal{S})$ will trivially be non-empty. Also, for global sections f and g of an arbitrary sheaf, the open set $\|f = g\|$ is not necessarily clopen, as follows for example from Proposition 8.20. Most of the following lemma on the basic properties of sheaves will not be used in the sequel, but the reader might appreciate some additional information.

8.13. LEMMA. *Let $\mathcal{S} = (S, \pi, X, (B_p)_{p \in X})$ be a sheaf.*

(a) *Each stalk B_p is a discrete space, in the topology induced by S.*

(b) *Assume $s \in S$ and v is an open canonical neighbourhood of s in S. Let $h = \pi \restriction v$, a homeomorphism from v onto $u = \pi[v]$. Then h^{-1} is a section of $\mathcal{S}$ over u.*

(c) *Each local section $f \in \Gamma_u(\mathcal{S})$ ($u \subseteq X$ open) is an open mapping from u into S.*

(d) *Assume the base space X is Boolean. Then for every $p^* \in X$ and $s^* \in B_p$, there is a global section f of $\mathcal{S}$ such that $f(p^*) = s^*$.*

(e) *Let X be Boolean. Then $\mathcal{S}$ is Hausdorff iff, for all global sections f and g of $\mathcal{S}$, the set $\|f = g\|$ is clopen.*

PROOF. (a) For $s \in B_p$, let v be a canonical neighbourhood of s. Then $v \cap B_p = \{s\}$, so s is isolated in B_p.

(b) An immediate consequence of the sheaf axioms: the set $u = \pi[v]$ is open since π is an open map, and h is a homeomorphism from v onto u. In particular, h^{-1} is a continuous map from u into S.

(c) Assume $u \subseteq X$ is open and f is a section over u. It suffices to prove that for every $p \in u$ there is an open neighbourhood u' of p such that $u' \subseteq u$ and $f[u']$ is open; so let $p \in u$ be given. Put $s = f(p)$ and choose, by axiom (c), an open canonical neighbourhood v of s in S such that $h = \pi \restriction v$ is a homeomorphism from v onto $w = \pi[v]$. Taking v small enough, we may assume that $w \subseteq u$. Both $f \restriction w$ and, by part (b), h^{-1} are sections over w, so by axiom (d),

$$u' = \|f \restriction w = h^{-1}\|$$

is an open neighbourhood of p. Moreover,

$$f[u'] = h^{-1}[u'] = v \cap \pi^{-1}[u']$$

is open in S.

(d) For every $p \in X$, fix a point s_p of B_p and assume $s_{p^*} = s^*$. By axiom (c) and part (b) of our lemma, there is for every $p \in X$ an open neighbourhood u_p of p and a section f_p over u_p such that $f_p(p) = s_p$. A global section f is obtained by patching together parts of the local sections f_p: let, by compactness of X, $u_{p(1)}, \ldots, u_{p(n)}$ be a finite subcover of $\{u_p \colon p \in X\}$; we may assume that $p(1) = p^*$. There are pairwise disjoint clopen subsets $c_{p(i)}$ of $u_{p(i)}$ such that $p^* \in c_{p(1)}$ and X is covered by $c_{p(1)}, \ldots, c_{p(n)}$ – see Exercise 3 in Section 7. So

$$f = f_{p(1)} \restriction c_{p(1)} \cup \cdots \cup f_{p(n)} \restriction c_{p(n)}$$

proves our claim.

(e) If $\mathscr{S}$ (i.e. S) is Hausdorff, then for any continuous functions $f, g: X \to S$, the set $\{x \in X: f(x) = g(x)\}$ is closed. So it is clopen by axiom (d). Conversely, assume $\|f = g\|$ is clopen for arbitrary global sections f and g of $\mathscr{S}$ and that s, s' are distinct points of S. If $p = \pi(s)$ and $p' = \pi(s')$ are distinct, choose disjoint neighbourhoods u of p and u' of p'; then $\pi^{-1}[u]$ and $\pi^{-1}[u']$ are disjoint neighbourhoods of s and s'. So assume $\pi(s) = \pi(s') = p$. By part (d) there are global sections f and f' such that $f(p) = s$ and $f'(p) = s'$. Then the set $c = X \setminus \|f = g\|$ is clopen in X, and $f[c]$, $f[c']$ are disjoint open neighbourhoods of s, s' in S by part (c). □

8.14. DEFINITION. A *sheaf of Boolean algebras* is a sequence $\mathscr{S} = (S, \pi, X, (B_p)_{p \in X})$ such that each of the following holds.

(a) S and X are topological spaces.

(b′) Each B_p is (the underlying set of) a Boolean algebra, the sets B_p are pairwise disjoint and $S = \bigcup_{p \in X} B_p$.

(c) The map $\pi: S \to X$ satisfying

$$\pi(s) = p \quad \text{iff } s \in B_p$$

is continuous, open and a local homeomorphism.

(d′) Let $u \subseteq X$ be open, $f_1, \ldots, f_n$ sections over u and $t(x_1 \ldots x_n)$, $t'(x_1 \ldots x_n)$ Boolean terms. Then the set

$$\{p \in u: t(f_1(p) \ldots f_n(p)) = t'(f_1(p) \ldots f_n(p))\}$$

is open.

8.15. PROPOSITION. *For every sheaf $\mathscr{S}$ of Boolean algebras, $\Gamma(\mathscr{S})$ is a subalgebra of $\prod_{p \in X} B_p$.*

PROOF. We show that for arbitrary global sections f and g of $\mathscr{S}$, the function $h: X \to S$ defined by $h(p) = f(p) + g(p)$ is continuous, hence an element of $\Gamma(\mathscr{S})$. It follows similarly that $\Gamma(\mathscr{S})$ is closed under all other Boolean operations and that the functions assigning to each $p \in X$ the unit (respectively the zero) element of B_p are global sections.

Let p be a point of X; it is enough to find a neighbourhood u of p such that h is continuous on u. Put $s = h(p)$ and fix a neighbourhood v of s in S such that $\pi \restriction v$ is a homeomorphism from v onto an open neighbourhood w of p. By Lemma 8.13(b), $h' = (\pi \restriction v)^{-1}$ is a section over w. Now

$$f(p) + g(p) = s = h'(p),$$

so by axiom (d′), there is a neighbourhood u of p such that $u \subseteq w$ and $f(q) + g(q) = h'(q)$ for all $q \in u$. Thus, h coincides with h' on u and is continuous on u. □

There is, of course, nothing particular about Boolean algebras in the preceding definition and proposition – if L is any language for first order predicate logic, then a sheaf of L-structures is defined by replacing in 8.14 the conditions (b′) and (d′) by

(b″) Each B_p is (the underlying set of) an L-structure, the sets B_p are pairwise disjoint and $S = \bigcup_{p \in X} B_p$.

(d″) Let $u \subseteq X$ be open, $f_1, \ldots, f_n$ sections over u and $\phi(x_1 \ldots x_n)$ an atomic L-formula. Then the set

$$\{p \in u: B_p \models \phi[f_1(p) \ldots f_n(p)]\}$$

is open.

We are ready to present our standard example for a sheaf of Boolean algebras.

8.16. CONSTRUCTION AND NOTATION (the sheaf associated with a pair of Boolean algebras). Assume B is a Boolean algebra and A a subalgebra of B. Then

$$X = \text{Ult}\, A$$

is a Boolean space. For $p \in X$, let $\bar{p}$ be the filter generated by p in B; it is, by Lemma 2.12, the proper filter

$$\bar{p} = \{b \in B: a \leq b \text{ for some } a \in p\}$$

of B. Let

$$\pi_p: B \to B_p = B/\bar{p}$$

be the canonical epimorphism – so $\pi_p(b) = \pi_p(b')$ iff $b \cdot a = b' \cdot a$ for some $a \in p$ (see the proof of 5.22).

The sets B_p, $p \in X$, are pairwise disjoint, for assume $s \in B_p$. Then since s is one of the equivalence classes modulo $\bar{p}$, $\bar{p}$ can be recovered from s by

$$\bar{p} = \{-(b \triangle b'): b, b' \in s\},$$

where $\triangle$ denotes symmetric difference, and p can be recovered from $\bar{p}$ by $p = \bar{p} \cap A$. Hence, we define

$$S = \bigcup_{p \in X} B_p,$$

$$\pi: S \to X, \qquad \pi(s) = p \quad \text{iff } s \in B_p.$$

For every $b \in B$, define a choice function by

$$f_b: X \to S, \qquad f_b(p) = \pi_p(b).$$

The following assertion will be useful in Theorem 8.17.

Claim. $\|f_b = f_{b'}\|$ is an open subset of X, for all b and b' in B. For let $p \in \|f_b = f_{b'}\|$. Then $\pi_p(b) = \pi_p(b')$, and there exists $a \in p$ such that $a \cdot b = a \cdot b'$.

Thus, for $s_A: A \to \text{Clop}\, X$ the Stone isomorphism, $s_A(a)$ is a clopen neighbourhood of p included in $\|f_b = f_{b'}\|$.

The sets $f_b[u]$, where $b \in B$ and u is open in X, constitute the base of a topology of S. To check this, assume $s \in f_b[u] \cap f_{b'}[u']$, where $b, b' \in B$ and u, u' are open in X. Put $p = \pi(s)$. Then $p \in u \cap u'$ and $s = f_b(p) = f_{b'}(p)$, i.e. $p \in \|f_b = f_{b'}\|$. By the Claim, there is an open neighbourhood u'' of p included in $u \cap u' \cap \|f_b = f_{b'}\|$. So

$$s \in f_b[u''] \subseteq f_b[u] \cap f_{b'}[u'] .$$

This finishes the construction of the sequence $\mathscr{S} = (S, \pi, X, (B_p)_{p \in X})$. In view of Theorem 8.17, it is called the sheaf associated with the pair (A, B).

8.17. THEOREM. *Let A be a subalgebra of a Boolean algebra B. Then the sequence $\mathscr{S} = (S, \pi, X, (B_p)_{p \in X}))$ constructed in* 8.16 *is a sheaf of Boolean algebras and the map*

$$e: B \to \prod_{p \in X} B_p ,$$

defined by

$$e(b) = f_b ,$$

is an isomorphism from B onto $\Gamma(\mathscr{S})$.

PROOF. The axioms (a) and (b′) of 8.14 are clearly satisfied. Also (c) is easily shown: π is an open map since the image of a basic open subset $f_b[u]$ of S is the open subset u of X. π is continuous, for let $s \in S$, $p = \pi(s)$ and u a neighbourhood of p in X. Since $s \in B_p = \pi_p[B]$, pick $b \in B$ such that $s = \pi_p(b)$. Then $s = f_b(p)$ and $f_b[u]$ is a neighbourhood of s mapped onto u by π. Finally, for every $s \in S$, say $s = \pi_p(b)$, where $p \in X$ and $b \in B$, $v = f_b[X]$ is a basic neighbourhood of s in S. π being continuous and open, $\pi \restriction v$ is a homeomorphism from v onto $\pi[v] = X$.

Before starting out on axiom (d′), let us prove three additional facts.

Claim 1. For $b \in B$, the map $f_b: X \to S$ is continuous. For let v be open in S. Then $f_b^{-1}[v] = f_b^{-1}[v']$, where $v' = v \cap f_b[X]$ is open in S. But $f_b^{-1}[v'] = \pi[v']$ is open in X since π is open.

Claim 2. Let $f \in \Pi_{p \in X} B_p$ be continuous and $p \in X$. Then there are $b \in B$ and a neighbourhood u of p such that $p \in u \subseteq \|f = f_b\|$. For $s = f(p)$ is a point of B_p, so $s = \pi_p(b) = f_b(p)$ for some $b \in B$; also $f_b[X]$ is a neighbourhood of s. By continuity of f, choose a neighbourhood u of p which is mapped into $f_b[X]$ under f. Obviously $u \subseteq \|f = f_b\|$.

Claim 3. The map e is a monomorphism from B into $\Pi_{p \in X} B_p$. e is clearly a homomorphism since each π_p is a homomorphism. It is one-to-one since every $b \neq 0$ is contained in an ultrafilter x of B and, for $p = x \cap A$, $f_b(p) = \pi_p(b) \neq 0$.

Now assume $u, f_1, \ldots, f_n$ and t, t' are given as in axiom (d′) and $p \in u$ is such that

$$t(f_1(p)\ldots f_n(p)) = t'(f_1(p)\ldots f_n(p)).$$

By Claim 2, fix a neighbourhood w of p and $b(1),\ldots,b(n)\in B$ such that $w\subseteq u$ and

$$p\in w\subseteq \|f_i = f_{b(i)}\|$$

for $i\in\{1,\ldots,n\}$. In B, consider the elements $b = t(b(1)\ldots b(n))$, $b' = t'(b(1)\ldots b(n))$. Then since $e\colon B\to\Pi_{p\in X} B_p$ is a homomorphism,

$$f_b = t(f_{b(1)}\ldots f_{b(n)}), \qquad f_{b'} = t'(f_{b(1)}\ldots f_{b(n)}).$$

Also, $f_b(p) = f_{b'}(p)$, so by the Claim in 8.16, fix a neighbourhood v of p satisfying $v\subseteq w\cap \|f_b = f_{b'}\|$. Thus,

$$p\in v\subseteq \{q\in u\colon t(f_1(q)\ldots f_n(q)) = t'(f_1(q)\ldots f_n(q))\}.$$

This concludes the proof that $\mathcal{S}$ is a sheaf of Boolean algebras.

We are left with showing that the monomorphism e from B into $\Gamma(\mathcal{S})$ is onto; so let $f\in\Gamma(\mathcal{S})$. For each $p\in X$, choose by Claim 2 some $b_p\in B$ and an open neighbourhood u_p of p such that

$$u_p\subseteq \|f = f_{bp}\|.$$

We apply once more the patching technique used in the proof of 8.13(d): by compactness of X, assume

$$X = u_{p(1)}\cup\cdots\cup u_{p(n)}.$$

Choose a finite clopen partition

$$X = s_A(a_1)\cup\cdots\cup s_A(a_n)$$

of X such that $s_A(a_i)\subseteq u_{p(i)}$; here $s_A\colon A\to \operatorname{Clop} X$ is the Stone map and $a_i\in A$. In B, define

$$b = a_1\cdot b_{p(1)} + \cdots + a_n\cdot b_{p(n)}.$$

Then for arbitrary $q\in X$, say $a_i\in q$, $f_b(q) = f_{b_{p(i)}}(q) = f(q)$ since $q\in s_A(a_i)\subseteq u_{p(i)}\subseteq \|f_{b_{p(i)}} = f\|$. Thus, $f = f_b$. $\square$

Suppose that in the situation of the preceding theorem, B is identified with $\Gamma(\mathcal{S})$ via e. Then an element a of A is identified with the characteristic function $\chi_c\colon X\to 2$, where c is the clopen set $s_A(a)$ of X – here, of course, the two-element subalgebra of B_p is identified with $2=\{0,1\}$. Thus, a global section of $\mathcal{S}$ corresponds to an element of A iff it attains only the values 0 and 1.

As an easy example for Theorem 8.17, let us consider the case that B is a simple extension of its subalgebra A.

8.18. EXAMPLE (simple extensions). Let B be generated over A by a single element t. There are two canonical ideals in A associated with t:

$$I_t = \{a \in A: a \leq t\}, \qquad I_{-t} = \{a \in A: a \leq -t\}.$$

The simple extension $A(t)$ is uniquely determined over A by the ideals I_t and I_{-t}; cf. Exercise 11 in Section 5. Now in the sheaf representation of $B = A(t)$ over A, every stalk B_p has at least two and at most four elements since B is generated by A and t and the epimorphism π_p maps A onto 2. For the global section f_t assigned to t in 8.16, axiom (d') of 8.14 implies that the set

$$u_t = \{p \in X: f_t(p) = 1\}$$

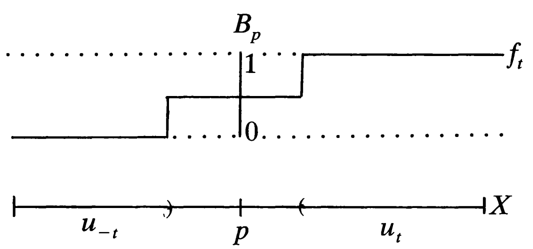
is open. In fact, u_t is the open set corresponding to the ideal I_t under Stone duality, since for $p \in X$,

$$f_t(p) = 1 \quad \text{iff } a \leq t \text{ for some } a \in p$$
$$\text{iff } p \cap I_t \neq \emptyset.$$

Similarly, the open set

$$u_{-t} = \{p \in X: f_t(p) = 0\}$$

corresponds to the ideal I_{-t}. Since $|B_p| = 4$ iff $f_t(p) \not\in 2$, we may represent the extension $A(t)$ over A by the following diagram:



The situation of Example 8.18 simplifies even more if we restrict our attention to Hausdorff sheafs.

8.19. DEFINITION. Let A be subalgebra of B. A is said to be *relatively complete* in B if for each $b \in B$ there is a greatest element $a \in A$ such that $a \leq b$. We write

$$\mathrm{pr}_A(b) = \mathrm{pr}(b) = \max\{a \in A: a \leq b\}.$$

8.20. PROPOSITION. *Let B be a Boolean algebra, A a subalgebra and $\mathcal{S} = (S, \pi, X, (B_p)_{p\in X})$ the associated sheaf. The following are equivalent:*

(a) *A is relatively complete in B,*
(b) *the continuous map* ϕ: Ult $B \to$ Ult A *dual to the inclusion homomorphism* $id_A: A \to B$ *is open,*
(c) *the subset* $\|f = g\|$ *of* Ult A *is clopen, for all global sections f and g,*
(d) $\mathscr{S}$ *is Hausdorff.*

Proof. Equivalence of (c) and (d) was proved in 8.13(e).

To prove equivalence of (a) and (c), we identify B with $\Gamma(\mathscr{S})$ and A with the set of continuous maps from X into 2. Then A is relatively complete in B iff for each $f \in \Gamma(\mathscr{S})$, the open subset $\|f = 1\|$ of X has a greatest clopen subset, i.e. iff $\|f = 1\|$ is clopen for each $f \in \Gamma(\mathscr{S})$. This is equivalent to (c) since for arbitrary $f, g \in \Gamma(\mathscr{S})$, $\|f = g\| = \|h = 1\|$, where $h = -(f \triangle g)$ in $\Gamma(\mathscr{S})$.

For the equivalence of (a) and (b), recall that ϕ: Ult $B \to$ Ult A is the map defined by

$$\phi(y) = y \cap A$$

for $y \in$ Ult B and denote by s_A and s_B the Stone isomorphisms of A (respectively B). Assume that A is relatively complete in B. Let $u \subseteq$ Ult B be open and $y \in u$; we find an open subset v of Ult A such that

$$\phi(y) \in v \subseteq \phi[u] .$$

To this end, pick $b \in B$ such that $y \in s_B(b) \subseteq u$, let $x = \phi(y)$, $a = -\mathrm{pr}(-b)$ and $v = s_A(a)$. So a is the least element of A satisfying $b \leq a$. We claim that $x \in s_A(a)$ (and thus $\phi(y) \in v$). Otherwise, $-a \in x \subseteq y$; also $a \in y$ since $b \in y$ and $b \leq a$, a contradiction. Next, $v = s_A(a) \subseteq \phi[s_B(b)]$. For assume $x' \in s_A(a)$. If there is no $y' \in s_B(b)$ satisfying $x' = y' \cap A$, then $x' \cup \{b\}$ does not have the finite intersection property; so $b \leq -c$ for some $c \in x'$. It follows that $a \leq -c$ and $a \cdot c = 0$, contradicting $a \in x'$ and $c \in x'$.

Conversely, suppose ϕ is open and let $b \in B$. The set $\phi[s_B(-b)]$ is open since ϕ is open and closed since ϕ (being a continuous map between compact Hausdorff spaces) is closed. So there is $a \in A$ such that

$$s_A(a) = X \setminus \phi[s_B(-b)] ;$$

we claim that $a = \mathrm{pr}(b)$. First, $a \leq b$ – otherwise $a \cdot -b$ is contained in an ultrafilter y of B. Then $x = \phi(y)$ is an element of $\phi[s_B(-b)]$ and $a \notin x$, a contradiction. Second, every $c \in A$ such that $c \leq b$ satisfies $c \leq a$: otherwise $c \cdot -a \in x$ for some $x \in$ Ult A. Then $x \notin s_A(a)$, so $x = y \cap A$ for some $y \in$ Ult B containing $-b$. It follows that $c \in x \subseteq y$ and $b \notin y$ which is absurd since $c \leq b$. □

It is particularly easy to visualize the sheaf representation of a simple extension B of A if A is relatively complete in B. Simple extensions of this type are the building blocks for projective Boolean algebras; see the survey chapter by Koppelberg [Ch. 20 in this Handbook] on projective algebras.

8.21. EXAMPLE (simple extensions with A relatively complete in B). Assume A is relatively complete in B and B is generated by t over A. Then for arbitrary $b \in B$, $s_A(\mathrm{pr}(b))$ is the open subset $\|f_b = 1\|$ of X and $s_A(\mathrm{pr}(-b)) = \|f_b = 0\|$. Letting, in A,

$$\mathrm{indp}(b) = -(\mathrm{pr}(b) + \mathrm{pr}(-b)),$$

the "independent part of b", we see that $\{\mathrm{pr}(b), \mathrm{pr}(-b), \mathrm{indp}(b)\}$ is a partition of unity in A and that

$$s_A(\mathrm{indp}(b)) = \{p \in \mathrm{Ult}\ A\colon f_b(p) \not\in 2\};$$

in particular, for the element t generating B over A:

$$s_A(\mathrm{indp}(t)) = \{p \in \mathrm{Ult}\ A\colon |B_p| = 4\}.$$

It should now be intuitively clear that, for arbitrary $b \in B = A(t)$, $\mathrm{indp}(b) \leq \mathrm{indp}(t)$ in A; moreover, $\mathrm{indp}(b) = \mathrm{indp}(t)$ iff b generates t over A, i.e. iff $A(b) = A(t)$. Thus, the simple extension $A(t)$ of A is determined, up to isomorphism over A, by A and the element $\mathrm{indp}(t)$ of A.

Exercises

1. Let $\phi\colon X \to Y$ be a continuous map of Boolean spaces.

(a) ϕ is one-to-one iff, for every Boolean space S and continuous maps $\alpha, \alpha'\colon S \to X$, $\phi \circ \alpha = \phi \circ \alpha'$ implies $\alpha = \alpha'$.

(b) ϕ is onto iff, for every Boolean space T and continuous maps $\beta, \beta'\colon Y \to T$, $\beta \circ \phi = \beta' \circ \phi$ implies $\beta = \beta'$.

Formulate and prove a dual statement for homomorphisms of Boolean algebras.

2. For a compact Hausdorff space X, let $G = \mathrm{Ult}\,\mathrm{RO}(X)$ and $f\colon G \to X$ the unique map satisfying $f(p) \in \bigcap \{\mathrm{cl}\, u\colon u \in p\}$ (cf. Exercise 2 in Section 2).

(a) f is a continuous map from G onto X; moreover f is *irreducible*, i.e. $f[H] \neq X$ for every proper closed subspace H of G.

(*b*) If $f'\colon G' \to X$ is another continuous irreducible map from an extremally disconnected Boolean space G' onto X, then there is a unique homeomorphism $h\colon G \to G'$ satisfying $f' \circ h = f$.

The pair (G, f) is called the *projective resolution* of X; G is the *Gleason space* or the *absolute* of X.

3. Let $(A_i)_{i \in I}$ be any family of Boolean algebras. It follows, for example from 6.2.6, 6.2.4, and 6.2.12 in ENGELKING [1977] that the Stone–Čech compactification of the disjoint union space $\bigcup_{i \in I} \mathrm{Ult}\ A_i$ is zero-dimensional. Use this fact, the universal property 6.3 of products and a duality argument to give another proof that $\mathrm{Ult}(\Pi_{i \in I} A_i) \cong \beta(\bigcup_{i \in I} \mathrm{Ult}\ A_i\}$.

4. Let A be a relatively complete subalgebra of B.

(a) $\mathrm{id}_A: A \to B$ preserves all sums and products existing in A.

(b) If B is complete, then so is A.

5. Let A be a Boolean algebra and $a \in A$. Prove (e.g. by using Exercise 12 in Section 5) that there is a simple extension $B = A(t)$ of A such that A is relatively complete in B and indp t, defined in 8.21, equals a. B is uniquely determined, over A, by A and a.

6. Let $A(t)$ and $A(t')$ be simple extensions of A such that A is relatively complete in both $A(t)$ and $A(t')$. Then $\mathrm{indp}(t) \leq \mathrm{indp}(t')$, in A, iff there is an embedding $e: A(t) \to A(t')$ such that $e \restriction A = \mathrm{id}_A$.

CHAPTER 4

Free Constructions

Sabine KOPPELBERG
Freie Universität Berlin

Contents

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Introduction

This chapter presents a construction of Boolean algebras generated in a particular way by some subset (respectively by the union of some prescribed family of subalgebras). The intuitive idea here is that the process of generation be as general as possible, i.e. the generators should satisfy no algebraic relations except those enforced by the laws of Boolean algebras (respectively by the structure of the given family of subalgebras). This idea is made precise by defining freeness by a somewhat technical condition on extendibility of homomorphisms which implies immediately the uniqueness of free algebras (respectively of free products).

It remains then to show existence of free algebras (respectively of free products). This follows from general principles of universal algebra: in each variety V, the free V-algebra over a set U is obtained by taking the set $\mathrm{Tm}(U)$ of all terms built up from the variables in U and the operations of V, and by then dividing $\mathrm{Tm}(U)$ by the relation of being equivalent in all V-algebras; it is this construction which lies at the background of the proof of Proposition 9.9. Stone duality, however, provides a much more natural construction of free algebras and free products: the free Boolean algebra over a set U is the clopen algebra of the Cantor space ${}^U2$, and the free product of a family $(A_i)_{i\in I}$ of Boolean algebras is the clopen algebra of the product space $\Pi_{i\in I}$ Ult A_i. This topological interpretation of free products explains their importance for the general structure theory of Boolean algebras.

Particular emphasis will be given to a special question: Given a Boolean algebra A, for which cardinals κ does A have a free subalgebra of size κ? The dual topological question (Given a Boolean space X, for which cardinals κ does X have the Cantor space ${}^\kappa 2$ as a continuous image?) has been intensively investigated in topology. Section 10 is devoted to several advanced results on the problem in its algebraic form. Let us mention a historically much earlier one, presented in Section 9 and due to Fichtenholz, Kantorovich and Hausdorff: the power set algebra of each infinite set X has a free subalgebra of size $2^{|X|}$. It will be generalized in Section 13 by the Balcar–Franěk theorem: each infinite complete algebra B has a free subalgebra of cardinality $|B|$.

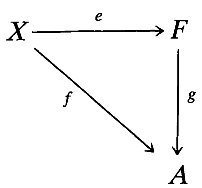
9. Free Boolean algebras

We would like to call a Boolean algebra F free over a subset X if X generates F and the elements of X do not satisfy any algebraic equations except those derivable from the axioms for Boolean algebras. This idea is formalized by defining F to be free over X if every map from X into an arbitrary Boolean algebra A extends to a homomorphism from F into A. It is quite obvious that free Boolean algebras, should they exist, are uniquely determined by the number of their free generators, and that every Boolean algebra is a homomorphic image of a free one. Also freeness of a Boolean algebra F over a subset X can be characterized by an internal algebraic property of X, independence. It is perhaps less obvious that free algebras exist at all; we prove this by applying the topological duality theory of Section 8 to the effect that the dual algebra of a generalized Cantor space ${}^{\kappa}2$ is free over κ generators. Another existence proof for free algebras which does not use Stone duality is given via algebras of formulas as considered in Example 1.12.

As free Boolean algebras are completely determined by the number of free generators, it is not surprising that their structure is very well understood. For example, every infinite free algebra is atomless, satisfies the countable chain condition and has, in some respect, a large automorphism group. The major combinatorial result on free algebras implies that every algebra of regular uncountable cardinality κ which is embeddable into a free one has a free subalgebra of size κ. This motivates the general question which Boolean algebras have large free subalgebras. We prove here that every infinite power set algebra $P(X)$ has a free subalgebra of cardinality $2^{|X|}$, postponing stronger results to the following section.

9.1. *General facts*

9.1. DEFINITION. Let X be an arbitrary set. A *free Boolean algebra over X* is a pair (e, F) such that F is a Boolean algebra and e is a map from X into F such that for every map f from X into a Boolean algebra A there is a unique homomorphism $g: F \to A$ satisfying $g \circ e = f$.



A Boolean algebra F is *free* if there are X and $e: X \to F$ such that (e, F) is free over X.

Thus, if the pair (e, F) is free over X and $x_1, \ldots, x_n$ are distinct elements of X, then every equation satisfied by $e(x_1), \ldots, e(x_n)$ in F is satisfied by arbitrary elements $a_1, \ldots, a_n$ of any Boolean algebra. For pick $f: X \to A$ such that $f(x_i) = a_i$ and let $g: F \to A$ be as guaranteed in the definition of freeness. If an equation is satisfied by the $e(x_i)$ in F, then so it is by their homomorphic images $g(e(x_i)) = f(x_i) = a_i$ in A.

We start out with the three basic results on uniqueness, characterization and existence of free algebras.

9.2. LEMMA (uniqueness). *Assume (e, F) is free over X, (e', F') is free over X' and $f: X \to X'$ is a bijection. Then there is a unique isomorphism $g: F \to F'$ such that $g \circ e = e' \circ f$.*

$$\begin{array}{ccc} X & \xrightarrow{\;e\;} & F \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X' & \xrightarrow[\;e'\;]{} & F' \end{array}$$

PROOF. Routine, like the proof of the uniqueness assertion in Proposition 6.3 from the universal property of products. □

Sikorski's extension criterion 5.5 characterizes, for U a set of generators of a Boolean algebra F, those maps from U into Boolean algebras which extend to homomorphisms. This gives an internal description of freeness.

9.3. DEFINITION. A subset U of a Boolean algebra A is *independent* if all non-trivial elementary products over U are non-zero, i.e. if for arbitrary disjoint finite subsets $\{u_1, \ldots, u_n\}$ and $\{v_1, \ldots, v_m\}$ of U,

$$u_1 \cdot \cdots \cdot u_n \cdot -v_1 \cdot \cdots \cdot -v_m > 0 .$$

The subalgebra of A generated by U is then said to be *independently generated* or, in view of 9.4, *freely generated* by U.

9.4. PROPOSITION (characterization). *Let e be a map from a set X into a Boolean algebra F. The pair (e, F) is free over X iff e is one-to-one and $e[X]$ independently generates F.*

PROOF. Assume first that (e, X) is free over X. If e is not one-to-one, pick x and y in X such that $x \neq y$ but $e(x) = e(y)$ and let f be a map from X into the two-element Boolean algebra 2 such that $f(x) \neq f(y)$. Clearly, there is no $g\colon F \to 2$ satisfying $g \circ e = f$, a contradiction.

In the rest of the proof, we may therefore assume that e is one-to-one; for simplicity let $X \subseteq F$ and e the identity map on X.

If X generates a proper subalgebra B of F, pick by Lemma 5.32 distinct ultrafilters p and p' of F satisfying $p \cap B = p' \cap B$ and let $g, g'\colon F \to 2$ be their characteristic homomorphisms. Then $g \neq g'$ but $g \restriction X = g' \restriction X$, contradicting the uniqueness assertion in the definition of free algebras.

If X is not independent, pick disjoint finite subsets $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_m\}$ of X such that $x_1 \cdot \cdots \cdot x_n \cdot -y_1 \cdot \cdots \cdot -y_m = 0$ and let $f\colon X \to 2$ be such that f maps each x_i onto 1 and each y_j onto 0. It follows from the trivial part of Sikorski's extension criterion 5.5 that f has no homomorphic extension to B, a contradiction.

Conversely, suppose that X is a set of independent generators of F. Then each map from X into any Boolean algebra has a homomorphic extension to F by the non-trivial part of Sikorski's extension criterion. This extension is unique by Lemma 5.4. □

9.5. THEOREM (existence). *For every set I, there is a free Boolean algebra over I.*

Proof. Given the set I, let F be the clopen algebra of the Cantor space ${}^I2$; for $i \in I$, define $e(i)$ to be the clopen subset

$$e(i) = u_i = \{x \in {}^I2: x(i) = 1\}$$

of ${}^I2$. Using Proposition 9.4, we prove that the pair (e, F) is free over I.

The following argument shows that e is one-to-one and $e[I]$ is independent in F: let $\{i_1, \ldots, i_n\}$ and $\{j_1, \ldots, j_m\}$ be disjoint finite subsets of I. Then any point x of ${}^I2$ such that $x(i) = 1$ for $i \in \{i_1, \ldots, i_n\}$ and $x(j) = 0$ for $j \in \{j_1, \ldots, j_m\}$ shows that

$$x \in e(i_1) \cap \cdots \cap e(i_n) \setminus (e(j_1) \cup \cdots \cup e(j_m)) \neq \emptyset .$$

Moreover, $e[I]$ generates F, for consider the subalgebra B of F generated by $e[I]$. B includes the canonical subbase

$$\{u_i: i \in I\} \cup \{{}^I2 \setminus u_i: i \in I\}$$

of ${}^I2$, hence the canonical base of ${}^I2$. Now $F \subseteq B$, since each clopen subset of ${}^I2$ is, by compactness, a finite union of basic sets. □

The uniqueness and existence assertions 9.2 and 9.5 allow us to speak about *the* free Boolean algebra (e, F) over a set X. By the characterization 9.4 we shall always assume that X is a set of independent generators of F. Since F depends only on the cardinality of X, we introduce the following notation.

9.6. Notation. For any cardinal κ, Fr κ is the free Boolean algebra over κ independent generators.

9.7. Corollary. (a) *The Stone space of* Fr κ *is homeomorphic to the generalized Cantor space* ${}^\kappa2$.

(b) *A finite Boolean algebra A is free iff A has cardinality* 2^{2^k}, *for some* $k \in \omega$.

Proof. (a) The construction of free algebras given in 9.5 shows that the free Boolean algebra over a set I has ${}^I2$ as its dual space.

(b) The Stone space X of A is discrete and finite, say of size n. By part (a), A is free over k independent generators iff $n = 2^k$. □

As an immediate consequence of Corollary 9.7, we obtain two cardinality observations. First, for any $k < \omega$, Fr k is generated by k elements and has size 2^{2^k} – the maximal cardinality of an algebra with k generators, by Corollary 4.5. Second, for an infinite cardinal κ, Fr κ is a Boolean algebra of size κ with Stone space of size 2^κ.

9.8. Corollary. *Every Boolean algebra A is a homomorphic image of a free one. More precisely, if $|A| \leq \kappa$, then A is a homomorphic image of* Fr κ.

PROOF. Assume $|A| \leq \kappa$. Let Fr κ be freely be generated by U, where $|U| = \kappa$ and fix a map f from U onto A. Then the homomorphic extension g of f to Fr κ is an epimorphism from Fr κ onto A. □

By Stone duality, the preceding corollary reproves a fact observed in Section 7: every Boolean space is homeomorphic to a closed subspace of a Cantor space.

One of our first examples of Boolean algebras was, in 1.12, the algebra $B(T)$ of (equivalence classes of) formulas in a propositional or first order language L modulo a fixed theory T. For suitably chosen propositional theories T, the algebras $B(T)$ are perfectly natural examples of free Boolean algebras; an easy generalization of the same construction shows that every Boolean algebra is isomorphic to $B(T)$, for some T. It follows along the same lines that every Boolean algebra is also representable as the Lindenbaum–Tarski algebra of some first order theory, as defined in 1.12. We postpone the proof of this to the exercises since it is basically the same one as for propositional logic, but looks somewhat artificial.

9.9. PROPOSITION. *For every set P, there is a propositional theory S such that $B(S)$ is free over $|P|$ independent generators.*

PROOF. Let L be the language of propositional logic having P as its set of propositional variables and let S be the theory in L consisting of all tautologies. Define

$$e: P \to B(S)$$

by letting

$$e(p) = [p],$$

where $[\alpha]$ denotes, as in 1.12, the equivalence class of a formula α with respect to S. Then $e[P]$ generates $B(S)$ since every formula arises from P by forming disjunctions, conjunctions and negations. We show that e is one-to-one and $e[P]$ is independent in $B(S)$: assume $\{p_1, \ldots, p_n\}$ and $\{q_1, \ldots, q_m\}$ are two finite disjoint subsets of P and let α denote the formula

$$\alpha = p_1 \wedge \cdots \wedge p_n \wedge \neg q_1 \wedge \cdots \wedge \neg q_m .$$

Then $\neg\alpha$ is not derivable from S, hence

$$0 \neq [\alpha] = e(p_1) \cdot \cdots \cdot e(p_n) \cdot -e(q_1) \cdot \cdots \cdot -e(q_m) . \quad \square$$

9.10. PROPOSITION. *For every Boolean algebra A, there is a propositional theory T such that $A \cong B(T)$.*

PROOF. Let P be any set of cardinality at least $|A|$; choose the language L and the theory S for this set P as in the proof of 9.9. Now $B(S)$ is free over a subset of size $|P|$, so let, by Corollary 9.8, f be an epimorphism from $B(S)$ onto A. Define a theory T in L by

$$\alpha \in T \quad \text{iff } f([\alpha]_S) = 1\,,$$

where $[\alpha]_S$ denotes the equivalence class of α with respect to S. Clearly, $S \subseteq T$; moreover, for any formula α in L,

$$(1) \qquad \alpha \in T \quad \text{iff } T \vdash \alpha\,.$$

For the non-trivial part of this, assume that $T \vdash \alpha$. Then there are finitely many elements of T, say $\alpha_1, \ldots, \alpha_n$, such that $\vdash \alpha_1 \wedge \cdots \wedge \alpha_n \rightarrow \alpha$. So

$$[\alpha_1]_S \cdot \cdots \cdot [\alpha_n]_S \leq [\alpha]_S\,,$$

and since $f([\alpha_i]_S) = 1$ for each i, we have $f([\alpha]_S) = 1$ and $\alpha \in T$.

Since $S \subseteq T$, there is a unique epimorphism

$$g\colon B(S) \rightarrow B(T)$$

satisfying

$$g([\alpha]_S) = [\alpha]_T$$

($[\alpha]_T$ the equivalence class of α with respect to T). By (1), the epimorphisms f and g both have $\{[\alpha]_S\colon \alpha \in T\}$ as their dual kernels, and thus by the homomorphism theorem 5.23, there is an isomorphism h from $B(T)$ onto A satisfying $h \circ g = f$. □

9.2. *Algebraic and combinatorial properties of free algebras*

By the characterization 9.4 and the normal form theorem 4.4, we know perfectly well how a free Boolean algebra arises from its free generators. As a consequence of this, infinite free algebras share many algebraic properties.

9.11. PROPOSITION. *Every infinite free Boolean algebra is atomless.*

PROOF. Let F be independently generated by an infinite subset U and let $0 < b$ in F with the aim of finding b' in F such that $0 < b' < b$. We may assume, by the normal form theorem 4.4, that b is an elementary product $b = u_1 \cdot \cdots \cdot u_n \cdot -v_1 \cdot \cdots \cdot -v_m$, where $u_1, \ldots, v_m$ are distinct elements of U. Since U is infinite, pick $u \in U \setminus \{u_1, \ldots, u_n, v_1, \ldots, v_m\}$ and let $b' = b \cdot u$. So $b' \leq b$ and, by independence of U, $0 < b'$. If $b = b'$, then $b \leq u$ and $b \cdot -u = 0$, contradicting independence of U. Thus, $b' < b$. □

9.12. DEFINITION. A Boolean algebra B is *homogeneous* if, for any non-zero element b of B, the relative algebra $B \upharpoonright b$ is isomorphic to B.

For every b in a Boolean algebra B, the Stone space of $B \upharpoonright b$ is homeomorphic to the clopen subset $s(b)$ of Ult B (cf. Example 7.26). Hence, B is homogeneous iff every non-empty clopen subset of Ult B is homeomorphic to Ult B.

The trivial Boolean algebra and the two-element algebra are clearly homogeneous. These are the only finite examples of homogeneous algebras since a homogeneous algebra with at least four elements must be atomless, hence infinite. In particular, the four-element algebra is not homogeneous. For all other Boolean algebras, there is a useful alternative description of homogeneity.

9.13. PROPOSITION. *Let B be a Boolean algebra with $|B| \neq 4$. Then B is homogeneous iff for any two elements a, b of B satisfying $0 < a, b < 1$, there is an automorphism of B mapping a onto b.*

PROOF. Let B be homogeneous and $0 < a, b < 1$. The relative algebras $B \upharpoonright a$, $B \upharpoonright -a$, $B \upharpoonright b$, $B \upharpoonright -b$ are all isomorphic; let $f: B \upharpoonright a \to B \upharpoonright b$ and $g: B \upharpoonright -a \to B \upharpoonright -b$ be isomorphisms. The map $h: B \to B$, defined by

$$h(x) = f(x \cdot a) + g(x \cdot -a),$$

is an automorphism of B mapping a onto b.

Conversely, let B satisfy the condition stated in the proposition; thus for $0 < a$, $b < 1$ in B, we have that $B \upharpoonright a \cong B \upharpoonright b$. Without loss of generality, $|B| \geq 8$ since every Boolean algebra with at most two elements is homogeneous. Let $0 < b < 1$ in B; we want to prove that $B \upharpoonright b \cong B$. Since $|B| \geq 8$, there are disjoint non-zero elements x and y in B such that $a = x + y$ is less than 1. Lemma 3.2 implies that $B \upharpoonright a \cong B \upharpoonright x \times B \upharpoonright y$. Thus,

$$\begin{aligned} B \upharpoonright b &\cong B \upharpoonright a && \text{by } 0 < a, b < 1 \\ &\cong B \upharpoonright x \times B \upharpoonright y \\ &\cong B \upharpoonright a \times B \upharpoonright -a && \text{by } 0 < x, y, a, -a < 1 \\ &\cong B && \text{by 3.2}. \quad \square \end{aligned}$$

9.14. PROPOSITION. *Every infinite free Boolean algebra is homogeneous.*

PROOF. We prove the topological dual: for an infinite set I, every non-empty clopen subset of the Cantor space $X = {}^I 2$ is homeomorphic to X.

Let B be the canonical base of X. Each element of B, say

$$b = \{x \in X: x(i_k) = \varepsilon_k \text{ for } 1 \leq k \leq n\},$$

where $i_1, \ldots, i_n \in I$ are distinct and $\varepsilon_1, \ldots, \varepsilon_n \in 2$, is a clopen subset of X homeomorphic to X, since I is infinite. In particular for $i \in I$, both

$$u_i = \{x \in X: x(i) = 1\}$$

and $X \backslash u_i$ are elements of B. Since $X = u_i \cup (X \backslash u_i)$, X is homeomorphic to the disjoint union space $X \dot\cup X$, defined in Section 8 (here $\dot\cup$ denotes the union of disjoint copies of X, for the sake of clarity).

Now let c be a non-empty clopen subset of X. Since Clop X is generated by $U = \{u_i: i \in I\}$, the normal form theorem 4.4 implies that c is the union of finitely many disjoint elementary products over U, i.e. of finitely many disjoint elements of the base B, say $b_1, \ldots, b_n$. So

$$c = b_1 \cup \cdots \cup b_n \cong X \dot\cup \cdots \dot\cup X \cong X . \quad \square$$

We now turn to combinatorial properties of free Boolean algebras. Here the main result is Theorem 9.16.

9.15. REMARK. Assume that U is an independent subset of a Boolean algebra A, $U_1, \ldots, U_n$ are pairwise disjoint subsets of U and $A_i = \langle U_i \rangle$, the subalgebra of A generated by U_i. Then for arbitrary non-zero elements a_i of A_i $(1 \leq i \leq n)$,

$$a_1 \cdot \cdots \cdot a_n > 0 .$$

To see this, pick for each i an elementary product p_i over U_i such that $0 < p_i \leq a_i$. Then $p_1 \cdot \cdots \cdot p_n \leq a_1 \cdot \cdots \cdot a_n$, and $p_1 \cdot \cdots \cdot p_n$ is non-zero by independence of U.

9.16. THEOREM. *Let κ be a regular uncountable cardinal. If F is a free Boolean algebra and $X \subseteq F$ has cardinality κ, then X has an independent subset of size κ.*

PROOF. Let F be independently generated by U and let $X \subseteq F$ have size κ. For each $x \in X$, pick a finite subset U_x of U generating x.

First, as the subalgebras $\langle U_x \rangle$ generated by U_x are finite, X has a subset X' of size κ such that $U_x \neq U_y$ for $x \neq y$ in X'. Next, by the Δ-system lemma, X' has a subset X'' of size κ such that the sets U_x, $x \in X''$, form a Δ-system, say with root V. Put $W_x = U_x \backslash V$ for x in X'' and let $B = \langle V \rangle$.

B is finite; let $b_1, \ldots, b_k$ be its atoms. Each element x of X'' is generated by $U_x = V \cup W_x \subseteq B \cup W_x$, hence by the remark following 4.7, there are $a_{1x}, \ldots, a_{kx}$ in $\langle W_x \rangle$ such that

$$x = a_{1x} \cdot b_1 + \cdots + a_{kx} \cdot b_k$$

and thus

$$-x = (-a_{1x}) \cdot b_1 + \cdots + (-a_{kx}) \cdot b_k .$$

Finally, there are subsets M and N of $\{1, \ldots, k\}$ and Y of X'' such that Y has cardinality κ and for each x in Y,

$$\{i \in \{1, \ldots, k\}: a_{ix} > 0\} = M, \qquad \{i \in \{1, \ldots, k\}: -a_{ix} > 0\} = N;$$

we show that Y is independent.

Note that $M \cap N \neq \emptyset$, for consider two distinct elements x and y in Y. Without loss of generality $x \cdot -y > 0$, and

$$x \cdot -y = \sum \{a_{ix} \cdot -a_{iy} \cdot b_i : 1 \leq i \leq k\}.$$

It follows that $a_{ix} \cdot -a_{iy} > 0$ for some i; so $i \in M \cap N$.

To prove independence of Y, let S and T be disjoint finite subsets of Y and consider the elementary product

$$p = \prod_{s \in S} s \cdot \prod_{t \in T} -t.$$

For each $i \in \{1, \ldots, k\}$,

$$p \geq b_i \cdot \prod_{s \in S} a_{is} \cdot \prod_{t \in T} -a_{it}.$$

But for i in $M \cap N$, 9.15 shows that the right-hand side is non-zero since $b_i \in \langle V \rangle$, $0 < a_{is} \in \langle W_s \rangle$, $0 < -a_{it} \in \langle W_t \rangle$ and the sets V, W_s for $s \in S$ and W_t for $t \in T$ are pairwise disjoint subsets of U. □

Since distinct elements of an independent set are neither comparable nor disjoint, we obtain the following corollaries.

9.17. COROLLARY. *Every chain in a free Boolean algebra is countable.*

9.18. COROLLARY. *Every free Boolean algebra satisfies the countable chain condition.*

The structure of free algebras being well known, they might be used for comparison with other Boolean algebras. For example, Corollary 9.18 says that no Boolean algebra with an uncountable pairwise disjoint family is embeddable into a free one. On the other hand, if Fr κ is embeddable into a Boolean algebra B, then the Stone space of B has the Cantor space ${}^{\kappa}2$ as a continuous image and hence $|\mathrm{Ult}\, B| \geq 2^{\kappa}$. We define the cardinal invariant ind B as a measure of which free algebras can be embedded into B.

9.19. DEFINITION. For B a Boolean algebra,

$$\mathrm{ind}\, B = \sup\{|U|: U \text{ an independent subset of } B\}$$

is the *independence* of B.

Thus, Theorem 9.16 implies that ind $B = |B|$ for every uncountable subalgebra

B of a free algebra. This is also true for countably infinite B since, for $k \in \omega$, any partition of unity in B of size 2^k generates a free subalgebra on k independent generators.

There are examples of Boolean algebras with large cardinality but small independence. For example, in Section 15 we prove that $\operatorname{ind} B \leq \omega$ for every interval algebra B. In the following section we shall define superatomic Boolean algebras and prove that an algebra is superatomic iff it has no infinite independent subset. As an easy example for the following proposition, consider the finite–cofinite algebra over an uncountable set X: it has cardinality $|X|$, is generated by the (pairwise disjoint) singletons $\{x\}$, $x \in X$, and has no infinite free subalgebra.

9.20. PROPOSITION (Argyros). *Let κ be a regular uncountable cardinal and A a Boolean algebra with a set G of generators such that no subset of G with cardinality κ has the finite intersection property. Then A has no free subalgebra of cardinality κ.*

PROOF. Assume that A has a free subalgebra F of size κ; we find a subset of G which has cardinality κ and the finite intersection property. By Corollary 5.10 to Sikorski's extension theorem, there is an epimorphism $h: A \to Q$, where Q has F as a dense subalgebra and h extends the inclusion map $\mathrm{id}_F: F \to Q$.

Now G generates A, so $h[G]$ generates Q. Since Q has F as a subalgebra and $|F| = \kappa$, also $|h[G]| \geq \kappa$. Thus, there is some $G' \subseteq G$ such that $|G'| = \kappa$, h is one-to-one on G' and $h(x) > 0$ for $x \in G'$. For $x \in G'$, pick by denseness of F in Q an element $a_x \in F$ such that $0 < a_x \leq h(x)$.

If G' has a subset G'' of size κ such that $a_x = a_y$ for all x and y in G'', then clearly G'' has the finite intersection property. Otherwise, by regularity of κ, there is some $G'' \subseteq G$ such that $|G''| = \kappa$ and $a_x \neq a_y$ for $x \neq y$ in G''. Then by Theorem 9.16, there is $G''' \subseteq G''$ such that $|G'''| = \kappa$ and the set $\{a_x: x \in G'''\}$ is independent in F; in particular, it has the finite intersection property. Thus, G''' has the finite intersection property in A. □

As another application of Proposition 9.20, consider a weak product $A = \prod^{\mathrm{w}}_{i \in I} A_i$ of Boolean algebras; suppose that κ is regular and uncountable and $|A_i| < \kappa$ for $i \in I$. A is generated by the subset

$$G = \{a \in A: a_i \neq 0 \text{ for at most one } i \in I\},$$

and no subset of G with cardinality κ has the finite intersection property. Hence, every independent subset of A has cardinality less than κ.

Let us finally give an example of Boolean algebras with large independent subsets.

9.21. EXAMPLE (Fichtenholz, Kantorovich, Hausdorff). Let A be a set with infinite cardinality κ. Then the power set algebra $P(A)$ has an independent subset of size 2^κ (in particular, $\operatorname{ind}(P(A)) = 2^{|A|}$) and 2^{2^κ} ultrafilters.

We may assume in the proof that A is (the underlying set of) the free Boolean algebra $\mathrm{Fr}\,\kappa$ on κ generators. Example 5.25 says that the set $U = \mathrm{Ult}\,A$, a subset of $P(A)$, is independent in $P(A)$; it has size 2^κ since Ult A is the Cantor space ${}^\kappa 2$. The subalgebra F of $P(A)$ generated by U is isomorphic to $\mathrm{Fr}\, 2^\kappa$; thus $|\mathrm{Ult}\, P(A)| \geq |\mathrm{Ult}\, F| = 2^{2^\kappa}$.

A considerably more general theorem, due to Balcar and Franěk, will be proved in Section 13: every infinite complete Boolean algebra B has an independent subset of size $|B|$.

Exercises

1. Find an infinite Boolean algebra A such that each algebra B with $|B| \leq |A|$ is a homomorphic image of A, but A is not free.
2. Show that for a finite Boolean algebra A with exactly n atoms, $\mathrm{ind}\, A = \max\{k \in \omega: 2^k \leq n\}$.
3. Prove that every free algebra F is *projective*, i.e. it has the following universal property. Let $p: A \rightarrow A'$ be an epimorphism of Boolean algebras and $g': F \rightarrow A'$ a homomorphism. Thus there is a homomorphism $g: F \rightarrow A$ such that $p \circ g = g'$.
4. Show that, for any two ultrafilters p and q of a free Boolean algebra F, there is an automorphism of F mapping p onto q. Equivalently, the topological space Ult F is homogeneous, i.e. for any two points p, q of Ult F there is a homeomorphism from Ult F onto itself mapping p to q.
5. Prove that every Boolean algebra is isomorphic to the Lindenbaum–Tarski algebra of some *first order* theory (9.10 says it is isomorphic to the algebra of formulas of a *propositional* theory).
6. For F free over U, show that there is a unique finitely additive measure $\mu: F \rightarrow [0, 1]$, as defined in Exercise 5 of Section 3, such that each elementary product $\varepsilon_1 u_1 \cdot \cdots \cdot \varepsilon_n u_n$ ($\varepsilon_i \in \{+1, -1\}$; $u_1, \ldots, u_n \in U$ pairwise distinct) has measure $1/2^n$. Conclude that each chain and also each pairwise disjoint family in F is countable – cf. 9.17 and 9.18.
7. Show that every infinite algebra with the countable separation property has an independent subset of size 2^ω.

10. Independence and the number of ideals

We study in some detail the problem of finding large independent subsets of Boolean algebras. Results of this type are also interesting to topologists since they guarantee that certain Boolean spaces have generalized Cantor spaces of large weight as continuous images.

More precisely, Shelah's theorem 10.1 says that the cardinal invariant ind A of a Boolean algebra A, i.e. the least upper bound of the sizes of independent subsets of A, is large if A satisfies, say, the κ-chain condition and the cardinality of A is large if compared with κ. Šapirovskii's theorem 10.16 then describes ind A in terms of a cardinal invariant for ultrafilters in homomorphic images of A. The crucial argument of both theorems is an inductive construction of independent families using somewhat advanced tools of set theory: stationary sets and Fodor's theorem. See the Appendix on Set Theory in this Handbook for a review of these.

As a consequence of Theorem 10.16, we prove in 10.17 that for A a subalgebra of B, ind B is essentially the supremum of ind A and the cardinals ind $B/\bar{p}$, where p is an ultrafilter of A and $\bar{p}$ the filter of B generated by p; $B/\bar{p}$ is then the stalk over p in the canonical sheaf representation of B over A developed in Section 8. The proof will, however, not use any terminology or results from sheaf theory.

Let us point out another substantial result on independence, the Balcar–Franěk theorem proved in Section 13: every infinite complete Boolean algebra of cardinality κ has an independent subset of size κ. A thorough discussion of the behaviour of the cardinal invariant ind under algebraic constructions such as products, quotients, subalgebras, etc. can be found in MONK [1983].

Using the main theorem and the methods of the first subsection, the second one presents a recent result by Shelah: the number id(A) of ideals in an infinite Boolean algebra A satisfies $\mathrm{id}(A)^{\omega} = \mathrm{id}(A)$.

10.1. Independence and chain conditions

The aim of this subsection is a theorem by Shelah claiming that the independence of a Boolean algebra A is large provided the cardinality of A is large if compared with the cellularity of A. Recall that A satisfies the κ-chain condition if every pairwise disjoint family in A has cardinality less than κ.

10.1. THEOREM (Shelah). *Assume κ and λ are regular infinite cardinals such that $\mu^{<\kappa} < \lambda$ for every cardinal $\mu < \lambda$, and that A is a Boolean algebra satisfying the κ-chain condition. Then every subset X of A of size λ has an independent subset Y of size λ.*

Before embarking on the proof, we collect some frequently used principles connecting chain conditions and the size of κ-complete subalgebras. In Definition 4.1, we called a subalgebra A of a Boolean algebra B a κ-complete subalgebra (a complete subalgebra) if for every $X \subseteq A$ of cardinality less than κ (respectively for every $X \subseteq A$) such that $\Sigma^B X$ exists, also $\Sigma^A X$ exists and $\Sigma^A X = \Sigma^B X$ – i.e. if A is closed under all sums of length less tan κ (respectively under arbitrary sums) which happen to exist in B. Strictly speaking, this is an abuse of language since the definition does not say that A is κ-complete in its own right. But if B is κ-complete and A is a κ-complete subalgebra of B, then A will be κ-complete, too.

10.2. LEMMA. *Assume B satisfies the κ-chain condition and $X \subseteq B$. Then there is some $Y \subseteq X$ of size less than κ such that X and Y have the same upper bounds in B. In particular, $\Sigma X = \Sigma Y$ if one of these sums exists.*

PROOF. This follows easily from Lemma 3.12 if B is $|X|$-complete. For arbitrary B, we argue as follows. Let D be a pairwise disjoint family in B maximal with respect to the condition that for each $d \in D$ there is some $x_d \in X$ satisfying $d \leq x_d$.

D and X have the same set of upper bounds: clearly every upper bound of X is an upper bound of D. Conversely, let b be an upper bound of D and suppose that $x \not\leq b$ for some $x \in X$. Then $D \cup \{x \cdot -b\}$ contradicts maximality of D. By the κ-chain condition, $|D| < \kappa$. So $Y = \{x_d : d \in D\}$ works for the lemma. □

As a consequence of 10.2, every σ-complete Boolean algebra satisfying the countable chain condition is complete. Similarly, a Boolean algebra is complete if it is κ-complete and satisfies the κ-chain condition.

10.3. LEMMA AND DEFINITION. Let κ be an infinite cardinal, B a Boolean algebra and $X \subseteq B$. Then

$$\langle X \rangle^{\kappa\text{-cm}} = \bigcap \{A : X \subseteq A \subseteq B,\ A \text{ a } \kappa\text{-complete subalgebra of } B\}$$

is the smallest κ-complete subalgebra of B including X, the *κ-complete subalgebra generated by X* (or: the subalgebra κ-completely generated by X). Similarly

$$\langle X \rangle^{\text{cm}} = \bigcap \{A : X \subseteq A \subseteq B,\ A \text{ a complete subalgebra of } B\}$$

is the smallest complete subalgebra of B including X, the *complete subalgebra generated by X* (or: the subalgebra completely generated by X).

Unfortunately there is no reasonable analogue of the normal form theorem 4.4 for the elements of $\langle X \rangle^{\kappa\text{-cm}}$ or $\langle X \rangle^{\text{cm}}$. In fact the lack of normal forms has the striking consequence that the complete subalgebra generated by a countable set X can have arbitrarily large cardinality – a result due to Gaifman and Hales which will be proved in Section 13. Thus, assertions on the size of $\langle X \rangle^{\text{cm}}$ have to rely on additional hypotheses.

10.4. LEMMA. *Let κ be a regular infinite cardinal and X a subset of a Boolean algebra B. Then*

$$|\langle X \rangle^{\kappa\text{-cm}}| \leq \max(\omega, |X|)^{<\kappa}.$$

PROOF. Define by induction subsets X_α of B for $\alpha < \kappa$: let

$$X_0 = X \cup \{0, 1\},$$

$$X_{\alpha+1} = X_\alpha \cup \{-x : x \in X_\alpha\} \cup \left\{\sum M : M \subseteq X_\alpha, |M| < \kappa, \sum M \text{ exists}\right\}$$

and for limit ordinals $\lambda < \kappa$,

$$X_\lambda = \bigcup_{\alpha < \lambda} X_\alpha .$$

It is easily checked by induction on $\alpha < \kappa$ that $X_\alpha \subseteq \langle X \rangle^{\kappa\text{-cm}}$ and

$|X_\alpha| \leq \max(\omega, |X|)^{<\kappa}$. By regularity of κ and the definition of $X_{\alpha+1}$, $\bigcup_{\alpha<\kappa} X_\alpha$ is a κ-complete subalgebra of B. Hence, $\langle X \rangle^{\kappa\text{-cm}} = \bigcup_{\alpha<\kappa} X_\alpha$. □

10.5. COROLLARY. *Assume B satisfies the κ-chain condition for a regular infinite cardinal κ. Then for every subset X of B,*

$$|\langle X \rangle^{\mathrm{cm}}| \leq \max(\omega, |X|)^{<\kappa} .$$

PROOF. By 10.2, $\langle X \rangle^{\mathrm{cm}}$ coincides with $\langle X \rangle^{\kappa\text{-cm}}$. □

The following Lemmas 10.6 and 10.8 allow us to construct independent sets in the proofs of both Theorem 10.1 and Šapirovskii's characterization 10.16 of independence.

10.6. LEMMA. *Assume T is a set of ordinals, $(A_\alpha)_{\alpha \in T}$ an increasing sequence of subalgebras of a Boolean algebra B, r a non-zero element of B and $(a_\alpha)_{\alpha \in T}$ a sequence in B such that*

(i) *$a_\beta \in A_\alpha$, for $\beta < \alpha$ in T,*

(ii) *$r \in A_\alpha$, for α in T,*

(iii) *$x \cdot a_\alpha > 0$ and $x \cdot -a_\alpha > 0$, for α in T and $x \in (A_\alpha \restriction r)^+ = A_\alpha \restriction r \setminus \{0\}$.*

Then $\{a_\alpha : \alpha \in T\}$ is an independent subset of B.

PROOF. We prove by induction on n that, for $\alpha(1) < \cdots < \alpha(n)$ in T and $\varepsilon_1, \ldots, \varepsilon_n$ in $\{+1, -1\}$,

$$p = r \cdot \varepsilon_1 a_{\alpha(1)} \cdot \cdots \cdot \varepsilon_n a_{\alpha(n)} > 0 .$$

This holds for $n = 0$ since $r > 0$. Suppose it holds for n, so $p > 0$. Now if $\varepsilon \in \{+1, -1\}$ and $\alpha \in T$ is such that $\alpha(n) < \alpha$, then (i) and (ii) imply that p is a non-zero element of $A_\alpha \restriction r$. Thus, both $p \cdot a_\alpha$ and $p \cdot -a_\alpha$ are non-zero, i.e. our assertion holds for $n+1$. □

We introduce a piece of notation for the proof of Theorem 10.1. If B is a complete Boolean algebra and C a complete subalgebra of B, let for fixed $x \in B$

$$\mathrm{lpr}(x, C) = \sum \{c \in C : c \leq x\} , \qquad \mathrm{upr}(x, C) = \prod \{c \in C : x \leq c\} ,$$

the *lower* and the *upper projection* of x with respect to C. So $\mathrm{lpr}(x, C)$ is the greatest element of C below x and $\mathrm{upr}(x, C)$ is the least element of C above x. (In the terminology of Definition 8.19, $\mathrm{lpr}(x, C) = \mathrm{pr}_C(x)$, but the above notation is more suggestive in the present context.) Clearly, $\mathrm{lpr}(x, C) \leq x \leq \mathrm{upr}(x, C)$, and $\mathrm{lpr}(x, C) = \mathrm{upr}(x, C)$ iff $x \in C$. The impact of these notions to independence comes from 10.6 and the following lemma.

10.7. LEMMA. *Let B be complete, A a complete subalgebra of B and $a \in B \setminus A$. Then*

$$r = \mathrm{upr}(a, A) \cdot -\mathrm{lpr}(a, A)$$

is a non-zero element of A *such that, for every non-zero element* x *of* $A \upharpoonright r$, $x \cdot a > 0$ *and* $x \cdot -a > 0$.

PROOF. r is non-zero since $a \not\in A$ and thus $\text{lpr}(a, A) < \text{upr}(a, A)$. Let $x \in A \upharpoonright r$ be non-zero.

If $x \cdot a = 0$, then $a \leq -x$, $\text{upr}(a, A) \leq -x$ since $-x \in A$, and $x \leq -\text{upr}(a, A) \leq -r$, which contradicts $0 < x \leq r$. Similarly, if $x \cdot -a = 0$, then $x \leq a$, $x \leq \text{lpr}(a, A)$ since $x \in A$, so again $x \leq -r$. □

Our final lemma for bridging the gap between 10.7 and 10.6 is a standard application of Fodor's theorem.

10.8. LEMMA. *Let* λ *be a regular uncountable cardinal and* $(A_\alpha)_{\alpha<\lambda}$ *an increasing sequence of Boolean algebras such that* $|A_\alpha| < \lambda$ *for* $\alpha < \lambda$ *and* $|\bigcup_{\alpha<\lambda} A_\alpha| = \lambda$. *Assume* S *is a stationary subset of* λ *and that, for* $\alpha \in S$, $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$ *and* r_α *is an element of* A_α. *Then there are* $r \in \bigcup_{\alpha<\lambda} A_\alpha$ *and a stationary subset* T *of* S *such that* $r_\alpha = r$ *for each* $\alpha \in T$.

PROOF. Fix a bijection

$$f: \lambda \to \bigcup_{\alpha<\lambda} A_\alpha .$$

A routine argument shows that

$$K = \left\{\alpha < \lambda: f[\alpha] = \bigcup_{\beta<\alpha} A_\beta\right\}$$

is closed and unbounded in λ, so $S \cap K$ is stationary. We obtain a regressive function

$$g: S \cap K \to \lambda$$

by letting

$$g(\alpha) = f^{-1}(r_\alpha) .$$

For r_α is an element of A_α; if $\alpha \in S$, then $r_\alpha \in \bigcup_{\beta<\alpha} A_\beta$ and if, in addition, $\alpha \in K$, then $f^{-1}(r_\alpha) < \alpha$.

Now by Fodor's theorem, g is constant on a stationary subset of $S \cap K$. □

Proof of Theorem 10.1. Suppose κ, λ, A and X are given as stated in 10.1. Note first that $\kappa \leq 2^{<\kappa} < \lambda$. Hence,

$$S = \{\alpha < \lambda: \kappa \leq \text{cf}\, \alpha\}$$

is a stationary subset of λ.

Let B be the completion of A and define, by induction on $\alpha < \lambda$, $a_\alpha \in X$ and a complete subalgebra A_α of B:

$$A_\alpha = \langle a_\beta \colon \beta < \alpha \rangle^{\mathrm{cm}}, \quad a_\alpha \in X \setminus A_\alpha .$$

This is possible since, by denseness of A in B, B satisfies the κ-chain condition; so

$$|A_\alpha| \le \max(\omega, |\alpha|)^{<\kappa} < \lambda$$

by Corollary 10.5 and our hypothesis on κ, λ. For every $\alpha \in S$, $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$, for by $\kappa \le \mathrm{cf}\,\alpha$, $\bigcup_{\beta<\alpha} A_\beta$ is a κ-complete subalgebra of B, hence a complete subalgebra by the κ-chain condition and Lemma 10.2. Moreover, $|\bigcup_{\alpha<\lambda} A_\alpha| = \lambda$ by $|A_\alpha| < \lambda$ and $a_\alpha \notin A_\alpha$.

For $\alpha \in S$, let

$$r_\alpha = \mathrm{upr}(a_\alpha, A_\alpha) \cdot -\mathrm{lpr}(a_\alpha, A_\alpha) .$$

So $r_\alpha \in A_\alpha$ and $r_\alpha > 0$ since $a_\alpha \notin A_\alpha$. Applying 10.8, we get a stationary subset T of S and $r > 0$ such that $r_\alpha = r$ for $\alpha \in T$. Lemma 10.7 says that $(A_\alpha)_{\alpha \in T}$, r and $(a_\alpha)_{\alpha \in T}$ satisfy the hypotheses of 10.6. Thus, $Y = \{a_\alpha \colon a \in T\}$ is an independent subset of X having cardinality λ. □

As an immediate consequence of 10.1, we obtain:

10.9. COROLLARY. *Let τ be an infinite cardinal. Then if A is a Boolean algebra such that $|A| > 2^\tau$ and A satisfies the τ^+-chain condition, then A has an independent subset of cardinality $(2^\tau)^+$. (In fact, every subset of A of size $(2^\tau)^+$ has an independent subset of size $(2^\tau)^+$.)*

PROOF. Apply Theorem 10.1 to $\kappa = \tau^+$ and $\lambda = (2^\tau)^+$. □

Letting $\tau = \omega$ in the corollary, we obtain that in an algebra A satisfying the countable chain condition, each subset of A of size $(2^\omega)^+$ has a subset of size $(2^\omega)^+$ which is independent, hence has the finite intersection property (cf. Section 2). This result is best possible in the following sense: in TODORČEVIĆ [1986], an algebra A is constructed (in ZFC) which satisfies the countable chain condition, but not every subset of A of size 2^ω has a subset of size 2^ω with the finite intersection property.

The technique of Theorem 10.1 is used in SHELAH [1980] to prove two additional facts:

(1) Let λ be a weakly compact cardinal and A a Boolean algebra of power λ satisfying the λ-chain condition. Then every subset X of A of size λ has an independent subset Y of size λ.

(2) Let λ be singular and κ regular such that $\mu^{<\kappa} < \lambda$ for every $\mu < \lambda$. Assume A is a Boolean algebra satisfying the κ-chain condition. Then every subset X of A of size λ^+ has an independent subset Y of size λ.

ARGYROS [1981] presents, under the generalized continuum hypothesis, a construction of a Boolean algebra A such that $|A| = \lambda^+$ where λ is singular, A satisfies the κ-chain condition where $\kappa = (\mathrm{cf}\,\lambda)^+$ and A has no independent subset of size λ^+ (note that these assumptions imply $\mu^{<\kappa} < \lambda$ for every $\mu < \lambda$). This shows that (2) above cannot be improved to give $|Y| = \lambda^+$.

10.2. *The number of ideals of a Boolean algebra*

We shall apply Shelah's theorem 10.1 and part of the techniques used in its proof to show the following theorem.

10.10. THEOREM (Shelah). *If A is an infinite Boolean algebra and* $\mathrm{id}(A)$ *the number of its ideals, then* $\mathrm{id}(A)^{\omega} = \mathrm{id}(A)$.

Thus, by the duality between ideals of Boolean algebras and open subsets of Boolean spaces (cf. 7.25), the number $o(X)$ of open subsets of an infinite Boolean space X satisfies $o(X)^{\omega} = o(X)$.

More generally, de Groot asked the question whether, for an infinite Hausdorff space X, $o(X)$ is necessarily a power of 2; asking whether $o(X)^{\omega} = o(X)$ considerably weakens this question. Let us note that in spaces X which fail to satisfy reasonable separation properties also the equation $o(X)^{\omega} = o(X)$ can fail badly: e.g. let the underlying set of X be any infinite cardinal κ; the set $o(X)$ of all initial segments of κ (i.e. $o(X) = \kappa + 1$) is a topology on X and has size κ. Concerning de Groot's problem, it was proved, for example, by Hajnal and Juhasz, assuming the generalized continuum hypothesis and non-existence of inaccessible cardinals, that $o(X)$ is a power of 2, for every Hausdorff space. See HODEL [1984] for a survey of de Groot's question. SHELAH [1986] shows, under an additional set-theoretical assumption ($0^{\#}$ does not exist), that $o(X)^{\omega} = o(X)$ if X is Hausdorff and $o(X) \geq 2^{2^{\omega}}$.

The major step towards Theorem 10.10 is Proposition 10.14 below. Call a subset D of a Boolean algebra A *ideal-independent* if no element d of D belongs to the ideal generated, in A, by $D\setminus\{d\}$. Similarly, a family $(d_i)_{i\in I}$ of elements of A is said to be ideal-independent if for no i in I, d_i belongs to the ideal generated, in A, by $\{d_j\colon j \neq i\}$. This notion is relevant to the proof of 10.10 since distinct subsets of any ideal-independent set D generate distinct ideals and hence $2^{|D|} \leq \mathrm{id}(A)$.

A translation of ideal-independence to topology runs as follows. For any topological space, the *spread* of X is the cardinal invariant

$$sX = \sup\{|Y|\colon Y \text{ a discrete subspace of } X\}\,.$$

Let

$$s^*X = \sup\{|Y|^{+}\colon Y \text{ a discrete subspace of } X\}\,.$$

10.11. LEMMA. (a) *An infinite Boolean algebra A has an ideal-independent subset of size κ iff its dual space has a discrete subspace of size κ. Hence,*

$$s^*(\mathrm{Ult}\, A) = \min\{\kappa\colon A \textit{ has no ideal-independent subset of size } \kappa\}\,.$$

(b) $\mathrm{cf}(s^*X) > \omega$, *for each infinite regular Hausdorff space X.*

PROOF. (a) If Y is a discrete subspace of $\mathrm{Ult}\, A$, then for $y \in Y$ choose a clopen subset a_y of Ult A such that $a_y \cap Y = \{y\}$. Clearly, $(a_y)_{y\in Y}$ is an ideal-independent family in Clop Ult A, an algebra isomorphic to A. Conversely, suppose D is an ideal-independent subset of Clop Ult $A \cong A$. No element d of D is covered by finitely many elements of $D\setminus\{d\}$; by compactness of Ult A, choose a point y_d in $d\setminus\bigcup (D\setminus\{d\})$. Then $Y = \{y_d : d \in D\}$ is a discrete subspace of X, since $d \cap Y = \{y_d\}$.

(b) A proof of this non-trivial result and of the set-theoretical background required is contained in JUHASZ [1971]. □

The proof of Proposition 10.14, and also of Lemma 10.13 preparing it, is facilitated by a slight extension of some notation in the preceding subsection. If B is a complete Boolean algebra, $b \in B$ and X an arbitrary subset of B, put

$$\mathrm{lpr}(b, X) = \mathrm{lpr}(b, \langle X\rangle^{\mathrm{cm}}), \qquad \mathrm{upr}(b, X) = \mathrm{upr}(b, \langle X\rangle^{\mathrm{cm}}).$$

10.12. LEMMA. (a) *Assume B is complete, $X \subseteq Y \subseteq B$ and $b \in B$ such that $\mathrm{lpr}(b, Y) \in X$. Then $\mathrm{lpr}(b, Y) = \mathrm{lpr}(b, X)$; similarly, $\mathrm{upr}(b, Y) = \mathrm{upr}(b, X)$ if $\mathrm{upr}(b, Y) \in X$.*

(b) *Assume B is complete, C and D are complete subalgebras of B such that $D \subseteq C \subseteq B$ and $d \in D$, $c \in C$, $b \in B$. Then*

$$\text{if } d \cdot b \cdot -c = 0 \text{ and } \mathrm{upr}(b, C) \in D, \text{ then } d \cdot \mathrm{upr}(b, C) \cdot -\mathrm{lpr}(c, D) = 0;$$

$$\text{if } d \cdot -(b \cdot -c) = 0, \text{ then } d \cdot -(\mathrm{lpr}(b, C) \cdot -\mathrm{upr}(c, D)) = 0.$$

PROOF. (a) Clearly, $\mathrm{lpr}(b, X) \leq \mathrm{lpr}(b, Y)$. On the other hand, $\mathrm{lpr}(b, Y)$ is an element of $\langle X\rangle^{\mathrm{cm}}$ lying below b, so $\mathrm{lpr}(b, Y) \leq \mathrm{lpr}(b, X)$ and $\mathrm{lpr}(b, Y) = \mathrm{lpr}(b, X)$.

(b) We prove the first assertion; the proof of the second one is similar but easier. If $d \cdot b \cdot -c = 0$, then $b \leq c + -d$, $\mathrm{upr}(b, C) \leq c + -d$ since $c + -d$ is in C, and $d \cdot \mathrm{upr}(b, C) \leq c$. Now if, in addition, $\mathrm{upr}(b, C) \in D$, then $d \cdot \mathrm{upr}(b, C) \leq \mathrm{lpr}(c, D)$; this gives $d \cdot \mathrm{upr}(b, C) \cdot -\mathrm{lpr}(c, D) = 0$. □

Let us say, in this subsection, that an infinite cardinal λ is *κ-closed* if $\mu^{<\kappa} < \lambda$ holds for each cardinal $\mu < \lambda$.

10.13. LEMMA. *Assume κ and λ are regular infinite cardinals and λ is κ-closed. Let B be a complete Boolean algebra satisfying the κ-chain condition and $(a_\alpha)_{\alpha<\lambda}$ a sequence of pairwise distinct elements of B. Then there are b_1, b_2 in B such that for any $X \subseteq B$ of size less than λ and containing b_1, b_2, the set*

$$\{\alpha < \lambda : \mathrm{lpr}(a_\alpha, X \cup \{a_\beta : \beta < \alpha\}) = b_1 \text{ and } \mathrm{upr}(a_\alpha, X \cup \{a_\beta : \beta < \alpha\}) = b_2\}$$

has cardinality λ.

PROOF. Suppose not. Then for every pair $p = (b_1, b_2)$ in B there is a counterexample $X(p) \subseteq B$ satisfying $|X(p)| < \lambda$, $b_1, b_2 \in X(p)$, but

$$|\{\alpha < \lambda: \operatorname{lpr}(a_\alpha, X(p) \cup \{a_\beta: \beta < \alpha\}) = b_1 \text{ and}$$
$$\operatorname{upr}(a_\alpha, X(p) \cup \{a_\beta: \beta < \alpha\}) = b_2\}| < \lambda .$$

We shall construct a complete subalgebra C of B which is closed under taking counterexamples and write C as the union of an increasing continuous chain of subalgebras C_α; a standard argument using stationary sets will then produce a contradiction.

To this end, define, for $p = (b_1, b_2) \in B \times B$,

$$X'(p) = X(p) \cup \{a_\alpha: \alpha < \lambda, \operatorname{lpr}(a_\alpha, X(p) \cup \{a_\beta: \beta < \alpha\}) = b_1,$$
$$\operatorname{upr}(a_\alpha, X(p) \cup \{a_\beta: \beta < \alpha\}) = b_2\}$$

and, for $\alpha < \lambda$,

$$X'_\alpha(p) = X'(p) \cup \{a_\beta: \beta < \alpha\} .$$

Then $X'(p)$ and $X'_\alpha(p)$ have cardinality less than λ. Moreover, application of Lemma 10.12(a) to $X(p) \cup \{a_\beta: \beta < \alpha\} \subseteq X'_\alpha(p)$ shows, together with b_1, $b_2 \in X(p)$, that

(3) if $\operatorname{lpr}(a_\alpha, X'_\alpha(p)) = b_1$ and $\operatorname{upr}(a_\alpha, X'_\alpha(p)) = b_2$, then $a_\alpha \in X'(p)$.

Let $C = \bigcup_{\alpha<\lambda} C_\alpha$, where the subalgebras C_α of B are defined, by induction, as follows: $C_0 = 2$, $C_\alpha = \bigcup_{\beta<\alpha} C_\beta$ for limit α, and

$$C_{\alpha+1} = \Big\langle C_\alpha \cup \{a_\beta: \beta < \alpha\} \cup \bigcup \{X'(p): p \in C_\alpha \times C_\alpha\}$$
$$\cup \Big\{\sum M: M \subseteq C_\alpha, |M| < \kappa\Big\}\Big\rangle .$$

Since B satisfies the κ-chain condition, λ is κ-closed and $|X'(p)| < \lambda$, each C_α has cardinality less than λ. Also, for each δ in the stationary set

$$S = \{\delta < \lambda: \operatorname{cf} \delta \geq \kappa\} ,$$

it follows, as in the proof of 10.1, that C_δ is a complete subalgebra of B. By continuity of the chain $(C_\alpha)_{\alpha<\lambda}$ and $|C_\alpha| < \lambda$, the set

$$S' = \{\delta < \lambda: \delta \text{ limit, and for } \alpha < \lambda: \alpha < \delta \text{ iff } a_\alpha \in C_\delta\}$$

is closed and unbounded in λ. We shall eventually reach a contradiction by fixing an element δ of $S \cap S'$. For this δ, C_δ is complete and $a_\delta \not\in C_\delta$. The pair

$$p = (b_1, b_2) = (\operatorname{lpr}(a_\delta, C_\delta), \operatorname{upr}(a_\delta, C_\delta))$$

is in $C_\delta \times C_\delta$; since δ is a limit ordinal, also $X'(p) \subseteq C_\delta$. It follows from

$$b_1, b_2 \in X(p) \subseteq X'(p) \subseteq X'_\delta(p) \subseteq C_\delta$$

and Lemma 10.12(a) that

$$b_1 = \mathrm{lpr}(a_\delta, C_\delta) = \mathrm{lpr}(a_\delta, X(p)) = \mathrm{lpr}(a_\delta, X'_\delta(p))$$

and similarly that $b_2 = \mathrm{upr}(a_\delta, X'_\delta(p))$. Thus by (3), $a_\delta \in X'(p) \subseteq C_\delta$, a contradiction. □

10.14. PROPOSITION. *Assume that κ is regular, λ is κ-closed, and that A is a Boolean algebra satisfying the κ-chain condition and $|A| \geq \lambda$. Then A has an ideal-independent subset of size λ.*

PROOF. If λ is regular, this follows from Theorem 10.1 since every independent set is ideal-independent. Thus assume that λ is singular. Note that

(4) for every infinite cardinal $\mu < \lambda$, $(\mu^{<\kappa})^+$ is κ-closed and satisfies
$$\mu \leq (\mu^{<\kappa})^+ < \lambda$$

since κ is regular and λ is κ-closed. By (4), we can write

$$\lambda = \sup_{\beta < \mathrm{cf}\,\lambda} \lambda_\beta ,$$

where $\lambda_\alpha < \lambda_\beta$ for $\alpha < \beta < \mathrm{cf}\,\lambda$, λ_β is a κ-closed successor cardinal and $\mathrm{cf}\,\lambda < \lambda_\beta$.

The following set of pairs of ordinals:

$$P = \{(\beta, \alpha) \colon \beta < \mathrm{cf}\,\lambda, \alpha < \lambda_\beta\} = \bigcup_{\beta < \mathrm{cf}\,\lambda} \{\beta\} \times \lambda_\beta$$

has cardinality λ and is well-ordered by the lexicographic order

$$(\beta, \alpha) < (\beta', \alpha') \quad \text{iff } \beta < \beta' \text{ or } (\beta = \beta' \text{ and } \alpha < \alpha') .$$

Since $|A| \geq \lambda$, choose pairwise distinct elements a_p, $p \in P$, in A and write $a_{\beta\alpha}$ for a_p, if $p = (\beta, \alpha) \in P$. We shall define, for $p \in P$, an element y_p of A and prove that $(y_p)_{p \in P}$ is ideal-independent.

Working in the completion $\bar{A}$ of A, we apply Lemma 10.13 to each of the sets $\{a_{\beta\alpha} \colon \alpha < \lambda_\beta\}$ to obtain $b_{1\beta}, b_{2\beta} \in \bar{A}$ such that for any $X \subseteq \bar{A}$ with $|X| < \lambda_\beta$ and $b_{1\beta}, b_{2\beta} \in X$, we have

(5) $$|\{\alpha < \lambda_\beta \colon \mathrm{lpr}(a_{\beta\alpha}, X \cup \{a_{\beta\nu} \colon \nu < \alpha\}) = b_{1\beta} \text{ and}$$
$$\mathrm{upr}(a_{\beta\alpha}, X \cup \{a_{\beta\nu} \colon \nu < \alpha\}) = b_{2\beta}\}| = \lambda_\beta .$$

Note that $b_{1\beta} < b_{2\beta}$, for otherwise each α in the set displayed in (5) satisfies $b_{1\beta} = b_{2\beta} = a_{\beta\alpha}$, i.e. the set has at most one element; contradiction. Hence,

$$b_\beta = b_{2\beta} \cdot -b_{1\beta}$$

is non-zero. For $p \in P$, define

$$E_p = \{b_\beta, b_{1\beta}, b_{2\beta}: \beta < \text{cf}\ \lambda\} \cup \{a_l: l \in P, 1 < p\};$$

so

(6) $\quad p < q$ implies $E_p \subseteq E_q$.

To define the elements y_p for $p \in \{\beta\} \times \lambda_\beta$, consider the subset

$$X = \{b_\beta, b_{1\beta}, b_{2\beta}: \beta < \text{cf}\ \lambda\} \cup \{a_l: l = (\gamma, \nu) \in P \text{ where } \gamma < \beta\}$$

of $\bar{A}$; it has cardinality less than λ_β and contains $b_{1\beta}$ and $b_{2\beta}$. If $p = (\beta, \alpha) \in \{\beta\} \times \lambda_\beta$, then $X \cup \{a_{\beta\nu}: \nu < \alpha\} = E_p$. By (5) above, the set

$$Q = \{p \in \{\beta\} \times \lambda_\beta: \text{lpr}(a_p, E_p) = b_{1\beta} \text{ and } \text{upr}(a_p, E_p) = b_{2\beta}\}$$

has size λ_β. Thus, by induction on the lexicographic order, assign to each pair $p \in \{\beta\} \times \lambda_\beta$ two pairs p', p'' such that

(7) $\quad p', p'' \in Q$ and $p' < p''$,

(8) $\quad$ if $p < q$, then $p' < p'' < q' < q''$;

note that, since for $p \in \{\beta\} \times \lambda_\beta$, also p' and p'' are in $\{\beta\} \times \lambda_\beta$, (8) will hold for arbitrary pairs p, q in P.

Put

$$y_p = a_{p''} \cdot -a_{p'},$$

an element of A. We are left with showing that the family $(y_p)_{p \in P}$ is ideal-independent. Suppose for contradiction that

$$y_p \cdot -y_{p(1)} \cdot \cdots \cdot -y_{p(n)} = 0,$$

where $p, p(1), \ldots, p(n)$ in P are distinct. We may assume that $n \in \omega$ is minimal for this situation and that $p(1) < \cdots < p(n)$.

Case 1. $n \geq 1$ and $p < p(n)$. Let then $p(n)''$ have the form (β, α) and consider the complete subalgebras

$$D = \langle E_{p(n)'} \rangle^{\text{cm}} \subseteq C = \langle E_{p(n)''} \rangle^{\text{cm}} \subseteq B = \langle E_{\beta,\alpha+1} \rangle^{\text{cm}}$$

of $\bar{A}$ and $d = y_p \cdot -y_{p(1)} \cdot \cdots \cdot -y_{p(n-1)}$, $c = a_{p(n)'}$, $b = a_{p(n)''}$. (8) shows that $d \in D$, $c \in C$, and $b \in B$. Now $\text{lpr}(b, C) = b_{1\beta}$, $\text{upr}(c, D) = b_{2\beta}$ and $0 = d \cdot -y_{p(n)} = d \cdot -(b \cdot -c)$; by Lemma 10.12(b) and $b_{1\beta} \leq b_{2\beta}$, we obtain $d = 0$ which contradicts the minimality of n.

Case 2. $n=0$ or $p(n)<p$. Let then p'' have the form (β, α) and consider the complete subalgebras

$$D=\langle E_{p'}\rangle^{\mathrm{cm}} \subseteq C=\langle E_{p''}\rangle^{\mathrm{cm}} \subseteq B=\langle E_{\beta,\alpha+1}\rangle^{\mathrm{cm}}$$

of $\bar{A}$ and the elements $d=-y_{p(1)}\cdot \cdots \cdot -y_{p(n)}$ of D, $c=a_{p'}$ of C, $b=a_{p''}$ of B. Then $\mathrm{upr}(b, C)=b_{2\beta}\in D$, $\mathrm{lpr}(c, D)=b_{1\beta}$ and $0=d\cdot y_p=d\cdot b\cdot -c$; again by Lemma 10.12(b), we obtain $d\cdot b_\beta=0$. If $n=0$, this implies $b_\beta=0$, a contradiction. Otherwise, consider $x=b_\beta\cdot -y_{p(1)}\cdot \cdots \cdot -y_{p(n-1)}$, an element of $\langle E_{p(n)'}\rangle^{\mathrm{cm}}$ disjoint from $-y_{p(n)}$. The proof of Case 1 then shows that $x=0$. But $a_{p''}\leq \mathrm{upr}(a_{p''}, E_{p''})=b_{2\beta}$ and similarly $-a_{p'}\leq -b_{1\beta}$, so $y_p\leq b_\beta$ and $y_p\cdot -y_{p(1)}\cdot \cdots \cdot -y_{p(n-1)}=0$, contradicting the minimality of n. □

Proof of Theorem 10.10. The proof is broken up into six steps defining certain sets and seven claims concerning these sets.

Let $\mathrm{Id}(B)$ denote the set of ideals of an arbitrary Boolean algebra B, $\mathrm{id}(B)$ its cardinality and, for b in B, put $\mathrm{id}(b)=\mathrm{id}(B \restriction b)$. For an arbitrary ideal I in B and b in B, define $I\restriction b=I\cap B\restriction b$.

Let A be an infinite Boolean algebra and assume for contradiction that $\mathrm{id}(A)<\mathrm{id}(A)^\omega$.

Step 1: Definition of λ and λ_n. Let λ be the least cardinal satisfying $\mathrm{id}(A)\leq\lambda^\omega$. By $\mathrm{id}(A)<\mathrm{id}(A)^\omega$ and minimality of λ,

$$\lambda\leq \mathrm{id}(A)<\lambda^\omega .$$

A has an infinite pairwise disjoint family and hence has at least 2^ω ideals; thus, $2^\omega<\mathrm{id}(A)$ and $2^\omega<\lambda$. By the minimal choice of λ, $\kappa<\lambda$ implies that $\kappa^\omega<\lambda$. It follows that $\mathrm{cf}\,\lambda=\omega$ – otherwise $\lambda^\omega=\Sigma_{\kappa<\lambda}\kappa^\omega\leq\lambda\leq\mathrm{id}(A)$, a contradiction. Hence, there is a sequence $(\lambda_n)_{n\in\omega}$ of cardinals satisfying

$$\lambda=\sup_{n\in\omega}\lambda_n\,,\quad 2^\omega\leq\lambda_0<\lambda_1<\cdots,\quad \lambda_n^\omega=\lambda_n\,.$$

Claim 1. There is (without loss of generality) no b in A such that $\mathrm{id}(b)\geq\lambda_0$ and $\mathrm{id}(-b)\geq\lambda$.

For, suppose that for each $n\in\omega$ and $a\in A$ with $\mathrm{id}(a)\geq\lambda$, there is $b\leq a$ such that $\mathrm{id}(b)\geq\lambda_n$ and $\mathrm{id}(a\cdot -b)\geq\lambda$. Then by induction we get a pairwise disjoint family $(a_n)_{n\in\omega}$ in A such that $\mathrm{id}(a_n)\geq\lambda_n$; it follows by disjointness of the a_n that $\Pi_{n\in\omega}\lambda_n\leq\Pi_{n\in\omega}\mathrm{id}(a_n)\leq\mathrm{id}(A)$, and $\Pi_{n\in\omega}\lambda_n=\lambda^\omega$ (e.g. see Theorem 1.6(i) in the Appendix on Set Theory); a contradiction. So there are $a\in A$ and $n\in\omega$ such that $\mathrm{id}(a)\geq\lambda$ but for no $b\leq a$ we have both $\mathrm{id}(b)\geq\lambda_n$ and $\mathrm{id}(a\cdot -b)\geq\lambda$. Replacing $(\lambda_k)_{k\in\omega}$ by $(\lambda_k)_{n\leq k<\omega}$ and A by $A\restriction a$, we may assume that $n=0$ and $a=1$.

Step 2: Definition of M. The set

$$M=\{a\in A: \mathrm{id}(a)<\lambda_0\}$$

is clearly an ideal of A; by Claim 1, it is maximal. In particular, $|A|=|M|$.

Claim 2. $|A| < \lambda_0$, without loss of generality.

To see this, assume first for contradiction that $\lambda \leq |A|$ and consider the countably generated ideals of A included in M. If such an ideal I is generated by the countable subset X of M, then

$$I = \bigcup \{A \restriction s: s \text{ a sum of finitely many elements of } X\} ,$$

and $|I| \leq \omega \cdot \lambda_0 = \lambda_0$ since there are only countably many sums of finite subsets of X, and if s is one of these sums, then $s \in M$ and $|A \restriction s| \leq \mathrm{id}(s) < \lambda_0$. Each ideal I as considered above has at most $\lambda_0^\omega = \lambda_0$ countable subsets. So there are at least $|M|^\omega \geq \lambda^\omega$ countably generated subideals of M and $\lambda^\omega \leq \mathrm{id}(A)$, a contradiction. Thus, $|A| < \lambda$ and, without loss of generality, $|A| < \lambda_0$.

Step 3: Definition of Id′ and Id″. We partition the set $Id(A)$ of all ideals of A into two subsets: let

$$Id' = \{I \in Id(A): I \subseteq M\} , \qquad Id'' = Id(A) \setminus Id' .$$

Claim 3. $|Id''| \leq \lambda_0$ and thus $|Id'| \geq \lambda$.

For if $I \in Id''$, then there is $a \in M$ such that $-a \in I$; since $-a \in I$, I is determined by $I \restriction a$. There are at most λ_0 choices for $a \in M$, by Claim 2; for each $a \in M$, there are at most λ_0 choices for $I \restriction a$, by definition of M. Thus, $|Id''| \leq \lambda_0$.

Step 4: Definition of s_I and $\alpha(I)$, for $I \in Id'$. For $I \in Id'$, choose by Zorn's lemma an ordinal $\alpha(I)$ and a sequence

$$s_I = (a_i)_{i<\alpha(I)}$$

in A satisfying

$$a_i \in M , \quad a_i \notin I , \quad a_i \cdot a_j \in I \text{ for } i < j < \alpha(I)$$

and such that, for $\pi: A \to A/I$ canonical, $\{\pi(a_i): i < \alpha(I)\}$ is maximally pairwise disjoint in the set $\{\pi(a): a \in M\}$.

Claim 4. If $s_I = s_J = (a_i)_{i<\alpha}$ and $I \restriction a_i = J \restriction a_i$ for all $i < \alpha$, then $I = J$.

For assume there exists $x \in I \setminus J$. Then $x \in M$, $x \notin J$, and $x \cdot a_i \in I \restriction a_i = J \restriction a_i \subseteq J$ holds for all $i < \alpha$, contradicting the maximal choice of $(a_i)_{i<\alpha} = s_J$.

Step 5: Definition of σ. Let σ be the cardinal $s^*(\mathrm{Ult}\, A)$. By Lemma 10.11, it is the least cardinal κ such that A has no ideal-independent subset of size κ; moreover $\mathrm{cf}\, \sigma > \omega$.

Claim 5. $2^{<\sigma} = (2^{<\sigma})^\omega < \lambda_0$, without loss of generality.

$2^{<\sigma} = (2^{<\sigma})^\omega$ follows from $\mathrm{cf}\, \sigma > \omega$. For each $\kappa < \sigma$, $2^\kappa \leq \mathrm{id}(A)$ holds by the very definition of σ; so $2^{<\sigma} \leq \mathrm{id}(A)$. Since $2^{<\sigma} = (2^{<\sigma})^\omega$ but $\mathrm{id}(A) < \mathrm{id}(A)^\omega$, we obtain $2^{<\sigma} < \mathrm{id}(A)$ and, without loss of generality, $2^{<\sigma} < \lambda_0$.

Claim 6. For $I \in Id'$, there are, without loss of generality, at most λ_0 elements J of Id' satisfying $s_I = s_J$.

In fact, fix $I \in Id'$ and let $s_I = (a_i)_{i<\alpha}$. For $i < \alpha$, consider

$$Id^i = \{K \in Id(A \restriction a_i): a_i \notin K \text{ but } a_i \cdot a_j \in K, \text{ for } j \in \alpha \setminus \{i\}\} ,$$
$$\mu_i = |Id^i| .$$

Then

$$|\{J \in Id' : s_I = s_J\}| \le \prod_{i<\alpha} \mu_i$$

since, by Claim 4, each $J \in Id'$ satisfying $s_I = s_J$ is determined by the sequence $(J \restriction a_i)_{i<\alpha}$, and $J \restriction a_i \in Id^i$ for all $i < \alpha$. Next,

$$\prod_{i<\alpha} \mu_i \le \mathrm{id}(A) ,$$

since the map assigning, to each sequence $(K_i)_{i<\alpha}$ where $K_i \in Id^i$, the ideal generated by $\bigcup_{i<\alpha} K_i$, is one-to-one. We claim that the cardinal $\mu = \prod_{i<\alpha} \mu_i$ satisfies $\mu < \lambda$.

This is trivial if μ is finite. Also, if $\mu = \mu_i$ for some i, then $\mu < \lambda_0$ since, by $a_i \in M$, $\mu_i \le \mathrm{id}(a_i) < \lambda_0$. If μ is infinite and $\mu > \mu_i$ for all $i < \alpha$, then $\mu^\omega = \mu$ (see, for example, Theorem 1.6(iii) in the Appendix on Set Theory). Since $\mu \le \mathrm{id}(A)$ and $\mathrm{id}(A) < \mathrm{id}(A)^\omega$, it follows that $\mu < \lambda$.

We have thus assigned, to each $I \in Id'$, a cardinal $\mu(I) < \lambda$ having the form $\prod_{i<\alpha(I)} \mu_{i,I}$ where (by ideal-independence of $(a_i)_{i<\alpha(I)}$) $\alpha(I) < \sigma$ and $\mu_{i,I} < \lambda_0$. Application of Theorem 1.6(iv) in the Appendix on Set Theory to σ and $\chi = \lambda_0$ shows that the set $\{\mu(I): I \in Id', \lambda_0 < \mu(I)\}$ is finite. So there is $n \in \omega$ such that $\mu(I) < \lambda_n$ for all $I \in Id'$; without loss of generality, $n = 0$. This proves Claim 6.

Claim 7. There is some $\kappa < \sigma$ such that $\lambda \le |A|^\kappa$.

The set Id' defined in Step 3 has, by Claim 3, at least λ elements. By Claim 6, there are at least λ elements I of Id' with pairwise distinct sequences s_I. Since $(a_i)_{i<\alpha(I)}$ is an ideal-independent sequence in A, the length $\alpha(I)$ of s_I is less than σ. Thus, there are at least λ sequences of length less than σ in A, and $\lambda \le |A|^{<\sigma}$. It follows from $\mathrm{cf}\,\lambda = \omega$ and $\mathrm{cf}\,\sigma > \omega$ that $\lambda \le |A|^\kappa$, for some $\kappa < \sigma$.

Step 6: *Definition of δ and χ.* Fix $\kappa < \sigma$ as guaranteed by Claim 7 and put

$$\delta = \max(\kappa^+, \mathrm{sat}\, A)$$

(sat A has been defined, in Section 3, to be the least cardinal ρ such that A has no pairwise disjoint family of size ρ). Let χ be minimal with respect to the property that

$$|A| \le \chi^{<\delta} .$$

We shall reach a final contradiction by applying Proposition 10.14 to the cardinals δ and χ. First, δ is regular by Corollary 3.11 of the Erdös–Tarski theorem. Also, A satisfies the δ-chain condition and, by sat $A \le \sigma$, we obtain $\delta \le \sigma$. By regularity of δ and the minimal choice of χ, χ is δ-closed, in the sense defined preceding 10.13. The definition of χ implies that $\chi \le |A|$. So by Proposition 10.14, A has an ideal-independent subset of size χ; this gives $\chi < \sigma$ and $2^\chi \le \mathrm{id}(A)$. Some cardinal arithmetic now shows that also $\mathrm{id}(A) \le 2^\chi$ and thus that $\mathrm{id}(A) = 2^\chi$ satisfies $\mathrm{id}(A)^\omega = \mathrm{id}(A)$, finishing the proof by contradiction:

$$
\begin{aligned}
2^{<\delta} &\le 2^{<\sigma} && \text{by } \delta \le \sigma \\
&< \lambda_0 && \text{by Claim 5} \\
&< \lambda \\
&\le |A|^{<\delta} && \text{by } \kappa < \delta \text{ and Claim 7}.
\end{aligned}
$$

So $2^{<\delta} < \chi$ by our choice of χ. It follows that $\lambda \le |A|^{<\delta} \le \chi^{<\delta}$, $\delta \le 2^{<\delta} \le \chi$, $\lambda \le \chi^{<\delta} \le \chi^{<\chi} \le 2^\chi$ and $\mathrm{id}(A) \le \lambda^\omega \le 2^\chi$. □

10.3. A characterization of independence

The two major results of this subsection, Theorems 10.16 and 10.17, are stated in terms of a cardinal invariant ind* defined as follows.

10.15. DEFINITION. Let A be a Boolean algebra. For D and X subsets of A, D is *dense in* X if for every $x \in X \backslash \{0\}$, there is some $d \in D$ such that $0 < d \le x$ (D is not required to be a subset of X). For any filter q in A,

$$\pi\chi_A(q) = \min\{|D|: D \subseteq A,\ D \text{ dense in } q\}$$

is the *pseudo-character* of q in A.

$$\pi\chi A = \min\{\pi\chi_A(p): p \in \mathrm{Ult}\ A\}$$

is the *pseudo-character* of A. Finally, let

$$\mathrm{ind}^*\ A = \sup\{\pi\chi A': A' \text{ a non-trivial homomorphic image of } A\}.$$

10.16. THEOREM (Šapirovskii). *If A has an infinite independent subset* (*i.e. by* 10.19, *if* $\mathrm{ind}^*\ A \ge \omega$), *then* $\mathrm{ind}\ A = \mathrm{ind}^*\ A$.

10.17. THEOREM (Šapirovskii). *Let A be a subalgebra of B; for $p \in \mathrm{Ult}\ A$ let $\bar{p}$ be the filter of B generated by p. Then*

$$\mathrm{ind}^*\ B = \max(\mathrm{ind}^*\ A,\ \sup\{\mathrm{ind}^*(B/\bar{p}): p \in \mathrm{Ult}\ A\}).$$

In the terminology of sheafs (see Section 8), Theorem 10.17 implies that if $\mathcal{S} = (S, \pi, X, (B_p)_{p \in X})$ is the sheaf associated with the pair (A, B), then the value of ind* for the algebra $\Gamma(\mathcal{S})$ of global sections is determined by the value of ind* for Clop X and the stalks B_p. Our proof will, however, not use any sheaf theory.

The term pseudo-character for $\pi\chi_A(q)$ comes from the dual topological situation where the filter q corresponds to a closed subset of Ult A – cf. the remarks following Theorem 7.25. ind* A could be more properly written as $h\pi\chi A$ (the hereditary pseudo-character of A), but in view of Theorem 10.16, we prefer the notation ind*.

If p is an ultrafilter of A, then clearly $\pi\chi_A(p) = 1$ iff p is generated by an atom of A, and $\pi\chi_A(p) \geq \omega$ otherwise. Hence, $\pi\chi A = 1$ iff A has at least one atom, and $\pi\chi A \geq \omega$ otherwise. It follows that ind* $A = 1$ iff every non-trivial homomorphic image of A has an atom, and ind* $A \geq \omega$ otherwise.

10.18. DEFINITION. A Boolean algebra A is *superatomic* if every non-trivial homomorphic image of A has an atom.

Superatomic algebras are investigated in greater detail in Section 17 where, in particular, several attractive equivalences to superatomicity are given. Cf. also the survey chapter by ROITMAN [Ch. 19 in this Handbook] for special questions on superatomic Boolean algebras.

Let us recall two obvious facts on independence: first, a Boolean algebra A has an independent subset of cardinality κ iff Fr κ, the free algebra on κ generators, embeds into A. Second, if $f\colon A \to A'$ is a homomorphism of Boolean algebras, f is one-to-one on $X \subseteq A$ and $f[X]$ is independent in A', then X is independent in A.

10.19. LEMMA. (a) *If A has no infinite independent subset, then A is superatomic.*

(b) *If A has an independent subset of size $\kappa \geq \omega$, then A has a homomorphic image A' such that $\pi\chi A' \geq \kappa$.*

In particular, ind* $A = 1$ *iff A is superatomic iff A has no infinite independent subset.*

PROOF. (a) Suppose A' is an atomless non-trivial homomorphic image of A. It is not difficult to see that A' has a countably infinite atomless subalgebra F'. By 9.11 and 5.16, F' is isomorphic to Fr ω; so each of F', A' and A has an infinite independent subset.

(b) Assume that Fr κ is a subalgebra of A. By the Corollary 5.10 to Sikorski's extension theorem, there is an epimorphism π from A onto an algebra A' having Fr κ as a dense subalgebra. Let p be an arbitrary ultrafilter of A'; we show that $\pi\chi_{A'}(p) \geq \kappa$. Otherwise, pick $Y \subseteq A'$ such that $0 \notin Y$, $|Y| < \kappa$ and Y is dense in p. By denseness of Fr κ in A', we may assume that $Y \subseteq$ Fr κ. Let U be a set of free generators for Fr κ, and for $y \in Y$, let U_y be a finite subset of U generating y. Then $U' = \bigcup_{y \in Y} U_y$ has cardinality less than κ, so pick $u \in U \setminus U'$. Now either $u \in p$ or $-u \in p$. If $u \in p$, then by denseness of Y in p, $0 < y \leq u$ for some $y \in Y$, a contradiction since y is generated by U', $u \in U \setminus U'$ and U is independent. Similarly, $-u \in p$ leads to a contradiction. □

10.20. LEMMA. *Let C be a subalgebra of A such that $|C| < \pi\chi A$. Then there are $a \in A$ and $r \in C^+$ such that for each $x \in (C \restriction r)^+$, $x \cdot a > 0$ and $x \cdot -a > 0$.*

PROOF. Every ultrafilter p of A has an element a_p such that, for every $x \in C^+$, $x \cdot -a_p > 0$; this is because C^+ is not dense in p. Since $a_p \in p$, the set $\{-a_p\colon p \in$ Ult $A\}$ is not included in any ultrafilter of A and does, by the Boolean prime ideal theorem 2.16, not have the finite intersection property. So there is a finite subset Y of $\{-a_p : p \in \mathrm{Ult}\, A\}$ such that $\prod Y = 0$; note that, by definition of the elements a_p, $x \cdot y > 0$ for x in C^+ and y in Y. Let Z be a subset of Y which is minimal with respect to the property that, for some r in C^+,

$$r \cdot \prod Z = 0 . \tag{9}$$

Z is non-empty since $r > 0$. Fix an element a of Z; by (9), $r \cdot a \cdot \prod (Z \setminus \{a\}) = 0$ and hence

$$r \cdot \prod (Z \setminus \{a\}) \leq -a .$$

Consider an arbitrary element x of $(C \restriction r)^+$. Then $x \cdot a > 0$ since $a \in Z \subseteq Y$. Also,

$$\begin{aligned} 0 &< x \cdot \prod (Z \setminus \{a\}) \qquad \text{by minimality of } Z \\ &= x \cdot r \cdot \prod (Z \setminus \{a\}) \\ &\leq x \cdot -a . \quad \square \end{aligned}$$

10.21. LEMMA. *Let λ be a regular uncountable cardinal such that $\lambda \leq \pi\chi A$. Then A has an independent subset of cardinality λ.*

PROOF. We define by induction elements a_α and r_α of A and subalgebras A_α of A, for $\alpha < \lambda$, as follows. Given a_β for $\beta < \alpha$, let

$$A_\alpha = \langle a_\beta : \beta < \alpha \rangle .$$

Since $|A_\alpha| < \lambda \leq \pi\chi A$, choose by Lemma 10.20 $a_\alpha \in A$ and $r_\alpha \in (A_\alpha)^+$ such that

$$x \cdot a_\alpha > 0 , \qquad x \cdot -a_\alpha > 0 \quad \text{for } x \in (A_\alpha \restriction r_\alpha)^+ .$$

In particular, $a_\alpha \notin A_\alpha$. So $|\bigcup_{\alpha<\lambda} A_\alpha| = \lambda$; also $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$ for each limit ordinal α. Application of Lemma 10.8 gives an $r \in A^+$ and a stationary subset T of λ such that $r_\alpha = r$ for each $\alpha \in T$. By Lemma 10.6, $\{a_\alpha : \alpha \in T\}$ is an independent subset of A of size λ. $\square$

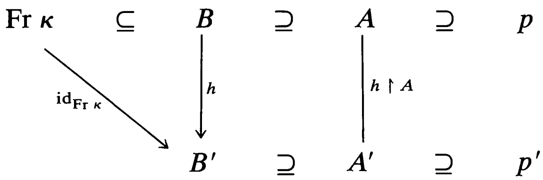
Proof of Theorem 10.16. Assume that A has an infinite independent subset, i.e. that $\mathrm{ind}^* A \geq \omega$. Then Lemma 10.19(b) implies that $\mathrm{ind}\, A \leq \mathrm{ind}^* A$. The converse, $\mathrm{ind}^* A \leq \mathrm{ind}\, A$, will follow if we can prove that for every non-limit cardinal $\lambda \leq \mathrm{ind}^* A$, also $\lambda \leq \mathrm{ind}\, A$ holds. But if $\lambda \leq \mathrm{ind}^* A$ is a successor cardinal, pick a homomorphic image A' of A such that $\lambda \leq \pi\chi A'$. By Lemma 10.21, A' and hence A has an independent subset of size λ. $\square$

Let us note, as a consequence of 10.16 and 10.19, that if A is a subalgebra or a homomorphic image of B, then $\mathrm{ind}^* A \leq \mathrm{ind}^* B$. This is trivial for homomorphic images; so suppose A is a subalgebra of B. If $\text{ind}^* B = 1$, then B is superatomic and so is A, by 10.19; thus $\text{ind}^* A = 1$. If $\text{ind}^* B \geq \omega$, then $\text{ind}^* A \geq (\text{ind}^* B)^+$ would give, by 10.16, an independent subset of A having size $(\text{ind}^* B)^+$; so $\text{ind}\, B \geq (\text{ind}^* B)^+$, a contradiction.

Proof of Theorem 10.17. It follows from the preceding remark that $\text{ind}^* A \leq \text{ind}^* B$ and, for each p in Ult A, $\text{ind}^*(B/\bar{p}) \leq \text{ind}^* B$. So our theorem follows if $\text{ind}^* B = 1$.

Thus, assume that $\text{ind}^* B \geq \omega$. It suffices, by Theorem 10.16, to prove that for each infinite cardinal κ, if $\text{Fr}\,\kappa$ embeds into B, then $\kappa \leq \text{ind}^* A$ or $\kappa \leq \text{ind}^*(B/\bar{p})$ for some p in Ult A.

Suppose $\text{Fr}\,\kappa$ is a subalgebra of B but $\text{ind}^* A < \kappa$. By Corollary 5.10, we find a Boolean algebra B' and an epimorphism $h: B \to B'$ such that B' has $\text{Fr}\,\kappa$ as a dense subalgebra and h extends the identity map on $\text{Fr}\,\kappa$.



Let $A' = h[A]$ and fix an ultrafilter p' of A' such that

$$\pi\chi_{A'}(p') = \pi\chi A' ;$$

we will show that the ultrafilter $p = (h \upharpoonright A)^{-1}[p']$ of A satisfies $\text{ind}^*(B/\bar{p}) \geq \kappa$.

To this end, let $\overline{p'}$ be the filter of B' generated by p'; then

$$\pi\chi_{B'}(\overline{p'}) \leq \pi\chi_{A'}(p') = \pi\chi A' \leq \text{ind}^* A < \kappa .$$

Here the first inequality holds since every subset of A' dense in p' is also, being a subset of B', dense in $\overline{p'}$, and the second one since A' is a homomorphic image of A. Since $\text{Fr}\,\kappa$ is dense in B', fix $D \subseteq (\text{Fr}\,\kappa)^+$ such that $|D| < \kappa$ and D is dense in $\overline{p'}$.

In B', fix an independent set U of generators for $\text{Fr}\,\kappa$. There is $V \subseteq U$ such that $|V| < \kappa$ and $D \subseteq \langle V \rangle$; let $W = U \setminus V$. So $|W| = \kappa$ and the subalgebra $\langle W \rangle$ of $\text{Fr}\,\kappa$ is free over W. We claim that the canonical epimorphism

$$k: B' \to B'/\overline{p'}$$

is a monomorphism on $\langle W \rangle$. For let x be a non-zero element of $\langle W \rangle$. If $k(x) = 0$, then $-x \in \overline{p'}$; by denseness of D in $\overline{p'}$, there is $d \in D$ such that $0 < d \leq -x$, i.e. $d \cdot x = 0$. This is impossible since $d \in \langle V \rangle$, $x \in \langle W \rangle$ and U is independent. So we have shown that $B'/\overline{p'}$ has an independent subset of size κ.

Finally, by the homomorphism theorem 5.23, $B'/\overline{p'}$ is a homomorphic image of $B/\bar{p}$, since $\bar{p}$ is contained in the dual kernel of $k \circ h$. By 10.16 again, it follows that

$$\kappa \leq \text{ind}(B'/\overline{p'}) \leq \text{ind}(B/\bar{p}) = \text{ind}^*(B/\bar{p}) . \quad \square$$

Exercises

1. Assume B is a complete algebra satisfying the countable chain condition and $(B_\alpha)_{\alpha<\rho}$ an increasing sequence of subalgebras such that B_0 is a complete subalgebra of B, $B_\lambda = \bigcup_{\alpha<\lambda} B_\alpha$ if $\lambda < \rho$ is a limit ordinal, and for each successor ordinal $\alpha < \rho$, B_α is a complete subalgebra of B. Show that, for $\lambda < \rho$ limit, B_λ is a complete subalgebra of B iff $\mathrm{cf}\,\lambda > \omega$.

Hint. Use the proof of 4.22.

2. Let A be the subalgebra of $P(\omega)$ generated by the singletons $\{n\}$, $n \in \omega$, plus 2^ω almost disjoint subsets of ω (see the proof of 5.28). Prove that A is superatomic. Thus, $|A| = 2^\omega$, A satisfies the countable chain condition but has no infinite independent subset.

3. Show that every Boolean algebra of size at least $(2^\omega)^+$ which satisfies the countable chain condition has a complete algebra of size $(2^\omega)^+$ as a homomorphic image.

4. Prove that for every Boolean algebra A,

$$\begin{aligned} s(\mathrm{Ult}\,A) &= \sup\{|\mathrm{At}\,A'|\colon A' \text{ a homomorphic image of } A\} \\ &= \sup\{|\mathrm{At}\,A'|\colon A' \text{ an atomic homomorphic image of } A\} \\ &= \sup\{\mathrm{c}A'\colon A' \text{ a homomorphic image of } A\}\,; \end{aligned}$$

here sX is the spread of a topological space X as defined before 10.11 and $\mathrm{c}A$ is the cellularity of A.

5. Compute the spread of Ult A for
(a) A a free Boolean algebra,
(b) A a power set algebra,
(c) A the interval algebra of the real line.

6. Let A be a Boolean algebra generated by the union of two subalgebras A_1 and A_2. Show that if A_1 and A_2 are superatomic, then so is A.

11. Free products

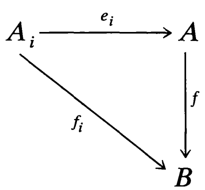
This section introduces another construction of new Boolean algebras from old ones, the free product of a family $(A_i)_{i\in I}$. The most natural interpretation of free products is given by Stone duality and shows their bearing on topology – the dual space of the free product of $(A_i)_{i\in I}$ is simply the cartesian product of the spaces Ult A_i. The denotation of "free product" comes from universal-algebraic considerations similar to those for free Boolean algebras: the free product of $(A_i)_{i\in I}$ has the A_i as subalgebras and is generated by their union in such a way that only those non-trivial algebraic equations can hold for the elements of $\bigcup_{i\in I} A_i$ which are forced to hold by the internal structure of the A_i. In the final subsection we consider a generalization, the amalgamated free product of a family $(A_i)_{i\in I}$ over a common subalgebra C. It is generated by the A_i in such a way that non-trivial algebraic equations can hold for the elements of $\bigcup_{i\in I} A_i$ only if forced to hold by the structure of the A_i and the way how C lies, as a subalgebra, in A_i.

Free products provide a natural tool for the construction of embeddings with prescribed properties. For example, every Boolean algebra B embeds into an atomless one (respectively a homogeneous one), simply by considering B as a subalgebra of a free product $B \oplus C$ for suitable C (cf. Exercise 1 and Theorem 11.10). Also free products are a source of counterexamples, in particular for questions on cellularity; they show that for each weakly inaccessible cardinal κ, there is a Boolean algebra A with $\mathrm{c}A = \kappa$ not attained (Example 11.14), which complements the Erdös–Tarski theorem 3.10. It is an attractive problem whether the free product of a family $(A_i)_{i \in I}$ satisfies the κ-chain condition if each of the A_i does. This question readily reduces to free products of finitely many factors; see Exercises 3 and 4. Even the most simple special case is known to be independent from the axioms of ZFC set theory: under Martin's axiom plus the negation of the continuum hypothesis, a free product $B \oplus C$ satisfies the countable chain condition if both B and C do (cf. KUNEN [1980]); under the continuum hypothesis however, GALVIN [1980] gives a counterexample. TODORČEVIĆ [1986] constructs, in ZFC, an example of an uncountable cardinal κ and a Boolean algebra B such that $\mathrm{c}(B) = \kappa$ and $\mathrm{c}(B \oplus B) > \kappa$.

11.1. Free products

The free product of a family of Boolean algebras is defined, similar to the notion of a free Boolean algebra, by a universal property concerning extendibility of maps to homomorphisms.

11.1. DEFINITION. Let $(A_i)_{i \in I}$ be a family of Boolean algebras. A pair $((e_i)_{i \in I}, A)$ is a *free product of* $(A_i)_{i \in I}$ if A is a Boolean algebra, each e_i is a homomorphism from A_i into A and, for every family $(f_i)_{i \in I}$ of homomorphisms from A_i into any Boolean algebra B, there is a unique homomorphism $f: A \to B$ such that $f \circ e_i = f_i$ for $i \in I$.



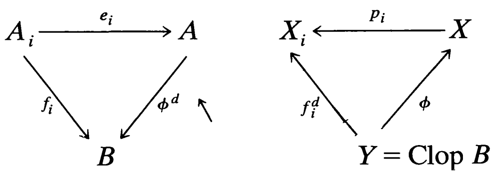
The defining property of free products is the category-theoretic dual of that one stated in Proposition 6.3 as being characteristic of products; so the free product of a family of Boolean algebras is its coproduct in the category of Boolean algebras. Existence and uniqueness of free products now follow immediately from standard arguments of category theory plus some Stone duality.

11.2. THEOREM (existence and uniqueness). *Every family of Boolean algebras has, up to isomorphism, a unique free product.*

PROOF. To be definite, let us formulate the uniqueness assertion: if $((e_i)_{i \in I}, A)$

and $((e'_i)_{i\in I}, A')$ are free products of $(A_i)_{i\in I}$, then there is a unique isomorphism $h: A \to A'$ such that $h \circ e_i = e'_i$ for $i \in I$. This follows like the uniqueness assertion 9.2 on free Boolean algebras or the second part of Proposition 6.3 on the characterization of products.

For existence, let $(A_i)_{i\in I}$ be a family of Boolean algebras. Let X_i be the dual space of A_i, X the product space of the X_i and p_i: $X \to X_i$ the projection map. It is a standard fact of topology that the pair $((p_i)_{i\in I}, X)$ has, in the category of Boolean spaces and continuous maps, the universal property of products stated in 6.3 for product algebras. Now let $A = \mathrm{Clop}\, X$ be the dual algebra of X, e_i: $\mathrm{Clop}\, X_i \to A$ the homomorphism p_i^d dual to p_i (cf. Theorem 8.2), and identify A_i with $\mathrm{Clop}\, X_i$. By the duality theorem 8.2, $((e_i)_{i\in I}, A)$ is a coproduct of $(A_i)_{i\in I}$ in the category of Boolean algebras and Boolean homomorphisms, as indicated in the diagrams below.



□

For many practical purposes, the abstract characterization of free products has to be replaced by a more down-to-earth one. This is done along the lines of the characterization 9.4 of free algebras.

11.3. DEFINITION. A family $(B_i)_{i\in I}$ of subalgebras of a Boolean algebra A is *independent* if, for arbitrary $n \in \omega$, pairwise distinct $i(1), \ldots, i(n) \in I$ and non-zero elements $b_{i(k)}$ of $B_{i(k)}$,

$$b_{i(1)} \cdot \cdots \cdot b_{i(n)} > 0 \quad \text{in } A\,.$$

Independent families of subalgebras occur naturally in free algebras: assume F is free over $U \subseteq F$ and $(U_i)_{i\in I}$ is a family of pairwise disjoint subsets of U. Then Remark 9.15 shows that the subalgebras $\langle U_i \rangle$ of F constitute an independent family.

The main and most intuitive example of an independent family of Boolean algebras runs as follows. Assume that, for $i \in I$, X_i is a set and g_i: $A_i \to P(X_i)$ is a monomorphism. Let X be the cartesian product of the sets X_i, p_i: $X \to X_i$ the projection map and define embeddings

$$e_i\colon A_i \to P(X)$$

by

$$e_i(a_i) = p_i^{-1}[g_i(a_i)]\,.$$

Then the subalgebras $e_i[A_i]$ of $P(X)$ are independent. Moreover, Proposition 11.4 asserts that $((e_i)_{i\in I}, A)$ is a free product of $(A_i)_{i\in I}$, where A is the subalgebra of $P(X)$ generated by the union of the $e_i[A_i]$.

11.4. PROPOSITION (characterization). *Let A be a Boolean algebra and, for $i \in I$, e_i: $A_i \to A$ a homomorphism; assume that no A_i is trivial. The pair $((e_i)_{i\in I}, A)$ is a free product of $(A_i)_{i\in I}$ iff each of (a) through (c) holds:*
(a) *each e_i: $A_i \to A$ is one-to-one,*
(b) *$(e_i[A_i])_{i\in I}$ is an independent family of subalgebras of A,*
(c) *A is generated by $\bigcup_{i\in I} e_i[A_i]$.*
Moreover, if $((e_i)_{i\in I}, A)$ is a free product of $(A_i)_{i\in I}$, then
(d) *$e_i[A_i] \cap e_j[A_j] = 2$, for $i \neq j$.*

PROOF. Assume first that $((e_i)_{i\in I}, A)$ is a free product of $(A_i)_{i\in I}$. If (a) fails, i.e. if e_j is not one-to-one for some j, then consider, in Definition 11.1, the algebra $B = A_j$ and an arbitrary family of homomorphisms f_i: $A_i \to B$ such that $f_j = \mathrm{id}_{A_j}$. Clearly, there is no homomorphism f: $A \to B$ satisfying $f \circ e_j = f_j = \mathrm{id}_{A_j}$; a contradiction.

For the rest of the proof, we may assume that each A_i is a subalgebra of A and that e_i: $A_i \to A$ is the inclusion map.

If (c) fails, then by Lemma 5.32 there are distinct homomorphisms f and f' from A into 2 coinciding on $\langle \bigcup_{i\in I} A_i \rangle$; let f_i: $A_i \to 2$ be the restriction of f to A_i. Then f and f' are distinct homomorphisms from A into 2 extending each f_i, which contradicts the uniqueness assertion in Definition 11.1.

If (b) fails, assume $a_{i(1)} \cdot \cdots \cdot a_{i(n)} = 0$ provides a counterexample where $a_{i(k)} \in A^+_{i(k)}$ and the $i(k)$ are distinct. For each $i \in I$, fix a homomorphism f_i: $A_i \to 2$ in such a way that $f_{i(k)}(a_{i(k)}) = 1$; this is possible since $a_{i(k)} > 0$. If f: $A \to 2$ is a homomorphism extending every f_i, then

$$\begin{aligned} 0 &= f(a_{i(1)} \cdot \cdots \cdot a_{i(n)}) \\ &= f_{i(1)}(a_{i(1)}) \cdot \cdots \cdot f_{i(n)}(a_{i(n)}) \\ &= 1\,, \end{aligned}$$

a contradiction.

Conversely, suppose $((e_i)_{i\in I}, A)$ satisfies conditions (a) through (c) and that f_i: $A_i \to B$ are homomorphisms. Since A is generated by $\bigcup_{i\in I} A_i$, there is at most one homomorphism f: $A \to B$ extending each f_i. To prove existence of such a homomorphism let $r \subseteq A \times B$ be the relation defined by

$$(x, b) \in r \quad \text{iff for some } i \in I \text{ and } a_i \in A_i,\ x = a_i \text{ and } b = f_i(a_i)\,.$$

We apply the version 5.6 of Sikorski's extension criterion: let $(x_1, b_1), \ldots, (x_n, b_n) \in r$ and $\varepsilon_1, \ldots, \varepsilon_n \in \{+1, -1\}$ such that

$$p = \varepsilon_1 x_1 \cdot \cdots \cdot \varepsilon_n x_n = 0$$

with the aim of proving that

$$q = \varepsilon_1 b_1 \cdot \cdots \cdot \varepsilon_n b_n = 0\,.$$

Say $x_k = a_{i(k)} \in A_{i(k)}$ and $b_k = f_{i(k)}(a_{i(k)})$. Since each $A_{i(k)}$ is a subalgebra of A and

$f_{i(k)}$ is a homomorphism, we may collect factors of p (respectively q) arising from the same $A_{i(k)}$ and thus assume the $i(k)$ to be pairwise distinct. By independence of the subalgebras A_i, $p=0$ implies that some $\varepsilon_k a_{i(k)}$ is zero. So, $f_{i(k)}$ being a homomorphism, $\varepsilon_k b_k = 0$ and $q=0$.

Finally, condition (b) immediately implies (d): suppose for contradiction that there are i, j in I and $b \in e_i[A_i] \cap e_j[A_j]$ such that $i \neq j$ and $0<b<1$ (we do not assume (a) here and thus have to distinguish A_i from $e_i[A_i]$). Letting $i(1)=i$, $i(2)=j$, $b_{i(1)}=b$ and $b_{i(2)}=-b$ then gives $b_{i(1)} \cdot b_{i(2)}=0$, contradicting independence of the subalgebras $e_i[A_i]$ of A. □

Given existence and uniqueness of free products, we set up the following notation.

11.5. NOTATION AND CONVENTIONS. In a free product $((e_i)_{i\in I}, A)$ of a family $(A_i)_{i\in I}$ of Boolean algebras, we denote the algebra A by $\oplus_{i\in I} A_i$. For $I = \{1, \ldots, n\}$ finite, we write $A = A_1 \oplus \cdots \oplus A_n$. Since each $e_i\colon A_i \to A$ is a monomorphism, we generally identify A_i with $e_i[A_i]$ if $A_i \cap A_j = 2$ for $i \neq j$, and simply call the algebra $\oplus_{i\in I} A_i$ the free product of $(A_i)_{i\in I}$.

We will henceforth generally assume that each A_i is a subalgebra of $A = \oplus_{i\in I} A_i$ and can then characterize A by the fact that $(A_i)_{i\in I}$ is a family of independent subalgebras of A whose union generates A. Even if $A_i \cap A_j \neq 2$ for some $i \neq j$, we might replace the A_i by isomorphic copies A_i' satisfying $A_i' \cap A_j' = 2$ for $i \neq j$ and then identify $\oplus_{i\in I} A_i$ with $\oplus_{i\in I} A_i'$, assuming each A_i' is a subalgebra of $\oplus_{i\in I} A_i'$. In particular, this applies to notation as $C \oplus C$, etc. Of course all this is an abuse of notation, and if desperate, one should use the correct notation $((e_i)_{i\in I}, A)$.

It should be clear that formation of free products is an associative and commutative operation, up to isomorphism. For example, if I is the union of a disjoint family $(I(k))_{k\in K}$, then

$$\bigoplus_{i\in I} A_i \cong \bigoplus_{k\in K} \Big(\bigoplus_{i\in I(k)} A_i \Big),$$

etc. We list some additional elementary facts on isomorphism of free products. Since, by the existence proof for free products,

$$\text{(1)} \qquad \operatorname{Ult}\Big(\bigoplus_{i\in I} A_i \Big) \cong \prod_{i\in I} \operatorname{Ult} A_i \,,$$

results of this type are often easily proved by applying the topological duality of Section 8.

11.6. EXAMPLES. (a) If each A_i is a four-element algebra, then $\bigoplus_{i\in I} A_i$ is the free Boolean algebra on $|I|$ independent generators.

(b) $B \oplus 2 \cong B$ and $B \oplus 1 \cong 1$ for any Boolean algebra B; here 1 denotes the trivial Boolean algebra.

(c) $(B \times C) \oplus D \cong (B \oplus D) \times (C \oplus D)$, for arbitrary Boolean algebras B, C and D.

(d) $B \oplus 2^n \cong B^n$, for any Boolean algebra B.

PROOF. (a) The Stone space of each A_i is a two-point discrete space. So $\mathrm{Ult}(\oplus_{i \in I} A_i)$ is by (1) homeomorphic to the Cantor space ${}^I 2$, and $\oplus_{i \in I} A_i \cong \mathrm{Fr}\,|I|$ by 9.7(a).

(b) follows from (1) since the Stone space of 2 is a one-point space and the Stone space of 1 is empty.

(c) For any Boolean algebras A_1 and A_2, $\mathrm{Ult}(A_1 \times A_2)$ is the disjoint union space $\mathrm{Ult}\, A_1 \cup \mathrm{Ult}\, A_2$, by Proposition 8.7, and $(X \cup Y) \times Z \cong (X \times Z) \cup (Y \times Z)$ holds for arbitrary topological spaces.

(d) follows from (c) and (b).

Let us have a closer look at the free product $B \oplus C$ of just two Boolean algebras B and C. We assume that B and C are independent subalgebras of $B \oplus C$ and that $B \cup C$ generates $B \oplus C$. By a remark following 4.7, each element x of $B \oplus C$ has a representation

(2) $$x = b_1 \cdot c_1 + \cdots + b_n \cdot c_n ,$$

where $n \in \omega$, $b_i \in B$ and $c_i \in C$. Moreover, by independence, $b \cdot c = 0$, where $b \in B$ and $c \in C$, implies that $b = 0$ or $c = 0$. The following lemma is frequently used when dealing with the normal form (2) in $B \oplus C$.

11.7. LEMMA. *Let $b_i, b'_j \in B$ and $c_i, c'_j \in C$ for $1 \leq i \leq n$, $1 \leq j \leq m$. Then*

$$b_1 \cdot c_1 + \cdots + b_n \cdot c_n \leq b'_1 \cdot c'_1 + \cdots + b'_m \cdot c'_m \quad \textit{in } B \oplus C$$

iff, for every $i \in \{1, \ldots, n\}$ and every $J \subseteq \{1, \ldots, m\}$,

$$b_i \leq \Sigma \{b'_j : j \in J\} \quad \textit{or} \quad c_i \leq \Sigma \{c'_j : j \notin J\} .$$

PROOF. Denote by l (respectively r) the left-hand side and the right-hand side of the inequality under consideration. Since $l \leq r$ iff $b_i \cdot c_i \leq r$ for every i, fix $i \in \{1, \ldots, n\}$. Now

$$\begin{aligned} b_i \cdot c_i \leq r \quad & \text{iff } b_i \cdot c_i \cdot -r = 0 \\ & \text{iff } b_i \cdot c_i \cdot \Pi \{-b'_j + -c'_j : 1 \leq j \leq m\} = 0 . \end{aligned}$$

Evaluating this last product by distributivity, we find that $b_i \cdot c_i \leq r$ iff, for each $J \subseteq \{1, \ldots, m\}$,

$$b_i \cdot c_i \cdot \Pi \{-b'_j : j \in J\} \cdot \Pi \{-c'_j : j \notin J\} = 0 .$$

By independence of B and C in $B \oplus C$, this means that, for each J,

$b_i \cdot \Pi\{-b'_j\colon j \in J\} = 0$ or $c_i \cdot \Pi\{-c'_j\colon j \not\in J\} = 0$, i.e. that $b_i \leq \Sigma\{b'_j\colon j \in J\}$ or $c_i \leq \Sigma\{c'_j\colon j \not\in J\}$. □

In 8.19, a Boolean algebra B was said to be relatively complete in D if B is a subalgebra of D and for each d in D there is a greatest b in B satisfying $b \leq d$ or, equivalently, for each d in D there is a least b in B satisfying $d \leq b$. B is a regular subalgebra of D (cf. Definition 1.29) if the inclusion map from B into D preserves all sums existing in B.

11.8. PROPOSITION. *B is relatively complete in $B \oplus C$, hence a regular subalgebra of $B \oplus C$.*

PROOF. The first assertion can be derived from the equivalence of (a) and (b) in Proposition 8.20 (the proof of this equivalence didn't use any sheaf theory), since the continuous map dual to the inclusion homomorphism from B into $B \oplus C$ is the projection from $\mathrm{Ult}(B \oplus C) \cong \mathrm{Ult}\, B \times \mathrm{Ult}\, C$ onto the first coordinate, an open map. For a purely algebraic proof, we indicate how to find, for every $x \in B \oplus C$, a least element β of B satisfying $x \leq \beta$. Writing x in the normal form (2), we may assume that $c_i \neq 0$ for $1 \leq i \leq n$. Then for $b \in B$,

$$\begin{aligned} x \leq b \quad & \text{iff } b_i \cdot c_i \leq b \text{ for every } i \\ & \text{iff } b_i \cdot -b \cdot c_i = 0 \text{ for every } i \\ & \text{iff } b_i \cdot -b = 0 \text{ for every } i \quad \text{by independence of } B \text{ and } C \\ & \text{iff } b_1 + \cdots + b_n \leq b\,. \end{aligned}$$

Thus, $\beta = b_1 + \cdots + b_n$ works for our claim. The second assertion follows immediately. □

11.9. PROPOSITION. *If both B and C are infinite, then $B \oplus C$ is not σ-complete.*

PROOF. By Proposition 3.4, let $(b_n)_{n\in\omega}$ and $(c_n)_{n\in\omega}$ be countably infinite pairwise disjoint families in B (respectively C) and assume that $\{b_n \cdot c_n\colon n \in \omega\}$ has a least upper bound in $B \oplus C$, say

$$\Sigma\{b_n \cdot c_n\colon n \in \omega\} = \beta_1 \cdot \gamma_1 + \cdots + \beta_r \cdot \gamma_r\,,$$

where $\beta_i \in B$ and $\gamma_i \in C$. For every $n \in \omega$, there is some $j \in \{1, \ldots, r\}$ such that $b_n \cdot c_n \cdot \beta_j \cdot \gamma_j > 0$. Hence, there are distinct k, $l \in \omega$ such that for some $j \in \{1, \ldots, r\}$,

$$b_k \cdot c_k \cdot \beta_j \cdot \gamma_j > 0 \quad \text{and} \quad b_l \cdot c_l \cdot \beta_j \cdot \gamma_j > 0\,.$$

So $b_k \cdot \beta_j > 0$ and $c_l \cdot \gamma_j > 0$ which implies, by independence of B and C, that $b_k \cdot \beta_j \cdot c_l \cdot \gamma_j > 0$ and hence $b_k \cdot c_l \cdot (\beta_1 \cdot \gamma_1 + \cdots + \beta_r \cdot \gamma_r) > 0$. This is absurd since, for $k \neq l$, the elements $b_k \cdot c_l$ and $\Sigma\{b_n \cdot c_n\colon n \in \omega\}$ are disjoint. □

11.2. Homogeneity, chain conditions, and independence in free products

The first aim of this subsection is the following theorem. Recall from Section 9 that a Boolean algebra A is said to be homogeneous if $A \upharpoonright a \cong A$, for every non-zero element a of A.

11.10. THEOREM (Grätzer). *Let B be a non-trivial Boolean algebra and κ an infinite cardinal such that $|B| \leq \kappa$. Then there is homogeneous Boolean algebra C of power κ such that $B \oplus C \cong C$.*

Since B is a subalgebra of $B \oplus C$, this shows that every Boolean algebra embeds into a homogeneous one. Moreover, the natural embedding from B into $B \oplus C$ has two additional pleasant features: it is complete by Proposition 11.8, and by Proposition 11.11 below, every automorphism of B extends to an automorphism of $B \oplus C$. We shall encounter similar theorems in Sections 13 and 14: every complete Boolean algebra is completely embeddable into a complete homogeneous algebra with a countable set of complete generators. In fact, the construction in Section 14 is a variation of the natural embedding from B into $B \oplus C$.

11.11. PROPOSITION. *For arbitrary Boolean algebras B and C, every automorphism of B extends to an automorphism of $B \oplus C$.*

PROOF. For any automorphisms f_B of B and f_C of C, there is a unique automorphism of $B \oplus C$ extending both f_B and f_C, by the universal property of $B \oplus C$ – consider f_B and f_C as being homomorphisms from B (respectively C) into $B \oplus C$.

A similar statement holds, of course, for free products of arbitrarily many factors A_i: if f_i is an automorphism of A_i, for $i \in I$, then there is a unique automorphism of $\oplus_{i \in I} A_i$ extending each f_i. □

11.12. LEMMA. *Let A be the free product $\oplus_{i \in I} A_i$; assume that $i(1), \ldots, i(n) \in I$ are distinct, $a_{i(k)} \in A_{i(k)}$ and $a = a_{i(1)} \cdot \cdots \cdot a_{i(n)}$. For $i \in I \setminus \{i(1), \ldots, i(n)\}$, put $a_i = 1$. Then*

$$A \upharpoonright a \cong \bigoplus_{i \in I} (A_i \upharpoonright a_i) .$$

PROOF. We may assume that $A = \operatorname{Clop} X$, where $X = \Pi_{i \in I} X_i$, each X_i is a Boolean space and without loss of generality,

$$A_i = \left\{ u \times \prod_{i \neq j} X_j : u \in \operatorname{Clop} X_i \right\} .$$

For $i \in I$, there is $u_i \in \operatorname{Clop} X_i$ such that

$$a_i = u_i \times \prod_{i \neq j} X_j .$$

Then the clopen subset $a = a_{i(1)} \cap \cdots \cap a_{i(n)}$ of X is the cartesian product of the clopen subsets u_i of X_i, and

$$\begin{aligned} A \restriction a &\cong \text{Clop}\, a \\ &\cong \bigoplus_{i\in I} \text{Clop}\, u_i \\ &\cong \bigoplus_{i\in I} (A_i \restriction a_i)\,. \quad \square \end{aligned}$$

Proof of Theorem 11.10. If B is the two-element algebra, let C by any homogeneous algebra of size κ, e.g. a free one (cf. Proposition 9.14); then $B \oplus C \cong C$ holds by Example 11.6(b).

So assume that B has at least four elements. Let $(A_i)_{i\in I}$ be a family of Boolean algebras satisfying:

(a) $|I| = \kappa$,

(b) for every $i \in I$, there is $b \in B^+$ such that $A_i \cong B \restriction b$,

(c) for every $b \in B^+$, there are κ different elements $i \in I$ such that $A_i \cong B \restriction b$, and define $C = \bigoplus_{i\in I} A_i$.

Clearly, $B \oplus C \cong C$ since C has infinitely many free factors isomorphic to $B \restriction 1 \cong B$. Also, $|C| = \kappa$ since $|I| = \kappa$, $|A_i| \le \kappa$ for every i and, on the other hand, there are κ different elements $i \in I$ such that $4 \le |A_i|$. So we are left with proving that C is homogeneous.

As a first step, note that if $i(1), \ldots, i(n) \in I$ are distinct, $a_{i(k)} \in A_{i(k)}$ is non-zero for $1 \le k \le n$ and $c = a_{i(1)} \cdot \cdots \cdot a_{i(n)}$, then $C \restriction c \cong C$, by Lemma 11.12 and the above choice of the algebras A_i. In the second step, we prove that $C \times C \cong C$. For pick $i \in I$ and $a_i \in A_i$ such that $0 < a_i < 1$; then by the first step and Lemma 3.2,

$$C \cong C \restriction a_i \times C \restriction -a_i \cong C \times C\,.$$

Finally, an arbitrary non-zero element x of C is the sum of finitely many pairwise disjoint non-zero products with factors in $\bigcup_{i\in I} A_i$, say $x = c_1 + \cdots + c_k$. By the previous steps,

$$C \restriction x \cong C \restriction c_1 \times \cdots \times C \restriction c_k \cong C^k \cong C\,. \quad \square$$

11.13. COROLLARY. *For any set $\boldsymbol{K}$ of non-trivial Boolean algebras, there is a homogeneous algebra C such that $K \oplus C \cong C$ for every K in $\boldsymbol{K}$.*

PROOF. Define a family $(A_{iK})_{i\in\omega, K\in\boldsymbol{K}}$ such that $A_{iK} \cong K$, for $i \in \omega$ and $K \in \boldsymbol{K}$, and let

$$B = \bigoplus_{i\in\omega, K\in\boldsymbol{K}} A_{iK}\,.$$

So $K \oplus B \cong B$ for every K in $\boldsymbol{K}$. For this algebra B, choose C by Theorem 11.10. Then, for $K \in \boldsymbol{K}$,

$$K \oplus C \cong K \oplus (B \oplus C) \cong (K \oplus B) \oplus C \cong B \oplus C \cong C\,. \quad \square$$

We next give an example on cellularity in free products. Recall that in 3.8, the cellularity cA of a Boolean algebra A was defined to be the least upper bound of the cardinals $|X|$, where X is a pairwise disjoint family in A. This least upper bound is trivially attained if it is a successor cardinal; it is also attained, by the Erdös–Tarski theorem 3.10, if it is singular. For regular limit cardinals, i.e. for weakly inaccessible cardinals, there is a counterexample.

11.14. EXAMPLE (Erdös–Tarski). Let κ be a weakly inaccessible cardinal and, for each cardinal $\alpha < \kappa$, let A_α be the power set algebra of α (so $cA_\alpha = \alpha$). Then $A = \bigoplus_{\alpha<\kappa} A_\alpha$ has the property that $cA = \kappa$ and cA is not attained.

For a proof of this, note that for $\alpha < \kappa$, $\alpha = cA_\alpha \leq cA$ since A_α is a subalgebra of A. Thus, $\kappa \leq cA$ and we have to prove that $|X| < \kappa$ for every pairwise disjoint family X in A. This follows by a standard application of the Δ-lemma: assume $|X| = \kappa$. For each x in X, pick a finite subset $I(x)$ of $I = \{\alpha: \alpha < \kappa, \alpha \text{ a cardinal}\}$ such that x is generated by $\bigcup_{\alpha\in I(x)} A_\alpha$. By the Δ-lemma, there is an $Y \subseteq X$ of size κ such that $\{I(y): y \in Y\}$ is a Δ-system, say with root J.

We may assume that each $y \in Y$ is a product

$$y = \prod_{\alpha\in I(y)} a_{y\alpha} ,$$

where $a_{y\alpha} \in (A_\alpha)^+$, since the products $\prod_{\alpha\in I(y)} a_\alpha$, $a_\alpha \in (A_\alpha)^+$, constitute a dense subset of $\bigoplus_{\alpha\in I(y)} A_\alpha$. It follows from independence of the subalgebras A_α in A that the elements

$$p_y = \prod_{\alpha\in J} a_{y\alpha}$$

of A are pairwise disjoint and non-zero; thus $c(\bigoplus_{\alpha\in J} A_\alpha) = \kappa$. This is impossible, for A_α is atomic with α atoms, and $\bigoplus_{\alpha\in J} A_\alpha$ is atomic with $\max\{\alpha: \alpha \in J\}$ atoms (cf. Exercise 1).

The rest of the subsection is devoted to a computation of the cardinal invariant $\text{ind}^*(\bigoplus_{i\in I} A_i)$, defined in 10.15, in terms of the cardinals $\text{ind}^* A_i$.

11.15. THEOREM. *Let B, C, and A_i, for $i \in I$, be Boolean algebras. Then*

$$\text{ind}^*(B \oplus C) = \max(\text{ind}^* B, \text{ind}^* C) ;$$

if I is infinite and $|A_i| \geq 4$ for all i, then

$$\text{ind}^*\Big(\bigoplus_{i\in I} A_i\Big) = \max(|I|, \sup\{\text{ind}^* A_i: i \in I\}) .$$

We proved in Section 10 that $\text{ind}^* B = 1$ iff B is superatomic. Hence,

11.16. COROLLARY. *If both B and C are superatomic, then so is $B \oplus C$.*

If $\text{ind}^* B \neq 1$, however, then $\text{ind}^* B$ is infinite and coincides with $\text{ind}\, B$, by Šapirovskii's first theorem 10.16. We use Šapirovskii's second theorem 10.17 and the following lemma to derive the first part of Theorem 11.15. Recall the notation of Section 10: for a subalgebra B of D and an ultrafilter p of B, $\bar{p}$ is the filter of D generated by p.

11.17. LEMMA. *Let B and C be arbitrary Boolean algebras. For every ultrafilter p of B,*

$$(B \oplus C)/\bar{p} \cong C .$$

PROOF. Consider the situation

$$C \xrightarrow{e} B \oplus C \xrightarrow{\pi} (B \oplus C)/\bar{p} ,$$

where e is the inclusion map and π is canonical. We claim that $f = \pi \circ e$ is an isomorphism.

f is one-to-one, since $f(c) = 0$ implies that $-c \in \bar{p}$, so $b \leq -c$ for some $b \in p$. By independence of B and C and $b > 0$, it follows that $c = 0$. Also, f is onto, for if

$$x = b_1 \cdot c_1 + \cdots + b_n \cdot c_n$$

is an arbitrary element of $B \oplus C$, where $b_i \in B$ and $c_i \in C$, then

$$\pi(x) = \sum \{\pi(c_i)\colon b_i \in p\} = \sum \{f(c_i)\colon b_i \in p\}$$

is in ran f. □

Proof of Theorem 11.15. The first assertion is immediate, by Theorem 10.17 and the preceding lemma.

For the second one, let

$$\kappa = \max(|I|, \sup\{\text{ind}^* A_i\colon i \in I\}) ,$$

$$A = \bigoplus_{i \in I} A_i .$$

It follows from Theorem 10.16 that $|I| \leq \text{ind}^* A$, since if A_i' is a four-element subalgebra of A_i for each $i \in I$, then $\oplus_{i \in I} A_i'$ is, by Example 11.6(a), a free subalgebra of A with $|I|$ independent generators. Also, $\text{ind}^* A_i \leq \text{ind}^* A$ for each i, since A_i is a subalgebra of A (cf. the remark following the proof of 10.16). We have thus proved that $\kappa \leq \text{ind}^* A$.

Now suppose that $\kappa < \text{ind}^* A$. Then Theorem 10.16 gives an independent subset X of A of size κ^+. For each x in X, choose a finite subset $I(x)$ of I such that x is generated by $\bigcup_{i \in I(x)} A_i$. Since $|I| < \kappa^+$, there are a subset Y of X of size κ^+ and a finite subset J of I such that $I(y) = J$ for $y \in Y$. Thus, $\oplus_{i \in J} A_i$ has an independent subset of power κ^+ and

$$\begin{aligned} \kappa^+ &\leq \text{ind}^*\Big(\bigoplus_{i \in J} A_i\Big) \\ &= \max\{\text{ind}^* A_i\colon i \in J\} \end{aligned}$$

by Theorem 10.16 and the first assertion of our theorem. This contradiction shows that ind* $A \leq \kappa$. □

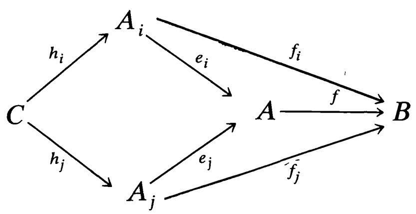
11.3. *Amalgamated free products*

11.18. DEFINITION. Let $(A_i)_{i\in I}$ be a family of Boolean algebras, C another Boolean algebra and, for $i \in I$, h_i: $C \to A_i$ a monomorphism. A pair $((e_i)_{i\in I}, A)$ is an *amalgamated free product* of $((h_i)_{i\in I}, (A_i)_{i\in I})$ over C if

(3) A is a Boolean algebra,

(4) e_i: $A_i \to A$ is a homomorphism and $e_i \circ h_i = e_j \circ h_j$ for all $i, j \in I$,

(5) for every family $(f_i)_{i\in I}$ of homomorphisms f_i from A_i into any Boolean algebra B satisfying $f_i \circ h_i = f_j \circ h_j$ for all $i, j \in I$, there is a unique homomorphism f: $A \to B$ such that $f \circ e_i = f_i$ for $i \in I$.



Letting C be the two-element algebra shows that free products, as defined in 11.1, are a special case of amalgamated free products, since the commutativity requirements in (4) and (5) are trivially satisfied.

As in the theory of free products, we have a uniqueness, an existence and a characterization theorem. Given the existence theorem 11.2 for free products, an algebraic proof for existence of amalgamated free products is no more difficult than a topological one.

11.19. THEOREM (existence and uniqueness). *Every family* $((h_i)_{i\in I}, (A_i)_{i\in I})$ *has, up to isomorphism, a unique free product.*

PROOF. Uniqueness is formulated and proved as in Theorem 11.2. For existence, let $((e'_i)_{i\in I}, F)$ be the free product of the family $(A_i)_{i\in I}$, and let M be the ideal of F generated by the set

$$\{h_i(c) \triangle h_j(c): c \in C, i, j \in I\}.$$

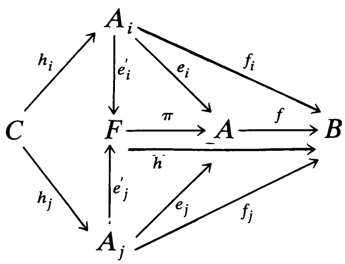
Then define

$$A = F/M$$

and

$$e_i = \pi \circ e_i'$$

for $i \in I$, where $\pi: F \to A$ is canonical. We prove that $((e_i)_{i\in I}, A)$ is an amalgamated free product of the family $((h_i)_{i\in I}, (A_i)_{i\in I})$.



Clearly, $e_i \circ h_i = e_j \circ h_j$ for all $i, j \in I$, by the very definition of M.

To prove the universal property (5), let $f_i: A_i \to B$ be homomorphisms such that $f_i \circ h_i = f_j \circ h_j$ for $i, j \in I$. Since $((e_i')_{i\in I}, F)$ is the free product of the A_i, there is a unique homomorphism $h: F \to B$ satisfying $h \circ e_i' = f_i$ for $i \in I$. By $f_i \circ h_i = f_j \circ h_j$, the kernel of h includes M, so there is a homomorphism $f: A \to B$ such that $f \circ \pi = h$. It follows that $f \circ e_i = f_i$ for $i \in I$. Moreover, f is uniquely determined by the requirement that $f \circ e_i = f_i$, since F is generated by the union of the $e_i'[A_i]$ and A is generated by the union of the $e_i[A_i]$. □

11.20. PROPOSITION (characterization). *Let A and $e_i: A_i \to A$ be given such that $e_i \circ h_i = e_j \circ h_j$ for $i, j \in I$. Then $((e_i)_{i\in I}, A)$ is an amalgamated free product of $((h_i)_{i\in I}, (A_i)_{i\in I})$ over C iff each of* (a) *through* (c) *holds*:

(a) *each e_i is one-to-one*;

(b) *if*

$$e_{i(1)}(a_{i(1)}) \cdot \cdots \cdot e_{i(n)}(a_{i(n)}) = 0\,,$$

where $i(1), \ldots, i(n) \in I$ are pairwise distinct and $a_{i(k)} \in A_{i(k)}$, then there are $c_1, \ldots, c_n \in C$ such that

$$a_{i(k)} \leq h_{i(k)}(c_k) \quad \text{and} \quad c_1 \cdot \cdots \cdot c_n = 0\,;$$

(c) *A is generated by $\bigcup_{i\in I} e_i[A_i]$.*

Moreover, if $((e_i)_{i\in I}, A)$ is an amalgamated free product of $((h_i)_{i\in I}, (A_i)_{i\in I})$ over C, then:

(d) *for $i \neq j$ in I, $e_i[A_i] \cap e_j[A_j] = e[C]$, where $e: C \to A$ is the homomorphism $e_i \circ h_i$, for any $i \in I$.*

PROOF. First assume that $((e_i)_{i\in I}, A)$ is an amalgamated free product. Assertion (c) has already been shown at the end of the preceding proof. For a proof of (a), consider a non-zero element a_i of A_i with the aim of showing that $e_i(a_i)\neq 0$. For each $j\in I$, choose a homomorphism $f_j\colon A_j\to 2$ as follows: for $j=i$, let f_i be such that $f_i(a_i)=1$. For $j\neq i$, let f_j be such that $f_i\circ h_i=f_j\circ h_j$ – to find such an f_j, note that $p=\{c\in C\colon f_i(h_i(c))=1\}$ is an ultrafilter of C and that, h_j being one-to-one, $h_j[p]$ generates a proper filter of A_j; choose $f_j\colon A_j\to 2$ such that $f_j(h_j(c))=1$ for $c\in p$. Let, by the universal property (5), $f\colon A\to 2$ be the homomorphism satisfying $f\circ e_j=f_j$ for $j\in I$. Then $f(e_i(a_i))=f_i(a_i)=1$ and $e_i(a_i))\neq 0$.

To prove (b), assume $a_{i(1)},\ldots,a_{i(n)}$ constitute a counterexample. Then the set

$$q_0=\{c\in C\colon a_{i(k)}\leq h_{i(k)}(c) \text{ for some } k\in\{1,\ldots,n\}\}$$

has the finite intersection property in C; let q be an ultrafilter of C including q_0. Also, for $1\leq k\leq n$, the set

$$q_k=h_{i(k)}[q]\cup\{a_{i(k)}\}$$

has the finite intersection property in $A_{i(k)}$, since otherwise, $a_{i(k)}\cdot h_{i(k)}(c)=0$ for some $c\in q$ which implies $a_{i(k)}\leq h_{i(k)}(-c)$ and $-c\in q_0\subseteq q$. So there is a homomorphism $f_{i(k)}\colon A_{i(k)}\to 2$ mapping q_k onto 1. For $i\in I\setminus\{i(1),\ldots,i(n)\}$, let $f_j\colon A_j\to 2$ map $h_j[q]$ onto 1. Thus, $f_i\circ h_i=f_j\circ h_j$ for $i, j\in I$; let $f\colon A\to 2$ be a homomorphism satisfying $f\circ e_i=f_i$. It follows that

$$0=f(e_{i(1)}(a_{i(1)})\cdot\cdots\cdot e_{i(n)}(a_{i(n)}))=f_{i(1)}(a_{i(1)})\cdot\cdots\cdot f_{i(n)}(a_{i(n)})=1\,,$$

a contradiction.

Conversely, suppose that the pair $((e_i)_{i\in I}, A)$ satisfies the conditions (a) through (c) and that homomorphisms $f_i\colon A_i\to B$ are given such that $f_i\circ h_i=h$ for each $i\in I$, where $h\colon C\to B$ does not depend on i. By Sikorski's extension criterion 5.5 and (c), we have to prove that, for $i(k)\in I$, $a_{i(k)}\in A_{i(k)}$ and $\varepsilon_k\in\{+1,-1\}$,

$$\varepsilon_1 e_{i(1)}(a_{i(1)})\cdot\cdots\cdot\varepsilon_n e_{i(n)}(a_{i(n)})=0$$

implies

$$\varepsilon_1 f_{i(1)}(a_{i(1)})\cdot\cdots\cdot\varepsilon_n f_{i(n)}(a_{i(n)})=0\,.$$

The e_i and f_i are homomorphisms, so we may assume that each $\varepsilon_k=1$ and that $i(1),\ldots,i(n)$ are distinct. Choosing $c_1,\ldots,c_n$ in C as guaranteed by (b), we find that

$$\begin{aligned} f_{i(1)}(a_{i(1)})\cdot\cdots\cdot f_{i(n)}(a_{i(n)}) &\leq f_{i(1)}(h_{i(1)}(c_1))\cdot\cdots\cdot f_{i(n)}(h_{i(n)}(c_n))\\ &=h(c_1)\cdot\cdots\cdot h(c_n)\\ &=h(c_1\cdot\cdots\cdot c_n)\\ &=0\,. \end{aligned}$$

Finally, condition (d) is a consequence of (b): suppose $i \neq j$ and $e_i(a_i) = e_j(a_j)$, where $a_i \in A_i$, $a_j \in A_j$. Then $e_i(a_i) \cdot e_j(-a_j) = 0$, so pick c and c' in C such that $c \cdot c' = 0$, $a_i \leq h_i(c)$ and $-a_j \leq h_j(c')$. This implies

$$e_i(a_i) \leq e(c) \leq e(-c') \leq e_j(a_j) ,$$

hence $e_i(a_i) = e(c) \in e[C]$. □

11.21. NOTATION AND CONVENTIONS. Let $((e_i)_{i \in I}, A)$ be an amalgamated free product of $((h_i)_{i \in I}, (A_i)_{i \in I})$ over C. We may assume, by 11.20(a) and (d), that C is a subalgebra of A_i and A_i a subalgebra of A, that h_i and e_i are inclusion maps and that $A_i \cap A_j = C$, for $i \neq j$ in I. (These assumptions should, of course, be used with the same precautions as those in 11.5.) The algebra A is denoted by

$$A = \bigoplus_{\substack{C \\ i \in I}} A_i ;$$

for $I = \{1, \ldots, n\}$ finite, we write

$$A = A_1 \underset{C}{\oplus} \cdots \underset{C}{\oplus} A_n .$$

We finally give a reformulation of the characterization 11.20 for the free amalgamated product of two Boolean algebras and a couple of examples.

11.22. COROLLARY. *Assume A_1, A_2 are subalgebras of a Boolean algebra A and C is a subalgebra of both A_1 and A_2. Then $A = A_1 \underset{C}{\oplus} A_2$ iff $A_1 \cup A_2$ generates A and for any two disjoint elements a_1 of A_1 and a_2 of A_2, there is some c in C such that $a_1 \leq c$ and $a_2 \leq -c$.*

11.23. EXAMPLES. Assume the notation of 11.21 and let

$$A = \bigoplus_{\substack{C \\ i \in I}} A_i .$$

(a) For every c in C,

$$A \restriction c \cong \bigoplus_{\substack{C \restriction c \\ i \in I}} (A_i \restriction c) .$$

(b) Let C be finite and X the set of its atoms. Then

$$A \cong \prod_{c \in X} \left(\bigoplus_{i \in I} (A_i \restriction c) \right) .$$

PROOF. (a) follows by checking that the unique homomorphism

$$f \colon \bigoplus_{\substack{C \restriction c \\ i \in I}} (A_i \restriction c) \to A \restriction c$$

such that $f(x) = x$ for $x \in \bigcup_{i \in I} (A_i \restriction c)$ is an isomorphism.

(b) A consequence of part (a) plus

$$A \cong \prod_{c \in X} (A \upharpoonright c), \qquad \bigoplus_{\substack{2 \\ i \in I}} A_i \cong \bigoplus_{i \in I} A_i. \quad \square$$

Exercises

1. Describe the atoms of $B \oplus C$ in terms of the atoms of B, respectively C. Use this to show that for each algebra B, there is an algebra C with $B \oplus C$ atomless.

2. Let A be a subalgebra of B and C another Boolean algebra. The following equivalent:

(a) $B = A \oplus C'$, for some subalgebra C' of B isomorphic to C;

(b) in the sheaf representation $\mathcal{S} = (S, \pi, X, (B_p)_{p \in X})$ of B over A (cf. 8.16), there is a homeomorphism $\phi: S \to X \times C$ such that $\pi = \mathrm{pr} \circ \phi$. Here C has the discrete and $X \times C$ the product topology; pr: $X \times C \to X$ is the projection onto the first coordinate.

3. Let A and B be Boolean algebras and κ an infinite cardinal such that the cellularities of A and B are not greater than κ. Then $\mathrm{c}(A \oplus B) \leq 2^\kappa$.

Hint. Use the Erdös–Rado theorem 7.2 in the Appendix on Set Theory.

4. Assume that, for each finite subset J of I, the cellularity of $\oplus_{i \in J} A_i$ is not greater than a regular uncountable cardinal κ. Show that $\mathrm{c}(\oplus_{i \in I} A_i) \leq \kappa$.

5. (a) For I an infinite set and $|J| \geq 2$, the partial order $\mathrm{Fn}(I, J, \omega)$ defined in 4.10 and the free product of $|I|$ copies of the power set algebra $P(J)$ have isomorphic completions.

(b) Let the infinite cardinal κ have the discrete topology and ${}^\omega\kappa$ the product topology. Then $\mathrm{RO}({}^\omega\kappa)$ is isomorphic to the completion of the free product of ω copies of $P(\kappa)$.

(c) Use Exercise 4 to show that the cellularity of $\mathrm{RO}({}^\omega\kappa)$ is attained and equals κ. Hence, $\mathrm{RO}({}^\omega\kappa)$ has cardinality 2^κ.

6. Show that if each A_i, $i \in I$, is an infinite homogeneous algebra, then so is $\oplus_{i \in I} A_i$.

7. (a) Show that the free product of two infinite Boolean algebras does not satisfy the countable separation property.

(b) Let βX denote the Stone–Čech compactification of a completely regular space X and $X^* = \beta X \setminus X$ the Stone–Čech remainder. Conclude from (a) that $\beta\omega \times \beta\omega$ is not homeomorphic to $\beta(\omega \times \omega)$ and $\omega^* \times \omega^*$ is not homeomorphic to $(\omega \times \omega)^*$.

8. If A is generated by the union of a family $(A_i)_{i \in I}$ of subalgebras, then

$$\mathrm{ind}^* A \leq \max(|I|, \sup\{\mathrm{ind}^* A_i : i \in I\}).$$

9. Let C be a Boolean algebra and, for $i \in I$, $h_i: C \to A_i$ a monomorphism. Consider the continuous surjection p_i: Ult $A_i \to$ Ult C dual to h_i and the subspace

$$X = \left\{(x_i)_{i \in I} \in \prod_{i \in I} \mathrm{Ult}\, A_i : p_i(x_i) = p_j(x_j) \text{ for all } i, j \in I\right\}$$

of $\prod_{i \in I} \mathrm{Ult}\, A_i$. Show that X is a closed subspace of $\prod_{i \in I} \mathrm{Ult}\, A_i$ and is, in fact, the dual space of the amalgamated free product of $((h_i)_{i \in I}, (A_i)_{i \in I})$.

CHAPTER 5

Infinite Operations

Sabine KOPPELBERG

Freie Universität Berlin

Contents

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Introduction

Most of the preceding chapters were devoted to a presentation of the basic notions and results on Boolean algebras; we shall now use the theory built up so far to study several special classes of Boolean algebras. In this chapter, we concentrate on Boolean algebras which are complete or satisfy assumptions weaker than completeness.

Part of our results might be described, in a somewhat formal setting, as speaking about the categories of complete (respectively of κ-complete) Boolean algebras and homomorphisms. It turns out that the category of κ-complete algebras behaves, in some aspects, like the category of all Boolean algebras but differs considerably from the category of complete algebras. Here the principal positive result is that amalgamated free products, and consequently also free products and free algebras, exist in the category of κ-complete algebras. There are, however, no free algebras over infinitely many free generators in the category of complete algebras. This is shown by the astounding example, for each infinite cardinal κ, of a complete algebra, the collapsing algebra $\mathrm{RO}({}^{\omega}\kappa)$, which is countably completely generated and has cardinality greater than κ.

The other main result on complete algebras, and in fact one of the highlights of their theory, is the Balcar–Franěk theorem: every infinite complete algebra A has a free subalgebra of maximal size $|A|$. This theorem is not only interesting in its own right and because of its applications, but also for its proof which applies a number of combinatorial techniques established in the previous sections.

In the final section of this chapter we are concerned with a property of Boolean algebras, infinite distributivity, which does not explicitly assume completeness, but is adequately studied in complete algebras. Distributive laws can be viewed as classifying complete (respectively κ-complete) algebras by their similarity to power set algebras (respectively κ-algebras of sets). This classification is generally very rough, but for some special types of complete algebras turns out to be quite successful. For example, $(\omega, 2)$-distributivity plus the countable chain condition characterize Souslin algebras which are well known in set theory by their close connection with Souslin trees and Souslin lines. On the other hand, the collapsing algebra $\mathrm{RO}({}^{\omega}\kappa)$ can be described by (ω, κ, κ)-nowhere distributivity plus a denseness assumption.

12. κ-complete algebras

We present here results on Boolean algebras which depend on σ-completeness or only on the countable separation property but do not require full completeness.

For definiteness, let C_σ, C_κ (for $\kappa > \omega$) and C_∞ denote, in this section, the classes of σ-complete, κ-complete, respectively complete Boolean algebras. In Section 5, a Boolean algebra A was said to have the countable separation property if, for any countable subsets M, N of A such that $m \cdot n = 0$ for all $m \in M$

and $n \in N$, there is an $a \in A$ satisfying $m \leq a$ and $n \leq -a$ for $m \in M$ and $n \in N$; let us denote by C_{csp} the class of all algebras with the countable separation property. Then, for $\kappa > \omega$,

$$C_\infty \subseteq C_\kappa \subseteq C_\sigma \subseteq C_{\text{csp}} ;$$

here the first inclusion is proper for arbitrary κ, the second one for $\kappa > \omega_1$, and the third one by Example 5.28 (the algebra $P(\omega)/fin$). This examples also shows, together with Lemma 5.27, that C_{csp} is closed under homomorphic images, but none of the classes C_σ, C_κ, C_∞ is. By abuse of notation, we also denote by C_σ the category of all σ-complete Boolean algebras and σ-complete homomorphisms; similarly for C_κ and C_∞.

From a systematic point of view inspired by Stone's theorem, it is the central question on κ-complete algebras how they are related to κ-complete algebras of sets, as defined in 1.29. Exercise 3 in Section 2, however, says that even a complete algebra is not necessarily isomorphic to a σ-complete algebra of sets. The only general result available here is the Loomis–Sikorski theorem 12.7 asserting that every σ-complete Boolean algebra is the image of a σ-complete algebra of sets under a σ-complete homomorphism. A detailed discussion of representability questions for κ-complete algebras can be found in SIKORSKI [1964]; also see Section 14.

As a consequence of the Loomis–Sikorski theorem, there is an explicit construction of the free σ-complete algebra on any number of generators. A general but less explicit construction using principles of universal algebra proves that amalgamated free products exist in each of the categories C_κ.

We further characterize the cardinalities and the Stone spaces of algebras lying in the classes C_∞ through C_{csp}. If C is one of these classes, then for each infinite cardinal κ, $\kappa = |A|$ for some infinite member A of C iff $\kappa^\omega = \kappa$; the Stone spaces of algebras with the countable separation property are exactly the Boolean F-spaces. The dual spaces of σ-complete (κ-complete, complete) algebras were described, in Proposition 7.21, as being those Boolean spaces in which the union of countably many (less than κ, arbitrarily many) clopen sets has open closure.

Theorem 12.4 expresses another attractive property of σ-complete Boolean algebras: two σ-complete algebras are isomorphic if each of them is isomorphic to a direct factor (i.e. a relative algebra) of the other one. This assertion does not hold for arbitrary Boolean algebras, as shown by Hanf's example 6.5.

12.1. The countable separation property

We describe here the Stone spaces and the possible cardinalities of Boolean algebras with the countable separation property. As a consequence, we are able to characterize the cardinalities of algebras in the classes C_σ, C_κ, and C_∞.

Let us recall some definitions from topology. A subset of a topological space is an *F_σ-set* if it is representable as the union of countably many closed sets. In a Boolean space X, this is equivalent, for open sets, to being a union of countably many clopen sets. For assume $u = \bigcup_{n \in \omega} c_n$ is open and each c_n is closed. For

$n \in \omega$, c_n and $X \backslash u$ are disjoint closed subsets of X, so by 7.6(c), there is a clopen subset a_n of X satisfying $c_n \subseteq a_n \subseteq u$, and $u = \bigcup_{n\in\omega} a_n$ proves our claim.

Two subsets u and v of a topological space X are *completely separated* if there is a continuous function $f: X \to [0, 1]$ sending u to 0 and v to 1. For X Boolean, this amounts to saying that there is a clopen set c such that $u \subseteq c$ and $v \subseteq X \backslash c$ – for the non-trivial part of this, note that u and v have disjoint closures if they are completely separated, and use Lemma 7.6(c).

A subset u of a topological space X is *co-zero* if there is a continuous function $f: X \to [0, 1]$ such that $u = \{x \in X: f(x) \neq 0\}$. u is *C^*-embedded* in X if every continuous function from u to $[0, 1]$ extends to a continuous function from X to $[0, 1]$. X is an *F-space* if every co-zero set in X is C^*-embedded. F-spaces have gained some interest in topology; they arise, for example, naturally in the theory of compactifications.

12.1. PROPOSITION. *A Boolean algebra has the countable separation property iff its dual space is an F-space.*

PROOF. We use two facts whose proofs can be found in textbooks on topology (e.g. WALKER [1974]): a completely regular space is an F-space iff disjoint co-zero sets are completely separated; in a normal space, an open subset is co-zero iff it is F_σ.

Let X be the dual space of a Boolean algebra A, a compact and hence normal Hausdorff space. Combining the preceding remarks, we find that X is an F-space iff any two disjoint open F_σ-sets are completely separated iff disjoint countable unions

$$u = \bigcup_{n\in\omega} a_n\,, \qquad v = \bigcup_{n\in\omega} b_n$$

of clopen sets a_n, b_n are completely separated. By the above equivalence of complete separation in Boolean spaces, this means that $\operatorname{Clop} X \cong A$ has the countable separation property. □

12.2. THEOREM (S. Koppelberg). *If A is an infinite Boolean algebra with the countable separation property, then $|A|^\omega = |A|$.*

12.3. COROLLARY. *Let $\boldsymbol{C}$ be one of the classes $\boldsymbol{C}_{\mathrm{csp}}$, $\boldsymbol{C}_\kappa$ (for $\kappa > \omega$), or $\boldsymbol{C}_\infty$, and let λ be an infinite cardinal. There is an algebra of cardinality λ in $\boldsymbol{C}$ iff $\lambda^\omega = \lambda$.*

PROOF. If A is an infinite member of $\boldsymbol{C}$ having size λ, then $\lambda^\omega = \lambda$ holds by $\boldsymbol{C} \subseteq \boldsymbol{C}_{\mathrm{csp}}$ and 12.2. Conversely, let $\lambda^\omega = \lambda$. The algebra $A = \overline{\mathrm{Fr}\,\lambda}$, the completion of the free Boolean algebra on λ generators, is complete and hence in $\boldsymbol{C}$; we show that $|A| = \lambda$. Trivially, $\lambda = |\mathrm{Fr}\,\lambda| \leq |A|$. On the other hand, since $\mathrm{Fr}\,\lambda$ satisfies the countable chain condition (cf. 9.18), it follows from Corollary 10.5 that $|A| \leq |\mathrm{Fr}\,\lambda|^\omega = \lambda^\omega = \lambda$. □

Proof of Theorem 12.2. Let $\kappa = |A|$ and assume that κ provides a minimal counterexample, i.e. that $\kappa < \kappa^\omega$ but $|B|^\omega = |B|$ holds for each infinite algebra B with the countable separation property and size less than κ.

Claim 1. If $(a_n)_{n\in\omega}$ is a disjoint sequence in A, then $|\Pi_{n\in\omega} A \upharpoonright a_n| \leq \kappa$. This holds because the function

$$f: A \to \prod_{n\in\omega} A \upharpoonright a_n\,, \qquad f(x) = (x \cdot a_n)_{n\in\omega}$$

is onto. For let $(x_n)_{n\in\omega}$ be an element of $\Pi_{n\in\omega} A \upharpoonright a_n$. Put, for $n \in \omega$, $y_n = a_n \cdot -x_n$. By the countable separation property, there is x in A such that $x_n \leq x$ and $y_n \leq -x$ for $n \in \omega$. So $f(x) = (x_n)_{n\in\omega}$. In particular, since there exists an infinite pairwise disjoint family in A (cf. 3.4), we have

$$(1) \qquad 2^\omega \leq \kappa\,.$$

Call an ideal J of A *σ-bounded* if every countable subset of J has an upper bound in J.

Claim 2. Assume J is a σ-bounded ideal and, for each $a \in J$ such that $A \upharpoonright a$ is infinite, $|A \upharpoonright a|^\omega = |A \upharpoonright a|$. Then $|J| < \kappa$. For otherwise, using (1) we obtain:

$$\kappa^\omega = |{}^\omega J| = \left|\bigcup_{a\in J} {}^\omega(A \upharpoonright a)\right| \leq |J| \cdot 2^\omega \cdot \kappa = \kappa\,,$$

a contradiction.

For the rest of the proof, let

$$I = \{a \in A: |A \upharpoonright a| < \kappa\}\,.$$

So I is a proper ideal of A. By minimality of κ and since each relative algebra of A has the countable separation property, we have

$$(2) \qquad \text{if } a \in I \text{ and } A \upharpoonright a \text{ is infinite, then } |A \upharpoonright a|^\omega = |A \upharpoonright a|\,.$$

Claim 3. A/I is finite and I is, without loss of generality, a prime ideal. For, if A/I is infinite, choose by 3.4 a pairwise disjoint family $(q_n)_{n\in\omega}$ in A/I and pick a_n in A such that $\pi(a_n) = q_n$, where $\pi: A \to A/I$ is canonical. Let $a'_n = a_n \cdot -\Sigma_{m<n} a_m$; then the a'_n are pairwise disjoint and $\pi(a'_n) = q_n > 0$. Hence $a'_n \notin I$ and

$$\kappa^\omega = \left|\prod_{n\in\omega} A \upharpoonright a'_n\right| \leq \kappa$$

by Claim 1; a contradiction. So A/I is a finite and non-trivial algebra; let $a \in A$ be such that $\pi(a)$ is an atom of A/I. Replacing A by $A \upharpoonright a$ (an algebra with the countable separation property and cardinality κ) and I by $I \cap A \upharpoonright a$, we may assume that I is prime.

It follows from Claim 3 that $|I| = \kappa$; (2) and Claim 2 show that I cannot be σ-bounded. Fix a set

$$\{b_n: n \in \omega\} \subseteq I$$

having no upper bound in I. Replacing b_n by $b_n \cdot -\Sigma_{m<n} b_m$, we may assume that the b_n are pairwise disjoint.

(3) $$\left|\prod_{n\in\omega} A \restriction b_n\right| < \kappa .$$

For otherwise, by Claim 1, $|\Pi_{n\in\omega} A \restriction b_n| = \kappa$, and

$$\begin{aligned}
\kappa^\omega &= \left|\prod_{n\in\omega} A \restriction b_n\right|^\omega \\
&\le 2^\omega \cdot \left|\prod_{n\in\omega} A \restriction b_n\right| \quad \text{by (2) and } b_n \in I \\
&\le \kappa \quad \text{by (1)} .
\end{aligned}$$

Define an ideal K of A by

$$K = \{x \in A: x \cdot b_n = 0 \text{ for every } n \in \omega\} .$$

(4) $|A/K| < \kappa$ and hence $|K| = \kappa$.

This follows from (3) and the fact that

$$g: A \to \prod_{n\in\omega} A \restriction b_n , \quad g(x) = (x \cdot b_n)_{n\in\omega}$$

is a homomorphism with kernel K; so A/K is isomorphic to a subalgebra of $\Pi_{n\in\omega} A \restriction b_n$.

(5) $K \subseteq I$.

For, if $x \in K$, then $-x$ is an upper bound of $\{b_n: n \in \omega\}$. So $-x \notin I$ and, I being a prime ideal, $x \in I$.

We are ready to reach a final contradiction: the ideal K is σ-bounded by its very definition and by the countable separation property for A. (5) and (2) show that the assumptions of Claim 2 apply to K, implying $|K| < \kappa$. This contradicts (4). □

12.2. *A Schröder–Bernstein theorem*

The classical Schröder–Bernstein theorem of set theory states that, given one-to-one maps $f: X \to Y$ and $g: Y \to X$ between two sets X and Y, there is a bijection $h: X \to Y$; in fact, h can be constructed from f and g without any use of the axiom of choice. We prove an analogue for σ-complete algebras. Recall that, by the characterization 6.4 of product decompositions, the direct factors of a Boolean algebra A are, up to isomorphism, its relative algebras $A \restriction a$.

12.4. THEOREM (Tarski). *If each of two σ-complete algebras is isomorphic to a factor of the other one, then they are isomorphic.*

PROOF. It suffices to prove the following

Claim. If $b \le a$ in a σ-complete algebra A, then $A \restriction b \cong A$ implies $A \restriction a \cong A$. For suppose A and B are arbitrary σ-complete algebras and $f: A \to B \restriction \beta$, $g: B \to A \restriction \alpha$ are isomorphisms. Letting $a = \alpha$ and $b = g(\beta)$ shows that $b \le a$ in A and $A \cong A \restriction b$ via $g \circ f$. Then $B \cong A \restriction a \cong A$, as desired.

For a proof of the claim, let A be σ-complete, $b \le a$ in A and $f: A \to A \restriction b$ an isomorphism. The proof follows closely the lines of the Schröder–Bernstein theorem in set theory: define inductively elements a_n and b_n of A by

$$a_0 = 1, \quad b_0 = a, \quad a_{n+1} = f(a_n), \quad b_{n+1} = f(b_n).$$

Then

$$(6) \qquad 1 = a_0 \ge b_0 \ge a_1 \ge b_1 \ge \cdots :$$

trivially $b_0 \le a_0 = 1$ and $a_1 = f(a_0) = b \le a = b_0$. If $a_{n+1} \le b_n \le a_n$ holds, then application of f yields $a_{n+2} \le b_{n+1} \le a_{n+1}$.

By σ-completeness of A, define

$$r = \prod_{n \in \omega} a_n = \prod_{n \in \omega} b_n .$$

(6) shows that

$$1 = \sum_{n \in \omega} (a_n \cdot -b_n) + \sum_{n \in \omega} (b_n \cdot -a_{n+1}) + r,$$

$$a = b_0 = \sum_{n \in \omega} (a_{n+1} \cdot -b_{n+1}) + \sum_{n \in \omega} (b_n \cdot -a_{n+1}) + r,$$

where the terms in both sums are pairwise disjoint. Again by σ-completeness of A and Proposition 6.4,

$$A \cong \prod_{n \in \omega} A \restriction (a_n \cdot -b_n) \times \prod_{n \in \omega} A \restriction (b_n \cdot -a_{n+1}) \times A \restriction r,$$

$$A \restriction a \cong \prod_{n \in \omega} A \restriction (a_{n+1} \cdot -b_{n+1}) \times \prod_{n \in \omega} A \restriction (b_n \cdot -a_{n+1}) \times A \restriction r.$$

For each $n \in \omega$, the isomorphism f maps $a_n \cdot -b_n$ in A to $a_{n+1} \cdot -b_{n+1}$ in $A \restriction b$, so

$$A \restriction (a_n \cdot -b_n) \cong (A \restriction b) \restriction (a_{n+1} \cdot -b_{n+1})$$
$$= A \restriction (a_{n+1} \cdot -b_{n+1}),$$

and it follows that $A \cong A \restriction a$. $\square$

It may be worth noticing that the classical Schröder–Bernstein theorem can be recovered from Tarski's theorem. For let $f: X \to Y$ and $g: Y \to X$ be one-to-one mappings. Then $P(X) \cong P(Y) \restriction f[X]$ and $P(Y) \cong P(X) \restriction g[Y]$, and Theorem 12.4 gives an isomorphism $h: P(X) \to P(Y)$. But h maps the atoms of $P(X)$ onto the atoms of $P(Y)$, thus inducing a bijection from X onto Y.

Tarski's theorem can often be applied to prove homogeneity of complete algebras. Recall from Section 4 that each separative partial order $(P, \leq)$ is a dense subset of a unique complete Boolean algebra B, its completion; we may think about B as being the regular open algebra of P, in the partial order topology. If $P = A \setminus \{0\}$, for a Boolean algebra A, then $B = \mathrm{RO}(P)$ is the completion $\bar{A}$ of A as defined in 4.18. Generalizing Definition 9.12, let us call, for a moment, a partial order $(P, \leq)$ *homogeneous* if it has a greatest element and, for every $p \in P$, the partially ordered subset $P \restriction p = \{x \in P: x \leq p\}$ is isomorphic to P.

12.5. COROLLARY. *The completion of a homogeneous separative partial order is homogeneous; in particular, the completion of a homogeneous Boolean algebra is homogeneous.*

PROOF. Let B be a complete Boolean algebra and P a dense subset of B which is homogeneous, in the partial order induced by B. We prove that $B \restriction b \cong B$, for any non-zero element b of B. By denseness of P in B, pick $p \in P$ such that $0 < p \leq b$. Now $P \restriction p$ is dense in $B \restriction p$ and isomorphic to P, thus

$$\begin{aligned} B \restriction p &\cong \text{the completion of } P \restriction p \\ &\cong \text{the completion of } P \\ &\cong B\,. \end{aligned}$$

So the Claim in the proof of 12.4 gives $B \restriction b \cong B$. □

As an application of the preceding corollary and homogeneity of infinite free algebras (cf. 9.14), we find that the completion of every infinite free Boolean algebra is homogeneous, and so is the completion of the partial order $\mathrm{Fn}(I, J, \lambda)$ defined in 4.10 (the partial functions mapping less than λ elements of I to J). In fact, the first observation is a special case of the second one since the partial order $\mathrm{Fn}(\kappa, 2, \omega)$ is isomorphic to the dense set of elementary products, in $\mathrm{Fr}\,\kappa$, over the κ free generators. So $\mathrm{Fn}(\kappa, 2, \omega)$ and $\mathrm{Fr}\,\kappa$ have the same completion.

12.3. The Loomis–Sikorski theorem

Regarding algebras of sets as the standard examples for Boolean algebras, we proved in Section 2 Stone's theorem that every Boolean algebra is isomorphic to an algebra of sets. In Definition 1.29, a Boolean algebra B was said to be a κ-algebra of sets if $B \subseteq P(X)$, for some set X, and B is closed under complementation and under intersections of length less than κ. Exercise 3 in Section 2 shows that the analogue of Stone's theorem fails to hold for κ-complete algebras, if $\kappa > \omega$. The Loomis–Sikorski theorem 12.7 seems to be the best available description of σ-complete algebras via σ-algebras of sets. (D. Maharam-Stone kindly informed me that this theorem had been proved, previous to its publication by Loomis and Sikorski, by J. von Neumann.)

12.6. DEFINITION. Let κ be an infinite cardinal. A Boolean algebra A is *κ-representable* if there exists a κ-complete epimorphism from a κ-complete algebra of sets onto A. A is *σ-representable* if it is ω_1-representable.

12.7. THEOREM (Loomis–Sikorski). *Every σ-complete Boolean algebra is σ-representable.*

In Section 14 we will find a close connection between κ-representability and validity of infinite distributive laws, and it will turn out that the Loomis–Sikorski theorem does not generalize to cardinals $\kappa > \omega_1$.

The proofs of the three subsequent results rely on the idea of dividing the Borel algebra of a Boolean space by the ideal of meager sets, plus Baire's theorem. The Borel algebra $\mathrm{Bor}(X)$ of an arbitrary topological space X was defined, in 1.30, to be the σ-algebra of subsets of X generated by the open subsets of X.

12.8. DEFINITION AND LEMMA. Let X be a topological space and $a \subseteq X$. a is *nowhere dense* in X if $\mathrm{int\,cl}\, a = \emptyset$. a is *meager* (or of first category) if it is the union of countably many nowhere dense sets. *X satisfies Baire's theorem* if no non-empty open subset of X is meager. The meager subsets of X constitute an ideal in $P(X)$ wich is closed under countable unions.

PROOF. Every subset of a nowhere dense set is nowhere dense; consequently a subset of a meager set is meager. It follows immediately that the set of meager subsets of X is a σ-complete ideal of $P(X)$, in the sense of Definition 5.19. □

The terminology of this definition comes from the famous Baire theorem in topology: in every locally compact Hausdorff space and in every completely metrizable space, non-empty open sets are non-meager. Since, in an arbitrary topological space, the Borel algebra is σ-complete and has the meager Borel sets as a σ-complete ideal, one should expect the quotient algebra to be exactly σ-complete. The following proposition, however, asserts full completeness of this algebra, for many interesting spaces. In the special case of a Boolean space X, we can then conclude from the classical Baire theorem for X and Proposition 7.17 that the quotient algebra is the completion of $\mathrm{Clop}\, X$.

12.9. PROPOSITION. *Let X be a space satisfying Baire's theorem and let M be the set of meager Borel sets, a σ-complete ideal of* $\mathrm{Bor}(X)$. *Then*

$$\mathrm{Bor}(X)/M \cong \mathrm{RO}(X)\,.$$

PROOF. We proved in Theorem 1.37 that, for any regular open subsets u, v of X,

$$u \cdot_{\mathrm{RO}(X)} v = u \cap v$$

and for $U \subseteq \mathrm{RO}(X)$,

$$\sum^{\mathrm{RO}(X)} U = \operatorname{int}\operatorname{cl}\left(\bigcup U\right).$$

Its elements being open sets, $\mathrm{RO}(X)$ is a subset, though generally not a subalgebra, of $\mathrm{Bor}(X)$.

To prove our proposition, let

$$\pi\colon \mathrm{Bor}(X) \to \mathrm{Bor}(X)/M$$

be canonical. The restriction of π to $\mathrm{RO}(X)$ is a mapping

$$f\colon \mathrm{RO}(X) \to \mathrm{Bor}(X)/M$$

which will be shown to be an isomorphism.

It is plain that $f(1)=1$ and $f(0)=0$. In both $\mathrm{RO}(X)$ and $\mathrm{Bor}(X)$, finite products coincide with set-theoretical intersections and are thus preserved by f. Also, f preserves countable sums, for let $U \subseteq \mathrm{RO}(X)$ be countable, put $v = \bigcup U$ and $r = \operatorname{int}\operatorname{cl} v$. Thus, $v \subseteq r$ and $\operatorname{cl} v$, being the closure of an open set, is regular closed, i.e. $\operatorname{cl}\operatorname{int}\operatorname{cl} v = \operatorname{cl} v$. It follows that

$$r \backslash v \subseteq \operatorname{cl} r \backslash v = \operatorname{cl}\operatorname{int}\operatorname{cl} v \backslash v = \operatorname{cl} v \backslash v .$$

So $\operatorname{cl} v \backslash v$ and $r \backslash v$ are nowhere dense, $r \,\triangle\, v$ is in M, and

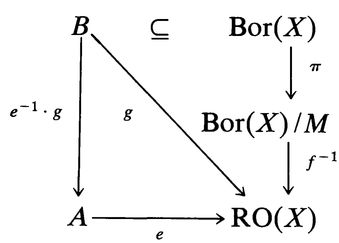
$$f\left(\sum^{\mathrm{RO}(X)} U\right) = f(r) = \pi(r) = \pi(v) = \sum^{\mathrm{Bor}(X)/M} \pi[U] = \sum^{\mathrm{Bor}(X)/M} f[U]$$

since π is σ-complete.

Preserving 0, 1 and finite sums and products, f is, by a remark at the beginning of Section 5, a homomorphism; we have shown above that it is σ-complete. Validity of Baire's theorem in X entails that f is a monomorphism, for if u is regular open and $f(u)=0$, then u is meager and hence empty.

To prove that f is onto, consider the set $\boldsymbol{O}$ of open subsets of X and note that, for u in $\boldsymbol{O}$, $\operatorname{int}\operatorname{cl} u \,\triangle\, u = \operatorname{int}\operatorname{cl} u \backslash u$ is nowhere dense, hence $\pi(u) = \pi(\operatorname{int}\operatorname{cl} u) = f(\operatorname{int}\operatorname{cl} u)$. Thus, $\pi[\boldsymbol{O}] \subseteq \operatorname{ran} f$. But $\boldsymbol{O}$ σ-generates $\mathrm{Bor}(X)$ and π is σ-complete, so $\pi[\boldsymbol{O}]$ σ-generates $\mathrm{Bor}(X)/M$. Moreover, since f is σ-complete, $\operatorname{ran} f$ is a σ-complete subalgebra of $\mathrm{Bor}(X)/M$. It follows that $\operatorname{ran} f = \mathrm{Bor}(X)/M$. □

Proof of Theorem 12.7. Let A be σ-complete. Its Stone space X, being compact and Hausdorff, satisfies Baire's theorem. So consider the following diagram.



Here M, π, and the isomorphism f^{-1}: $\mathrm{Bor}(X)/M \to \mathrm{RO}(X)$ are as in the proof of 12.9 and e is the canonical embedding from A into its completion $\mathrm{RO}(X) \supseteq \mathrm{Clop}\, X$ (cf. Proposition 7.17). Now π, f^{-1}, and e are σ-complete, hence the preimage B of $e[A]$ under $f^{-1} \circ \pi$ is a σ-algebra of sets. If g denotes the restriction of $f^{-1} \circ \pi$ to B, then $e^{-1} \circ g$ is well-defined and a σ-complete epimorphism from B onto A. $\square$

The Loomis–Sikorski theorem allows an explicit construction of the free σ-complete algebra over a set I.

12.10. DEFINITION. Let κ be an infinite cardinal and I any set. A *free κ-complete algebra over I* is a pair (e, F) such that F is a κ-complete Boolean algebra, e: $I \to F$ and for every κ-complete algebra A and every map f: $I \to A$, there is a unique κ-complete homomorphism g: $F \to A$ satisfying $g \circ e = f$.

12.11. PROPOSITION. *Let, for a set I, X be the Cantor space ${}^I 2$ and F the algebra of sets over X σ-generated by* Clop X. *Define*

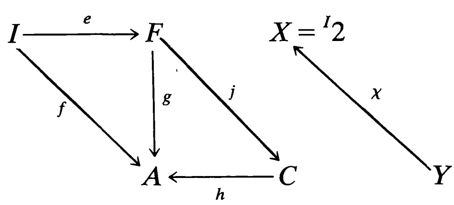
$$e: I \to \mathrm{Clop}\, X \subseteq F$$

by

$$e(i) = \{x \in X: x(i) = 1\}\ .$$

Then (e, F) is a free σ-complete algebra over I.

PROOF. Assume A is σ-complete and f: $I \to A$.



By the Loomis–Sikorski theorem, choose a σ-complete epimorphism h: $C \to A$, where C is a σ-algebra of sets, say over Y. For $i \in I$, fix $c_i \in C$ such that $f(i) = h(c_i)$. Let $\chi_i \in {}^Y 2$ be the characteristic function of $c_i \subseteq Y$ and define χ: $Y \to X = {}^I 2$ by $\chi(y) = (\chi_i(y))_{i \in I}$. Each b in F is a subset of X, thus j: $F \to P(Y)$ defined by $j(b) = \chi^{-1}[b]$ is a σ-complete homomorphism.

We prove that j maps F into C. It is sufficient to show that j maps some set of σ-complete generators of F into C. But $e[I]$ generates Clop X and σ-generates F; moreover, for $i \in I$, $j(e(i)) = c_i$ is in C.

Now $g = h \circ j$ is a σ-complete homomorphism from F into A satisfying $g \circ e = f$ since, for $i \in I$,

$$g(e(i)) = h(j(e(i))) = h(c_i) = f(i)\ . \quad \square$$

As in Proposition 9.8, we can prove that every σ-complete algebra is a

σ-homomorphic image of a free σ-complete one, and 12.11 tells us that free σ-complete algebras are σ-algebras of sets. This can be conceived as a particularly lucid version of the Loomis–Sikorski theorem. Note, however, that the Loomis–Sikorski theorem was heavily used in our proof of 12.11.

In the next subsection, we shall generalize Proposition 12.11 to prove that, for every infinite cardinal κ, amalgamated free products, in particular free products and free algebras, exist in the category of κ-complete Boolean algebras and κ-complete homomorphisms. No satisfactory internal characterization like 11.20, 11.4 or 9.4, however, is known for these algebras, and for $\kappa > \omega_1$, there does not even exist an explicit construction of free κ-complete algebras similar to 12.11.

12.4. Amalgamated free products and injectivity in the category of κ-complete Boolean algebras

Let us consider two questions concerning the categories $\boldsymbol{C}_\kappa$ (respectively $\boldsymbol{C}_\infty$). Here $\boldsymbol{C}_\kappa$ is, for κ an infinite cardinal, the category of all κ-complete Boolean algebras and κ-complete homomorphisms; similarly $\boldsymbol{C}_\infty$ is the category of complete Boolean algebras and complete homomorphisms. So

$$\boldsymbol{C}_\infty \subseteq \boldsymbol{C}_\kappa \subseteq \boldsymbol{C}_\omega ,$$

and $\boldsymbol{C}_\omega$ is the category of all Boolean algebras and homomorphisms. We proved, in Section 11, that amalgamated free products exist in $\boldsymbol{C}_\omega$. Also, the injective algebras in $\boldsymbol{C}_\omega$ were nicely characterized in Theorem 5.13 as being the complete Boolean algebras. In particular, plenty of injectives exist in $\boldsymbol{C}_\omega$, in the sense that every Boolean algebra can be embedded into an injective one.

12.12. THEOREM. *For every infinite cardinal κ, amalgamated free products, in particular free products and free algebras, exist in the category of κ-complete Boolean algebras and κ-complete homomorphisms.*

12.13. PROPOSITION. *For uncountable κ, there are no injectives in $\boldsymbol{C}_\kappa$. Similarly, no injectives exist in $\boldsymbol{C}_\infty$.*

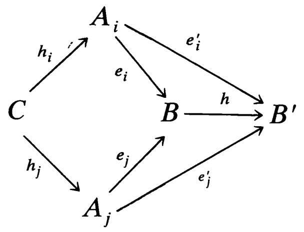
For a proper statement of these assertions, we need a couple of definitions.

12.14. DEFINITION. Let κ be an infinite cardinal.

A *cone* in $\boldsymbol{C}_\kappa$ is a pair $((h_i)_{i\in I}, (A_i)_{i\in I})$ such that, for some fixed algebra C, the algebras C and A_i and the homomorphisms h_i: $C \to A_i$ are in $\boldsymbol{C}_\kappa$ and the maps h_i are one-to-one.

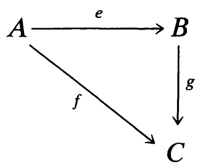
A pair $s = ((e_i)_{i\in I}, B)$ is a *commutative square* over the cone $c = ((h_i)_{i\in I}, (A_i)_{i\in I})$ if B and e_i: $A_i \to B$ are in $\boldsymbol{C}_\kappa$ and $e_i \circ h_i = e_j \circ h_j$ for all $i, j \in I$.

For commutative squares $s = ((e_i)_{i\in I}, B)$ and $s' = ((e_i')_{i\in I}, B')$ over $c = ((h_i)_{i\in I}, (A_i)_{i\in I})$, a *homomorphism* (respectively *isomorphism*) from s to s' is a homomorphism (respectively isomorphism) h: $B \to B'$ in $\boldsymbol{C}_\kappa$ satisfying $h \circ e_i = e_i'$ for $i \in I$.



An *amalgamated free product* of $c = ((h_i)_{i \in I}, (A_i)_{i \in I})$ over C, in $\mathbf{C}_\kappa$, is a commutative square s over c such that, for every commutative square s' over c, there is a unique homomorphism h from s to s'.

12.15. DEFINITION. A morphism $e: A \to B$ in a category $\mathbf{C}$ is *mono* if, for any object D in $\mathbf{C}$ and any morphisms $h, k: D \to A$ in $\mathbf{C}$, $e \circ h = e \circ k$ implies $h = k$.
An object C in $\mathbf{C}$ is *injective* in $\mathbf{C}$ if, for any morphisms $e: A \to B$ and $f: A \to C$ in $\mathbf{C}$, where e is mono, there is some $g: B \to C$ in $\mathbf{C}$ satisfying $g \circ e = f$.



Thus, if the morphisms of a category $\mathbf{C}$ are mappings and their composition, in $\mathbf{C}$, is the set-theoretical one, then every one-to-one morphism is mono; in some categories, e.g. in the category $\mathbf{C}_\omega$ of all Boolean algebras and homomorphisms and in the category of all Boolean spaces and continuous maps, the converse holds – see Exercise 1 in Section 8. And the injective objects of $\mathbf{C}_\omega$ are, by Theorem 5.13, the complete Boolean algebras.

Theorem 12.12 follows rather easily from abstract reasoning, provided we can show that over each cone c in $\mathbf{C}_\kappa$, there exists at least one commutative square. This is seen by letting B be the trivial (one-element) algebra and $e_i: A_i \to B$ the unique map from A_i to B. Our preliminary lemma 12.16, however, says that there exists a commutative square $s = ((e_i)_{i \in I}, B)$ over c in which each e_i is one-to-one; it follows that in every amalgamated free product $s = ((e_i)_{i \in I}, B)$ of c over C, the e_i are one-to-one.

12.16. LEMMA (LaGrange). *Let $c = ((h_i)_{i \in I}, (A_i)_{i \in I})$ be a cone in $\mathbf{C}_\kappa$ and let $((e_i)_{i \in I}, A)$ be an amalgamated free product of c over C, in the category $\mathbf{C}_\omega$ of all Boolean algebras and homomorphisms. Then each of the homomorphisms $e_i: A_i \to A$ is κ-complete.*

Hence, for the κ-complete subalgebra B of $\bar{A}$ generated by $\bigcup_{i \in I} e_i[A_i]$, $((e_i)_{i \in I}, B)$ is a commutative square over c in $\mathbf{C}_\kappa$.

PROOF. By the remarks in 11.21, we may assume that each A_i is a subalgebra of A, that C is a κ-complete subalgebra of each A_i and that the union of the A_i generates A; we have to show that A_i is a κ-complete subalgebra of A.

Fix $i \in I$ and a subset X of A_i of power less than κ. Now $\Pi^{A_i} X \le \Pi^A X$ since A_i is a subalgebra of A; it suffices to consider the special case that $\Pi^{A_i} X = 0$ and to prove that $\Pi^A X = 0$. Otherwise, by the remarks on normal forms following 4.7, there are $a_i \in A_i$ and $a_{i(1)} \in A_{i(1)}, \ldots, a_{i(n)} \in A_{i(n)}$ such that

$$0 < p = a_i \cdot a_{i(1)} \cdot \cdots \cdot a_{i(n)} \le x \quad \text{for } x \in X ,$$

where i and $i(1), \ldots, i(n)$ are distinct in I. For $x \in X$, $p \cdot -x = 0$, so by (b) in the characterization theorem 11.20 for amalgamated free products in $\boldsymbol{C}_\omega$, pick elements $c_{xi}, c_{xi(1)}, \ldots, c_{xi(n)}$ of C such that

$$a_i \cdot -x \le c_{xi} , \qquad a_{i(k)} \le c_{xi(k)} \quad \text{for } 1 \le k \le n$$

and

$$c_{xi} \cdot c_{xi(1)} \cdot \cdots \cdot c_{xi(n)} = 0 . \tag{7}$$

By κ-completeness of C, we can define

$$c_i = \Sigma^C \{c_{xi}: x \in X\} , \qquad c_k = \Pi^C \{c_{xi(k)}: x \in X\} .$$

Then $a_{i(k)} \le c_k$ for $1 \le k \le n$, since C is a κ-complete subalgebra of $A_{i(k)}$. Similarly for $x \in X$, we obtain $a_i \cdot -c_{xi} \le x$ and hence

$$a_i \cdot \Pi^C \{-c_{xi}: x \in X\} \le \Pi^{A_i} X = 0 .$$

Thus, $a_i \le c_i$ and, by (7), $p \le c_i \cdot c_1 \cdot \cdots \cdot c_n = 0$, a contradiction. □

Proof of Theorem 12.12. Let $c = ((h_i)_{i\in I}, (A_i)_{i \in I})$ be a cone in $\boldsymbol{C}_\kappa$. Consider the following class of commutative squares over c:

$$Sq = \left\{ s: s = ((e_i)_{i \in I}, D) \text{ a commutative square over } c,\ D\ \kappa\text{-completely generated by } \bigcup_{i \in I} e_i[A_i] \right\} .$$

This class is non-empty, by the preceding lemma. If $s = ((e_i)_{i\in I}, D)$ is in Sq, then Lemma 10.4, applied to the regular cardinal κ^+, gives

$$\lambda = \Big(\sum_{i \in I} |A_i| \Big)^\kappa$$

as an upper bound for the cardinality of D. Defining

$$sq = \{ s \in Sq: s = ((e_i)_{i \in I}, D),\ D \text{ a subset of } \lambda \} ,$$

we obtain a set, rather than a proper class, of representatives for the isomorphism classes in Sq. For each s in sq, we write

$$s = ((e_{si})_{i\in I}, B_s)\,.$$

Both the product algebra $B = \Pi_{s\in sq} B_s$ and, for $i \in I$, the mapping $e_i\colon A_i \to B$ defined by

$$e_i(a) = (e_{si}(a))_{s\in sq}$$

are in $\boldsymbol{C}_\kappa$, and it is easily checked that $((e_i)_{i\in I}, B)$ is a commutative square over c. Let B^* be the κ-complete subalgebra generated by $\bigcup_{i\in I} e_i[A_i]$; we prove that the commutative square

$$s^* = ((e_i)_{i\in I}, B^*)$$

is the amalgamated free product of c, in $\boldsymbol{C}_\kappa$.

So let $s' = ((e'_i)_{i\in I}, A)$ be a commutative square over c; we claim that there is a unique homomorphism h from s^* to s', in the sense of Definition 12.14. Uniqueness is trivial since B^* is κ-completely generated by $\bigcup_{i\in I} e_i[A_i]$. Let B' be the κ-complete subalgebra of A generated by $\bigcup_{i\in I} e'_i[A_i]$. Then $((e'_i)_{i\in I}, B')$ is in the class Sq, hence isomorphic to some s in sq. Without loss of generality, assume that

$$((e'_i)_{i\in I}, B') = ((e_{si})_{i\in I}, B_s)\,.$$

The restriction h of the projection map $\mathrm{pr}_s\colon B \to B_s$ to B^* works for our claim, since, for $i \in I$ and $a \in A_i$,

$$h(e_i(a)) = \mathrm{pr}_s((e_{si}(a))_{s\in sq}) = e_{si}(a) = e'_i(a)\,. \quad \square$$

Proof of Proposition 12.13. Suppose C is injective in $\boldsymbol{C}_\kappa$. Let $\lambda = \max(\omega, |C|)^+$ and consider, in $\boldsymbol{C}_\kappa$, the diagram

$$\begin{array}{ccc} 2 & \xrightarrow{\;e\;} & B = \overline{\mathrm{Fr}\,\lambda} \\ & \searrow{\scriptstyle f} & \big\downarrow{\scriptstyle g} \\ & & C \end{array}$$

where B is the completion of the free Boolean algebra on λ generators and e (respectively f) are the unique morphisms from 2 to B (respectively C); trivially e is mono. If g is in $\boldsymbol{C}_\kappa$ and $g \circ e = f$, then the kernel I of g is a proper κ-complete ideal of B; by completeness of B, let $b = \Sigma^B I$. B satisfies the countable chain condition, by 9.18, so by Lemma 10.2 there is a countable subset I' of I such that $b = \Sigma^B I'$. Thus, $b \in I$, I is the principal ideal generated by b and

$$\operatorname{ran} g \cong B/I \cong B \restriction -b\,,$$

as shown by the remarks following 5.23. But $b \neq 1$ and B is a homogeneous algebra, by 12.5. So $B \upharpoonright -b \cong B$ and ran g has cardinality greater than $|C|$, a contradiction.

In this proof, we see that B, 2, and e are actually in the category $\boldsymbol{C}_\infty$; also, if C is complete, then f is in $\boldsymbol{C}_\infty$. Hence, the same proof shows that there are no injectives in $\boldsymbol{C}_\infty$. □

Exercises

1. Imitating the proof of 4.22, show that an infinite algebra with the countable separation property has no irredundant set of generators and is not the union of a countable strictly increasing sequence of subalgebras.

2. Assume that $2^\omega = \omega_1$ and that A and Q are algebras with the countable separation property and with cardinality 2^ω. Prove that Q is a homomorphic image of A. In particular, if Q satisfies the countable separation property and $|Q| = 2^\omega = \omega_1$, then Q is the homomorphic image of a σ-complete algebra.

3. An ultrafilter p of a σ-complete Boolean algebra A is said to be σ-complete if $\Pi M \in p$ for each countable subset M of p. Show that for σ-complete A, the following are equivalent:

(a) A is isomorphic to a σ-algebra of sets,

(b) each non-zero element of A is contained in a σ-complete ultrafilter of A,

(c) no non-empty open subset of Ult A is included in

$$\bigcup \{n \subseteq \text{Ult } A\text{: } n \text{ a nowhere dense intersection of countably many clopen sets}\}.$$

(This equivalence does, of course, generalize to higher cardinals.)

4. If A is a σ-algebra of sets over X, we say that an ultrafilter p of A is determined by $x \in X$ if $p = \{a \in A\colon x \in a\}$. Prove that each σ-complete ultrafilter in the Borel algebra of the reals is determined by a point of $\boldsymbol{R}$.

Hint. Show that each σ-complete prime ideal of Bor($\boldsymbol{R}$) contains a greatest open set. (The assertion generalizes immediately to every second countable T_1-space, and it holds in fact for each metric space whose cardinality is non-measurable; cf. Theorem 27.1 in Sikorski [1964].)

5. Consider the space $X = \omega_1$ of all countable ordinals, equipped with the order topology.

(a) For each Borel subset b of X, either b or $X \setminus b$ includes a closed unbounded subset of ω_1.

(b) $\{b \in \text{Bor}(X)\colon b$ includes a closed unbounded set$\}$ is the only σ-complete ultrafilter of Bor(X) not determined by a point of X.

6. (for logicians) Let κ, λ be infinite cardinals, κ regular; we present a construction for the free κ-complete Boolean algebra $\text{Fr}_\kappa(\lambda)$ on λ free generators. Denote by *Fm* the set of all formulas built up from a set V of λ distinct propositional variables and negation, disjunctions and conjunctions of less than κ formulas. For $\alpha, \beta \in Fm$, write $\alpha \sim \beta$ if each map $h\colon V \to 2 = \{\text{false}, \text{true}\}$ assigns the same truth value to α and β. $Fm/\sim$ is a Boolean algebra in an obvious way, and it is isomorphic to $\text{Fr}_\kappa(\lambda)$.

7. (D. Maharam; a slight generalization of the Loomis–Sikorski theorem) Assume A is a σ-complete Boolean algebra and A_0 is a subalgebra of A σ-generating A. Then A is the quotient, under a σ-complete epimorphism, of a σ-complete subalgebra of Bor(Ult A_0).

13. Complete algebras

The class of complete Boolean algebras has been intensively investigated and it is probably the most important special class of Boolean algebras. This is partially due to its role in the theory of forcing, via Boolean-valued models of set theory. In fact, many results on complete algebras translate into relevant assertions on Boolean-valued models or were originally motivated by this translation. Let us point out that the results of this section will be complemented, in the next one, by characterizations, via distributivity (respectively non-distributivity) conditions, of two special kinds of complete Boolean algebras: Souslin algebras and the collapsing algebras RO($^{\omega}\kappa$) considered below.

This section centers around three results: the Gaifman–Hales theorem 13.2, the Balcar–Franěk theorem 13.6, and Frolík's theorem 13.23. The first of these states that, contrary to the situation for κ-complete algebras, there are no free complete algebras over infinitely many generators. This follows from a theorem of Solovay's which is interesting in its own right: the regular open algebra of the product space $^{\omega}\kappa$ (κ with the discrete topology) has a countable set of complete generators. We then present Stavi's proof of a theorem due to Kripke: every complete Boolean algebra is completely embeddable into a countably generated one.

The Balcar–Franěk theorem says that every infinite complete Boolean algebra A has an independent subset of cardinality $|A|$. Its proof uses different combinatorial techniques, including the non-trivial results on pairwise disjoint families from Section 3, to produce large independent subsets. We proceed to give two applications, by McKenzie (respectively Monk), of the Balcar–Franěk theorem, dealing with independent complete generators for infinite complete Boolean algebras (respectively with independence in algebras having the countable separation property).

Frolík's theorem reveals part of the structure of automorphisms (in fact, of particular endomorphisms) of a complete Boolean algebra A: if f is an automorphism, then A has a partition of unity $\{a_0, a_1, a_2, a_3\}$ such that f is the identity on $A \upharpoonright a_0$ and $f(a_i)$ is disjoint from a_i for $i \geq 1$. It follows that the set of fixed points of the dual of f, a continuous map from Ult A into itself, is clopen in Ult A. Combination with the Balcar–Franěk theorem implies that the dual space of an infinite Boolean algebra is not homogeneous, i.e. that there are points x and y in Ult A such that no homeomorphism from Ult A onto itself maps x to y.

13.1. Countably generated complete algebras

We have shown in Section 12 that, for an arbitrary set I and any infinite cardinal κ, there exists a free algebra over I, in the category $\boldsymbol{C}_\kappa$ of κ-complete

Boolean algebras and κ-complete homomorphisms. This was part of the more general theorem 12.12 on the existence of amalgamated free products in C_κ, the essential point of the proof being that the proper *class* of commutative squares over a cone in C_κ has a *set* of representatives. The existence of this set, in turn, followed from the fact that, for B in C_κ and $X \subseteq B$, Lemma 10.4 gives an upper bound on the cardinality of $\langle X \rangle^{\kappa\text{-cm}}$ (the κ-complete subalgebra of B generated by X) depending only on $|X|$ and κ but not on B.

This fact does not generalize to the category of complete Boolean algebras and complete homomorphisms: even for countable X, there is no upper bound for the size of $\langle X \rangle^{\text{cm}}$, by the following theorem. The proof uses the regular open algebra $\mathrm{RO}({}^\omega\kappa)$, where the infinite cardinal κ is given the discrete and ${}^\omega\kappa$ the product topology. This algebra is called a *collapsing algebra* since the effect of forcing with $\mathrm{RO}({}^\omega\kappa)$ is to collapse κ onto ω.

13.1. THEOREM (Solovay). *Let κ, an infinite cardinal, have the discrete and ${}^\omega\kappa$ the product topology. Then the collapsing algebra* $\mathrm{Ro}({}^\omega\kappa)$ *is countably completely generated and has cardinality at least κ.*

Similar to Definition 12.10, call a pair (e, F) a *free complete Boolean algebra* over I if F is a complete Boolean algebra, e maps I into F and for each map f from I into a complete Boolean algebra B, there exists a unique complete homomorphism $g: F \to B$ such that $g \circ e = f$. Solovay's theorem immediately implies non-existence of free complete algebras over infinite sets, a result originally proved by Gaifman and Hales, independently, by arguments involving infinitary logic.

13.2. COROLLARY (Gaifman, Hales). *There are no free complete Boolean algebras over infinite sets.*

PROOF. Assume that I is infinite and (e, F) is a free complete algebra over I. Let κ be any cardinal greater than $|F|$ and consider the collapsing algebra $B = \mathrm{RO}({}^\omega\kappa)$. By Solovay's theorem, B has a countable set X of complete generators; let $f: I \to X$ be onto. Now if $g: F \to B$ is complete and satisfies $g \circ e = f$, then ran g is a complete subalgebra of B including X. So g is onto and $\kappa \leq |B| \leq |F| < \kappa$, a contradiction. $\square$

Proof of Theorem 13.1. Denote by X the product space ${}^\omega\kappa$ and by B the algebra $\mathrm{RO}({}^\omega\kappa)$.

First note that each subset u of X depending only on finitely many coordinates is clopen, hence an element of B. For let $u = \{x \in X: x \restriction k \in M\}$, where $k \in \omega$ and $M \subseteq {}^k\kappa$. Now ${}^k\kappa$ is a discrete space, so M is clopen in ${}^k\kappa$ and u is clopen in X.

Thus, for $\alpha < \kappa$ and $n, m \in \omega$, the sets defined by

$$a_{n\alpha} = \{x \in X: x(n) = \alpha\},$$

$$b_{mn} = \{x \in X: x(m) < x(n)\}$$

are elements of B. Since $\{a_{n\alpha}: \alpha < \kappa\}$ is a pairwise disjoint family in B for arbitrary n, it follows that $|B| \geq 2^\kappa$ (Exercise 5 in Section 11 shows that, actually, $|B| = 2^\kappa$).

Being a subbase for the product topology, the set $\{a_{n\alpha} : n \in \omega, \alpha < \kappa\}$ completely generates B. We shall prove that

$$(1) \qquad a_{n\alpha} = \Big(-\sum_{\beta<\alpha} a_{n\beta}\Big) \cdot \prod_{m\in\omega} \Big(-b_{mn} + \sum_{\beta<\alpha} a_{m\beta}\Big).$$

Then the countable set $\{b_{mn} : m, n \in \omega\}$ completely generates each $a_{n\alpha}$, by induction on α, and hence all of B. To prove (1), we note that

$$(2) \qquad \sum_{\beta<\alpha} a_{n\beta} = \{x \in X : x(n) < \alpha\}.$$

This holds because the right-hand side of (2) equals $\bigcup_{\beta<\alpha} a_{n\beta}$ and this set is clopen, hence regular open, depending only on the nth coordinate. (2) gives:

$$(3) \qquad -b_{mn} + \sum_{\beta<\alpha} a_{m\beta} = \{x \in X : x(m) < x(n) \text{ implies } x(m) < \alpha\},$$

since both sets depend only on the coordinates m and n and are clopen.

Denote by r the right-hand side of (1). By (2), $a_{n\alpha} \cdot \Sigma_{\beta<\alpha} a_{n\beta} = 0$ and by (3), $a_{n\alpha} \leq -b_{mn} + \Sigma_{\beta<\alpha} a_{m\beta}$ for $m \in \omega$; so $a_{n\alpha} \leq r$. If $a_{n\alpha} < r$, let $u \in B$ such that $0 < u \leq r \cdot -a_{n\alpha}$. We may assume that u is a set in the canonical base of X, say

$$u = \{x \in X : x(0) = \alpha(0), \ldots, x(k-1) = \alpha(k-1)\},$$

where $k \in \omega$ and $\alpha(0), \ldots, \alpha(k-1) < \kappa$. It follows from $u \cdot a_{n\alpha} = 0$ that $n < k$ and $\alpha \neq \alpha(n)$. If $\alpha(n) < \alpha$, then $u \leq a_{n\alpha(n)} \leq \Sigma_{\beta<\alpha} a_{n\beta}$ and $u \leq r \leq -\Sigma_{\beta<\alpha} a_{n\beta}$, a contradiction. If $\alpha < \alpha(n)$, pick $m \in \omega \setminus k$ and $x \in u$ such that $x(m) = \alpha$. Then $x(m) < \alpha(n) = x(n)$; so

$$x \in u \subseteq r \subseteq -b_{mn} + \sum_{\beta<\alpha} a_{m\beta},$$

and (3) implies that $x(m) < \alpha$; a contradiction. Thus, $a_{n\alpha} = r$. □

The collapsing algebra $\mathrm{RO}({}^\omega\kappa)$ has very interesting properties. It is homogeneous by Corollary 12.5, being the completion of the homogeneous partial order $\mathrm{Fn}(\omega, \kappa, \omega)$. In Section 14, $\mathrm{RO}({}^\omega\kappa)$ is characterized, up to isomorphism, by a non-distributivity condition. As a consequence of this characterization, many naturally defined complete Boolean algebras turn out to be isomorphic to collapsing algebras (cf. Exercise 3 in Section 14). Finally, the characterization gives a result due to Kripke and proved in Section 14: every complete Boolean algebra of size at most κ is completely embeddable into $\mathrm{RO}({}^\omega\kappa)$. This, in particular, shows:

13.3. THEOREM (Kripke). *Every complete Boolean algebra can be completely embedded into a countably generated one.*

We give here a proof of 13.3 due to Stavi, which is interesting in its own right for a reader acquainted with infinitary logic. Let us recall some definitions from the logic $L_{\infty\omega}$.

Consider a language $L = \{\varepsilon, U\}$, where ε is a binary and U a unary predicate symbol; ε will denote the membership relation of set theory in the formal language being set up. Let X be a fixed countable set of individual variables. The formulas

$$x\varepsilon y\,, \qquad U(x)$$

where x and y are in X, are called atomic. $L_{\infty\omega}$ is the least class F satisfying:

each atomic formula is in F,

for ϕ in F and x in X, $\neg\phi$ and $\exists x\phi$ are in F,

if Φ is a subset of F and $\bigcup\,\{\mathrm{Fv}(\phi)\colon \phi \in \Phi\}$ is finite, then $\bigvee \Phi$ (the disjunction of the formulas in Φ) is in F.

Here $\mathrm{Fv}(\phi)$, the set of free variables of ϕ, is defined as usual. $\bigwedge \Phi$, $\forall x\Phi$, $\phi \vee \psi$, $\phi \wedge \psi$, $\phi \to \psi$, $\phi \leftrightarrow \psi$ are abbreviations for the formulas $\neg \bigvee \{\neg\phi\colon \phi \in \Phi\}$, $\neg\exists x\neg\phi$, $\bigvee \{\phi, \psi\}$, $\bigwedge \{\phi, \psi\}$, $\neg\phi \vee \psi$, $(\phi \to \psi) \wedge (\psi \to \phi)$. It follows by induction on the complexity of ϕ that each ϕ in $L_{\infty\omega}$ has only finitely many free variables; we write $\phi(x_1 \ldots x_n)$ if $\mathrm{Fv}(\phi) \subseteq \{x_1, \ldots, x_n\}$. A sentence of $L_{\infty\omega}$ is a formula without any free variables. $L_{\infty\omega}$ is, of course, a proper class. But for each infinite cardinal κ, we can consider $L_{\kappa\omega}$, the least subset of $L_{\infty\omega}$ including all atomic formulas and closed under negation, existential quantification and disjunctions of length less than κ, as far as they are feasible in $L_{\infty\omega}$.

The formulas of $L_{\infty\omega}$ can be naturally interpreted in Boolean-valued structures. To this end, let A be a complete Boolean algebra. An *A-valued L-structure* is a sequence

$$\boldsymbol{M} = (M, \varepsilon^{\boldsymbol{M}}, U^{\boldsymbol{M}})\,,$$

where M, the underlying set of $\boldsymbol{M}$, is non-empty and $\varepsilon^{\boldsymbol{M}}$, $U^{\boldsymbol{M}}$ are functions with "truth"-values in A:

$$\varepsilon^{\boldsymbol{M}}\colon M \times M \to A\,, \qquad U^{\boldsymbol{M}}\colon M \to A\,.$$

Given such a structure $\boldsymbol{M}$, a formula $\phi(x_1 \ldots x_n)$ in $L_{\infty\omega}$ and elements $m_1, \ldots, m_n$ of M, we define by induction the Boolean truth-value $\|\phi[m_1 \ldots m_n]\|$ of ϕ in $\boldsymbol{M}$ under the assignment of m_i to x_i, an element of A:

$$\|(x_i \varepsilon x_j)[m_1 \ldots m_n]\| = \varepsilon^{\boldsymbol{M}}(m_i, m_j) \quad \text{if } 1 \le i, j \le n\,,$$

$$\|U(x_i)[m_1 \ldots m_n]\| = U^{\boldsymbol{M}}(m_i) \quad \text{if } 1 \le i \le n\,,$$

$$\|(\neg\phi)[m_1 \ldots m_n]\| = -\|\phi[m_1 \ldots m_n]\|\,,$$

$$\|(\bigvee \Phi)[m_1 \ldots m_n]\| = \textstyle\sum^A \{\|\phi[m_1 \ldots m_n]\|\colon \phi \in \Phi\}\,,$$

$$\|(\exists x\phi(x\,x_1 \ldots x_n))[m_1 \ldots m_n]\| = \textstyle\sum^A \{\|\phi[m\,m_1 \ldots m_n]\|\colon m \in M\}\,.$$

(We used completeness of A in the last two clauses.) For application in Lemma 13.5, let us define the following $L_{\infty\omega}$-formulas; intuitively, $\alpha_c(x)$ defines the position of the set c in the set-theoretical universe $(V, \in)$.

13.4. DEFINITION. For every set c, the formula $\alpha_c(x)$ and the sentence β_c of $L_{\infty\omega}$ are defined by induction on the set-theoretical rank of c:

$$\alpha_c(x): \quad \forall y(y\varepsilon x \to \bigvee \{\alpha_d(y): d \in c\}) \wedge \bigwedge \{\exists y(y\varepsilon x \wedge \alpha_d(y)): d \in c\}$$

$$\beta_c: \quad \exists x(U(x) \wedge \alpha_c(x)) .$$

13.5. LEMMA (Stavi). *Let A be a complete Boolean algebra. Then there is an A-valued L-structure $\boldsymbol{M} = (M, \ldots)$ such that $A \subseteq M$ and $\|\beta_a\| = a$ for each a in A.*

PROOF. Let M be any transitive set including A. We define two functions ε^M and U^M, coding the membership relation on M and the subset A of M, by

$$\varepsilon^M(m, m') = \begin{cases} 1 & \text{if } m \in m' , \\ 0 & \text{if } m \notin m' ; \end{cases} \qquad U^M(m) = \begin{cases} m & \text{if } m \in A , \\ 0 & \text{if } m \notin A . \end{cases}$$

For an arbitrary set c and $m \in M$, $\|\alpha_c[m]\| \in \{0, 1\}$ since the symbol U does not occur in the formula $\alpha_c(x)$ and ε^M attains only values in $\{0, 1\}$. We prove by induction on the rank of c that

$$(4) \qquad \|\alpha_c[m]\| = 1 \quad \text{iff } m = c .$$

Suppose that for $d \in c$ and $n \in M$, $\|\alpha_d[n]\| = 1$ iff $n = d$. Then

$$\|\alpha_c[m]\| = 1 \quad \text{iff (for } n \in M \text{ such that } n \in m, \text{ there is } d \in c \text{ such that } n = d) \text{ and (for } d \in c, \text{ there is } n \in M \text{ such that } n \in m \text{ and } n = d) .$$

By transitivity of M, this amounts to saying that $m \subseteq c$ and $c \subseteq m$, i.e. that $m = c$.
Now for a in A, (4) implies that

$$\|\beta_a\| = \Sigma \{U^M(m) \cdot \|\alpha_a(m)\|: m \in M\} = U^M(a) = a . \quad \square$$

Proof of Theorem 13.3. Given a complete Boolean algebra A, fix an infinite cardinal κ such that $|A| < \kappa$ and each of the sentences β_a defined in 13.4, for $a \in A$, is in $L_{\kappa\omega}$. By Lemma 13.5, fix an A-valued L-structure $\boldsymbol{M}$ such that $\|\beta_a\| = a$ for each a in A.

Starting out with $\boldsymbol{M}$ and $L_{\kappa\omega}$, we define a Boolean algebra of (equivalence classes of) formulas, as in Example 1.12: for ϕ and ψ in $L_{\kappa\omega}$, say $\phi = \phi(x_1 \ldots x_n)$ and $\psi = \psi(x_1 \ldots x_n)$, let $\phi \sim \psi$ iff ϕ and ψ are equivalent in $\boldsymbol{M}$, i.e.

$$\phi \sim \psi \quad \text{iff } \|\forall x_1 \ldots x_n(\phi(x_1 \ldots x_n) \leftrightarrow \psi(x_1 \ldots x_n))\| = 1 .$$

It is a routine matter to check that the equivalence classes

$$\bar{\phi} = \{\psi \in L_{\kappa\omega} : \phi \sim \psi\},$$

for ϕ in $L_{\kappa\omega}$, constitute a Boolean algebra C under the operations:

$$-\bar{\phi} = \overline{\neg\phi}, \qquad \bar{\phi} + \bar{\psi} = \overline{\phi \vee \psi},$$

etc. and that

$$\bar{\phi} \leq \bar{\psi} \quad \text{iff } \|\forall x_1 \dots x_n(\phi \rightarrow \psi)\| = 1,$$

(5) $$\textstyle\sum^C \{\bar{\phi} : \phi \in \Phi\} = \overline{\bigvee \Phi}$$

if $\bigvee \phi \in L_{\kappa\omega}$. Moreover,

(6) $$\overline{\exists x\phi(x\, x_1 \dots x_n)} = \sum_{y \in X}^{C} \{\overline{\phi(y\, x_1 \dots x_n)} : y \in X\}$$

for every formula $\phi(x\, x_1 \dots x_n)$ in $L_{\kappa\omega}$ – here X is the set of variables in $L_{\infty\omega}$ and $\phi(y\, x_1 \dots x_n)$ denotes the result of substituting y for x in $\phi(x\, x_1 \dots x_n)$. To prove (6), note that trivially

$$\overline{\phi(y\, x_1 \dots x_n)} \leq \overline{\exists x\phi(x\, x_1 \dots x_n)}.$$

For the converse, let $\bar{\psi}$ be any upper bound of $\{\overline{\phi(y\, x_1 \dots x_n)} : y \in X\}$ in C; by taking the list $\{x_1, \dots, x_n\}$ of variables big enough, we may assume that also $\psi = \psi(x_1 \dots x_n)$. Let y be a variable occurring neither in ψ nor in $\phi(x\, x_1 \dots x_n)$. Then

$$\|\forall y\, \forall x_1 \dots x_n(\phi(y\, x_1 \dots x_n) \rightarrow \psi(x_1 \dots x_n))\| = 1,$$

and hence

$$\|\forall x_1 \dots x_n(\exists y\phi(y\, x_1 \dots x_n) \rightarrow \psi(x_1 \dots x_n))\| = 1,$$

$$\|\forall x_1 \dots x_n(\exists x\phi(x\, x_1 \dots x_n) \rightarrow \psi(x_1 \dots x_n))\| = 1;$$

so $\overline{\exists x\phi(x\, x_1 \dots x_n)} \leq \bar{\psi}$, which proves (6).

By (5) and (6), both C and its completion B are completely generated by the countable set $\{\bar{\phi} : \phi \text{ atomic}\}$. We claim that the assignment

$$e: A \rightarrow B, \qquad e(\|\phi\|) = \bar{\phi},$$

for each sentence ϕ of $L_{\kappa\omega}$, is a complete embedding from A into B; note that actually every element a of A is the truth-value of some sentence in $L_{\kappa\omega}$, e.g. $a = \|\beta_a\|$.

e is well-defined and one-to-one since, for arbitrary $L_{\kappa\omega}$-sentences ϕ and ψ, $\|\phi\| = \|\psi\|$ iff $\|\phi \leftrightarrow \psi\| = 1$ iff $\bar{\phi} = \bar{\psi}$. It preserves sums of arbitrary length, for let $Y \subseteq A$, say $Y = \{\|\phi\| : \phi \in \Phi\}$ for some set Φ of $L_{\kappa\omega}$-sentences satisfying $|\Phi| =$

$|Y|$. So $|\Phi| \leq |A| < \kappa$, $\bigvee \Phi$ is a sentence of $L_{\kappa\omega}$ and

$$\begin{aligned}
e\left(\sum^A Y\right) &= e\left(\sum^A \{\|\phi\|: \phi \in \Phi\}\right) \\
&= e\left(\|\bigvee \Phi\|\right) \\
&= \overline{\bigvee \Phi} \qquad \text{by definition of } e \\
&= \sum^C \{\bar{\phi}: \phi \in \Phi\} \text{ by (5)} \\
&= \sum^C e[Y] \\
&= \sum^B e[Y] .
\end{aligned}$$

A similar but simpler argument shows that e preserves complements. □

13.2. The Balcar–Franěk theorem

We have seen in Example 9.21 that the power set algebra $P(X)$ of an infinite set X has an independent subset of maximal cardinality $2^{|X|}$. This subsection is devoted to a vast generalization.

13.6. THEOREM (Balcar–Franěk). *Every infinite complete Boolean algebra A has an independent subset of cardinality $|A|$.*

Before embarking on the proof, let us indicate two easy consequences. Three less obvious applications will be found in the following subsections.

13.7. COROLLARY. *For every infinite complete Boolean algebra A, $|\mathrm{Ult}\, A| = 2^{|A|}$.*

PROOF. Let U be an independent subset of A of size $|A|$. The subalgebra of A generated by U is free over U, hence has $2^{|A|}$ ultrafilters by Corollary 9.7, and each of these extends to an ultrafilter of A. □

13.8. COROLLARY. *Let A and B be complete Boolean algebras and $|B| \leq |A|$. Then there is an epimorphism from A onto B.*

PROOF. This is trivial if A is finite. Otherwise, let $U \subseteq A$ be independent and of size $|A|$ and let f be a map from U onto B. By independence, f extends to an epimorphism g from the subalgebra generated by U onto B, and g extends to an epimorphism h from A onto B, by Sikorski's extension theorem 5.9. □

The proof of the Balcar–Franěk theorem is based on a number of lemmas which fall naturally into four groups. The first group deals with decompositions of a complete Boolean algebra into a product of simpler factors and states that the theorem holds for the product if it holds for each factor.

For a statement of the next lemma, call a subset F of a product $\Pi_{i\in I} X_i$ of arbitrary sets X_i *finitely distinguished* if, for each finite subset $\{f_1, \ldots, f_n\}$ of F with $f_1, \ldots, f_n$ pairwise distinct, there is some $i \in I$ such that $f_1(i), \ldots, f_n(i)$ are pairwise distinct.

13.9. LEMMA. *For any family $(X_i)_{i\in I}$ of infinite sets, there is a finitely distinguished subset of $\Pi_{i\in I} X_i$ of cardinality $|\Pi_{i\in I} X_i|$.*

PROOF. Denote by P the cartesian product $\Pi_{i\in I} X_i$. Let I be well-ordered by a relation $<$ in such a way that

(7) $\qquad i \le j$ implies $|X_i| \le |X_j|$.

We may assume that I is an ordinal, say γ, and prove the lemma by induction on γ. There is nothing to prove if $\gamma = 0$ or $\gamma = 1$.

First, consider the case that γ is representable as the ordinal sum $\alpha + \beta$ where α and β are ordinals less than γ. Then one of the products $\Pi_{i<\alpha} X_i$ and $\Pi_{\alpha\le i<\gamma} X_i$ has cardinality $|P|$; without loss of generality, assume that $\Pi_{i<\alpha} X_i$ does. By induction hypothesis, let $G \subseteq \Pi_{i<\alpha} X_i$ be finitely distinguished and of size $|P|$; fix some h in $\Pi_{\alpha\le i<\gamma} X_i$ and let

$$F = \{f \in P\colon f \restriction \alpha \in G \text{ and } f \restriction (\gamma\backslash\alpha) = h\} .$$

Clearly, F works for the lemma.

So we are left with the case that, for each $\alpha < \gamma$, the set $\gamma\backslash\alpha$ has order type γ. In particular, for $\kappa = |\gamma|$,

(8) $\qquad$ for each $\alpha < \gamma$, $\gamma\backslash\alpha$ has cardinality κ ,

and so has the set E of all finite non-empty subsets of γ. Using an enumeration of E of type κ, define by induction and (8), for each $e \in E$, an element $i(e)$ of γ such that $\max(e) \le i(e)$ and the $i(e)$ are pairwise distinct. For $e \in E$, put

$$P_e = \prod_{i\in e} X_i .$$

The sets X_i are infinite; so by (7) and our choice of $i(e)$, there is a one-to-one map

$$m_e\colon P_e \to X_{i(e)} .$$

Then let the map

$$*\colon P \to P$$

be such that, for $g \in P$,

$$g^*(i) = m_e(g \restriction e) \quad \text{if } i = i(e) \text{ for some } e \in E\,.$$

The following argument shows that this map is one-to-one and that

$$F = \{g^*\colon g \in P\}$$

is finitely distinguished: assume $g_1, \ldots, g_n \in P$ are pairwise distinct. Pick $e \in E$ large enough such that $g_1 \restriction e, \ldots, g_n \restriction e$ are distinct. Then, for $i = i(e)$, $g_1^*(i), \ldots, g_n^*(i)$ are distinct since $g_k^*(i) = m_e(g_k \restriction e)$ and m_e is one-to-one. □

13.10. COROLLARY. *Let, for $i \in I$, U_i be an infinite independent subset of a Boolean algebra A_i. Then $\Pi_{i\in I} A_i$ has an independent subset of size $|\Pi_{i\in I} U_i|$.*

PROOF. By Lemma 13.9, let U be a finitely distinguished subset of $\Pi_{i\in I} U_i$; we show that U is independent in $\Pi_{i\in I} A_i$. For pairwise distinct elements $u_1, \ldots, u_n$ of U and arbitrary $\varepsilon_1, \ldots, \varepsilon_n \in \{+1, -1\}$, pick $i \in I$ such that $u_1(i), \ldots, u_n(i)$ are all distinct. Then by independence of U_i,

$$(\varepsilon_1 u_1 \cdot \cdots \cdot \varepsilon_n u_n)(i) = \varepsilon_1 u_1(i) \cdot \cdots \cdot \varepsilon_n u_n(i) > 0\,,$$

so $\varepsilon_1 u_1 \cdot \cdots \cdot \varepsilon_n u_n > 0$. □

By this corollary, an infinite complete Boolean algebra A has an independent subset of size $|A|$ if A is isomorphic to a product $\Pi_{i\in I} A_i$, where each A_i is infinite and has an independent subset of size $|A_i|$. Now the easiest way of decomposing A into a product of simpler factors is to write

$$A \cong A \restriction a \times A \restriction -a\,,$$

where a is the sum of all atoms of A. Then $A \restriction a$ is atomic and hence, being complete, isomorphic to the power set algebra $P(\lambda)$ for some cardinal λ, by Corollary 2.7. If λ is infinite, then the Balcar–Franěk theorem holds for $A \restriction a \cong P(\lambda)$ by Example 9.21. Thus, the proof of the Balcar–Franěk theorem essentially reduces to the case where A is atomless. The next lemma gives a more sophisticated method for decomposing a complete Boolean algebra into simpler factors.

13.11. DEFINITION. Let A be a Boolean algebra. An *order preserving cardinal function* on A is a function ϕ which assigns a cardinal $\phi(A \restriction a)$ to each relative algebra $A \restriction a$ of A such that $\phi(A \restriction b) \leq \phi(A \restriction a)$ if $b \leq a$ in A. We write $\phi(a)$ for $\phi(A \restriction a)$. A is *ϕ-homogeneous* if $\phi(a) = \phi(1)$ for each $a \in A \setminus \{0\}$.

There are several important examples of order preserving cardinal functions on a complete Boolean algebra A, e.g. those defined by

$$\operatorname{card}(A \upharpoonright a) = |A \upharpoonright a| \qquad \text{(cardinality)},$$

$$c(A \upharpoonright a) = \sup\{|D|: D \subseteq A \upharpoonright a \text{ a pairwise disjoint family}\} \qquad \text{(cellularity)},$$

$$\pi(A \upharpoonright a) = \min\{|Y|: Y \subseteq A \upharpoonright a \text{ dense in } A \upharpoonright a\} \text{(density)},$$

$$\operatorname{ind}(A \upharpoonright a) = \sup\{|U|: U \subseteq A \upharpoonright a \text{ independent}\} \quad \text{(independence)},$$

$$\tau(A \upharpoonright a) = \min\{|X|: X \subseteq A \upharpoonright a \text{ a set of complete generators for } A \upharpoonright a\} \qquad \text{(complete generation)}.$$

The last one of these functions will be considered in McKenzie's theorem 13.19; it is order preserving since, for $b \leq a$, the canonical projection p_b defined by $p_b(x) = x \cdot b$ is a complete epimorphism from $A \upharpoonright a$ onto $A \upharpoonright b$.

13.12. LEMMA. *Let $\phi_1, \ldots, \phi_n$ be finitely many order preserving cardinal functions on a complete Boolean algebra A. Then A is decomposable into a product*

$$A \cong \prod_{i \in I} A_i ,$$

where each A_i is homogeneous for $\phi_1, \ldots, \phi_n$.

PROOF. It suffices to show that

$$D = \{x \in A: x > 0,\ A \upharpoonright x \text{ homogeneous for } \phi_1, \ldots, \phi_n\}$$

is a dense subset of A; then let $\{a_i: i \in I\}$ be a partition of unity in A included in D and put $A_i = A \upharpoonright a_i$. To prove denseness of D, let $a > 0$ in A. We construct a descending sequence

$$a = a_0 \geq a_1 \geq \cdots \geq a_n > 0$$

such that $A \upharpoonright a_i$ is ϕ_i-homogeneous: given a_i, a_{i+1} exists since ϕ_{i+1} is order preserving and there is no infinite strictly descending sequence of cardinals. Clearly, a_n is an element of D. □

It is an essential part of our strategy for the proof of the Balcar–Franěk theorem to generalize the notion of independence from subsets of A to sets of partitions in A.

13.13. DEFINITION. Let A be a Boolean algebra. A set $\boldsymbol{P}$ of partitions of unity in A is *independent* if, for $n \in \omega$, pairwise distinct elements $P_1, \ldots, P_n$ of $\boldsymbol{P}$ and arbitrary $p_1 \in P_1, \ldots, p_n \in P_n$,

$$p_1 \cdot \cdots \cdot p_n > 0 .$$

This notion is closely related to that of independence for subsets (respectively for families of subalgebras) of A. For example, let $U \subseteq A$ be such that $0, 1 \notin U$ and $-u \notin U$ for $u \in U$; then U is independent, in the sense of Definition 9.3, iff $\{\{u, -u\}: u \in U\}$ is an independent set of partitions. On the other hand, if $(P_i)_{i \in I}$ is a family of distinct partitions, consider the subalgebras $B_i = \langle P_i \rangle$ and $A_i = \langle P_i \rangle^{\mathrm{cm}}$ of A. Both B_i and A_i are atomic subalgebras of A with P_i as their set of atoms, and $\{P_i: i \in I\}$ is an independent set of partitions iff $(B_i)_{i \in I}$ (respectively $(A_i)_{i \in I}$) is an independent family of subalgebras of A, as defined in 11.3.

Our second group of lemmas produces a large independent subset of A out of a possibly small one plus one additional large partition, and reproves, in Corollary 13.15, the statement of Example 9.21.

13.14. LEMMA. *Let U be an infinite independent subset of a complete Boolean algebra A and P a partition such that $\{P\} \cup \{\{u, -u\}: u \in U\}$ is an independent set of partitions. Then A has an independent subset of size $|U|^{|P|}$.*

PROOF. For each $p \in P$, $\{u \cdot p: u \in U\}$ is an independent subset of $A \upharpoonright p$, by independence of $\{P\} \cup \{\{u, -u\}: u \in U\}$, and has size $|U| \geq \omega$. So by Lemma 13.10, $A \cong \prod_{p \in P} A \upharpoonright p$ has an independent subset of size $|U|^{|P|}$. □

13.15. COROLLARY. *For each set X of cardinality $\kappa \geq \omega$, $P(X)$ has an independent subset of size 2^κ.*

PROOF. By Lemma 13.14, it is sufficient to find an independent set of partitions $\{P\} \cup \{\{u, -u\}: u \in U\}$ in $P(X)$ such that $|P| = \kappa$ and $|U| = \omega$. We may assume that

$$X = \{(x, f) \in \kappa \times {}^{\omega}2: f(i) = 0 \text{ for all but finitely many } i \in \omega\} ,$$

since the set on the right-hand side has cardinality κ. Then let

$$P = \{p_\alpha: \alpha < \kappa\} , \qquad U = \{u_n: n \in \omega\} ,$$

where

$$p_\alpha = \{(x, f) \in X: x = \alpha\} , \qquad u_n = \{(x, f) \in X: f(n) = 1\} . \quad \square$$

The third preparatory step for the Balcar–Franěk theorem consists in one single lemma, producing independent subsets of complete Boolean algebras by essentially the technique of Lemma 10.7.

13.16. LEMMA (Vladimirov). *Let A be complete and B a complete subalgebra of A; assume that for no $b \in B^+$, $B \cap A \restriction b$ is dense in $A \restriction b$. Then there is an element u of A such that $b \cdot u > 0$ and $b \cdot -u > 0$ for each $b \in B^+$ (i.e. such that B and $\langle u \rangle$ are independent subalgebras of A).*

PROOF. We claim that the set

$$D = \{d \in A^+ : B^+ \cap A \restriction d = \emptyset\}$$

is dense in A. For let $a > 0$ in A. If $B^+ \cap A \restriction a = \emptyset$, then $a \in D$, and we are finished. Otherwise pick $b \in B$ such that $0 < b \leq a$. Since $B \cap A \restriction b$ is not dense in $A \restriction b$, there is $a' \in A$ such that $0 < a' \leq b$ and $B^+ \cap A \restriction a' = \emptyset$, i.e. $a' \in D$.

For $a \in A$, define as in Lemma 10.7

$$\mathrm{upr}(a) = \prod \{b \in B : a \leq b\},$$

the least element of B lying above a. Then $\{\mathrm{upr}(d) : d \in D\}$ is dense in B. To prove this, let $b \in B^+$. By denseness of D, let $d \in D$ such that $d \leq b$; then $\mathrm{upr}(d) \leq \mathrm{upr}(b) = b$.

By Zorn's lemma, fix $E \subseteq D$ such that $\{\mathrm{upr}(e) : e \in E\}$ is a partition of unity in B and, for $e \neq e'$ in E, $\mathrm{upr}(e) \neq \mathrm{upr}(e')$. We show that $u = \sum E$ works for the lemma. For let $b \in B^+$ and fix $e \in E$ such that $b \cdot \mathrm{upr}(e) > 0$. If $b \cdot u = 0$, then $b \cdot e = 0$, $e \leq -b$, $\mathrm{upr}(e) \leq -b$ and $b \cdot \mathrm{upr}(e) = 0$, a contradiction. On the other hand, if $b \cdot -u = 0$, then using

$$-u = \sum_{e \in E} \mathrm{upr}(e) \cdot -e,$$

we conclude tht $b \cdot \mathrm{upr}(e) \cdot -e = 0$ and $b \cdot \mathrm{upr}(e) \leq e$. But $b \cdot \mathrm{upr}(e) \in B$ and $e \in D$, so $b \cdot \mathrm{upr}(e) = 0$ by definition of D, a contradiction. □

The proof of the Balcar–Franěk theorem splits into two cases. In the first one, the cellularity cA of A is attained and a large independent subset of A is constructed from a single partition P of size cA and the preceding lemmas. Our fourth and last preliminary step is a lemma which, in case cA is not atained, replaces the partition P by a family $\boldsymbol{P}$ of independent partitions satisfying $\sup\{|P| : P \in \boldsymbol{P}\} = \mathrm{c}A$. This lemma heavily relies on the Balcar–Vojtáš theorem 3.14 on disjoint refinements.

13.17. LEMMA. *Let A be a complete Boolean algebra such that $\kappa = \mathrm{c}A$ is not attained and A is cellularity-homogeneous. Then there is an independent set $\boldsymbol{P}$ of partitions of A such that $|\boldsymbol{P}| = \kappa$ and $\sup\{|P| : P \in \boldsymbol{P}\} = \kappa$.*

PROOF. We show that for any set $\boldsymbol{Q}$ of independent partitions such that $|\boldsymbol{Q}| < \kappa$ and for any cardinal $\tau < \kappa$, there is a partition P of size at least τ such tht $\boldsymbol{Q} \cup \{P\}$ is still independent. A straightforward induction then gives $\boldsymbol{P}$ with the desired properties.

Given $\boldsymbol{Q}$, let T be the closure of $\bigcup \boldsymbol{Q}$ under finite products. Now since $\kappa = \mathrm{c}A$ is not attained, $|Q| < \kappa$ for each $Q \in \boldsymbol{Q}$. The Erdös–Tarski theorem 3.10 shows that κ is regular. So $\bigcup \boldsymbol{Q}$ and T have size less than κ, and we can write

$$T\setminus\{0\} = \{a_\alpha \colon \alpha < \mu\}$$

for some cardinal $\mu < \kappa$. Since $\mu^+ \leq \mathrm{c}A$ and A is cellularity-homogeneous, the Balcar–Vojtáš theorem 3.14 gives a disjoint refinement $\{b_\alpha \colon \alpha < \mu\}$ of $\{a_\alpha \colon \alpha < \mu\}$, i.e. $0 < b_\alpha \leq a_\alpha$ and the b_α are pairwise disjoint. For each $\alpha < \mu$, pick by $\mathrm{c}(A \restriction b_\alpha) = \mathrm{c}A = \kappa > \tau$ a partition $\{c_{\alpha\beta} \colon \beta < \tau\}$ of unity in $A \restriction b_\alpha$.

We define a partition P of unity of size τ by letting

$$P = \{x_\beta \colon \beta < \tau\} .$$

where, for $0 < \beta < \tau$,

$$x_\beta = \sum_{\alpha<\mu} c_{\alpha\beta}$$

and $\Sigma\, P = 1$ is guaranteed by putting

$$x_0 = \Big(-\sum_{\alpha<\mu} b_\alpha\Big) + \sum_{\alpha<\mu} c_{\alpha 0} .$$

To prove independence of $\boldsymbol{Q} \cup \{P\}$, assume $Q_1, \ldots, Q_n \in \boldsymbol{Q}$ are distinct and $q_i \in Q_i$ for $1 \leq i \leq n$; let $x_\beta \in P$. Then by independence of $\boldsymbol{Q}$, $q_1 \cdot \cdots \cdot q_n$ is a non-zero element of T, say $q_1 \cdot \cdots \cdot q_n = a_\alpha$, and

$$0 < c_{\alpha\beta} \leq b_\alpha \cdot x_\beta \leq a_\alpha \cdot x_\beta = q_1 \cdot \cdots \cdot q_n \cdot x_\beta . \quad \square$$

Proof of Theorem 13.6. Let A be an infinite complete Boolean algebra; we shall find an independent subset of A of cardinality $|A|$.

As in the remark following Corollary 13.10, write $A \cong B \times C$, where B is atomic and C is atomless. If $|B| = |A|$, then B is infinite; by Corollary 2.7 and Corollary 13.15, B has an independent subset of size $|A|$ and so has A, B being a homomorphic image of A. Otherwise $|C| = |A|$, and it suffices to find an independent subset of C of size $|C|$. We can thus assume A to be atomless; in particular $A \restriction a$ is infinite for each a in $A^+ = A\setminus\{0\}$. We also assume, by 13.12 and 13.10, that A is homogeneous for the order preserving cardinal functions c (cellularity) and π (density). Let, for the rest of the proof,

$$\kappa = \mathrm{c}A , \qquad \lambda = \pi A .$$

Case 1. cA is attained.

Then $|A| = \lambda^\kappa$. For $|A| \leq \lambda^\kappa$ follows from 4.9 and 10.5. To show $|A| \geq \lambda^\kappa$, pick a partition P of size κ and note that, for each $a \in P$, $\lambda = \pi A = \pi(A \restriction a) \leq |A \restriction a|$.

Let μ be minimal satisfying $\mu^\kappa = \lambda^\kappa$. So $\mu \le \lambda$, and $\nu < \mu$ implies $\nu^\kappa < \mu$ – otherwise, $\lambda^\kappa = \mu^\kappa \le (\nu^\kappa)^\kappa = \nu^\kappa$, a contradiction.

Let us now find an independent subset of size $|A| = \lambda^\kappa$, breaking up the proof into the cases $\mu = 2$ and $\mu > 2$. This is easy if $\mu = 2$. In this case, fix a partition P of A of size κ and consider the complete subalgebra D of A generated by P. Then D is isomorphic to $P(\kappa)$, hence has an independent subset of cardinality $2^\kappa = |A|$ by 13.15, and we are finished.

Now assume that $2 < \mu$ and thus that $2^\kappa < \mu$. By Lemma 13.14, it suffices to find an independent set $\boldsymbol{P} \cup \{Q\}$ of partitions such that $|Q| = \kappa$, $|\boldsymbol{P}| = \mu$ and $|P| = 2$ for $P \in \boldsymbol{P}$. To do this, first pick Q satisfying $|Q| = \kappa$. $\boldsymbol{P}$ is then constructed by induction: suppose we have already constructed a set $\boldsymbol{Q}$ of independent partitions such that $|\boldsymbol{Q}| < \mu$; we claim that there is an element u of A such that $\boldsymbol{Q} \cup \{\{u, -u\}\}$ is still independent. Consider the complete subalgebra D of A generated by $\bigcup \boldsymbol{Q}$; then by 10.5,

$$|D| \le (|\boldsymbol{Q}| \cdot \kappa)^\kappa = |\boldsymbol{Q}|^\kappa \cdot 2^\kappa < \mu \le \lambda = \pi A .$$

By Vladimirov's lemma 13.16 and π-homogeneity of A, there is some $u \in A$ such that $d \cdot u > 0$ and $d \cdot -u > 0$ for each non-zero d in D. Clearly, this u works for our claim.

Case 2. cA is not attained.

It follows, as in Case 1, that $|A| = \lambda^{<\kappa}$; let μ be minimal satisfying $\mu^{<\kappa} = \lambda^{<\kappa}$. Again $\nu < \mu$ implies $\nu^{<\kappa} < \mu$ – here we use regularity of κ, ensured by the Erdös–Tarski theorem 3.10. By Lemma 13.17, fix an independent set $\boldsymbol{P}$ of partitions such that $|\boldsymbol{P}| = \kappa$, say

$$\boldsymbol{P} = \{P_\alpha : \alpha < \kappa\}$$

and, for $\kappa_\alpha = |P_\alpha|$,

$$\kappa = \sup_{\alpha<\kappa} \kappa_\alpha .$$

If $\mu = 2$, consider for $\alpha < \kappa$ the complete subalgebra D_α of A generated by P_α. Then D_α is isomorphic to $P(\kappa_\alpha)$ and $(D_\alpha)_{\alpha<\kappa}$ is an independent family of subalgebras of A. By 13.15, let U_α be an independent subset of D_α such that $|U_\alpha| = 2^{\kappa_\alpha}$. Clearly,

$$U = \bigcup_{\alpha<\kappa} U_\alpha$$

is an independent subset of A and has cardinality

$$\sum_{\alpha<\kappa} 2^{\kappa_\alpha} = 2^{<\kappa} = \lambda^{<\kappa} = |A| .$$

Now assume that $2 < \mu$ and thus that $\kappa \le 2^{<\kappa} < \mu$. We indicate how to find, for $\alpha < \kappa$, a set $\boldsymbol{Q}_\alpha$ of partitions such that $|\boldsymbol{Q}_\alpha| = \mu$, each $Q \in \boldsymbol{Q}_\alpha$ has size 2 and, with the partitions P_α chosen above, $\{P_\alpha: \alpha < \kappa\} \cup \bigcup_{\alpha<\kappa} Q_\alpha$ is independent. Starting out with $\{P_\alpha: \alpha < \kappa\}$, construct as in Case 1 by induction and using Vladimirov's lemma 13.16, a set Q of partition of size 2 such that $|Q| = \mu$ and $\{P_\alpha: \alpha < \kappa\} \cup Q$ is independent. Then choose pairwise disjoint subsets Q_α of Q, for $\alpha < \kappa$, such that each Q_α has cardinality μ and $Q = \bigcup_{\alpha<\kappa} Q_\alpha$.

For $\alpha < \kappa$, consider the subalgebra A_α of A generated by $P_\alpha \cup \bigcup Q_\alpha$; $(A_\alpha)_{\alpha<\kappa}$ is an independent family of (non-complete) subalgebras of A. It was mentioned as being one of the consequences of 5.11 that there is, for each $\alpha < \kappa$, a subalgebra D_α of A such that $A_\alpha \subseteq D_\alpha$ and D_α is isomorphic, over A_α, to the completion of A_α; note that D_α is complete in its own right but not necessarily a complete subalgebra of A. By denseness of A_α in D_α, the family $(D_\alpha)_{\alpha<\kappa}$ of subalgebras of A is still independent. Inside of D_α, we apply Lemma 13.14 to $\{P_\alpha\} \cup Q_\alpha$ and obtain an independent subset U_α of D_α of size

$$|U_\alpha| = |Q_\alpha|^{|P_\alpha|} = \mu^{\kappa_\alpha} .$$

Again

$$U = \bigcup_{\alpha<\kappa} U_\alpha$$

is independent and has cardinality

$$\sum_{\alpha<\kappa} \mu^{\kappa_\alpha} = \mu^{<\kappa} = |A| . \quad \square$$

13.3. Two applications of the Balcar–Franěk theorem

We apply the Balcar–Franěk theorem to prove a theorem on independent sets of complete generators in complete algebras, and another one on independence in Boolean algebras satisfying the countable separation property. For the first one, let us recall the cardinal invariant τ (complete generation) quoted in the proof of the Balcar–Franěk theorem as an example of an order preserving cardinal function.

13.18. DEFINITION. For every complete Boolean algebra A,

$$\tau A = \min\{|X|: X \subseteq A \text{ a set of complete generators for } A\} .$$

The Balcar–Franěk theorem states that in every infinite complete Boolean algebra A there is an independent subset U which is large in the sense of cardinality, i.e. $|U| = |A|$. Does A also have an independent subset X which is large in the sense that X completely generates A? Indeed it has, and we can prescribe the cardinality of X, the obvious restriction being that $\tau A \leq |X| \leq |A|$.

13.19. THEOREM (McKenzie). *Let A be an infinite complete Boolean algebra and λ a cardinal such that $\tau A \leq \lambda \leq |A|$. Then A has an independent subset X of size λ which completely generates A.*

PROOF. Let us first prove the following Claim. By the way, this gives an easy proof, which *is* in fact Pierce's original proof from PIERCE [1958], of Theorem 12.2 for complete algebras.

Claim. There is a partition of unity $\{a_n: n \in \omega\}$ in A such that $|A| = |A \upharpoonright a_n|$ for all $n \in \omega$.

To prove this, we may assume that $A = B \times C$, where B is atomic, say $B = P(Y)$, and C is atomless. If $|B| = |A|$, then let $Y = \bigcup_{n\in\omega} Y_n$ be a partition of Y such that $|Y_n| = |Y|$ for each $n \in \omega$, and define $a_0 = (Y_0, 1)$, $a_n = (Y_n, 0)$ for $n > 0$. Otherwise $|C| = |A|$. In this case, fix by Lemma 13.12 a partition of unity $\{c_i: i \in I\}$ in C such that each $C \upharpoonright c_i$ is homogeneous for the order preserving cardinal function assigning $|C \upharpoonright c|$ to $c \in C$. For each $i \in I$, there is a partition $\{c_{in}: n \in \omega\}$ in $C \upharpoonright c_i$, since C is atomless. Put $c_n = \Sigma_{i\in I} c_{in}$ for $n \in \omega$, and let $a_0 = (1, c_0)$, $a_n = (0, c_n)$ for $n > 0$. This choice of the a_n finishes the proof of the Claim.

To prove our theorem, choose by $\tau A \leq \lambda$ a set

$$\{b_\alpha: 1 \leq \alpha < \lambda\}$$

of complete generators of A, possibly with repetitions. Take a partition of unity $\{a_n: n \in \omega\}$ as guaranteed in the Claim. By the Balcar–Franěk theorem and $\lambda \leq |A| = |A \upharpoonright a_n|$, each $A \upharpoonright a_n$ has an independent subset

$$\{u_{ni\alpha}: i \in \omega, \alpha < \lambda\} \subseteq A \upharpoonright a_n$$

consisting of pairwise distinct elements. Let then

$$X = \{x_{i\alpha}: i \in \omega, \alpha < \lambda\} ,$$

where

$$x_{i0} = a_i + \Sigma \{u_{ni0}: n \in \omega, n > i\} ,$$

$$x_{i\alpha} = b_\alpha \cdot a_i + \Sigma \{u_{ni\alpha}: n \in \omega, n > i\} \quad \text{for } 1 \leq \alpha .$$

We show that X is a set of complete generators of A: it generates each a_i, since for $n > 0$,

$$\Sigma \{a_i: i \in \omega, i \geq n\} = \Sigma \{x_{i0}: i \in \omega, i \geq n\}$$

and the a_i form a partition of unity. Next, it generates $b_\alpha \cdot a_i$ for $1 \leq \alpha < \lambda$, since $b_\alpha \cdot a_i = x_{i\alpha} \cdot a_i$. Thus, it generates each b_α, hence all of A.

Finally, X is independent: assume that the pairs $(i(1), \alpha(1)), \ldots, (i(r), \alpha(r))$

are pairwise distinct in $\omega \times \lambda$ and that $\varepsilon_1, \ldots, \varepsilon_r \in \{+1, -1\}$. Then for $n > \max\{i(1), \ldots, i(r)\}$,

$$a_n \cdot \varepsilon_1 x_{i(1)\alpha(1)} \cdot \cdots \cdot \varepsilon_r x_{i(r)\alpha(r)} = a_n \cdot \varepsilon_1 u_{n\, i(1)\alpha(1)} \cdot \cdots \cdot \varepsilon_r u_{n\, i(r)\alpha(r)} > 0 . \quad \square$$

We proceed to the second major result of this subsection.

13.20. THEOREM (Monk). *Every infinite Boolean algebra A with the countable separation property satisfies* $\operatorname{ind} A = (\operatorname{ind} A)^\omega$.

13.21. LEMMA. *If A is a complete Boolean algebra satisfying the countable chain condition and B is a dense subalgebra of A with the countable separation property, then $B = A$.*

PROOF. Let $a \in A$ be arbitrary. By denseness of B in A, there are $X, Y \subseteq B$ such that $a = \Sigma\, X$ and $-a = \Sigma\, Y$, in A. By Lemma 10.2, we may assume X and Y to be countable. The countable separation property gives $b \in B$ such that $x \leq b$ and $y \leq -b$, for all x in X and y in Y. But then $a \leq b$ and $-a \leq -b$, i.e. $a = b \in B$. $\square$

13.22. LEMMA. *Let κ be an infinite cardinal and A, B Boolean algebras such that neither A nor B has an independent subset of size κ. Then $A \times B$ has no independent subset of size κ.*

PROOF. This can be concluded from Šapirovskii's second theorem 10.17; we give a simple direct proof. Let $\{(a_\alpha, b_\alpha): \alpha < \kappa\}$ be any subset of $A \times B$ of size κ. Since A has no independent subset of size κ, there are distinct $\alpha(1), \ldots, \alpha(n)$ in κ and $\varepsilon_1, \ldots, \varepsilon_n \in \{+1, -1\}$ such that $\varepsilon_1 a_{\alpha(1)} \cdot \cdots \cdot \varepsilon_n a_{\alpha(n)} = 0$ holds in A. Similarly, there are distinct $\beta(1), \ldots, \beta(m)$ in $\kappa \setminus \{\alpha(1), \ldots, \alpha(n)\}$ and $e_1, \ldots, e_m \in \{+1, -1\}$ such that $e_1 b_{\beta(1)} \cdot \cdots \cdot e_m b_{\beta(m)} = 0$ holds in B. Thus, in $A \times B$,

$$\varepsilon_1(a_{\alpha(1)}, b_{\alpha(1)}) \cdot \cdots \cdot \varepsilon_n(a_{\alpha(n)}, b_{\alpha(n)}) \cdot e_1(a_{\beta(1)}, b_{\beta(1)}) \cdot \cdots \cdot e_m(a_{\beta(m)}, b_{\beta(m)}) = 0 . \quad \square$$

Proof of Theorem 13.20. Let A be an infinite algebra with the countable separation property and put $\kappa = \operatorname{ind} A$.

Claim. For every infinite cardinal $\lambda < \kappa$, A has an independent subset of size λ^ω.

For, $\lambda < \kappa = \operatorname{ind} A$ implies that $\operatorname{Fr} \lambda$, the free Boolean algebra on λ free generators, embeds into A. By 5.10, there is an epimorphism $f: A \to Q$, where Q has $\operatorname{Fr} \lambda$ as a dense subalgebra. Now Q has the countable separation property by 5.27 and satisfies the countable chain condition; so by Lemma 13.21, Q is isomorphic to $\overline{\operatorname{Fr} \lambda}$. Since $\overline{\operatorname{Fr} \lambda}$ has cardinality λ^ω (by 12.2 and the countable chain condition), the Balcar–Franěk theorem shows that Q and hence A has an independent subset of size λ^ω.

The Claim implies that $\lambda^\omega \leq \kappa$ for each infinite cardinal $\lambda < \kappa$. Thus, the theorem holds if cf $\kappa > \omega$. Assume, for contradiction, that cf $\kappa = \omega$, i.e.

$$\kappa = \sup\{\kappa_n : n \in \omega\},$$

where $\omega \leq \kappa_1 < \kappa_2 < \cdots$.

We shall construct, by induction, a disjoint sequence $(a_n)_{n\in\omega}$ in A such that $A \upharpoonright a_n$ has an independent subset U_n of size κ_n and $\mathrm{ind}(A \upharpoonright -(a_0 + \cdots + a_n)) = \kappa$. Given such a sequence, we obtain a contradiction as follows. By Lemma 13.9, let F be a finitely distinguished subset of the cartesian product $\Pi_{n\in\omega} U_n$ of size $\Pi_{n\in\omega} \kappa_n$. For each element $f = (x_n)_{n\in\omega}$ of F, the countable separation property for A gives an element u_f in A such that $u_f \cdot a_n = x_n$, for all $n \in \omega$ (this was explicitly proved in Claim 1 of Theorem 12.2). As in the proof of 13.10, it follows that $\{u_f : f \in F\}$ is an independent subset of A of cardinality $\Pi_{n\in\omega} \kappa_n$. But $\Pi_{n\in\omega} \kappa_n > \kappa$ by cf $\kappa = \omega$; a contradiction.

To find the elements a_n, assume $a_0, \ldots, a_{n-1}$ have already been constructed and that $r = -(a_0 + \cdots + a_{n-1})$ satisfies $\mathrm{ind}(A \upharpoonright r) = \kappa$ (for $n = 0$, put $r = 1$). Pick an independent subset X of $A \upharpoonright r$ of cardinality κ_n, and an element x of X. Both $A \upharpoonright x$ and $A \upharpoonright (r \cdot -x)$ have independent subsets of size κ_n – e.g. in $A \upharpoonright x$, consider $\{y \cdot x : y \in X\backslash\{x\}\}$. By Lemma 13.22, at least one of the algebras $A \upharpoonright x$ and $A \upharpoonright (r \cdot -x)$, say $A \upharpoonright (r \cdot -x)$, has independence κ; let then $a_n = x$. □

13.4. Automorphisms of complete algebras: Frolík's theorem

We concentrate on a result of automorphisms, more generally on surjective endomorphisms, of complete Boolean algebras.

13.23. THEOREM (Frolík). *Let A be a complete Boolean algebra and $f: A \to A$ a homomorphism from A onto A. Then there is a partition of unity $\{a_0, a_1, a_2, a_3\}$ in A such that f is the identity on $A \upharpoonright a_0$, and $a_i \cdot f(a_i) = 0$ holds for $1 \leq i \leq 3$.*

By Proposition 7.21, the dual spaces of complete algebras are the extremally disconnected Boolean spaces. Frolík's theorem therefore gives a powerful method for investigating the structure of these spaces and their homeomorphisms.

13.24. COROLLARY. *Let X be an extremally disconnected Boolean space and $\phi: X \to X$ a continuous one-to-one mapping. Then the set $\{x \in X : \phi(x) = x\}$ of fixed points of ϕ is clopen in X.*

PROOF. Let $A = \mathrm{Clop}\, X$ be the dual algebra of X. By the duality theorem 8.2 between homomorphisms and continuous maps, ϕ gives rise to an endomorphism f from A onto A defined by $f(a) = \phi^{-1}[a]$. Let $X = a_0 \cup a_1 \cup a_2 \cup a_3$ be the clopen partition of X guaranteed by Frolík's theorem. Then ϕ is the identity on a_0 and for $1 \leq i \leq 3$, $a_i \cdot f(a_i) = 0$ implies that $a_i \cap \phi^{-1}[a_i] = \emptyset$. Thus, the set of fixed points of ϕ is a_0, a clopen subset of X. □

Combination of Frolík's theorem with the Balcar–Franěk theorem gives a non-homogeneity statement for extremally disconnected Boolean spaces. A topological space X is called *homogeneous* if, for arbitrary points x and y in X, there is a homeomorphism of X onto itself mapping x to y. Let us point out a generalization of 13.25 below whose proof, however, uses more sophisticated methods: it follows from FROLÍK [1967] plus a result on the Rudin–Keisler order on the ultrafilters of $P(\omega)$ established later by Kunen (cf. EFIMOV [1972]) that no infinite compact F-space is homogeneous.

13.25. COROLLARY. *No infinite extremally disconnected Boolean space is homogeneous.*

PROOF. Let X be infinite and extremally disconnected and $A = \operatorname{Clop} X$, a complete Boolean algebra. We first construct an epimorphism $k: A \to A$ such that the kernel $\{x \in A: k(x) = 0\}$ of k is dense in A; this will enable us to apply Corollary 13.24.

Let $k = h \circ g \circ f$, where the epimorphisms f, g, and h are chosen as follows. Put $\kappa = |A|$; then $\kappa^{\omega} = \kappa$ by Theorem 12.2. The completion $\overline{\operatorname{Fr} \kappa}$ of the free Boolean algebra over κ free generators has cardinality κ, as shown in the proof of 12.3. So by Corollary 13.8, there are epimorphisms $f: A \to \overline{\operatorname{Fr} \kappa}$ and $h: \overline{\operatorname{Fr} \kappa} \to A$; this is where the Balcar–Franěk theorem enters the picture. g has to be constructed a bit more carefully: let U be a set of independent generators of $\operatorname{Fr} \kappa$ such that $|U| = \kappa$; fix two disjoint subsets V and W of U such that $V \cup W = U$ and $|V| = |W| = \kappa$, and let B (respectively C) be the subalgebras of $\operatorname{Fr} \kappa$ generated by V (respectively W). Then $\operatorname{Fr} \kappa$ is the free product $B \oplus C$; by independence of B and C in $B \oplus C$, there is a homomorphism $g_0: B \oplus C \to B$ extending the identity on B and mapping C onto $\{0, 1\}$. Sikorski's extension theorem gives a homomorphism $g: \overline{B \oplus C} \to \bar{B}$ extending g_0 and Lemma 13.21 shows that g is onto, since $\bar{B}$ satisfies the countable chain condition and the subalgebra ran g of $\bar{B}$ includes B and has, by 5.27, the countable separation property. Identifying $\bar{B}$ with $\overline{\operatorname{Fr} \kappa}$, we have constructed an epimorphism $g: \overline{\operatorname{Fr} \kappa} \to \overline{\operatorname{Fr} \kappa}$.

It is now easy to show that the kernel $\ker(k)$ of k is dense: let $a > 0$ in A. If $f(a) = 0$, then $a \in \ker(k)$ and we are finished. Otherwise, by denseness of $B \oplus C$ in $\overline{B \oplus C}$, pick $b \in B$ and $c \in C$ such that $0 < b \cdot c \leq f(a)$. Since C is atomless, pick disjoint non-zero elements c' and c'' of C such that $c = c' + c''$. Now g maps C onto $\{0, 1\}$, so at least one of the elements c' and c'' is mapped onto 0, say c'. It follows that $0 < b \cdot c' \leq f(a)$ and $g(b \cdot c') = 0$. Pick $x \in A$ such that $f(x) = b \cdot c'$ and define $a' = a \cdot x$. Then $f(a') = b \cdot c'$ (hence $a' > 0$) and $g(f(a')) = 0$; thus $0 < a' \leq a$ and $a' \in \ker(k)$.

Returning to the space X, we see that the continuous map $\phi: X \to X$ dual to the epimorphism k is one-to-one and that, by denseness of $\ker(k)$, the image $X' = \phi[X]$ of X is a closed nowhere dense subspace of X. Consider an arbitrary point x in X and the point $y = \phi(x)$ in X'; we show non-homogeneity of X by proving that there is no homeomorphism $g: X \to X$ mapping y onto x. For otherwise, $\phi \circ g$ is a continuous one-to-one map from X onto X'. By Corollary 13.24, the set F of fixed points of $\phi \circ g$ is clopen; being a subset of the nowhere dense subspace X', it must be empty. This is a contradiction, since $y \in F$. $\square$

Proof of Theorem 13.23. Let A be complete and $f\colon A \to A$ an epimorphism. For b in A, call $(b_1, b_2, b_3) \in A^3$ a *3-partition* of b (with respect to f) if $b = b_1 + b_2 + b_3$, the b_i are pairwise disjoint, and $b_i \cdot f(b_i) = 0$, for $1 \le i \le 3$. Define

$$P = \{(b, b_1, b_2, b_3)\colon (b_1, b_2, b_3) \text{ a 3-partition of } b, \text{ and } b \le f(b)\}\ ;$$

this set is certainly non-empty, having the zero element $(0, 0, 0, 0)$. P is partially ordered by letting

$$(b, b_1, b_2, b_3) \le (c, c_1, c_2, c_3) \quad \text{iff } b \le c \text{ and } b_i \le c_i\ (1 \le i \le 3)\ .$$

Claim 1. Every non-empty chain in $(P, \le)$ has an upper bound.

For suppose $\{p_k\colon k \in K\}$ is a chain in P, where

$$p_k = (b_k, b_{k1}, b_{k2}, b_{k3})\ .$$

Define

$$s_i = \sum_{k\in K} b_{ki} \quad (1 \le i \le 3)\ , \qquad s = s_1 + s_2 + s_3 = \sum_{k\in K, 1\le i\le 3} b_{ki}\ ;$$

the sequence $p = (s, s_1, s_2, s_3)$ will be an upper bound of $\{p_k\colon k \in K\}$ if we can prove that $p \in P$.

Clearly, the elements s_i are pairwise disjoint. Since, for $k \in K$ and $1 \le i \le 3$,

$$b_{ki} \le b_k \le f(b_k) \le f(s)\ ,$$

also $s \le f(s)$ holds. We still have to prove that $s_i \cdot f(s_i) = 0$, for $1 \le i \le 3$; consider, for example, $i = 1$. By distributivity and the very definition of s_1, this amounts to proving that, for all $k \in K$, $b_{k1} \cdot f(s_1) = 0$. Now $b_{k1} \le b_k \le f(b_k)$ and $f(b_k)$ is the disjoint sum

$$f(b_k) = f(b_{k1}) + f(b_{k2}) + f(b_{k3})\ ,$$

where $b_{k1} \cdot f(b_{k1}) = 0$. So

$$b_{k1} \le f(b_{k2}) + f(b_{k3}) \le f(s_2) + f(s_3) \le -f(s_1)\ ,$$

as desired.

Claim 2. If f is not the identity map, then P has a non-zero element.

To prove this, we first construct a disjoint sequence $(c_n)_{n\in\omega}$ in A such that, for $n \in \omega$,

$$(9) \qquad f(c_{n+1}) = c_n \cdot -r\ , \quad \text{where } r = f(c_0)\ .$$

At the beginning of the construction, fix some $c \in A$ such that $c > 0$ and $c \cdot f(c) = 0$: pick $b \in A$ such that $b \ne f(b)$. Then either $b \not\le f(b)$, and we let $c = b \cdot -f(b)$; or $f(b) \not\le b$, and we let $c = f(b) \cdot -b$. Let

$$c_1 = c\,, \qquad c_0 = f(c)\,.$$

Then $c_0 \cdot c_1 = 0$, $f(c_1) = f(c)$, and $c_0 \cdot -r = f(c) \cdot -f^2(c) = f(c)$ since $c \cdot f(c) = 0$ and hence $f(c) \cdot f^2(c) = 0$. Suppose we have constructed pairwise disjoint elements $c_0, \ldots, c_k$ satisfying (9) for all $n < k$; note that

$$\begin{aligned} f(c_0 + \cdots + c_k) &= r + \sum_{i<k} c_i \cdot -r \\ &= c_0 + \cdots + c_{k-1} + r\,. \end{aligned}$$

Since f is onto, there is an $x \in A$ such that $f(x) = c_k \cdot -r$; let

$$c_{k+1} = x \cdot -(c_0 + \cdots + c_k)\,.$$

Then c_{k+1} is disjoint from $c_0, \ldots, c_k$ and

$$\begin{aligned} f(c_{k+1}) &= c_k \cdot -r \cdot -(c_0 + \cdots + c_{k-1} + r) \\ &= c_k \cdot -r \end{aligned}$$

by disjointness of $c_0, \ldots, c_k$; so (9) holds for $n = k$. This finishes the construction of the elements c_n.

We check that the sequence (b, b_1, b_2, b_3) is in P, where

$$b_1 = c_0\,, \qquad b_2 = \sum \{c_n : 2 \leq n,\ n \text{ even}\}\,, \qquad b_3 = \sum \{c_n : n \text{ odd}\}\,,$$

$$b = b_1 + b_2 + b_3 = \sum \{c_n : n \in \omega\}\,.$$

By disjointness of the c_n, $\{b_1, b_2, b_3\}$ is a partition of b. It follows from (9) that, for $n \in \omega$,

$$c_n \leq f(c_{n+1}) + r = f(c_{n+1}) + f(c_0) \leq f(b)$$

and thus that $b \leq f(b)$. We still have to show that $b_i \cdot f(b_i) = 0$ for $1 \leq i \leq 3$. This holds for $i = 1$ since $b_1 = c_0 = f(c)$ and $c \cdot f(c) = 0$. To show it for $i = 2$, consider c_n, where $n \geq 2$ is even. Then

$$c_n \leq f(c_{n+1}) + r \leq f(b_3) + r\,.$$

Now $f(b_3) \leq -f(b_2)$ since $b_2 \cdot b_3 = 0$ and $f(b_2) \cdot f(b_3) = 0$. Also, $r \leq -f(b_2)$ since $c_0 \cdot b_2 = b_1 \cdot b_2 = 0$ and $r \cdot f(b_2) = f(c_0) \cdot f(b_2) = 0$. Thus, $c_n \leq -f(b_2)$ holds for every $n \geq 2$ which is even, i.e. $b_2 \cdot f(b_2) = 0$. Similarly, $b_3 \cdot f(b_3) = 0$.

Claim 3. If (a, a_1, a_2, a_3) is a maximal element of $(P, \leq)$, then $f(a) = a$.

Otherwise, let $b = f(a)$; so $a < b$. We check that (b, b_1, b_2, b_3) is an element of P greater than (a, a_1, a_2, a_3), where

$$b_1 = a_1 + f(a_3) \cdot -a\,, \qquad b_2 = a_2 + f(a_1) \cdot -a\,, \qquad b_3 = a_3 + f(a_2) \cdot -a\,.$$

It is easily seen that $\{b_1, b_2, b_3\}$ is a partition of b; moreover, $a \leq b$ implies that $b = f(a) \leq f(b)$. We show that $b_i \cdot f(b_i) = 0$ for $i = 1$, the cases $i = 2$ and $i = 3$ being similar. Now

$$f(b_1) = f(a_1) + f^2(a_3) \cdot -f(a)$$

and $b_1 \cdot f(b_1) = 0$ follows from the subsequent observations:

$a_1 \cdot f(a_1) = 0$, since $(a, a_1, a_2, a_3) \in P$;

$a_1 \cdot f^2(a_3) \cdot -f(a) = 0$, since $a_1 \leq a \leq f(a)$;

$f(a_3) \cdot -a \cdot f(a_1) = 0$, since $a_1 \cdot a_3 = 0$ and $f(a_1) \cdot f(a_3) = 0$;

$f(a_3) \cdot -a \cdot f^2(a_3) \cdot -f(a) = 0$, since $a_3 \cdot f(a_3) = 0$ and $f(a_3) \cdot f^2(a_3) = 0$.

The theorem now follows without difficulty from the above claims: by Claim 1, P has a maximal element (a, a_1, a_2, a_3); put $a_0 = -a$. So $\{a_0, a_1, a_2, a_3\}$ is a partition of unity in A. Claim 3 implies that $f(a) = a$, hence $f(a_0) = a_0$, and it remains to show that f is the identity on $A \upharpoonright a_0$. Otherwise, application of Claim 2 to the epimorphism $f \upharpoonright (A \upharpoonright a_0)$ from $A \upharpoonright a_0$ onto itself gives $0 < b \leq a_0$ and a partition $\{b_1, b_2, b_3\}$ of b such that $b \leq f(b) \leq f(a_0) = a_0$ and $b_i \cdot f(b_i) = 0$. Then the sequence $(a + b, a_1 + b_1, a_2 + b_2, a_3 + b_3)$ is in P and contradicts maximality of (a, a_1, a_2, a_3). □

Exercises

1. Let G be a subset of a power set algebra $P(X)$. The complete subalgebra $\langle G \rangle^{\mathrm{cm}}$ generated by G coincides with $P(X)$ iff G separates points, i.e. iff for $x \neq y$ in X, there is $g \in G$ containing exactly one of the points x, y. Hence, $\tau(P(X))$ is the least cardinal λ with $|X| \leq 2^{\lambda}$.

2. Let, for an infinite cardinal κ, B be the completion of the free Boolean algebra over a set U of κ independent generators. Then for every subset V of U, $\langle V \rangle^{\mathrm{cm}} = \{\Sigma^B X : X \subseteq \langle V \rangle\}$. It follows that $\tau B = \kappa$.

3. Let *Fm* be the class of all formulas built up from an infinite set V of propositional variables, negation, and conjunction (respectively disjunction) of arbitrary sets of formulas. For $\alpha, \beta \in Fm$, define $\alpha \approx \beta$ iff, for every assignment $h: V \to A$ of "truth-values" in a complete Boolean algebra A, α and β are assigned the same truth value. Show that $Fm/\approx$ is a proper class, i.e. it does not have a *set* of representatives.

Hint. Apply the proof of Theorem 13.1.

4. For a cardinal $\kappa > 2^{\omega}$, let A be the σ-complete algebra consisting of all countable or cocountable subsets of κ. Then $|A| = \kappa^{\omega}$, but ind $A = 2^{\omega}$.

5. Let X be an arbitrary set and $\phi: X \to X$ any permutation. Give an elementary proof that there is a partition $\{a_0, a_1, a_2, a_3\}$ of X such that ϕ is the identity on a_0 and $\phi[a_i]$ is disjoint from a_i, for $1 \leq i \leq 3$.

6. The following examples show that several assumptions in Frolik's theorem cannot be dispensed with.

(a) Let (as in Exercise 4) A be the countable–cocountable algebra over an uncountable cardinal κ (thus A is not complete); let f be an automorphism of A

moving every atom of A. The homeomorphism of Ult A dual to f has a nowhere dense but non-empty set of fixed points.

(b) Let A be complete and atomless and $f: A \to 2 \subseteq A$ an endomorphism (thus f is not onto). If f is the identity on $A \upharpoonright a_0$ for some $a_0 \in A$, then $a_0 = 0$. Also, if $a \cdot f(a) = 0$ for some $a \in A$, then a is in the kernel of f.

(c) For A a non-trivial Boolean algebra and f the identity on A, $a \cdot f(a) = 0$ implies that $a = 0$. So there is no partition $\{a_1, a_2, a_3\}$ of A such that $f(a_i) \cdot a_i = 0$, for $1 \leq i \leq 3$.

(d) Consider the permutation $\phi: \omega \to \omega$ with $\phi(n) = n + 1$ if $n \equiv 0$ or $n \equiv 1$ modulo 3 and $\phi(n) = n - 1$ if $n \equiv 2$ modulo 3; let f be the automorphism of $P(\omega)$ defined by $f(a) = \phi[a]$. Then there is no partition $\{a_0, a_1, a_2\}$ of X such that f is the identity on $A \upharpoonright a_0$ and $f(a_i) \cdot a_i = 0$ for $1 \leq i \leq 2$.

14. Distributive laws

This section is devoted to the study of a kind of infinitary equations holding in some, but not all, Boolean algebras. The most general distributive law proved, in Section 1, to hold in every Boolean algebra was 1.33(c): the product of finitely many sums $s_i = \Sigma_{j \in J} a_{ij}$, $i \in I$, can be computed by picking, with a choice function $f: I \to J$, one term $a_{if(i)}$ from each sum, and summing up the products $\Pi_{i \in I} a_{if(i)}$ over all choice functions $f \in {}^I J$. The distributive laws considered below generalize this evaluation of $\Pi_{i \in I} \Sigma_{j \in J} a_{ij}$ to infinitely many sums $\Sigma_{j \in J} a_{ij}$. Simple examples, however, show that already for I of size ω and J of size 2, the resulting distributive law fails to hold in some complete algebras. There is another, less computational, way of interpreting distributive laws: Application of 1.33(c) immediately shows that finitely many partitions of unity in an arbitrary Boolean algebra have a common refinement. In sufficiently complete algebras, the infinite distributive laws are equivalent to the assertion that also infinite families of partitions have a common refinement.

Infinite distributive laws can be considered as a measure of how close a κ-complete Boolean algebra comes to being a κ-algebra of sets. For example, every κ^+-algebra of sets is (κ, κ)-distributive and a complete Boolean algebra is completely distributive iff it is isomorphic to a power set algebra, as shown in 14.4 and 14.5. Proposition 14.12 exhibits an intimate connection between distributivity and representability by κ-algebras of sets.

The conjunction of sufficiently strong distributive laws and chain conditions defines Souslin algebras; existence of these is shown, in Theorem 14.20, to be equivalent to existence of Souslin trees, and is thus independent from the axioms of ZFC set theory.

We finally deal with two variations of the distributive laws considered so far: weak distributivity and three-parameter distributivity. Our main results here are the observation that measure algebras are weakly (ω, ω)-distributive (Theorem 14.30) and a characterization of the collapsing algebras $\mathrm{RO}({}^{\omega}\kappa)$, considered in Section 13 in the proof of the Gaifman–Hales theorem, via three-parameter distributivity (Theorem 14.17). As a consequence of the characterization, many complete algebras turn out to be isomorphic to collapsing algebras – e.g. the regular open algebra of every regular ω_1-Aronszajn tree is isomorphic to $\mathrm{RO}({}^{\omega}\kappa)$, for $\kappa = \omega_1$.

For more detailed investigations of distributive laws, the reader might consult Sikorski's book SIKORSKI [1964]. Also see the chapter by JECH [Ch. 8 in this Handbook].

14.1. Definitions and examples

We introduce here the distributive laws most frequently considered. More general distributivity conditions, and also weaker ones, will be studied in later subsections.

14.1. DEFINITION. Let κ and λ be cardinals and A a Boolean algebra.

(a) A is (κ, λ)-*distributive* (or satisfies the (κ, λ)-*distributive law*) if, for any sets I and J such that $|I| \leq \kappa$ and $|J| \leq \lambda$ and for any family $(a_{ij})_{i\in I, j\in J}$ in A,

$$(1) \qquad \prod_{i\in I} \sum_{j\in J} a_{ij} = \sum \left\{ \prod_{i\in I} a_{if(i)} \colon f \in {}^{I}J \right\}$$

provided that

(2) each of the sums (respectively products) $\Sigma_{j\in J}\, a_{ij}$ for $i \in I$, $\Pi_{i\in I}\, \Sigma_{j\in J}\, a_{ij}$ and $\Pi_{i\in I}\, a_{if(i)}$ for $f \in {}^{I}J$ (i.e. $f\colon I \to J$) exists in A.

(b) A is (κ, ∞)-*distributive* if it is (κ, λ)-distributive for every cardinal λ.

(c) A is *completely distributive* if it is (κ, λ)-distributive for all cardinals κ and λ.

Of course, if A is complete or just $\max(\kappa, \lambda)^{+}$-complete, then (2) in the definition of (κ, λ)-distributivity is superfluous.

A (κ, λ)-distributive algebra is, a fortiori, (κ', λ')-distributive for $\kappa' \leq \kappa$ and $\lambda' \leq \lambda$. Each Boolean algebra trivially satisfies the (κ, λ)-distributive law for finite κ, by 1.33(c); also, even more trivially, the $(\kappa, 1)$-distributive law holds in every Boolean algebra. We shall see in Example 14.3 that the weakest non-trivial law, the $(\omega, 2)$-distributive law, fails to hold in some algebras.

The following remark reduces the distributive law (1) to its essential content.

14.2. REMARK. For every Boolean algebra A and arbitrary elements a_{ij} of A,

$$\sum \left\{ \prod_{i\in I} a_{if(i)} \colon f \in {}^{I}J \right\} \leq \prod_{i\in I} \sum_{j\in J} a_{ij}\,,$$

provided all sums and products displayed exist. This holds since, for $i \in I$ and $f \in {}^{I}J$,

$$\prod_{i\in I} a_{if(i)} \leq a_{if(i)} \leq \sum_{j\in J} a_{ij}\,.$$

14.3. EXAMPLE. In 4.10, we defined the partial order $\mathrm{Fn}(I, J, \lambda)$ for any sets I and J and any cardinal λ. The elements of $\mathrm{Fn}(I, J, \lambda)$ are the partial functions p from I into J satisfying $|\mathrm{dom}\ p| < \lambda$, and they are ordered by reverse inclusion. Assume that $|I| \geq \lambda$, $|J| \geq 2$, λ is infinite and regular, and let $B = \mathrm{RO}(\mathrm{Fn}(I, J, \lambda))$ be the completion of $\mathrm{Fn}(I, J, \lambda)$ in the sense of Definition 4.12, where $\mathrm{Fn}(I, J, \lambda)$ is given the partial order topology. It follows from Proposition 14.7 below that B is (μ, ∞)-distributive for every $\mu < \lambda$; we now show that it is not $(\lambda, 2)$-distributive.

Fix a subset I' of I of size λ and some $j^* \in J$. In B, define

$$b_{i0} = \{p \in \mathrm{Fn}(I, J, \lambda): i \in \mathrm{dom}\ p, \text{ and } p(i) = j^*\},$$
$$b_{i1} = -b_{i0} = \{p \in \mathrm{Fn}(I, J, \lambda): i \in \mathrm{dom}\ p, \text{ and } p(i) \neq j^*\},$$

for $i \in I'$. Then $b_{i0} + b_{i1} = 1$ and $\Pi_{i\in I'} \Sigma_{n\in 2}\, b_{in} = 1$. For each $f: I' \to 2$, however, $\Pi_{i\in I'}\, b_{if(i)} = 0$, for otherwise there is $p \in \mathrm{Fn}(I, J, \lambda)$ such that $p \in b_{if(i)}$, for all $i \in I'$. It follows that $f \subseteq p$ and thus that $|\mathrm{dom}\ p| \geq \lambda$, contrary to the definition of $\mathrm{Fn}(I, J, \lambda)$.

If $\lambda = \omega$, then $\mathrm{RO}(\mathrm{Fn}(I, J, \lambda))$ is isomorphic to $\mathrm{RO}({}^IJ)$, where J is discrete and ${}^IJ$ has the product topology, since these algebras have isomorphic dense subsets. An argument similar to that given above shows that the regular open algebra of a product space $\Pi_{i\in I} X_i$ is not $(\omega, 2)$-distributive if each X_i has two disjoint non-empty open subsets, thus generalizing the example of $\mathrm{RO}(\mathrm{Fn}(I, J, \omega))$.

We now give two standard examples of algebras sastisfying infinite distributive laws: μ-algebras of sets and algebras having a dense κ-closed subset. A Boolean algebra A is defined, in 1.29, to be a μ-algebra of sets if μ is an infinite cardinal, A is an algebra of sets and, for each $M \subseteq A$ of size less than μ, $\bigcup M$ and $\bigcap M$ are in A.

14.4. PROPOSITION. *Suppose κ, $\lambda < \mu$ and A is a μ-algebra of sets. Then A is (κ, λ)-distributive.*

PROOF. Let $a_{ij} \in A$ for $i \in I$, $j \in J$, where $|I| \leq \kappa$ and $|J| \leq \lambda$. Then

$$\begin{aligned}\prod_{i\in I} \sum_{j\in J} a_{ij} &= \bigcap_{i\in I} \bigcup_{j\in J} a_{ij} \\ &= \bigcup \left\{\bigcap_{i\in I} a_{if(i)}: f \in {}^IJ\right\} \\ &= \bigcup \left\{\prod_{i\in I} a_{if(i)}: f \in {}^IJ\right\} \\ &= \sum \left\{\prod_{i\in I} a_{if(i)}: f \in {}^IJ\right\};\end{aligned}$$

here the second equality uses the axiom of choice and the last one holds since $\bigcup \{\Pi_{i\in I}\, a_{if(i)}: f \in {}^IJ\}$ is an element of A, hence is the least upper bound of the set $\{\Pi_{i\in I}\, a_{if(i)}: f \in {}^IJ\}$. □

The same argument proves that if A is a regular subalgebra of B (i.e. $\Sigma^A M = \Sigma^B M$ for each $M \subseteq A$ such that $\Sigma^A M$ exists) and B is (κ, λ)-distributive, then so is A. Example 14.22 shows that validity of a distributive law in A does not necessarily carry over to the completion of A.

14.5. THEOREM. *A Boolean algebra A is completely distributive iff it is atomic. In particular, a complete Boolean algebra is (by 2.7) completely distributive iff it is isomorphic to a power set algebra.*

PROOF. If A is completely distributive, consider the product

$$1 = \prod_{x \in A} (x + -x) = \prod_{x \in A} \sum_{j \in 2} a_{xj} ,$$

where $a_{x0} = x$ and $a_{x1} = -x$. For any $f \in {}^A2$, the element

$$p_f = \prod_{x \in A} a_{xf(x)}$$

exists and is either zero or an atom of A – for if $\{a_{xf(x)}: x \in A\}$ has a non-zero lower bound, say b, then b is an atom of A by Lemma 2.4(b). $(|A|, 2)$-distributivity of A gives

$$1 = \sum \{p_f: f \in {}^A2\} ,$$

and hence A is atomic.

Conversely, for A atomic, the canonical monomorphism $f: A \to P(\mathrm{At}\, A)$ ($\mathrm{At}\, A$ the set of atoms of A) considered in the proof of Proposition 2.6 and given by

$$f(a) = \{x \in \mathrm{At}\, A: x \leq a\}$$

preserves all sums and products which happen to exist in A, i.e. A is isomorphic to a regular subalgebra of $P(\mathrm{At}\, A)$. Thus, by Proposition 14.4 and the remark following it, A is (κ, λ)-distributive for arbitrary κ and λ. □

14.6. DEFINITION. Let κ be an infinite cardinal. A partial order $(P, \leq)$ is κ*-closed* if for every ordinal $\rho < \kappa$, each decreasing sequence $(p_\alpha)_{\alpha<\rho}$ in P has a lower bound, i.e. there exists $q \in P$ satisfying $q \leq p_\alpha$ for $\alpha < \rho$.

For example, the partial order $\mathrm{Fn}(I, J, \lambda)$ defined in 4.10 is λ-closed, for each regular infinite cardinal λ. Also, the non-zero elements of the quotient algebra $P(\omega)/fin$ constitute, by 5.28(e), an ω_1-closed dense subset of $P(\omega)/fin$.

14.7. PROPOSITION. *If A is a Boolean algebra with a* κ*-closed dense subset, then A is* (μ, ∞)*-distributive for every cardinal* $\mu < \kappa$.

PROOF. Suppose $|I| = \mu < \kappa$ and $(a_{ij})_{i \in I, j \in J}$ satisfies (2) in Definition 14.1; for the sake of simplicity, assume $I = \mu$. Let $P \subseteq A$ be dense and κ-closed; note that by the definition of denseness in 4.8, $0 \notin P$.

If (1) in 14.1 fails, then by 14.2,

$$a = \prod_{\alpha \in \mu} \sum_{j \in J} a_{\alpha j}$$

is an upper bound of $\{\prod_{\alpha \in \mu} a_{\alpha f(\alpha)}: f \in {}^{\mu}J\}$, but not the least one; so let b be an upper bound with $b < a$. By denseness of P, pick $p \in P$ such that $p \leq a \cdot -b$.

We construct a function $f: \mu \to J$ and a decreasing sequence $(p_\alpha)_{\alpha < \mu}$ in P such that $p_\alpha \leq p$: if $f(\beta)$ and p_β have been constructed for $\beta < \alpha$, pick by κ-closedness an element r of P with $r \leq p_\beta$ for $\beta < \alpha$; for $\alpha = 0$, let $r = p$. Now

$$0 < r \leq p \leq a \leq \sum_{j \in J} a_{\alpha j} ,$$

so choose $f(\alpha) \in J$ satisfying $r \cdot a_{\alpha f(\alpha)} > 0$. Then let $p_\alpha \in P$ such that $p_\alpha \leq r \cdot a_{\alpha f(\alpha)}$.

Again by κ-closedness, let $q \in P$ be a lower bound for all p_α. Thus, by our choice of f and b,

$$q \leq \prod_{\alpha \in \mu} a_{\alpha f(\alpha)} \leq b ,$$

contradicting

$$q \leq p_\alpha \leq p \leq -b . \quad \square$$

14.2. Equivalences to distributivity

We shall reformulate distributivity in terms of refinement properties of partitions and then prove an equivalence between different distributive laws. The following proposition can be viewed as being a preliminary version of the first subject.

14.8. PROPOSITION. *Let κ be infinite and $\mu = (2^\kappa)^+$. Then a μ-complete Boolean algebra A is $(\kappa, 2)$-distributive iff for each subset X of A of size at most κ, the μ-complete subalgebra of A generated by X is atomic.*

PROOF. Let A be μ-complete; thus all of the sums and products below exist in A.

First assume that A is $(\kappa, 2)$-distributive and that $X \subseteq A$ has cardinality at most κ. For $x \in X$, let $a_{x0} = x$ and $a_{x1} = -x$. Then

$$1 = \prod_{x \in X} \sum_{j \in 2} a_{xj} = \sum \{p_f: f \in {}^{X}2\} ,$$

where $p_f = \prod_{x \in X} a_{xf(x)}$. Let B be the μ-complete subalgebra of A generated by X. Each of the products p_f is in B and, for $x \in X$,

$$x = \sum \{p_f: f \in {}^{X}2, f(x) = 0\} .$$

Thus, B is also μ-generated by the pairwise disjoint elements p_f; in particular B is atomic, having $\{p_f: f \in {}^{X}2\} \setminus \{0\}$ as its set of atoms.

Conversely, suppose that for each $X \subseteq A$ of size at most κ, the μ-complete subalgebra of A generated by X is atomic. Then for every family $(a_{ij})_{i\in I, j\in 2}$ in A with $|I| \leq \kappa$, the set $X = \{a_{ij}: i \in I,\ j \in 2\}$ is included in an atomic μ-complete subalgebra B of A. By Theorem 14.5, the elements a_{ij} satisfy the $(\kappa, 2)$-distributive law (1) in B, hence in A. □

Part (c) of the next proposition characterizes distributivity via partitions of unity, as promised above. Parts (a) and (b) are technically useful, since they reduce checking of distributive laws to quite special families. To abbreviate their formulation, let us tacitly assume that the sums and products required in (2) of Definition 14.1 exist.

14.9. PROPOSITION. (a) *Suppose λ is infinite and A is* $\max(\kappa, \lambda)^+$*-complete. Then A is (κ, λ)-distributive iff the equation*

$$(1) \qquad \prod_{i\in I} \sum_{j\in J} a_{ij} = \sum \left\{ \prod_{i\in I} a_{if(i)}: f \in {}^I J \right\}$$

holds for families $(a_{ij})_{i\in I, j\in J}$ satisfying $|I| \leq \kappa$, $|J| \leq \lambda$, plus, for each $i \in I$,

$$\sum_{j\in J} a_{ij} = 1, \qquad a_{ij} \cdot a_{ij'} = 0 \quad \text{for } j \neq j'.$$

(b) *A is (κ, λ)-distributive iff equation* (1) *holds for families $(a_{ij})_{i\in I, j\in J}$ satisfying $|I| \leq \kappa$, $|J| \leq \lambda$, plus, for every $f \in {}^I J$,*

$$\prod_{i\in I} a_{if(i)} = 0.$$

(c) *For infinite λ, A is (κ, λ)-distributive iff every set of at most κ partitions of A, each of size at most λ, has a common refinement.*

PROOF. Necessity of the conditions stated in (a) (respectively (b)) is clear. For sufficiency, assume that $|I| \leq \kappa$, $|J| \leq \lambda$ and that some family $(a_{ij})_{i\in I, j\in J}$ in A violates (1). Let then

$$a = \prod_{i\in I} \sum_{j\in J} a_{ij}.$$

By Remark 14.2, the set $\{\prod_{i\in I} a_{if(i)}: f \in {}^I J\}$ has an upper bound b strictly smaller than a. We shall construct a family $(b_{ij})_{i\in I, j\in J}$ in A satisfying the conditions in (a) (respectively (b)) and still violating (1).

(a) We may assume that $J = \lambda \setminus \{0\}$. For $i \in I$ and $j \in \lambda$, define $b_{ij} \in A$ by

$$b_{i0} = -a, \qquad b_{ij} = a_{ij} \cdot a \cdot - \sum_{0<k<j} a_{ik} \quad \text{for } 0 < j < \lambda.$$

These elements b_{ij} are as required: clearly $b_{ij} \cdot b_{ij'} = 0$ for $j \neq j'$. By

$$a = a \cdot \sum_{0<j<\lambda} a_{ij} = \sum_{0<j<\lambda} b_{ij},$$

we have $\Sigma_{j<\lambda} b_{ij} = 1$. In particular, $\Pi_{i\in I} \Sigma_{j\in J} b_{ij} = 1$. We prove that, for every $f \in {}^I\lambda$,

$$(3) \qquad a \cdot -b \cdot \prod_{i\in I} b_{if(i)} = 0$$

and thus, using $a \cdot -b > 0$, that $\Sigma \{\Pi_{i\in I} b_{if(i)}: f \in {}^I\lambda\} < 1$: if $f(i) = 0$ for some i, then (3) holds since $b_{i0} = -a$. Otherwise $f \in {}^IJ$ and (3) holds by $\Pi_{i\in I} b_{if(i)} \leq \Pi_{i\in I} a_{if(i)} \leq b$.

(b) For $i \in I$ and $j \in J$, define

$$b_{ij} = a_{ij} \cdot a \cdot -b\,.$$

Then (2) of Definition 14.1 holds with the a_{ij} replaced by b_{ij},

$$0 < a \cdot -b = a \cdot -b \cdot \prod_{i\in I} \sum_{j\in J} a_{ij} = \prod_{i\in I} \sum_{j\in J} b_{ij}\,,$$

and for each $f \in {}^IJ$,

$$\prod_{i\in I} b_{if(i)} = a \cdot -b \cdot \prod_{i\in I} a_{if(i)} = 0\,.$$

(c) follows easily from (a): note that a family $\{P_i: i \in I\}$ of partitions of A, say $P_i = \{a_{ij}: j \in J_i\}$, has a common refinement iff

$$P = \left\{\prod_{i\in I} a_{if(i)}: f: I \to \bigcup_{i\in I} J_i,\ f(i) \in J_i \text{ for } i \in I\right\} \setminus \{0\}$$

is a partition of unity – in fact, P is then the coarsest partition refining each P_i. Now P is certainly a pairwise disjoint family, so it is a partition iff $\Sigma P = 1$. □

We now set up a surprising connection between different distributive laws. The proof relies on the idea of coding, for an infinite cardinal κ, elements of $\kappa \times \kappa$ by elements of κ, hence functions in ${}^\kappa({}^\kappa 2)$ by functions in ${}^\kappa 2$.

14.10. THEOREM. *Let κ be an infinite cardinal. Then every $(\kappa, 2)$-distributive Boolean algebra is (κ, κ)-distributive. If $\kappa \leq \lambda \leq 2^\kappa$ and A is $(\kappa, 2)$-distributive and λ^+-complete, then it is (κ, λ)-distributive. In particular, a $(2^\kappa)^+$-complete Boolean algebra is $(\kappa, 2)$-distributive iff it is $(\kappa, 2^\kappa)$-distributive.*

PROOF. In the whole proof, fix a set I of size κ and a bijection

$$\pi: I \times I \to I\,.$$

For the first assertion, suppose that $(a_{ij})_{i,j\in I}$ violates the (κ, κ)-distributive law. We may assume by 14.9(b) that

$$\prod_{i\in I} a_{if(i)} = 0$$

for each $f \in {}^I I$, and, by Remark 14.2, that

$$a = \prod_{i \in I} \sum_{j \in I} a_{ij} > 0 ,$$

and we will show that the family $(b_{kn})_{k \in I, n \in 2}$, where

$$b_{\pi(i,j),0} = a \cdot a_{ij} , \qquad b_{\pi(i,j),1} = a \cdot -a_{ij} ,$$

violates the $(\kappa, 2)$-distributive law. Obviously,

$$0 < a = \prod_{k \in I} \sum_{n \in 2} b_{kn} .$$

And for each $g \in {}^I 2$, $\Pi_{k \in I} b_{kg(k)} = 0$: otherwise, pick $y \in A$ such that

(4) $$0 < y \leq b_{kg(k)}$$

for all $k \in I$; we shall find a function $f \in {}^I I$ such that $0 < y \leq \Pi_{i \in I} a_{if(i)}$, thus obtaining a contradiction. Let $i \in I$; to define $f(i)$, note that $a \leq \Sigma_{j \in I} a_{ij}$ and hence

(5) $$0 = \prod_{j \in I} a \cdot -a_{ij} = \prod_{j \in I} b_{\pi(i,j),1} .$$

By (4) and (5), there is some $j \in I$ such that $g(\pi(i, j)) = 0$; let $f(i)$ be such an element j. This choice of $f(i)$ gives

$$\begin{aligned} y \leq b_{\pi(i,f(i)),g(\pi(i,f(i)))} &= b_{\pi(i,f(i)),0} \\ &= a \cdot a_{if(i)} \\ &\leq a_{if(i)} . \end{aligned}$$

So f is as required.

To prove the second assertion, assume that A is $(\kappa, 2)$-distributive and λ^+-complete and that $(a_{if})_{i \in I, f \in \Phi}$ is a family in A, where $\kappa \leq |\Phi| = \lambda \leq 2^\kappa$. To check (κ, λ)-distributivity for such a family, we may assume by 14.9(a) that for fixed $i \in I$, the elements a_{if} $(f \in \Phi)$ are pairwise disjoint, $\Sigma_{f \in \Phi} a_{if} = 1$, and, by $|\Phi| \leq 2^\kappa$, that $\Phi \subseteq F$, where F is the set ${}^I 2$. To facilitate notation, define $a_{if} = 0$ for $i \in I$ and $f \in F \setminus \Phi$ and consider the family $(a_{if})_{i \in I, f \in F}$.

For i, $j \in I$ and $\varepsilon \in 2$, let

$$s_{ij\varepsilon} = \Sigma \, \{a_{if} \colon f \in F, \, f(j) = \varepsilon\} .$$

Then clearly for every j,

(6) $$s_{ij0} + s_{ij1} = \sum_{f \in F} a_{if}$$

and, for $h \in F$,

(7) $\qquad h(j) \neq \varepsilon$ implies $a_{ih} \cdot s_{ij\varepsilon} = 0$,

by disjointness of $\{a_{if}: f \in F\}$. Also

(8) $$a_{if} = \prod_{j \in I} s_{ijf(j)}:$$

trivially, $a_{if} \leq \prod_{j \in I} s_{ijf(j)}$. For arbitrary j and ε, $s_{ij\varepsilon} \leq \sum_{h \in F} a_{ih}$; so

$$\prod_{j \in I} s_{ijf(j)} \leq \sum_{h \in F} a_{ih}.$$

By (7), $f \neq h$ implies that $a_{ih} \cdot \prod_{j \in I} s_{ijf(j)} = 0$, hence

$$\prod_{j \in I} s_{ijf(j)} \leq -\sum \{a_{ih}: h \in F \setminus \{f\}\}$$

and (8) follows.

Denote by l and r the unique functions from I into I such that, for our bijection $\pi: I \times I \to I$ above and every $k \in I$,

$$\pi^{-1}(k) = (l(k), r(k)).$$

This induces a bijection $*$ from ${}^I F = {}^I({}^I 2)$ onto $F = {}^I 2$, e.g. let

$$g^*(k) = g(l(k))(r(k))$$

for $g \in {}^I F$. Then for $g: I \to F$,

(9) $$\prod_{i \in I} a_{ig(i)} = \prod_{k \in I} s_{l(k)r(k)g^*(k)},$$

since by (8),

$$\begin{aligned}\prod_{i \in I} a_{ig(i)} &= \prod_{i \in I} \prod_{j \in I} s_{ijg(i)(j)} \\ &= \prod_{k \in I} s_{l(k)r(k)g(l(k))(r(k))} \\ &= \prod_{k \in I} s_{l(k)r(k)g^*(k)}.\end{aligned}$$

The following computation now establishes (κ, λ)-distributivity of A; here the second equality holds since l is onto and the last one since, by our choice of $a_{if} = 0$ for $f \notin \Phi$, $\prod_{i \in I} a_{ig(i)} = 0$ if $g \in {}^I F \setminus {}^I \Phi$.

$$\begin{aligned}\prod_{i \in I} \sum_{f \in \Phi} a_{if} &= \prod_{i \in I} \sum_{f \in F} a_{if} \\ &= \prod_{i \in I} \sum_{f \in F} a_{l(i)f} \\ &= \prod_{i \in I} \sum_{\varepsilon \in 2} s_{l(i)r(i)\varepsilon} \qquad \text{by (6)}\end{aligned}$$

$$= \sum_{f\in F} \prod_{i\in I} s_{l(i)r(i)f(i)} \quad \text{by } (\kappa, 2)\text{-distributivity}$$

$$= \sum_{g:\, I\to F} \prod_{i\in I} a_{ig(i)} \quad \text{by (9)}$$

$$= \sum_{g:\, I\to \Phi} \prod_{i\in I} a_{ig(i)} . \quad \square$$

Example 14.22 below shows that the implication from $(\kappa, 2)$-distributivity to $(\kappa, 2^\kappa)$-distributivity cannot be proved without a completeness assumption on the algebra under consideration.

14.3. Distributivity and representability

In Section 12, a κ-complete Boolean algebra A was called κ-representable if there exists a κ-algebra of sets B and a κ-complete epimorphism from B onto A; ω_1-representability was called σ-representability. We shall establish, in Proposition 14.12, a close connection between representability and distributivity; half of this connection comes from the fact, proved in Proposition 14.4, that κ-algebras of sets are (λ, λ)-distributive for $\lambda < \kappa$.

Another application of 14.4 shows, in the following example, that the Loomis–Sikorski theorem 12.7 (every σ-complete Boolean algebra is σ-representable) does not generalize to κ-complete algebras, if $\kappa \geq \omega_2$.

14.11. EXAMPLE (Karp). For $\kappa = \omega_1$, the collapsing algebra $\mathrm{RO}({}^\omega\kappa)$ is not ω_2-representable. For consider the inequality

$$(10) \qquad \prod_{\alpha\in\kappa} \sum_{n\in\omega} x_{\alpha n} \leq \sum \{x_{\alpha n} \cdot x_{\beta n} \colon n \in \omega;\ \alpha, \beta \in \kappa;\ \alpha \neq \beta\} ;$$

we will show that it holds in every ω_2-algebra of sets, is preserved by every ω_2-complete homomorphism, but does not hold in $\mathrm{RO}({}^\omega\kappa)$.

Preservation under ω_2-complete homomorphisms is obvious, since the sums and products displayed in (10) have length at most ω_1. In $\mathrm{RO}({}^\omega\kappa)$, the elements

$$a_{\alpha n} = \{x \in {}^\omega\kappa \colon x(n) = \alpha\}$$

($n \in \omega$, $\alpha \in \kappa = \omega_1$) satisfy the equation $\prod_{\alpha\in\kappa} \sum_{n\in\omega} a_{\alpha n} = 1$ since, for each $\alpha \in \kappa$, the set $\bigcup_{n\in\omega} a_{\alpha n}$ is dense in ${}^\omega\kappa$; also $\sum \{a_{\alpha n} \cdot a_{\beta n} \colon n \in \omega,\ \alpha \neq \beta\} = 0$ since $a_{\alpha n}$ and $a_{\beta n}$ are disjoint, for $\alpha \neq \beta$.

Now let B be any ω_2-algebra of sets and $b_{\alpha n} \in B$ for $\alpha \in \kappa = \omega_1$, $n \in \omega$. B is (ω_1, ω_1)-distributive by Proposition 14.4, thus

$$\prod_{\alpha\in\kappa} \sum_{n\in\omega} b_{\alpha n} = \sum \left\{ \prod_{\alpha\in\kappa} b_{\alpha f(\alpha)} \colon f \in {}^\kappa\omega \right\} .$$

Put $r = \sum \{b_{\alpha n} \cdot b_{\beta n} \colon n \in \omega,\ \alpha, \beta \in \kappa,\ \alpha \neq \beta\}$; we show that, for each function $f\colon \kappa \to \omega$, $\prod_{\alpha\in\kappa} b_{\alpha f(\alpha)} \leq r$. This holds since f cannot be one-to-one, so there are

$\alpha \neq \beta$ in κ and $n \in \omega$ such that $f(\alpha) = f(\beta) = n$. Hence, $\Pi_{\alpha\in\kappa} b_{\alpha f(\alpha)} \leq b_{\alpha n} \cdot b_{\beta n} \leq r$.

14.12. PROPOSITION. *Every* $(2^\kappa)^+$*-representable Boolean algebra is* $(\kappa, 2)$*-distributive. Every* κ^+*-complete and* $(\kappa, 2)$*-distributive algebra is* κ^+*-representable.*

PROOF. The first assertion is a straightforward consequence of Proposition 14.4: if $g: B \to A$ is a $(2^\kappa)^+$-complete epimorphism and B is a κ^+-algebra of sets, then B is $(\kappa, 2)$-distributive by 14.4 and so is A, since for $|I| \leq \kappa$ and $|J| \leq 2$, all sums and products displayed in the distributive law (1) are preserved by g.

Now assume A is κ^+-complete and $(\kappa, 2)$-distributive. Our proof consists in generalizing the notions and arguments relevant to the proof of the Loomis–Sikorski theorem 12.7. Denote by X the Stone space of A and, for the sake of simplicity, identify A with Clop X. If $M \subseteq A$ is such that $\Pi^A M$ exists, then

$$\Pi^A M = \operatorname{int}\left(\bigcap M\right)$$

(by 7.21) will generally not coincide with $\bigcap M$. Call a subset u of X *κ-nowhere dense* if there is some $M \subseteq A$ such that $|M| \leq \kappa$, $\Pi^A M = 0$ and $u \subseteq \bigcap M$; and call u *κ-meager* if $u \subseteq \bigcup_{i\in I} u_i$ for some family $\{u_i: i \in I\}$ of κ-nowhere dense subsets of X such that $|I| \leq \kappa$. Let B be the κ^+-algebra of sets generated by $A = \text{Clop}\, X$. Then

$$M = \{b \in B: b \text{ is } \kappa\text{-meager}\}$$

is a κ^+-complete ideal of B. We claim that the restriction of the canonical epimorphism

$$\pi: B \to B/M$$

is an isomorphism from A onto B/M, thus proving κ^+-representability of A.

To prove that $\pi[A] = B/M$, it suffices to show that

$$B' = \{b \in B: \pi(b) = \pi(a) \text{ for some } a \in A\}$$

coincides with B. Clearly, B' is a subalgebra of B including A. It is also closed under unions of length at most κ, for if $b_\alpha \in B'$ for $\alpha < \kappa$, say $\pi(b_\alpha) = \pi(a_\alpha)$, where $a_\alpha \in A$, then also $\bigcup_{\alpha<\kappa} b_\alpha \in B$ since

$$\pi\Big(\bigcup_{\alpha<\kappa} b_\alpha\Big) = \pi\Big(\bigcup_{\alpha<\kappa} a_\alpha\Big) = \pi\Big(\sum_{\alpha<\kappa}^A a_\alpha\Big);$$

here the second equality follows from the fact that

$$\bigcup_{\alpha<\kappa} a_\alpha \,\triangle\, \sum_{\alpha<\kappa}^A a_\alpha = \sum_{\alpha<\kappa}^A a_\alpha \setminus \bigcup_{\alpha<\kappa} a_\alpha$$

is a κ-nowhere dense set. Thus, $B' = B$ holds.

We finally show that π is one-to-one on A; here the $(\kappa, 2)$-distributivity of A will replace Baire's theorem for X, used in the proof of 12.9 and the Loomis–Sikorski theorem. Assume $a \in A \cap M$ with the aim of proving that $a = 0$. Since a is κ-meager,

$$a \subseteq \bigcup_{i \in I} \bigcap_{j \in J} a_{ij} ,$$

where $|I| \leq \kappa$, $|J| \leq \kappa$ and $\Pi^{A}_{j \in J} a_{ij} = 0$ for each $i \in I$. A is (κ, κ)-distributive by Theorem 14.10, thus

$$1 = \prod_{i \in I} \sum_{j \in J} -a_{ij} = \Sigma \left\{ \prod_{i \in I} -a_{if(i)} \colon f \in {}^{I}J \right\} .$$

If $a > 0$, then there are $f\colon I \rightarrow J$ and $x \in X$ such that $x \in a \cdot \Pi_{i \in I} - a_{if(i)}$. By $x \in a$, there is $i \in I$ such that $x \in \bigcap_{j \in J} a_{ij}$. So $x \in a_{if(i)}$, a contradiction. □

14.4. Three-parameter distributivity

This subsection is devoted to a generalization of the two-parameter distributive laws considered so far. Our principal application of the generalized notion will be a characterization of the collapsing algebras $\mathrm{RO}({}^{\omega}\kappa)$ via a strong form of (ω, κ, κ)-non-distributivity. This enables us to prove Kripke's theorem that every complete Boolean algebra is completely embeddable into a collapsing algebra.

For convenience, let us call in this subsection a family $(a_j)_{j \in J}$ in a Boolean algebra A a *quasipartition* of A if the elements a_j are pairwise disjoint and $\Sigma_{j \in J} a_j = 1$; unlike the definition of a partition given in 3.7, we do not require here the a_j to be non-zero. $(a_j)_{j \in J}$ is a λ-*quasipartition* if it is a quasipartition with $|J| = \lambda$. To save notation, we will not always distinguish between a family $p = (a_j)_{j \in J}$ and the set $\{a_j\colon j \in J\}$. For example, $y \in p$ means that $y = a_j$ for some $j \in J$, etc.

14.13. DEFINITION. Let A be a Boolean algebra and κ, λ, μ cardinals. A is (κ, λ, μ)-*distributive* if for each set P of at most κ λ-quasipartitions, there is a quasipartition q such that each element of q meets less than μ elements of each $p \in P$, i.e. such that, for $x \in q$ and $p \in P$,

$$|\{y \in p\colon x \cdot y > 0\}| < \mu .$$

A is (κ, λ, μ)-*nowhere distributive* if there is a set P of at most κ λ-quasipartitions in A such that for each $x \in A^{+}$ there is some $p \in P$ such that x meets at least μ elements of p.

It might be helpful to notice that the notion of three-parameter distributivity becomes stronger when increasing the first and second and decreasing the third argument: if A is (κ, λ, μ)-distributive and $\kappa' \leq \kappa$, $\lambda' \leq \lambda$, $\mu \leq \mu'$, then A is $(\kappa', \lambda', \mu')$-distributive. A (κ, λ, μ)-nowhere distributive algebra is, of course, not (κ, λ, μ)-distributive, the exact relationship between both conditions being clarified by the following proposition.

14.14. PROPOSITION. (a) *Assume* λ *is infinite and* A *is* $\max(\kappa, \lambda)^+$*-complete. Then* A *is* $(\kappa, \lambda, 2)$*-distributive iff it is* (κ, λ)*-distributive.*

(b) *Assume* A *is complete. Then* A *is* (κ, λ, μ)*-nowhere distributive iff none of the relative algebras* $A \upharpoonright a$ *(where* $0 < a$ *in* A*) is* (κ, λ, μ)*-distributive.*

PROOF. (a) For any quasipartitions p and q of A, let us say that q refines p if for each $x \in q$ there is some $y \in p$ such that $x \leq y$. This is equivalent to saying that each element of q meets at most one element of p. Thus, by Proposition 14.9(c), $(\kappa, \lambda, 2)$-distributivity is equivalent to (κ, λ)-distributivity.

(b) Suppose first that (κ, λ, μ)-nowhere distributivity of A is exemplified by a set P of at most κ λ-quasipartitions of A. Given $a \in A^+$, we show that $A \upharpoonright a$ is not (κ, λ, μ)-distributive by considering the set $P' = \{p' \colon p \in P\}$ of λ-quasipartitions of $A \upharpoonright a$, where for $p = (a_j)_{j \in J}$ in P,

$$p' = \{a_j \cdot a \colon j \in J\} .$$

For if q' is any quasipartition of $A \upharpoonright a$, then each non-zero element of q' meets at least μ elements of some $p \in P$, hence of p'.

Conversely, suppose no relative algebra $A \upharpoonright a$ of A is (κ, λ, μ)-distributive, for $a > 0$. Define

$$D = \{d \in A^+ \colon A \upharpoonright d \text{ is } (\kappa, \lambda, \mu)\text{-nowhere distributive}\} .$$

Claim. D is a dense subset of A.

For otherwise, assume that $a > 0$ in A and no element of $A \upharpoonright a$ is in D. Then

(11) if P is a set of at most κ λ-quasipartition of $A \upharpoonright a$, then $D' = \{c \in (A \upharpoonright a)^+ \colon c$ meets less than μ elements of each $p \in P\}$ is dense in $A \upharpoonright a$.

For if failure of (11) is demonstrated by P, let $b \in A \upharpoonright a$ such that $b > 0$ and no element of $A \upharpoonright b$ is in D'. Then the relativization of P to $A \upharpoonright b$ shows (κ, λ, μ)-nowhere distributivity of $A \upharpoonright b$ and $b \in D$, a contradiction proving (11). Now (11) implies that $A \upharpoonright a$ is (κ, λ, μ)-distributive, for given P as stated in (11), any partition of unity in $A \upharpoonright a$ consisting of elements of D' will work. This contradicts our assumption on A and proves the Claim.

By the Claim, let r be a partition of unity in A consisting of elements of D. For each $c \in r$, pick a family $(p_{c\alpha})_{\alpha < \kappa}$ of λ-quasipartitions of $A \upharpoonright c$, say $p_{c\alpha} = (a_{c\alpha j})_{j \in J}$, where $|J| = \lambda$, demonstrating (κ, λ, μ)-nowhere distributivity of $A \upharpoonright c$. Now (κ, λ, μ)-nowhere distributivity of A is shown by the quasipartitions p_α for $\alpha < \kappa$, where

$$p_\alpha = (a_{\alpha j})_{j \in J} ,$$

$$a_{\alpha j} = \sum_{c \in r} a_{c\alpha j} :$$

let $x > 0$ in A and fix $c \in r$ such that $x \cdot c > 0$. Then by our choice of the quasipartitions $p_{c\alpha}$, $x \cdot c$ meets at least μ elements of $p_{c\alpha}$, for some $\alpha < \kappa$. So x meets at least μ elements of p_α. $\square$

14.15. EXAMPLES. (a) Let κ be an infinite cardinal; we will see that the collapsing algebra $\mathrm{RO}({}^\omega\kappa)$ studied in Solovay's theorem 13.1 is (ω, κ, κ)-nowhere distributive. To this end, consider the elements

$$a_t = \{x \in {}^\omega\kappa : t \subseteq x\}$$

of $\mathrm{RO}({}^\omega\kappa)$, where $t: n \to \kappa$, for some $n \in \omega$. For each $n \in \omega$,

$$p_n = \{a_t : t \in {}^n\kappa\}$$

is a partition of size κ; moreover, the set

$$S = \bigcup_{n\in\omega} p_n$$

is dense in $\mathrm{RO}({}^\omega\kappa)$. The partitions p_n demonstrate (ω, κ, κ)-nowhere distributivity of $\mathrm{RO}({}^\omega\kappa)$, for let $a > 0$ in $\mathrm{RO}({}^\omega\kappa)$ and choose $t: n \to \kappa$ such that $a_t \leq a$. Clearly, a_t, and hence a, meets κ distinct elements of p_{n+1}.

(b) Since $\mathrm{RO}({}^\omega\kappa) \cong \mathrm{RO}(\mathrm{Fn}(\omega, \kappa, \omega))$, the following example gives an analogue of part (a). Let κ be uncountable and $B = \mathrm{RO}(\mathrm{Fn}(\omega_1, \kappa, \omega_1))$. For $\alpha < \omega_1$ and $t: \alpha \to \kappa$, let

$$a_t = \{x \in \mathrm{Fn}(\omega_1, \kappa, \omega_1) : t \subseteq \kappa\}\ ;$$

put

$$p_\alpha = \{a_t : t \in {}^\alpha\kappa\}\ , \qquad S = \bigcup \{p_\alpha : \alpha < \omega_1\}\ .$$

Similar to the situation in part (a), S is dense in B, and the family $\{p_\alpha : \alpha < \omega_1\}$ of partitions proves $(\omega_1, \kappa^\omega, \kappa^\omega)$-nowhere distributivity of B. By Proposition 14.7, however, B is (ω, ∞)-distributive. In particular it is $(\omega, \lambda, 2)$-distributive for every infinite cardinal λ, as shown by 14.14(a).

We proceed to give two characterizations of collapsing algebras using partitions of unity. The first of these is quite elementary; the second one works via nowhere distributivity. To elucidate the first one and also the preceding examples, let us collect some set-theoretical definitions concerning trees. Distributive laws in the regular open algebra of a tree will be more thoroughly considered in the next subsection.

A *tree* is a partial order $(T, \leq_T)$ such that for every $t \in T$, the set

$$\mathrm{pred}(t) = \{x \in T : x <_T t\}$$

of predecessors of t is well-ordered by $<_T$. For each ordinal α,

$$U_\alpha = \{t \in T\colon \text{pred}(t) \text{ has order type } \alpha \text{ under } <_T\}$$

is the αth level of T, and

$$\text{height}(T) = \sup\{\alpha + 1\colon U_\alpha \text{ non-empty}\}\ .$$

As a standard example of a tree, let J be any set, λ an ordinal and $T = \bigcup\{{}^\alpha J\colon \alpha < \lambda\}$, the set of all functions from ordinals $\alpha < \lambda$ into J; T is partially ordered by inclusion. Then $(T, \subseteq)$ is a tree of height λ, with ${}^\alpha J$ as its αth level.

A tree $(T, \leq_T)$ is *normal* if it satisfies the conditions

(N1) T has a single root, i.e. $|U_0| = 1$,

(N2) for $\alpha < \text{height}(T)$ and $x \in U_\alpha$, there are $y \neq y'$ in $U_{\alpha+1}$ such that $x <_T y$ and $x <_T y'$,

(N3) if $\beta \leq \alpha < \text{height}(T)$ and $x \in U_\beta$, then there is $y \in U_\alpha$ such that $x \leq_T y$,

(N4) if $\lambda < \text{height}(T)$ is a limit ordinal and $s \neq t$ in U_λ, then there are $\alpha < \lambda$ and $x \neq y$ in U_α such that $x <_T s$ and $y <_T t$.

For example, the tree $(\bigcup\{{}^\alpha J\colon \alpha < \lambda\}, \subseteq)$ considered above is normal, if λ is a limit ordinal.

For $(T, \leq_T)$ a tree, denote by $T^* = (T, \geq_T)$ the opposite partial ordering. In T^*, any two incomparable elements s and t are incompatible, i.e. there is no $r \in T$ such that $r \leq_{T^*} s, t$. It follows that if $(T, \leq_T)$ satisfies condition (N2), in particular if $(T, \leq_T)$ is a normal tree, then T^* is a separative partial order, as defined in Section 4, hence a dense subset of a unique complete Boolean algebra B, its completion, and that $B \cong \text{RO}(T^*)$. By abuse of notation, we also write $\text{RO}(T)$ for $\text{RO}(T^*)$.

For the formulation of 14.16, let us say that a partition q of a Boolean algebra A κ-refines a partition p if q refines p and for each $y \in p$, there are κ elements of q below y.

14.16. PROPOSITION. *For every complete Boolean algebra A, the following are equivalent*:

(a) $A \cong \text{RO}({}^\omega\kappa)$,

(b) *A has a dense subset isomorphic to T^*, for the normal tree $T = \bigcup\{{}^n\kappa\colon n \in \omega\}$*,

(c) *there is a family $(q_n)_{n\in\omega}$ of partitions in A such that $|q_0| = 1$, q_{n+1} κ-refines q_n and $\bigcup_{n\in\omega} q_n$ is dense in A.*

PROOF. (a) is equivalent to (b) since $\text{RO}({}^\omega\kappa) \cong \text{RO}(T)$, as shown in part (a) of Example 14.15: the set $S = \{a_t\colon t \in T\}$ considered there is dense in $\text{RO}({}^\omega\kappa)$ and isomorphic to T^*. The partitions $p_n = \{a_t\colon t \in {}^n\kappa\}$ in 14.15(a) show that (a) implies (c). Finally, if $(q_n)_{n\in\omega}$ is a family of partitions satisfying (c), then $\bigcup_{n\in\omega} q_n$ is isomorphic to T^*, and (b) holds. □

14.17. THEOREM (McAloon). *A complete Boolean algebra is isomorphic to the collapsing algebra $\text{RO}({}^\omega\kappa)$ iff it is (ω, κ, κ)-nowhere distributive and has a dense subset of size κ.*

PROOF. By part (b) of the preceding proposition, $\mathrm{RO}({}^{\omega}\kappa)$ has a dense subset of size κ, and (ω, κ, κ)-nowhere distributivity of $\mathrm{RO}({}^{\omega}\kappa)$ has been shown in part (a) of Example 14.15. Conversely, let A be a complete Boolean algebra with a dense subset D of size κ and let $(p_n)_{n\in\omega}$ be a family of κ-quasipartitions demonstrating (ω, κ, κ)-nowhere distributivity of A; we shall find partitions q_n satisfying (c) of Proposition 14.16.

It follows from our assumptions on A that for every non-zero a in A, the cellularity $\mathrm{c}(A \restriction a)$ of $A \restriction a$ equals κ and is attained. Define, for $n \in \omega$,

$$D_n = \{d \in D\colon d \text{ meets } \kappa \text{ elements of } p_n\}\ ;$$

by our choice of the partitions p_n, $D = \bigcup_{n\in\omega} D_n$. Since $|D_n| \leq \kappa$, there is a one-to-one function $f_n\colon D_n \to p_n$ such that $f_n(d)\cdot d > 0$ for $d \in D_n$. Let then r_n be the following partition of unity:

$$r_n = [(p_n \backslash \operatorname{ran} f_n) \cup \{f_n(d)\cdot d\colon d \in D_n\} \cup \{f_n(d)\cdot -d\colon d \in D_n\}] \backslash \{0\}\ .$$

By denseness of $D = \bigcup_{n\in\omega} D_n$, also $\bigcup_{n\in\omega} r_n$ is dense in A. It is now easy to construct the partitions q_n: let $q_0 = \{1\}$. Given q_n, let r be any common refinement of q_n and r_n; since $\mathrm{c}(A \restriction a) = \kappa$ is attained for each $a \in r$, there is a κ-refinement q_{n+1} of r. So q_{n+1} κ-refines q_n. Also $\bigcup_{n\in\omega} q_n$ is dense in A since q_{n+1} refines r_n and $\bigcup_{n\in\omega} r_n$ is dense. □

Using McAloon's characterization 14.17, we can reprove Kripke's theorem 13.3, in fact a somewhat stronger statement.

14.18. COROLLARY. *For any non-trivial Boolean algebra A of cardinality at most κ,*

$$\overline{A \oplus \mathrm{RO}({}^{\omega}\kappa)} \cong \mathrm{RO}({}^{\omega}\kappa)\ .$$

Thus, A is complete embeddable into the countably generated algebra $\mathrm{RO}({}^{\omega}\kappa)$ *in such a way that every automorphism of A extends to an automorphism of* $\mathrm{RO}({}^{\omega}\kappa)$.

PROOF. We verify the hypotheses of McAloon's theorem for the complete algebra $B = \overline{A \oplus \mathrm{RO}({}^{\omega}\kappa)}$. By the remarks following 11.5, we may assume that A and $\mathrm{RO}({}^{\omega}\kappa)$ are independent subalgebras of B.

Clearly, if D is a dense subset of size κ in $\mathrm{RO}({}^{\omega}\kappa)$, then $E = \{a \cdot d\colon a \in A^+, d \in D\}$ is a dense subset of size κ in B. And each family $(p_n)_{n\in\omega}$ of partitions in $\mathrm{RO}({}^{\omega}\kappa)$ verifying (ω, κ, κ)-nowhere distributivity of $\mathrm{RO}({}^{\omega}\kappa)$ also verifies (ω, κ, κ)-nowhere distributivity of B, by denseness of E and independence of A and $\mathrm{RO}({}^{\omega}\kappa)$.

The remaining assertion follows from Propositions 11.8 and 11.11. □

14.5. Distributive laws in regular open algebras of trees

Compared with arbitrary partial orders, trees have a particularly simple structure; e.g. each tree can be decomposed into the disjoint union of its levels. If $(T, \leq_T)$ is a normal tree, then we can think of T (or rather its opposite partial order $T^* = (T, \geq_T)$) as being a dense subset of the complete algebra $\mathrm{RO}(T)$, and each level U_α of T is a partition of unity in $\mathrm{RO}(T)$. Hence $\langle U_\alpha \rangle^{\mathrm{cm}}$ is a complete subalgebra of $\mathrm{RO}(T)$ isomorphic to $P(U_\alpha)$, for each $\alpha < \mathrm{height}(T)$, and the subalgebras $\langle U_\alpha \rangle^{\mathrm{cm}}$ constitute an increasing chain whose union is dense in $\mathrm{RO}(T)$. Atomic algebras being completely distributive, it seems natural to investigate distributive laws in $\mathrm{RO}(T)$. We shall do this for Aronszajn trees and, in particular, for Souslin trees. The regular open algebras of the latter are characterized by a powerful combination of chain conditions and distributivity; this combination is, in fact, so strong that in some models of set theory there are no ω_1-Souslin trees at all. In the following subsection, we will see natural examples of complete atomless algebras satisfying the countable chain condition plus a distributive law strictly weaker than (ω, ω)-distributivity.

We recall from set theory some definitions concerning trees, in addition to those given in the last subsection. A subset X of a tree $(T, \leq_T)$ is an *antichain* if any two distinct elements of X are incomparable in T (hence incompatible in T^* and disjoint in $\mathrm{RO}(T)$). For κ an infinite cardinal, $(T, \leq_T)$ is a *normal κ-Souslin tree* if T is normal, has height κ and every antichain of T has size less than κ. It is a non-trivial question in set theory for which cardinals κ there can exist κ-Souslin trees. For example, in the constructible universe L, for regular uncountable κ there is a κ-Souslin tree iff κ is not weakly compact (cf. JECH [1978]); but in models of $\mathrm{ZFC} + \omega_1 < 2^\omega$ + Martin's axiom, there are no ω_1-Souslin trees, as shown below in 14.21.

A subset b of a tree $(T, \leq_T)$ is a *branch* of T if any two elements of b are comparable (i.e. if b is totally ordered by $\leq_T$) and $x \in b$, $y \leq x$ in T implies that $y \in b$. $(T, \leq_T)$ is a *κ-Aronszajn tree* if it has height κ but every level and every branch of T has size less than κ. A normal κ-Souslin tree is Aronszajn, for consider a branch b of T. For each $x \in b$, say $x \in U_\alpha$, pick by normality a point y_x in $U_{\alpha+1}$ such that $x <_T y_x$ but $y_x \not\in b$. Then $\{y_x : x \in b\}$ is an antichain of T and $|b| = |\{y_x : x \in b\}| < \kappa$. Unlike Souslin trees, however, κ-Aronszajn trees can be shown, in ZFC, to exist for every inaccessible non-weakly compact cardinal and for every successor cardinal $\kappa = \lambda^+$ if λ satisfies $\lambda^{<\lambda} = \lambda$.

14.19. DEFINITION. A Boolean algebra A is a *κ-Souslin algebra* if A is complete, atomless, satisfies the κ-chain condition and, for each $\lambda < \kappa$, the (λ, ∞)-distributive law.

In 13.18 we defined, for a complete Boolean algebra A, τA to be the least cardinal μ such that A is completely generated by μ elements. Because of the following theorem, κ-Souslin algebras are sometimes assumed, in addition to the requirements of 14.19, to satisfy $\tau A \leq \kappa$.

14.20. THEOREM. *The following are equivalent, for every regular uncountable cardinal* κ:

(a) *there exists a normal* κ*-Souslin tree,*
(b) *there exists a* κ*-Souslin algebra* A *with* $\tau A \leq \kappa$,
(c) *there exists a* κ*-Souslin algebra.*

PROOF. Clearly (b) implies (c). The implications from (a) to (b) and from (c) to (a) will be proved by showing that for a normal κ-Souslin tree T, $A = \mathrm{RO}(T)$ is a κ-Souslin algebra with $\tau A \leq \kappa$, and by constructing a normal κ-Souslin tree T out of an arbitrary κ-Souslin algebra A. Moreover, if $\tau A \leq \kappa$, then T can be constructed in such a way that T^* (the partial order opposite to $(T, \leq)$) is dense in A, i.e. that $\mathrm{RO}(T) \cong A$.

First assume that $(T, \leq_T)$ is a normal κ-Souslin tree and T^* is dense in A. Then A is atomless since, by normality, T has no maximal element.

A subset X of T is an antichain in T iff it is a pairwise disjoint family in A. Moreover, by denseness of T^* in A, X is a maximal antichain in T iff it is a maximal pairwise disjoint family, i.e. a partition of unity, in A. In particular, $\Sigma^A U_\alpha = 1$ for each $\alpha < \kappa$. Thus, A satisfies the κ-chain condition, since T is a κ-Souslin tree. Also, $\tau A \leq \kappa$ since T^* has size κ and is dense in A.

To prove distributivity, let $(a_{ij})_{i\in I, j\in J}$ be a family in A, where $|I| < \kappa$; we may assume by 14.9 that $\{a_{ij}: j \in J\}$ is a partition of unity for each $i \in I$. By denseness of T^*, pick antichains X_{ij} of T such that $a_{ij} = \Sigma^A X_{ij}$. So for $i \in I$,

$$X_i = \bigcup_{j\in J} X_{ij}$$

is a maximal antichain of T. By the Souslin property of T and regularity of κ, there is some $\alpha < \kappa$ satisfying

$$\bigcup_{i\in I} X_i \subseteq \bigcup_{\nu<\alpha} U_\nu .$$

For each $t \in U_\alpha$, define a function $f(t) \in {}^IJ$ as follows: consider $i \in I$. Since X_i is a maximal antichain of T and by our choice of α, there is a unique $x \in X_i$ such that $x <_T t$; let then $f(t)(i)$ be the unique $j \in J$ such that $x \in X_{ij}$. This choice of $f(t)$ implies that, in A,

$$t \leq \prod_{i\in I} a_{i,f(t)(i)}$$

and

$$1 = \Sigma^A U_\alpha \leq \Sigma\left\{\prod_{i\in I} a_{i,f(t)(i)}: t \in U_\alpha\right\} \leq \Sigma\left\{\prod_{i\in I} a_{if(i)}: f \in {}^IJ\right\},$$

as desired.

Conversely, suppose A is a κ-Souslin algebra. Note first that $\tau A \geq \kappa$, since $\tau A < \kappa$ would imply by 14.8 and $(\tau A, 2)$-distributivity that A is atomic.

We define by induction an increasing chain $(A_\alpha)_{\alpha<\kappa}$ of complete atomic subalgebras of A such that $\tau(A_\alpha) \leq \max(|\alpha|, \omega)$; then, letting U_α be the set of atoms of A_α, we show that

$$T = \bigcup_{\alpha<\kappa} U_\alpha$$

is, with the partial order $\leq_T$ converse to $\leq_A$, a normal κ-Souslin tree. Let $A_0 = 2$. If A_α has been defined, let $A_{\alpha+1}$ be such that $A_\alpha \subseteq A_{\alpha+1}$ and each atom of A_α is the sum of at least two distinct atoms of $A_{\alpha+1}$. For $\lambda < \kappa$ a limit ordinal, let A_λ be the subalgebra of A completely generated by $\bigcup_{\alpha<\lambda} A_\alpha$; then by $\tau(A_\lambda) \leq |\lambda| \cdot \sup\{\tau(A_\alpha): \alpha < \lambda\} < \kappa$ and distributivity, A_λ is atomic. The atoms of A_λ can be described as follows. Since A_λ is the least atomic complete subalgebra including each A_α, U_λ is the coarsest partition of unity in A refining each U_α. Consequently,

$$U_\lambda = \left\{ \prod_{\alpha<\lambda}^A x_\alpha : (x_\alpha)_{\alpha<\lambda} \in {}^\lambda A, x_\alpha \in U_\alpha \right\} \setminus \{0\} . \tag{12}$$

This finishes our construction of the A_α, hence of T.

The following remark shows that $(T, \leq_T)$ is a tree with U_α as its αth level and that height $(T) = \kappa$: suppose that $\beta \leq \alpha < \kappa$ and that $s \in U_\beta$, $t \in U_\alpha$. Then since A_β is a subalgebra of A_α, either $t \leq s$ or $t \cdot s = 0$. In particular, if $t < s$ then $\beta < \alpha$ and s is the unique atom of A_β such that $t < s$. By our construction of the A_α, $(T, \leq_T)$ satisfies conditions (N1) through (N3) in the definition of a normal tree, in the previous subsection. To check (N4), suppose s and t are in U_λ (λ a limit ordinal) and for each $\alpha < \lambda$, the unique element x_α of U_α satisfying $x_\alpha \leq_T s$ also satisfies $x_\alpha \leq_T t$. Then by (12), $s = \prod_{\alpha<\lambda}^A x_\alpha = t$. Finally, a subset of T is an antichain in $(T, \leq_T)$ iff it is a pairwise disjoint family in A, so by the κ-chain condition for A, T is a κ-Souslin tree.

If the Souslin algebra A also satisfies $\tau A \leq \kappa$, then we construct the subalgebras A_α with a bit more care: fix a set

$$Y = \{y_\alpha : \alpha < \kappa\}$$

of complete generators for A. Take A_0 and A_λ ($\lambda < \kappa$ limit) as above and choose $A_{\alpha+1}$ so large that, in addition to the above requirements on $A_{\alpha+1}$, $y_\alpha \in A_{\alpha+1}$. Then $A' = \bigcup_{\alpha<\kappa} A_\alpha$ is a subalgebra of A which is closed under sums of arbitrary length. To prove this, it suffices by the κ-chain condition for A to check that $\Sigma^A M \in A'$ for each subset M of A' of size less that κ. But by regularity of κ, $M \subseteq A_\alpha$ for some $\alpha < \kappa$, so $\Sigma^A M \in A_\alpha$. Since A' includes the set Y of complete generators, $A' = \bigcup_{\alpha<\kappa} A_\alpha = A$. This shows that $T = \bigcup_{\alpha<\kappa} U_\alpha$ is dense in A, i.e. that $\mathrm{RO}(T) \cong A$. □

The above construction of κ-Souslin algebras from normal κ-Souslin trees and vice versa raises the following question: Let A be a κ-Souslin algebra; is it isomorphic to the regular open algebra of some κ-Souslin tree? The proof of the last theorem shows that this holds iff $\tau A = \kappa$. Let us briefly state what is known about the question for $\kappa = \omega_1$. If $A \cong \mathrm{RO}(T)$ for a normal ω_1-Souslin tree T and the continuum hypothesis $2^\omega = \omega_1$ holds, then clearly $|A| = 2^\omega = \omega_1$ by the countable chain condition for A. Jensen has exhibited, in the constructible universe L, an ω_1-Souslin algebra of size ω_2 (cf. DEVLIN and JOHNSBRÅTEN [1974]); note that the generalized continuum hypothesis holds in L, so this algebra is not the completion of an ω_1-Souslin tree. On the other hand, ω_1-Souslin algebras cannot be arbitrarily large, for Solovay has shown (in ZFC) that an ω_1-Souslin algebra has cardinality at most 2^{ω_1} (see JECH [1978] for a proof).

The connection between Souslin algebras and Souslin trees given in 14.20 makes it particularly easy to conclude that, in certain models of set theory, there are no ω_1-Souslin trees.

14.21. PROPOSITION. *Assume that $\omega_1 < 2^\omega$ and that Martin's axiom (cf. Section 2) holds. Then there are no ω_1-Souslin algebras (and, consequently, no ω_1-Souslin trees).*

PROOF. Let A be an ω_1-Souslin algebra. The proof of Theorem 14.20 shows that there is a sequence $(U_\alpha)_{\alpha<\omega_1}$ of partitions of unity in A such that U_α refines U_β, for $\beta \leq \alpha < \omega_1$, and each element of U_α splits into at least two distinct elements of $U_{\alpha+1}$. Since $\omega_1 < 2^\omega$, there is an ultrafilter p of A preserving each of the sums $\Sigma^A U_\alpha = 1$, for $\alpha < \omega_1$; let x_α be the unique element of $U_\alpha \cap p$. It follows that $\{x_\alpha : \alpha < \omega_1\}$ is a decreasing chain in A. But then the pairwise disjoint family $\{x_\alpha \cdot -x_{\alpha+1} : \alpha < \omega_1\}$ contradicts the countable chain condition for A. □

A tree T of height ω_1 with countable levels is called *regular* if it is the union of countably many antichains. Such a tree must be Aronszajn since every branch of T intersects every antichain of T in at most one element; on the other hand, it cannot be Souslin, since it has cardinality ω_1 and one of the countably many antichains covering T must be uncountable. The standard construction of an ω_1-Aronszajn tree produces a normal and regular tree; in fact a non-trivial application of Martin's axiom plus $\omega_1 < 2^\omega$ shows that every normal ω_1-Aronszajn tree is regular (see JECH [1978] for these facts).

14.22. EXAMPLE (the regular open algebra of a regular Aronszajn tree). Let $(T, \leq_T)$ be a normal and regular ω_1-Aronszajn tree. Let A be a complete Boolean algebra in which T^* is dense and let C be the subalgebra of A generated by T. Then C is dense in A and A is the completion of C. We will see that C is $(\omega, 2)$-distributive but not (ω, ω_1)-distributive and that A is $(\omega, \omega_1, \omega_1)$-nowhere distributive, in the sense of Definition 14.13.

These properties of A and C show, first, that the implication 14.10 from the $(\omega, 2)$-distributive law to the (ω, ω)-distributive law is best possible, for non-complete algebras, since C is not (ω, ω_1)-distributive. As a second consequence, the $(\omega, 2)$-distributive law is not inherited by the completion of C, since A is not $(\omega, \omega_1, 2)$-distributive by 14.14, hence not $(\omega, 2^\omega)$-distributive and, finally, not $(\omega, 2)$-distributive by 14.10. It might be interesting to note that, by $(\omega, \omega_1, \omega_1)$-nowhere distributivity, denseness of T in A and McAloon's characterization 14.17 of collapsing algebras, A is isomorphic to $\mathrm{RO}({}^\omega\omega_1)$.

To show that C is $(\omega, 2)$-distributive, let $(a_{ij})_{i\in\omega, j\in 2}$ be any family in C.

Denoting by U_α the αth level of T and by C_α the subalgebra of A generated by $\bigcup_{\beta<\alpha} U_\beta$, we see that C is the union of the increasing chain $(C_\alpha)_{\alpha<\omega_1}$ of subalgebras. Pick $\alpha < \omega_1$ so large that $\{a_{ij}: i \in \omega,\ j \in 2\} \subseteq C_{\alpha+1}$. It follows from normality of T that $C_{\alpha+1}$ is atomic with U_α as its set of atoms. So the $(\omega, 2)$-distribuive law holds for the elements a_{ij} by Theorem 14.5.

For $(\omega, \omega_1, \omega_1)$-nowhere distributivity of A, fix a family $(X_n)_{n\in\omega}$ of antichains of T such that $T = \bigcup_{n\in\omega} X_n$; we may assume by Zorn's lemma that each X_n is a maximal antichain and thus a partition of unity in both C and A. Note that $|X_n| \leq |T| = \omega_1$. We claim that for each $t \in T$, there is an $n \in \omega$ such that ω_1 elements of X_n lie above t, in the tree $(T, \leq_T)$; this demonstrates, by denseness of T^* in C and A, that both C and A are $(\omega, \omega_1, \omega_1)$-nowhere distributive and hence not (ω, ω_1)-distributive. Given $t \in T$, the claim holds since the set $\{x \in T: t \leq_T x\}$ has size ω_1 by normality of T and is covered by the X_n.

14.6. Weak distributivity

We consider a weaker notion of distributivity which is mainly interesting because of its validity in measure algebras – see Theorem 14.30 below.

Definition 14.1 of (κ, λ)-distributivity deals with expressions of the form $\prod_{i\in I} \sum_{j\in J} a_{ij}$. In equation (1) of 14.1, such an expression is evaluated by picking exactly one term $a_{if(i)}$ from each sum $\sum_{j\in J} a_{ij}$ and summing up the products $\prod_{i\in I} a_{if(i)}$. In the weak distributive laws below, we allow ourselves to pick finitely many terms of $\sum_{j\in J} a_{ij}$, for each $i \in I$.

14.23. DEFINITION AND NOTATION. For any sets I and J, let $F(I, J)$ be the set of all functions from I into the set of finite subsets of J. For $(a_{ij})_{i\in I, j\in J}$ a family in a Boolean algebra A and e a finite subset of J, write

$$s_{ie} = \sum_{j\in e} a_{ij}\ .$$

For any cardinals κ and λ, a Boolean algebra A is called *weakly* (κ, λ)-*distributive* if, for $|I| \leq \kappa$, $|J| \leq \lambda$ and $(a_{ij})_{i\in I, j\in J}$ a family in A,

$$\prod_{i\in I} \sum_{j\in J} a_{ij} = \sum \left\{ \prod_{i\in I} s_{ih(i)}: h \in F(I, J) \right\} \tag{13}$$

provided that

(14) each of the sums (respectively products) $\sum_{j\in J} a_{ij}$ for $i \in I$, $\prod_{i\in I} \sum_{j\in J} a_{ij}$, and $\prod_{i\in I} s_{ih(i)}$, for $h \in F(I, J)$, exist in A .

As an analogue of 14.2, we note the following fact.

14.24. REMARK. For every Boolean algebra A and arbitrary elements a_{ij} of A,

$$\sum \left\{ \prod_{i\in I} s_{ih(i)}: h \in F(I, J) \right\} \leq \prod_{i\in I} \sum_{j\in J} a_{ij}\ ,$$

provided that all sums and products displayed exist; this follows from the fact that, for all $h \in F(I, J)$ and $i \in I$, $s_{ih(i)} \leq \Sigma_{j \in J} a_{ij}$.

The denotation of *weak* distributivity is justified by the subsequent assertion.

14.25. PROPOSITION. *Each* (κ, λ)*-distributive algebra is weakly* (κ, λ)*-distributive.*

PROOF. Suppose that A is (κ, λ)-distributive and that I, J, and the family $(a_{ij})_{i \in I, j \in J}$ satisfy the assumptions of (14) above. We want to prove that

$$p = \prod_{i \in I} \sum_{j \in J} a_{ij}$$

is the least upper bound of $\{\Pi_{i \in I} s_{ih(i)} : h \in F(I, J)\}$. By Remark 14.24, p is an upper bound; let q be another one. Now

$$\left\{\prod_{i \in I} a_{if(i)} : f \in {}^I J\right\} \subseteq \left\{\prod_{i \in I} s_{ih(i)} : h \in F(I, J)\right\},$$

since for every $f \in {}^I J$, the function $h \in F(I, J)$ defined by $h(i) = \{f(i)\}$ satisfies $\Pi_{i \in I} a_{if(i)} = \Pi_{i \in I} s_{ih(i)}$. So q is an upper bound of $\{\Pi_{i \in I} a_{if(i)} : f \in {}^I J\}$, and by (κ, λ)-distributivity of A,

$$p = \Sigma \left\{\prod_{i \in I} a_{if(i)} : f \in {}^I J\right\} \leq q. \quad \square$$

The rest of this subsection is devoted to measure algebras, the main examples for weak (ω, ω)-distributivity.

14.26. DEFINITION. A *measure algebra* is a pair (A, μ) such that A is a σ-complete Boolean algebra and $\mu: A \to [0, 1]$ is a map into the real unit interval satisfying

(a) μ is *σ-additive*, i.e. $\mu(\Sigma_{n \in \omega} a_n) = \Sigma_{n \in \omega} \mu(a_n)$ for each family of pairwise disjoint elements of A,

(b) μ is *normed*, i.e. $\mu(1) = 1$,

(c) μ is *strictly positive*, i.e. $\mu(a) \neq 0$ for $a \in A^+$.

14.27. EXAMPLE. Assume that (X, B, ν) is a measure space, i.e. that B is a σ-algebra of sets over X and $\nu: B \to [0, 1]$ is a σ-additive measure, and that $\nu(1) = 1$. Then

$$N = \{b \in B : \nu(b) = 0\}$$

is a σ-complete ideal in B. Let $A = B/N$ be the quotient algebra and $\pi: B \to A$ the canonical map. It is easily checked that there is a unique map $\mu: A \to [0, 1]$ satisfying $\mu \circ \pi = \nu$, and that (A, μ) is a measure algebra. We call (A, μ) the measure algebra *associated* with the measure space (X, B, ν).

The most familiar special case of Example 14.27 is obtained by letting $X_1 = [0, 1]$, B_1 the algebra of Lebesgue measurable subsets of X_1 (cf. Example 1.32) and ν_1 the Lebesgue measure on these sets. Next, consider for any cardinal κ the product $(X_\kappa, B_\kappa, \nu_\kappa)$ of κ isomorphic copies of the space (X_1, B_1, ν_1), as defined in measure theory; this is a measure space with underlying set $X_\kappa = {}^\kappa[0, 1]$, and ν_κ is called the product measure of ν_1. Denote by (M_κ, μ_κ) the measure algebra associated, via Example 14.27, with the product measure space. A famous theorem by D. Maharam (cf. MAHARAM [1942]) says that this last example covers essentially all measure algebras: every measure algebra (A, μ) has a product decomposition $A \cong \Pi_{i\in I}(A \upharpoonright a_i)$ such that I is at most countable, $a_i > 0$ in A and either $A \upharpoonright a_i$ is a two-element algebra or there is some infinite cardinal κ (depending on i) such that $(A \upharpoonright a_i, \mu_i) \cong (M_\kappa, \mu_\kappa)$, where μ_i is the induced measure on $A \upharpoonright a_i$, defined by $\mu_i(x) = \mu(x)/\mu(a_i)$.

We list several easy consequences of the axioms (a) through (c) for measure algebras.

14.28. LEMMA. *Let (A, μ) be a measure algebra; let a, b, and a_n $(n \in \omega)$ be arbitrary elements of A.*

(a) $\mu(a_1 + \cdots + a_k) = \mu(a_1) + \cdots + \mu(a_k)$, *if $a_1, \ldots, a_k$ are pairwise disjoint.*

(b) *If $a \leq b$, then $\mu(a) \leq \mu(b)$ and $\mu(b \cdot -a) = \mu(b) - \mu(a)$.*

(c) *If $a < b$, then $\mu(a) < \mu(b)$.*

(d) $\mu(\Sigma_{n\in\omega} a_n) \leq \Sigma_{n\in\omega} \mu(a_n)$.

(e) *For each real number $\varepsilon > 0$, there is $k \in \omega$ such that*

$$\mu\Big(\sum_{n\in\omega} a_n\Big) - \mu(a_0 + \cdots + a_{k-1}) < \varepsilon\,.$$

PROOF. (a) follows from σ-additivity by letting, in (a) of Definition 14.26, $a_0 = 0$ and also $a_n = 0$ for $n > k$.

(b) follows from (a) by letting $k = 2$ and noting that $b = a + b \cdot -a$, where a and $b \cdot -a$ are disjoint.

(c) is a consequence of (b), since $b \cdot -a > 0$ for $a < b$ and hence $\mu(b \cdot -a) > 0$ by positivity of μ.

For a proof of (d) and (e), let $b_n = a_n \cdot -\Sigma_{i<n} a_i$. Then $b_n \leq a_n$, the b_n are pairwise disjoint, $a_0 + \cdots + a_n = b_0 + \cdots + b_n$, and (d) follows from

$$\mu\Big(\sum_{n\in\omega} a_n\Big) = \mu\Big(\sum_{n\in\omega} b_n\Big) = \sum_{n\in\omega} \mu(b_n) \leq \sum_{n\in\omega} \mu(a_n)\,.$$

In (e), choose $k \in \omega$ so large that

$$\sum_{n\in\omega} \mu(b_n) - \sum_{n<k} \mu(b_n) < \varepsilon$$

and use the equations $\mu(\Sigma_{n\in\omega} a_n) = \mu(\Sigma_{n\in\omega} b_n)$, $\mu(\Sigma_{n<k} a_n) = \mu(\Sigma_{n<k} b_n)$. □

14.29. PROPOSITION. *If (A, μ) is a measure algebra, then A satisfies the countable chain condition and hence is complete.*

PROOF. For any pairwise disjoint family X in A and $n \in \omega$, let

$$X_n = \{x \in X \colon \mu(x) \geq 1/(n+1)\} .$$

Then $X = \bigcup_{n\in\omega} X_n$ and, by part (a) of the preceding lemma, X_n has at most $n+1$ elements; thus X is countable.

Completeness of A now follows from σ-completeness, the countable chain condition and Lemma 10.2. □

14.30. THEOREM. *If (A, μ) is a measure algebra, then A is weakly (ω, ω)-distributive.*

PROOF. Let I, J be countable sets and $(a_{ij})_{i\in I, j\in J}$ a family in A; we may assume that $I = J = \omega$. Put

$$t_i = \sum_{j\in\omega} a_{ij} \quad \text{for } i \in \omega ,$$

$$p = \prod_{i\in\omega} t_i ,$$

$$r = \sum\left\{\prod_{i\in\omega} s_{ih(i)} \colon h \in F(\omega, \omega)\right\} .$$

By Remark 14.24, $r \leq p$; $r < p$ would imply $\mu(r) < \mu(p)$. Thus, it suffices to find, for each positive real number ε, some $h \in F(\omega, \omega)$ such that $\mu(p) - \mu(\prod_{i\in\omega} s_{ih(i)}) \leq \varepsilon$.

Given ε, pick by 14.28(e) for each $i \in \omega$ some $k(i) \in \omega$ large enough to guarantee that

$$\mu(t_i) - \mu(u_i) < \varepsilon/2^{i+1} ,$$

where

$$u_i = \sum_{j<k(i)} a_{ij} ;$$

define h by letting $h(i) = \{0, \ldots, k(i)-1\}$. Then $s_{ih(i)} = u_i$ and

$$\mu(p) - \mu\left(\prod_{i\in\omega} s_{ih(i)}\right) = \mu\left(\prod_{i\in\omega} t_i\right) - \mu\left(\prod_{i\in\omega} u_i\right)$$

$$= \mu\left(\prod_{i\in\omega} t_i \cdot - \prod_{i\in\omega} u_i\right) \qquad \text{by } \prod_{i\in\omega} u_i \leq \prod_{i\in\omega} t_i$$

$$\leq \mu\left(\sum_{j\in\omega} t_j \cdot - u_j\right)$$

$$\begin{aligned}
&\leq \sum_{j\in\omega} \mu(t_j \cdot -u_j) && \text{by 14.28(d)} \\
&= \sum_{j\in\omega} (\mu(t_j) - \mu(u_j)) && \text{by } u_j \leq t_j \text{ and 14.28(b)} \\
&\leq \varepsilon \,;
\end{aligned}$$

here the first inequality follows from

$$\prod_{i\in\omega} t_i \cdot - \prod_{i\in\omega} u_i \leq \sum_{j\in\omega} t_j \cdot -u_j \,. \quad \square$$

Exercises

1. Imitating the argument of Example 14.11, show that no collapsing algebra $\mathrm{RO}({}^{\omega}\kappa)$, where $\kappa \geq \omega_1$, is ω_2-representable.

2. Let κ be an infinite cardinal. Then a Boolean algebra is κ^+-representable iff, in its dual space, the intersection of at most κ dense open sets each of which is the union of at most κ clopen sets, is dense.

3. (D. Maharam) Let B be the regular open algebra of a metric space (X, d) and κ an infinite cardinal; assume that for each non-zero element b of B, the cellularity $\mathrm{c}(B \restriction b)$ is κ. By Exercise 8 in Section 4, $\mathrm{c}(B \restriction b)$ is attained, for every $b \in B^+$. Prove that B is isomorphic to the collapsing algebra $\mathrm{RO}({}^{\omega}\kappa)$.

Hint. Construct in B a sequence of partitions of unity verifying the assumptions of Proposition 14.16(c); this works as in Exercise 8 of Section 4.

4. Show that for a Souslin tree T, $T \times T$ with the componentwise partial order has an uncountable subset consisting of pairwise incompatible elements. Hence for a Souslin algebra A, the free product $A \oplus A$ does not satisfy the countable chain condition.

Note. It is independent of the axioms of ZFC set theory whether there is a Boolean algebra A such that A satisfies the countable chain condition but $A \oplus A$ does not. For more information, see the introduction to Section 11.

5. A dense linear order $(L, \leq)$ is a *Souslin continuum* if it is complete and has no uncountable set of pairwise disjoint open intervals and no countable dense subset.

(a) If A is an ω_1-Souslin algebra, then every maximal chain of A is a Souslin continuum.

(b) For every Souslin continuum L, equipped with its order topology, $\mathrm{RO}(L)$ is an ω_1-Souslin algebra with $\tau(\mathrm{RO}(L)) = \omega_1$.

Hence existence of ω_1-Souslin trees, ω_1-Souslin algebras and Souslin continua are equivalent.

6. Show that none of the following algebras is weakly (ω, ω)-distributive:

(a) the Borel algebra of the reals, divided by the ideal of nowhere dense Borel sets,

(b) the regular open algebra of the partial order $\mathrm{Fn}(I, J, \omega)$, where I is infinite and J has at least two elements.

7. The measure algebra associated with Lebesgue measure on the real unit interval is weakly (ω, ω)-distributive, by Theorem 14.30. Show that it is not $(\omega, 2)$-distributive.

8. Let κ, λ be infinite cardinals. Then a Boolean algebra is (κ, λ)-distributive iff it is both weakly (κ, λ)-distributive and $(\kappa, 2)$-distributive.

9. Let κ, λ be infinite cardinals. Then a Boolean algebra is weakly (κ, λ)-distributive iff, in its dual space, the intersection of at most κ dense open sets each of which is the union of at most λ clopen sets, includes a dense open set.

Hence, by Exercise 2, every weakly (κ, κ)-distributive algebra is κ^+-representable.

CHAPTER 6

Special Classes of Boolean Algebras

Sabine KOPPELBERG
Freie Universität Berlin

Contents

HANDBOOK OF BOOLEAN ALGEBRAS
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Introduction

We devote this chapter to a study of three particular classes of Boolean algebras: interval algebras, tree algebras, and superatomic algebras. Unlike the situation in the preceding chapter, these algebras are of an extremely finitary nature; e.g. they have no free uncountable subalgebras and, a fortiori, no σ-complete infinite subalgebras. Interval algebras and superatomic algebras have been intensively investigated in the literature. On the other hand, tree algebras have only recently been introduced by G. Brenner, their main feature being that they give fairly easy examples of algebras with certain paradoxical properties.

The classes of interval algebras and of tree algebras are closely related: every tree algebra embeds into an interval algebra. Another common property is that, by the very definition of both classes, every interval algebra and also every tree algebra has a canonical set of generators with the property that every element of the algebra is uniquely representable, over the generators, in a simple normal form. Consequently, combinatorial questions concerning interval algebras or tree algebras can often be solved by reduction to the finite set of generators involved in the normal form of each element and application of set-theoretical arguments to these finite sets.

Superatomic algebras are not readily describable by having a set of generators with particularly pleasent properties, but several aspects of their structure are well understood due to a process called the Cantor–Bendixson analysis. As an example, let us mention that countable superatomic algebras are completely characterized by a pair of ordinal invariants, the Cantor–Bendixson invariants, and the uncountable ones can be roughly classified by (possibly infinite) sequences of cardinal invariants.

15. Interval algebras

Interval algebras constitute a class of Boolean algebras which arises from an apparently simpler class of structures – linear orders – by a similarly simple construction – taking finite unions of half-open intervals. They should therefore provide a good testing ground for the question of how much the theory of Boolean algebras, in particular their combinatorics, has been understood. For example, interval algebras have been extensively used to construct Boolean algebras with peculiar features.

After a characterization of interval algebras and their Stone spaces, we show in this section that the class of (algebras isomorphic to) interval algebras is closed under taking quotients and finite products, but not under subalgebras and free products. It follows that the class of interval algebras is quite narrow, e.g. no uncountable free algebra embeds into an interval algebra. Finally we prove two particular and strong properties of subalgebras of interval algebras, found by M. Rubin. If B is such an algebra, then it is retractive, i.e. every homomorphic image of B is a retract of B. And if the cardinality κ of B is regular and uncountable, then B has either a chain or a pairwise incomparable family of size κ.

15.1. Characterization of interval algebras and their dual spaces

It is the aim of this subsection to give an algebraic characterization of interval algebras and a topological one of their Stone spaces: interval algebras are (up to isomorphism) exactly those Boolean algebras which have a set of generators which is a chain, i.e. totally ordered, under the Boolean ordering; their Stone spaces are those Boolean spaces whose topology is induced by a linear order.

We briefly recall (respectively introduce) some notation concerning linear orders and interval algebras. In a linear order $(X, \leq)$, we write

$$(-\infty, a] = \{x \in X: x \leq a\},$$
$$(a, +\infty) = \{x \in X: a < x\},$$

etc. More formally, we assume that $-\infty$ and $+\infty$ are distinct elements not contained in X, that $-\infty < a < +\infty$ for all $a \in X$, and that sets like $(-\infty, a]$, $(a, +\infty)$, etc. are being formed in $X \cup \{-\infty, +\infty\}$.

15.1. NOTATION. For M and N subsets of a linear order X, we write

$$M \ll N$$

if M lies left on N, i.e. if $m < n$ for all $m \in M$ and $n \in N$.

For $(L, \leq)$ a linear order with a least element 0_L, the interval algebra Intalg L of L was defined in Example 1.11 to be the algebra of sets over L consisting of all finite unions of half-open intervals $[x, y)$, where $x \leq y$ in L. Every element a of Intalg L has a *standard representation*

$$a = [x_1, y_1) \cup \cdots \cup [x_n, y_n),$$

where $0_L \leq x_1 < y_1 < \cdots < x_n < y_n \leq +\infty$, and this representation is unique since x_1 is the first point of L lying in a, y_1 is the first point of $L \cup \{+\infty\}$ greater than x_1 and not lying in a, etc.

It may be worth noticing that non-isomorphic linear orders L and L' can have isomorphic interval algebras. For instance, let, in the real line,

$$L = ([0, 1) \cap \mathbb{Q}) \cup [1,2),$$
$$L' = [0, 1) \cup ([1, 2) \cap \mathbb{Q});$$

Proposition 15.11 shows that both Intalg L and Intalg L' are isomorphic to the product algebra $\mathrm{Intalg}([0, 1) \cap \mathbb{Q}) \times \mathrm{Intalg}([1, 2))$.

The following special case of Sikorski's extension criterion 5.5 will be used in characterizing interval algebras, in 15.3.

15.2. REMARK. Let A and B be Boolean algebras, $C \subseteq A$ a chain (with respect to the Boolean ordering of A) and $f: C \to B$ any map. Then f extends to a homomorphism $g: \langle C \rangle \to B$ iff

(a) f is order-preserving, i.e. $x \leq y$ implies $f(x) \leq f(y)$,
(b) $0_A \in C$ implies $f(0_A) = 0_B$; similarly for 1_A.
Moreover, g is one-to-one iff, additionally,
(c) f is one-to-one,
(d) $x \in C \setminus \{0_A\}$ implies $f(x) \neq 0_A$; similarly for 1_A.

PROOF. This follows immediately from 5.5 (respectively (2′) in 5.6) because the elementary products over the chain C have the form $x \cdot -y$ (where $x, y \in C$), x (where $x \in C$) and $-y$ (where $y \in C$). □

15.3. THEOREM. *A Boolean algebra A is isomorphic to an interval algebra iff it is generated by a chain $C \subseteq A$. Moreover, if $A \cong \operatorname{Intalg} L$, then C may be taken to be isomorphic to L.*

PROOF. If $A = \operatorname{Intalg} L$, then

$$C = \{[0_L, x) \colon x \in L\}$$

is a chain generating A and isomorphic to L. Conversely, assume that A is generated by a chain C and that, without loss of generality, $0_A \in C$ but $1_A \notin C$. Then by Remark 15.2, there is a unique monomorphism g from A into $\operatorname{Intalg} C$ satisfying $g(x) = [0_A, x)$ for $x \in C$. g is onto since $g[C]$ generates $\operatorname{Intalg} C$. □

For a description of the dual spaces of interval algebras, we need some more definitions from the theory of linear orders. For $(X, \leq)$ a linear order, the "open" intervals

$$(x, y) = \{z \in X \colon x < z < y\},$$

where $x, y \in X \cup \{-\infty, +\infty\}$, constitute the base for a topology on X, the *order topology*. This topology is Hausdorff and the "closed" intervals

$$[x, y] = \{z \in X \colon x \leq z \leq y\},$$

$$(-\infty, x], \qquad [x, +\infty),$$

where $x, y \in X$, are topologically closed. $(X, \leq)$ is *complete* if for every subset M of X, the least upper bound $\sup M$ and the greatest lower bound $\inf M$ of M exist; in particular, $0_X = \inf X$ and $1_X = \sup X$ are then the least and the greatest elements of X.

We say that $x, y \in X$ constitute a *jump* in X and write $x \lessdot y$ if $x < y$ but there is no $z \in X$ satisfying $x < z < y$. Clearly, every jump $x \lessdot y$ defines a partition

$$X = (-\infty, x] \cup [y, +\infty)$$

of X into two clopen subsets; similarly, if $x \lessdot y < r \lessdot s$, then

$$X = (-\infty, x] \cup [y, r] \cup [s, +\infty)$$

is a clopen partition.

15.4. DEFINITION. A linear order $(X, \leq)$ is a *Boolean order* if it is complete and the jumps of X are dense in X, i.e. for $r < s$ in X there are $x, y \in X$ such that $r \leq x \lessdot y \leq s$.

We proceed to give the standard example of a Boolean order. The proof of Theorem 15.7 shows that, in fact, this example covers all Boolean orders, up to isomorphism.

15.5. EXAMPLE (the initial segments of a linear order). Let $(C, \leq)$ be a linear order. An *initial segment* of C is a subset e of C such that $x \in e$ and $y \in C$, $y \leq x$, implies $y \in e$. The set

$$C^* = \{e \subseteq C\colon e \text{ an initial segment of } C\}$$

is clearly a linear order under inclusion; we show that it is Boolean.

Every subset M of C^* has $\bigcap M$ as its greatest lower bound and $\bigcup M$ as its least upper bound in C^*, so C^* is complete. If $e < f$ in C^*, pick a point x of C such that $x \in f \setminus e$. Then, in C^*,

$$e \leq (-\infty, x) \lessdot (-\infty, x] \leq f .$$

The following proposition reduces topological properties of linearly ordered topological spaces to order-theoretic ones. Its first part is well known in topology.

15.6. PROPOSITION. *Let $(X, \leq)$ be a linear order and $\mathbf{O}$ its order topology.*

(a) *$(X, \mathbf{O})$ is a compact Hausdorff space iff the linear order $(X, \leq)$ is complete.*

(b) *$(X, \mathbf{O})$ is a Boolean space iff the linear order is Boolean.*

PROOF. In the whole proof, note that the order topology of an arbitrary linear order is Hausdorff.

(a) We first show that if $(X, \leq)$ is not complete, then $(X, \mathbf{O})$ is not compact. Assume, for example, that $M \subseteq X$ has no least upper bound. Then M has no greatest element, the set N of upper bounds of M has no least element, and since

$$X = \bigcup_{m \in M} (-\infty, m) \cup \bigcup_{n \in N} (n, +\infty) ,$$

there is an open cover of X without any finite subcover.

Conversely, let $(X, \leq)$ be complete and U an open cover of X. Define

$$s = \sup\{x \in X\colon [0_X, x] \text{ is covered by finitely many elements of } U\} .$$

Pick $u \in U$ such that $s \in u$ and then pick $a, b \in X \cup \{-\infty, +\infty\}$ be such that $s \in (a, b) \subseteq u$. Since $a < s$, there is a finite subset U' of U such that $[0_X, a] \subseteq \bigcup U'$ (if $a = -\infty$, let $U' = \emptyset$). Now if $b = +\infty$, then $U' \cup \{u\}$ is a finite subcover of U and we are finished. If $b \in X$, pick $v \in U$ such that $b \in v$. Then $[0_X, b]$ is covered by the finite subset $U' \cup \{u, v\}$ of U, which contradicts $s < b$ and the definition of s.

(b) If $(X, \leq)$ is a Boolean order, then $(X, \boldsymbol{O})$ is a compact Hausdorff space by part (a). To prove zero-dimensionality, assume that $x \in X$ and that (a, b) is an open interval containing x. Since $(X, \leq)$ is a Boolean linear order, there are jumps

$$a \leq s \lessdot t \leq x \leq r \lessdot p \leq b ;$$

then $[t, r]$ is a clopen subinterval of (a, b) containing x.

For a proof of the converse, assume that $(X, \boldsymbol{O})$ is a Boolean space; by part (a), the linear order $(X, \leq)$ is complete.

Claim. If a is a clopen subset of X, then both a and $-a = X \backslash a$ are unions of finitely many clopen intervals whose endpoints are the lower (respectively upper) points of jumps in X.

For let a be clopen, say $a = \bigcup U$ and $-a = \bigcup V$, where the elements of U (respectively V) are non-empty open intervals in X. By compactness of X and closedness of a and $-a$, we can write

$$a = u_1 \cup \cdots \cup u_n , \qquad -a = v_1 \cup \cdots \cup v_m ,$$

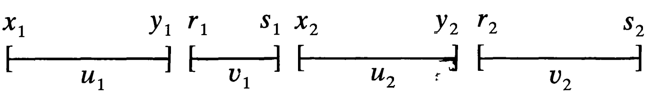
where $u_i \in U$ and $v_j \in V$. We may assume that, in the notation of 15.1, $u_1 \ll \cdots \ll u_n$ and $v_1 \ll \cdots \ll v_m$, and for the sake of definiteness, that $n = m$ and

$$u_1 \ll v_1 \ll \cdots \ll u_n \ll v_n .$$

Since $(X, \leq)$ is complete and both a and $-a$ are closed, $\sup u_1$ exists and is an element of a; similarly, $\inf v_1$ is in $-a$. Hence, $\sup u_1 \in u_1$, $\inf v_1 \in v_1$ and $\sup u_1 \lessdot \inf v_1$. Similar reasoning applies to the endpoints of u_2, v_2, etc. and shows that

$$u_i = [x_i, y_i] , \qquad v_i = [r_i, s_i] ,$$

where $x_1 = 0_X$, $s_n = 1_X$, $y_i \lessdot r_i$ and, for $i < n$, $s_i \lessdot x_{i+1}$.



This proves the claim.

Booleanness of the linear order $(X, \leq)$ now follows readily from the Claim: assume that $r < s$ in X. $(X, \boldsymbol{O})$ is a Boolean space, so there is a clopen subset a of X such that $r \in a$ but $s \notin a$. Using the notation in the proof of the Claim, assume that $r \in u_i$. Then clearly $r \leq y_i \lessdot r_i \leq s$. □

15.7. THEOREM. *A Boolean space is homeomorphic to the dual space of an interval algebra iff its topology is induced by a linear order.*

PROOF. Suppose first that $\leq$ is a linear order on a Boolean space X inducing the topology of X. Then $(X, \leq)$ is a Boolean order by 15.6(b) and the dual algebra Clop X of X is generated by the chain

$$C = \{[0_X, x]: x \lessdot y, \text{ for some } y \in X\},$$

as follows immediately from the Claim in the proof of 15.6(b). By 15.3, Clop X is isomorphic to an interval algebra.

Conversely, assume that X is the dual space Ult A of a Boolean algebra A generated by a chain C; we may assume that $0_A \in C$ and $1_A \not\in C$. Consider the Boolean order $(C^*, \subseteq)$ of initial segments of C constructed in Example 15.5. We show that the map

$$\phi: X \to C^*, \quad \phi(p) = C \setminus p$$

is a homeomorphism from X onto the Boolean space associated, by 15.6(b), with the linear order $(C^*, \subseteq)$.

For each ultrafilter p of A, $p \cap C$ is an end segment of C (i.e. if $x \in p \cap C$ and $y \in C$, $y \geq x$, then $y \in p \cap C$). The characteristic homomorphism of p from A onto 2 is completely determined by its action on the set C of generators, i.e. p is determined by $p \cap C$. Conversely, Remark 15.2 shows that for each end segment u of C there is an ultrafilter p of A such that $p \cap C = u$. Thus, assigning $p \cap C$ to p gives a bijection from $X = \text{Ult } A$ onto the set of end segments of C and ϕ defined as above is a bijection from X onto C^*.

Since both X with the Stone topology and C^* with the order topology are Boolean spaces, it suffices to prove that ϕ is an open map. It is enough to prove that $\phi[v]$ is clopen for each clopen subset v of X, say $v = s(c)$, where $c \in A$ and $s: A \to \text{Clop } X$ is the Stone isomorphism. Also, since ϕ is bijective and C generates A, it is enough to consider $c \in C$. For the elements $e = (-\infty, c)$ and $f = (-\infty, c]$ of C^*, we find that $e \lessdot f$ and thus

$$\begin{aligned} \phi[s(c)] &= \{\phi(p): c \in p\} \\ &= \{C \setminus p: c \in p\} \\ &= [0_{C^*}, e] \end{aligned}$$

is a clopen subset of C^*. □

15.2. *Closure properties of interval algebras*

We are going to consider the question under which algebraic operations the class of (isomorphic copies of) interval algebras is closed. It is fairly easy to prove that quotients of interval algebras and products of finite families of interval algebras are again interval algebras, but subalgebras of interval algebras are generally not. The main result, then, is the Treybig–Ward theorem 15.14 stating that free products are, as a rule, not embeddable into interval algebras. It follows that uncountable free algebras and infinite algebras satisfying the countable separation property do not embed into interval algebras.

Let us begin by describing the ideals of interval algebras. A subset u of a linear order C is *convex* if, for $x < y$ in u and $z \in C$ such that $x \leq z \leq y$, z is in u.

15.8. DEFINITION AND REMARK. An equivalence relation $\sim$ on a linear order C is *convex* if each of its equivalence classes is convex. In this case, the set $C/\sim$ of equivalence classes is linearly ordered by the relation $\leqslant$ defined in 15.1.

15.9. PROPOSITION. *If A is isomorphic to an interval algebra, then so is every quotient of A. Moreover, let $C \subseteq A$ be a chain generating A and containing 0_A and 1_A. Then there is a one-to-one correspondence between the ideals of A and the convex equivalence relations on C such that A/I is generated by a chain isomorphic to $C/\sim$, if $\sim$ is the equivalence relation corresponding to the ideal I.*

PROOF. We begin by sketching two short and natural proofs of the first assertion, although it is reproved by the second one. An algebraic proof is obtained by using 15.3 and noting that, if A is generated by a chain C, then for every epimorphism $\pi: A \twoheadrightarrow A'$, A' is generated by the chain $\pi[C]$. A topological proof follows from 15.7 and the fact that, if the topology of a space X is induced by a linear order $\leq$ and Y is a closed subspace of X, then the subspace topology of Y is induced by the restriction of $\leq$ to Y.

Now fix a chain $C \subseteq A$, as supposed in the second statement. Given any ideal I of A, let $\pi: A \twoheadrightarrow A/I$ be canonical and assign to I the equivalence relation $\sim_I$ on C defined by

$$x \sim_I y \quad \text{iff } \pi(x) = \pi(y);$$

Then $\sim_I$ is convex since π is order preserving. C generates A, so $\pi[C]$ generates A/I, and clearly $\pi[C]$ is isomorphic to the linear order $(C/\sim_I, \leqslant)$ introduced in 15.8.

Conversely, to each convex equivalence relation $\sim$ on C we can assign an ideal $I_\sim$ as follows. Let $a \in A$; since 0_A and 1_A are in C, a has a unique representation

$$a = y_1 \cdot -x_1 + \cdots + y_n \cdot -x_n ,$$

where $x_1 < y_1 < \cdots < x_n < y_n$ in C. Then define

$$a \in I_\sim \quad \text{iff } x_1 \sim y_1, \ldots, x_n \sim y_n .$$

It is a matter of routine to check that these assignments are inverses of each other. $\square$

15.10. COROLLARY. *The free Boolean algebra Fr ω on ω independent generators is isomorphic to an interval algebra, and so is every countable Boolean algebra.*

PROOF. Both Fr ω and the interval algebra of the rational unit interval $[0, 1) \cap \mathbb{Q}$ are countable and atomless (cf. Proposition 9.11), hence isomorphic by Corollary 5.16. Also, each countable algebra, being a homomorphic image of Fr ω, is isomorphic to an interval algebra by the preceding proposition. $\square$

15.11. PROPOSITION. *If A and B are isomorphic to interval algebras, then so is their product $A \times B$.*

PROOF. Suppose that $A \cong \operatorname{Intalg} K$ and $B \cong \operatorname{Intalg} L$ where, without loss of generality, K and L are disjoint and both have a least element. Let $(K + L, \leq)$ or, for short, $K + L$ be the linear order with $K \cup L$ as its underlying set and such that $K \ll L$. An isomorphism

$$f\colon \operatorname{Intalg}(K + L) \to \operatorname{Intalg} K \times \operatorname{Intalg} L$$

is obtained by letting

$$f(a) = (a \cap K, a \cap L) . \quad \square$$

Thus, the class of Boolean algebras isomorphic to interval algebras is closed under finite products. We will see in Corollary 15.16 that products of infinitely many non-trivial algebras are never isomorphic to, and not even embeddable into, interval algebras.

15.12. EXAMPLE. We exhibit a subalgebra of an interval algebra which is not (isomorphic to) an interval algebra.

Let κ be an uncountable cardinal; the elements of κ, i.e. the ordinals less than κ, are well ordered of order type κ. For each $\alpha \in \kappa$, the singleton $\{\alpha\} = [\alpha, \alpha + 1)$ belongs to $\operatorname{Intalg} \kappa$, and thus the finite–cofinite algebra F over κ (cf. Example 1.9) is a subalgebra of $\operatorname{Intalg} \kappa$. Assume that F is isomorphic to some interval algebra $\operatorname{Intalg} L$. Then L must have cardinality κ, since $|F| = \kappa$; so F has a chain C of size κ. But this is impossible; an arbitrary chain C in F contains, for each $n \in \omega$, at most one element of size n and at most one element whose complement has size n. Hence, C is countable.

Concerning the question whether the free product of two interval algebras B and C is isomorphic to an interval algebra, let us first handle two trivial cases. If both B and C are countable, then so is $B \oplus C$, hence by 15.10 it is isomorphic to an interval algebra. If one of B or C is finite, say $C \cong 2^n$, then 11.6(d) shows that $B \oplus C \cong B^n$, and $B \oplus C$ is isomorphic to an interval algebra by 15.11. In all remaining cases, $B \oplus C$ is not embeddable into an interval algebra, by the following theorem. The topological dual of this result is due, independently, to Treybig (cf. TREYBIG [1964]) and WARD (unpublished) and holds for arbitrary compact Hausdorff spaces.

Another piece of notation is useful in the proofs of the Treybig–Ward theorem and of Rubin's theorems 15.18 and 15.22.

15.13. NOTATION. Let a be an element of an interval algebra with standard representation

$$a = [x_1, y_1) \cup \cdots \cup [x_n, y_n) .$$

Then let

$$\mathrm{Int}(a) = \{[x_i, y_i): 1 \leq i \leq n\} ,$$

the set of intervals constituting a,

$$l(a) = |\mathrm{Int}(a)| = n$$

the *length* of a and

$$\mathrm{rel}(a) = \{x_1, y_1, \ldots, x_n, y_n\} \setminus \{0_L, +\infty\}$$

the set of *relevant points* of a.

15.14. THEOREM (Treybig, Ward). *Suppose that B and C are subalgebras of an interval algebra and that B is infinite and C is uncountable. Then B and C are not independent.*

PROOF. Let B and C be subalgebras of Intalg L. By 3.4, fix a countably infinite pairwise disjoint family $B' \subseteq B$. Let, for each $c \in C$,

$$B'_c = \{b \in B' : b \cap \mathrm{rel}(c) = \emptyset\} .$$

Then B'_c is cofinite in B' since rel(c) is finite and the elements of B' are disjoint. Now C is uncountable and B' has only countably many cofinite subsets, so there is an infinite $C' \subseteq C$ and some $B'' \subseteq B'$ such that $B'_c = B''$ for every c in C'.

Fix an element b of B'' for the rest of the proof. For every c in C', we have $b \in B'_c$; thus $b \cap \mathrm{rel}(c) = \emptyset$ which means that every interval u in Int(b) is either included in c or in $-c$. Since C' is infinite and Int(b) is finite, there are distinct c and c' in C' such that for every u in Int(b), $u \subseteq c$ iff $u \subseteq c'$. The symmetric difference $c \triangle c'$ is a non-zero element of C and satisfies $(c \triangle c') \cap b = \emptyset$, proving that B and C are not independent. □

15.15. COROLLARY. *No interval algebra has an uncountable independent subset.*

PROOF. Let F be a free Boolean algebra over an uncountable independent set U. If U is partitioned into two uncountable subsets V and W, then F is the free product of the subalgebras generated by V (respectively W) and both subalgebras are uncountable. Hence, F cannot be embedded into an interval algebra. □

15.16. COROLLARY. *Infinite Boolean algebras having the countable separation property and products of infinitely many non-trivial Boolean algebras cannot be embedded into interval algebras.*

PROOF. The second assertion follows from the first one, for if I is infinite and, for $i \in I$, A_i is an algebra with at least two elements, then $\Pi_{i \in I} A_i$ has the subalgebra

${}^I2$ isomorphic to $P(I)$, and this algebra has the countable separation property, hence does not embed into an interval algebra.

The first assertion is a consequence of Corollary 15.15 and the fact that each infinite algebra B with the countable separation property has an uncountable independent subset. To see the latter, pick by 3.4 countably many pairwise disjoint non-zero elements b_n in B, let U be an uncountable independent subset of $P(\omega)$ by 9.21 and choose for every u in U an element v_u of B such that $b_n \leq v_u$ if $n \in u$ and $b_n \leq -v_u$ if $n \notin u$; this is possible in view of the countable separation property. Clearly, the set $\{v_u : u \in U\}$ is independent in B. □

15.3. Retractive algebras

We have seen in Example 15.12 that a subalgebra of an interval algebra is not necessarily isomorphic to an interval algebra. Subalgebras of interval algebras do, however, have a very strong property, which is quite obvious for interval algebras, retractivity.

15.17. DEFINITION. A Boolean algebra B is *retractive* if for every epimorphism π from B onto a Boolean algebra B', there is a homomorphism $\varepsilon : B' \to B$ such that $\pi \circ \varepsilon = \mathrm{id}_{B'}$.

$$B \underset{\varepsilon}{\overset{\pi}{\rightleftarrows}} B'$$

15.18. THEOREM (M. Rubin). *Every subalgebra of an interval algebra is retractive.*

Loosely speaking, retractivity of B means that every homomorphic image of B is a retract of B, as defined in 5.12. Since $\pi \circ \varepsilon = \mathrm{id}_{B'}$ implies that ε is a monomorphism, every quotient of a retractive algebra B embeds into B. In particular, the cellularity of the quotient cannot exceed the cellularity of B; this shows that uncountable free algebras and, by 5.28(c), also the power set algebra $P(\omega)$ are not retractive.

In proving retractivity of an algebra B, it is of course suficient to consider canonical epimorphisms $\pi : B \to B/I$, where I is an ideal of B. A few definitions and lemmas are useful to do this in a systematic way. The first lemma is important in its own right. Its second part gives, via Zorn's lemma, another proof for the consequence 5.10 of Sikorski's extension theorem: for E a subalgebra of B, there is a quotient of B having (an isomorphic copy of) E as a dense subalgebra.

15.19. DEFINITION. Let E be a subalgebra of a Boolean algebra B and I an ideal of B. I is *disjoint from* E if $I \cap E = \{0\}$; it is *maximally disjoint from* E if it is maximal among the ideals of B disjoint from E.

15.20. LEMMA. *Let E be a subalgebra of B, I an ideal of B and $\pi : B \to B/I$ canonical.*

(a) I is disjoint from E iff π is one-to-one on E, hence an isomorphism from E onto $\pi[E]$.

(b) I is maximally disjoint from E iff π is one-to-one on E and $\pi[E]$ is a dense subalgebra of B/I.

$$\begin{array}{ccc} E & \subseteq & B \\ {\scriptstyle \pi \restriction E}\downarrow & & \downarrow{\scriptstyle \pi} \\ \pi[E] & \subseteq & B/I \end{array}$$

PROOF. (a) is trivial: $\pi \restriction E$ is a monomorphism iff its kernel $\ker(\pi \restriction E)$ is the trivial ideal $\{0\}$. But $\ker(\pi \restriction E) = I \cap E$.

(b) Assume first that I is maximally disjoint from E and let $b \in B$ satisfy $0 < \pi(b)$; we find $e \in E$ such that $0 < \pi(e) \leq \pi(b)$. By maximality of I and $b \not\in I$, the ideal of B generated by $I \cup \{b\}$ is not disjoint from E. So there are $e \in E$ and $i \in I$ such that $0 < e \leq i + b$, and it follows that $0 < \pi(e) \leq \pi(b)$.

Conversely, assume that I is disjoint from E and that $\pi[E]$ is dense in B/I. Let $b \in B \setminus I$; we show that the ideal J of B generated by $I \cup \{b\}$ is not disjoint from E. Since $0 < \pi(b)$, pick $e \in E$ such that $0 < \pi(e) \leq \pi(b)$. Then $i = e \cdot -b$ is in I, $e \leq i + b$ and thus $e \in (J \cap E)\setminus\{0\}$. □

15.21. LEMMA. *Let I be an ideal of B and π: $B \to B/I$ canonical. The following are equivalent*:

(a) *there is a monomorphism ε: $B/I \to B$ satisfying $\pi \circ \varepsilon = \mathrm{id}_{B/I}$,*

(b) *there is a subalgebra E of B such that I is disjoint from E and $\pi[E] = B/I$ (i.e. E is a subalgebra having exactly one element in common with each equivalence class of B under I).*

PROOF. If ε is a monomorphism as described in (a), then the range E of ε proves (b). Conversely, if E is a subalgebra of B as described in (b), then by 15.20(a), $\tau = \pi \restriction E$ is an isomorphism from E onto B/I and the monomorphism $\varepsilon = \tau^{-1}$ from B/I into B proves (a). □

Proof of Theorem 15.18. Suppose that $A = \mathrm{Intalg}\, L$, B is a subalgebra of A and I an ideal of B with canonical map π: $B \to B/I$. By the preceding lemmas, we have to find a subalgebra E of B such that I is disjoint from E and $\pi[E] = B/I$.

Every subset M of L defines a subalgebra $\{a \in A: \mathrm{rel}(a) \subseteq M\}$ of A; the set $\mathrm{rel}(a)$ of relevant points of a was defined in 15.13. Thus,

$$B_M = \{a \in A: \mathrm{rel}(a) \subseteq M\} \cap B$$

is a subalgebra of B. Choose $M \subseteq L$ maximal with respect to the property that I is disjoint from B_M and put $E = B_M$; we show that $\pi[E] = B/I$.

To this end, define for b in B an integer $n(b)$ by

$$n(b) = |\text{rel}(b) \setminus M| ;$$

we prove by induction on $n(b)$ that $\pi(b) \in \pi[E]$.

If $n(b) = 0$, then $\text{rel}(b) \subseteq M$, $b \in B_M = E$ and we are finished. If $n(b) > 0$, we construct b' in B such that $n(b') < n(b)$ and $\pi(b) = \pi(b')$ which, by induction, concludes the proof. Let b have standard representation

$$b = [x_1, y_1) \cup \cdots \cup [x_n, y_n) .$$

Since $n(b) > 0$, there is some $z \in \text{rel}(b) \setminus M$. We may assume that $z = x_k$ for some $k \in \{1, \ldots, n\}$. For otherwise, we pass from b to $-b$; then if $b' \in B$ is such that $\pi(b') = \pi(-b)$ and $n(b') < n(-b)$, we obtain $\pi(b) = -\pi(b')$ and (since $\text{rel}(a) = \text{rel}(-a)$ for all $a \in A$) $n(-b') = n(b') < n(-b) = n(b)$. Now $z \notin M$ and M was maximal, hence I is not disjoint from $B_{M \cup \{z\}}$. So pick $d \in I \setminus \{0\}$ such that $\text{rel}(d) \subseteq M \cup \{z\} = M \cup \{x_k\}$. By disjointness of I from B_M, $\text{rel}(d)$ is not included in M and thus $x_k \in \text{rel}(d)$. Write d in standard representation

$$d = [s_1, t_1) \cup \cdots \cup [s_m, t_m) .$$

If $x_k = s_l$ for some l, then $b' = b \cdot -d$ is in B and satisfies $\pi(b') = \pi(b)$. But x_k is a relevant left endpoint of one of the intervals constituting b and the same holds for x_k and d, so $x_k \notin \text{rel}(b')$. It follows that $\text{rel}(b') \setminus M \subseteq \text{rel}(b) \setminus (M \cup \{x_k\})$ and $n(b') < n(b)$.

If $x_k = t_l$ for some l, let $b' = b + d$. As above, b' is an element of B satisfying $\pi(b') = \pi(b)$ and $x_k \notin \text{rel}(b')$; so $n(b') < n(b)$. □

15.4. Chains and antichains in subalgebras of interval algebras

We prove a combinatorial property of subalgebras of interval algebras which looks quite natural but does not hold in every Boolean algebra.

In a Boolean algebra, and likewise in an arbitrary partial order, a subset X is called a chain if every two elements of X are comparable; in Definition 4.24, X is said to be a pairwise incomparable family if every two distinct elements of X are incomparable. For brevity, let us call here a pairwise incomparable family in a partially ordered set an antichain (warning: in part of the literature, an antichain is a family consisting of pairwise *incompatible* elements). It is a non-trivial question whether in certain types of uncountable partial orders there exist uncountable chains or antichains. For example, a normal tree of height ω_1 is Souslin iff it has no uncountable antichain; as remarked in Section 14, such a tree has no uncountable chain either.

In SHELAH [1981], a Boolean algebra of cardinality ω_1 is constructed under CH which has no uncountable chain or antichain. On the other hand, BAUMGARTNER [1980] gives a model of set theory in which Martin's axiom plus $2^\omega = \omega_2$ holds and every uncountable Boolean algebra has an uncountable chain or antichain.

15.22. THEOREM (M. Rubin). *Let κ be a regular uncountable cardinal and B a subalgebra of an interval algebra such that $|B| = \kappa$. Then B has a chain or an antichain of size κ.*

PROOF. Let B be a subalgebra of the interval algebra $A = \text{Intalg}\, L$ and assume that $|B| = \kappa$ but B has no antichain of size κ; we will eventually find a chain of size κ in B.

We may assume, in the rest of the proof, that B is dense in A. To this end, we apply Corollary 5.10 to obtain an epimorphism $h: A \twoheadrightarrow A'$, where A' has B as a dense subalgebra and note that A' is isomorphic to an interval algebra, by Proposition 15.9.

Since B has no antichain of size κ, Shelah's theorem 4.25 implies that B, and hence A, has a dense subset E of size less than κ. Consequently, if M is any subset of A of cardinality κ, then by denseness of E, $|E| < \kappa$ and regularity of κ, there are a subset M' of M and some $e \in E$ such that $|M'| = \kappa$ and $e \leq m$ for all $m \in M'$.

Pick any subset M of B having cardinality κ; we shall construct a chain of size κ from M. Passing, if necessary, to suitable subsets of M, we may assume that M has several homogeneity properties: without loss of generality, assume that $l(a) = n$ for every $a \in M$, say

$$\text{Int}(a) = \{u_{a1}, \ldots, u_{an}\},$$

where $u_{a1} \ll \cdots \ll u_{an}$, in the notation of 15.1, and

$$u_{ai} = [x_{ai}, y_{ai}).$$

Without loss of generality, assume that $0_L \in a$ for each $a \in M$, i.e. that $x_{a1} = 0_L$ – otherwise, we may assume that $0_L \not\in a$ for each $a \in M$ and pass to the set $\{-a: a \in M\}$ of complements. Without loss of generality, $l(-a) = m$ for every $a \in M$, say

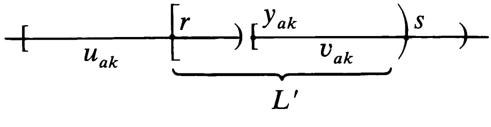
$$\text{Int}(-a) = \{v_{a1}, \ldots, v_{am}\},$$

where $v_{a1} \ll \cdots \ll v_{am}$.

Since M has cardinality κ, there is without loss of generality some $k \in \{1, \ldots, n\}$ such that the points y_{ak}, $a \in M$, are pairwise distince – otherwise, without loss of generality, the points x_{ak}, $a \in M$, are pairwise distinct for some k and an argument similar to that given below applies. Finally, the above remarks on E show that, without loss of generality, there are d and e in E such that

$$d \leq u_{ak}, \qquad e \leq v_{ak}$$

for every $a \in M$. Fix two points $r \in d$ and $s \in e$ and let L' be the subset $[r, s)$ of L.



We now use retractivity of B, proved in Rubin's theorem 15.18, to find a chain of size κ in B: consider the epimorphism

$$p: A = \text{Intalg } L \to \text{Intalg } L', \qquad p(a) = a \cap L'$$

and let $B' = p[B]$, the image of B under p. The set $C' = p[M]$ is a chain of size κ in B' since $C' = \{[r, y_{ak}): a \in M\}$. By retractivity of B, there is a monomorphism $\varepsilon: B' \to B$ satisfying $p \circ \varepsilon = \text{id}_{B'}$. So $\varepsilon[C']$ is a chain of size κ in B. □

Exercises

1. (a) The weak product $\Pi^w_{i \in I} A_i$ (cf. 6.2) of infinitely many non-trivial Boolean algebras is isomorphic to an interval algebra iff I is countable and each A_i is isomorphic to an interval algebra.

(b) If each A_i is isomorphic to an interval algebra, then $\Pi^w_{i \in I} A_i$ is embeddable into an interval algebra, regardless of the cardinality of I.

2. Give a direct proof that the finite–cofinite algebra over an arbitrary infinite set is retractive.

3. Using the description 15.9 of quotients of interval algebras, prove directly that every interval algebra is retractive.

4. Show that, in contrast to Theorem 15.22, an uncountable subset of an interval algebra does not necessarily have an uncountable subset which is either a chain or an antichain.

Hint. Let $L = \omega_1 + \boldsymbol{R}$ be the sum of the linear orders ω_1 and $\boldsymbol{R}$, as defined in the proof of Proposition 15.11; let a_α, $\alpha < \omega_1$, be pairwise distinct real numbers and consider the subset

$$X = \{[\alpha, a_\alpha): \alpha < \omega_1\}$$

of Intalg L.

5. Show that a quotient algebra of the interval algebra of the reals is either finite or countable or has cardinality 2^ω.

6. Suppose U is a finite non-empty independent subset of an interval algebra, $|U| = n$, and each $u \in U$ has length k (i.e. u is the union of exactly k disjoint intervals). Show that $2^{n-1}/n \leq k$. Deduce Corollary 15.15 from this.

7. For every Boolean algebra A, the set Sub(A) of all subalgebras of A is a lattice under set-theoretic inclusion with A (respectively 2) as its greatest (respectively least) elements; for $B, C \in \text{Sub}(A)$, $B \cap C$ is the greatest lower bound and $\langle B \cup C \rangle$ the least upper bound of B and C in Sub(A). C is a *complement* of B in Sub(A) if $B \cap C = 2$ and $\langle B \cup C \rangle = A$, and Sub($A$) is *complemented* if each element of Sub(A) has a complement.

(a) If Sub(A) is complemented, then A is retractive.

(b) Imitate the proof of Rubin's theorem 15.18 to show that Sub(A) is complemented if A is a subalgebra of an interval algebra.

16. Tree algebras

In this section we describe a simple method of associating a Boolean algebra, the so-called tree algebra, with every tree. Tree algebras sometimes provide easy examples of Boolean algebras with paradoxical properties which are otherwise difficult to obtain; e.g. we construct in Theorem 16.14 a rigid tree algebra, i.e. an algebra without any non-trivial automorphisms, and Exercise 6 gives an example of a tree algebra which is weakly homogeneous but has no homogeneous factor. There is a close connection between tree algebras and interval algebras inasmuch as every tree algebra is embeddable into an interval algebra; hence strong properties like retractivity, proved for subalgebras of interval algebras in the preceding section, are inherited by tree algebras.

Our first two subsections contain a detailed discussion of normal forms, several basic technical results on tree algebras, and the proof that tree algebras embed into interval algebras. We then concentrate on the construction of rigid tree algebras and investigate the closure properties of the class of tree algebras.

The results of this section, except Proposition 16.20, are due to G. Brenner and contained in his thesis BRENNER [1982]; see also BRENNER [1983] and BRENNER and MONK [1983]. The difficult original proof of Proposition 16.18 has been replaced here by a relatively simple one.

16.1. Normal forms

The tree algebra B_T associated with a tree $(T, \leq)$ is generated by a set $\{b_t: t \in T\}$ of canonical generators. By the normal form theorem 4.4, the elements of B_T are representable in additive normal form over the canonical generators. The partial order on T imposes several non-trivial relations on the generators; e.g. any two canonical generators are either comparable or disjoint. We can therefore develop a more special normal form for the elements of B_T which will be permanently used in results on tree algebras. Unfortunately, this normal form is more difficult to describe and to handle than the standard representation of elements of interval algebras.

Under additional restrictions on the normal form, we are able to prove its uniqueness. The reader is advised to postpone this second normal form lemma for some time since the proof is quite tedious and the result will not be applied until Proposition 16.18.

Concerning trees, we use the notation of Section 14; in particular, the αth level of a tree is denoted by U_α. We will, however, not distinguish between the partial order $\leq_T$ of a tree T and the Boolean partial order $\leq_B$ of any Boolean algebra occurring. The minimal elements, i.e. the elements in the lowest level U_0 of T, are called the *roots* of T.

16.1. DEFINITION. Let $(T, \leq)$ be a tree. For $t \in T$, put

$$b_t = \{s \in T: t \leq s\} ;$$

we write b_t^T for b_t if necessary, referring to the particular tree T in which b_t is formed. The *tree algebra* B_T of T is the algebra of sets over T generated by $\{b_t: t \in T\}$. The sets b_t, $t \in T$, are the *canonical generators* of B_T.

There are two trivial special cases of tree algebras. Each well-ordered set $(T, \leq)$ is a tree, its tree algebra being the interval algebra of the linear order

$(T, \leq)$. The normal form exhibited in the first normal form lemma below then coincides with the standard representation of elements of the interval algebra. On the other hand, if $(T, \leq)$ is a tree with height 1 (i.e. such that every element of T is a root of T), then its tree algebra is simply the finite–cofinite algebra on the set T.

The following fact on canonical generators will be used throughout the section. Let s and t be elements of T. Then either s and t are incomparable, hence incompatible, in T and $b_s \cdot b_t = 0$ in B_T; or s and t are comparable in T, say $t \leq s$, and then $b_s \leq b_t$, $b_s \cdot b_t = b_s > 0$ and $-b_s \cdot -b_t = -b_t$ in B_T.

Some notation and a careful formulation of the first normal form lemma make most of its statements obvious.

16.2. NOTATION. We define two sets of elementary products over the canonical generators by

$$E = \left\{ b_t \cdot - \sum_{s \in S} b_s : S \text{ a finite antichain in } T;\ t < s \text{ for } s \in S \right\},$$

$$F = \left\{ - \sum_{s \in S} b_s : S \text{ a finite antichain in } T \right\}.$$

The hatched area in Fig. 6.1 represents a typical element of E.

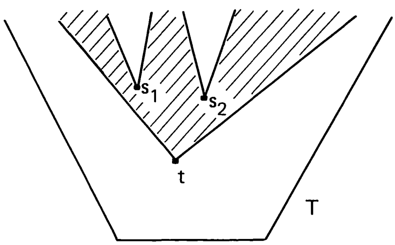


Fig. 6.1

16.3. LEMMA (first normal form lemma). (a) *The elements of E are non-zero.*

(b) *If T has a single root, then every non-zero element of F is in E.*

(c) *Let*

(1) $$p = b_{t(1)} \cdot \cdots \cdot b_{t(n)} \cdot -b_{s(1)} \cdot \cdots \cdot -b_{s(m)}$$

be an elementary product of canonical generators and assume that $n + m \neq 0$. Then $p = 0$ iff one of the following holds:

(2) $n = 0$ *and* $U_0 \subseteq \{s(1), \ldots, s(m)\}$,

(3) $n \geq 2$ *and there are distinct* $i, j \leq n$ *such that* $t(i), t(j)$ *are incomparable*,

(4) $n \geq 1$, $\{t(1), \ldots, t(n)\}$ *is a chain in* T, *and* $s(i) \leq \max(t(1), \ldots, t(n))$ *for some* i.

(d) *Every non-zero elementary product of canonical generators is in* $E \cup F$.
(e) *Every element* b *of* B_T *is a sum of pairwise disjoint non-zero elements*

$$b = e_1 + \cdots + e_n + f_1 + \cdots + f_m , \tag{5}$$

where $e_i \in E$ *and* $f_j \in F$. *Moreover, if*

$$e_i = b_{t(i)} \cdot - \sum_{s \in S(i)} b_s ,$$

where $S(i)$ *is a finite antichain in* T *and* $t(i) < s$ *for all* $s \in S(i)$, *we may assume that* $t(j) \not\in S(i)$ *for distinct* $i, j \in \{1, \ldots, n\}$.

PROOF. (a) If $e \in E$ has the form $b_t \cdot - \Sigma_{s \in S} b_s$ given in the definition of E, then $t \in e$ and $e \neq 0$.

(b) If 0_T is the single root of T, then b_{0_T} is the unit element of B_T. And if $f \in F \backslash \{0\}$ has the form $-\Sigma_{s \in S} b_s$ displayed in the definition of F, then $0_T < s$ for every $s \in S$. So $f = b_{0_T} \cdot - \Sigma_{s \in S} b_s$ is in E.

(c) If $n = 0$, then clearly $p = 0$ iff $b_{s(1)} \cup \cdots \cup b_{s(m)} = T$ iff $\{s(1), \ldots, s(m)\}$ includes U_0. So let $n \geq 1$. If $\{t(1), \ldots, t(n)\}$ contains two incomparable elements, then $p = 0$. Thus assume that $\{t(1), \ldots, t(n)\}$ is a chain in T with t as its maximal element; then $p = b_t \cdot -b_{s(1)} \cdot \cdots \cdot -b_{s(m)}$. Now if $s(i) \leq t$ for some i, then $p \leq b_t \cdot -b_{s(i)} = 0$; if $s(i) \not\leq t$ for every i, then $t \in p$ and $p \neq 0$.

(d) Assume that the elementary product p is non-zero and that, in the representation (1) of p, $n + m$ is minimal. Since $p > 0$, $\{t(1), \ldots, t(n)\}$ is a chain in T which, by minimality of n, is either empty or consists of a single element t. Also, by minimality of m, $\{s(1), \ldots, s(m)\}$ is an antichain of T and if t exists, then it is comparable with each $s(i)$; in fact $t < s(i)$ since $p > 0$. So $p \in E$ if t exists and $p \in F$ otherwise.

(e) By the normal form theorem 4.4 and (d), we need only check the "moreover" part. We assume that n is chosen minimal in the representation (5) of b. Now if $i \neq j$ and $t(j) \in S(i)$, then

$$e_i + e_j = b_{t(i)} \cdot - \sum \{b_s : s \in (S(i) \cup S(j)) \backslash \{t(j)\}\}$$

gives a representation of b with only $n - 1$ terms in E, a contradiction. □

Sikorski's extension criterion 5.5 and part (c) of the preceding lemma immediately imply the following result. In its formulation, note that the map from T into B_T assigning b_t to t is one-to-one.

16.4. COROLLARY. *Let* R *be a subset of the tree* T, A *a Boolean algebra and*

$$f : \{b_t : t \in R\} \to A$$

an arbitrary mapping.

(a) *f extends to a homomorphism from* $\langle b_t : t \in R\rangle$ *into A iff each of the following hold*:

(6) *if* $U_0 \subseteq \{s(1), \ldots, s(m)\} \subseteq R$, *then* $f(b_{s(1)}) + \cdots + f(b_{s(m)}) = 1$,

(7) *if* $t, t' \in R$ *are incomparable, then* $f(b_t) \cdot f(b_{t'}) = 0$,

(8) *if* $t \leq t'$ *in R, then* $f(b_{t'}) \leq f(b_t)$.

(b) *f extends to a monomorphism from* $\langle b_t : t \in R\rangle$ *into A iff* (6), (7), (8), *and each of the following hold*:

(9) *if* $\{s(1), \ldots, s(m)\} \subseteq R$ *and* $U_0 \not\subseteq \{s(1), \ldots, s(m)\}$, *then* $f(b_{s(1)}) + \cdots + f(b_{s(m)}) \neq 1$,

(10) *if* $t < s(1), \ldots, s(m)$ *in R, then* $f(b_{s(1)}) + \cdots + f(b_{s(m)}) < f(b_t)$,

(10′) $f(b_t) > 0$ *for every t in R.* □

The conditions stated in 16.4 are somewhat redundant: (10′) is the special case of (10) for $m = 0$; on the other hand, (10) follows from (7), (8), and (10′) if for every t in the αth level U_α of T there are infinitely many $x \in U_{\alpha+1}$ satisfying $t < x$. (6) is superfluous if T has infinitely many roots.

It is often convenient to work with a tree T having a single root – see 16.3(b), for example. In particular, normal forms are more easily described and even turn out to be unique in this case. We will see in 16.7 that every tree algebra is generated by a tree with a single root.

16.5. NOTATION. Let $e \in E$ and $b \in B_T$. A representation of e:

(11) $$e = b_t \cdot -\sum_{s \in S} b_s$$

is in *normal form* if S is a finite antichain in T and $t < s$ for all $s \in S$. A representation of b:

(12) $$b = e_1 + \cdots + e_n$$

is in *normal form* if the e_i are pairwise disjoint elements of E, say $e_i = b_{t(i)} \cdot -\Sigma_{s \in S(i)} b_s$ in normal form, and $t(j) \not\in S(i)$ for $i \neq j$.

By parts (e) and (b) of the first normal form lemma, every e in E and also, if T has a single root, every b in B_T can be written in normal form.

16.6. LEMMA (second normal form lemma). (a) *Let*

$$e = b_t \cdot -\sum_{s \in S} b_s, \qquad \varepsilon = b_\tau \cdot -\sum_{\sigma \in \Sigma} b_\sigma$$

be elements of E written in normal form. Then $\varepsilon \leq e$ *iff each of the following holds*:

(13) $t \leq \tau$,

(14) $\quad s \not\leq \tau$ *for* $s \in S$,

(15) $\quad$ *if* $s \in S$ *and* $\tau \leq s$, *then* $\sigma \leq s$ *for some* $\sigma \in \Sigma$.

(b) *Let* $b = e_1 + \cdots + e_n$ *be a normal form of* $b \in B_T$ *and let* $\varepsilon \in E$. *Then* $\varepsilon \leq b$ *iff* $\varepsilon \leq e_i$ *for some* i.

(c) *The normal forms* (11) *for elements of* E *and* (12) *for elements of* B_T *are uniquely determined.*

(d) *Assume* $b = e_1 + \cdots + e_n$ *and* $\beta = \varepsilon_1 + \cdots + \varepsilon_m$, *written in normal form, are comparable* (*i.e.* $b \leq \beta$ *or* $\beta \leq b$) *in* B_T *and* $\varepsilon_i < e_j$ *for some pair* i, j. *Then* $\beta < b$.

PROOF. (a) Suppose that $\varepsilon \leq e$. Then $\tau \in \varepsilon \leq e \leq b_t$ and hence (13) holds. If (14) fails, say $s \leq \tau$, where $s \in S$, then $e \leq -b_s$ and $\varepsilon \leq b_\tau \leq b_s$ imply $e \cdot \varepsilon = 0$, contradicting $0 < \varepsilon \leq e$. To prove (15), assume $\tau \leq s$ where $s \in S$. By $s \notin e$ and $\varepsilon \leq e$, we find that $s \notin \varepsilon$. But since $s \in b_\tau$, there is some $\sigma \in \Sigma$ such that $s \in b_\sigma$, i.e. $\sigma \leq s$.

Conversely, assume (13) through (15) hold; we have to prove that $\varepsilon \leq b_t$ and $\varepsilon \cdot b_s = 0$ for every $s \in S$. Now $\varepsilon \leq b_\tau \leq b_t$ follows from (13). Let $s \in S$. If s and τ are incomparable, then clearly $\varepsilon \cdot b_s \leq b_\tau \cdot b_s = 0$. Otherwise by (14), $\tau < s$, and by (15), $\sigma \leq s$ for some $\sigma \in \Sigma$. Then $b_s \leq b_\sigma \leq -\varepsilon$ and $\varepsilon \cdot b_s = 0$.

(b) For $x \leq y$ in T, let

$$[x, y] = \{z \in T: x \leq z \leq y\}$$

be the segment from x to y. Note that each e in E is a convex subset of T in the sense that, if $x \leq y$ in e, then $[x, y]$ is included in e.

For the non-trivial direction of (b), assume $\varepsilon \leq e_1 + \cdots + e_n$ and write

$$\varepsilon = b_\tau \cdot - \sum_{\sigma \in \Sigma} b_\sigma , \qquad e_i = b_{t(i)} \cdot - \sum_{s \in S(i)} b_s$$

in normal form. Let i be the unique element of $\{1, \ldots, n\}$ such that $\tau \in e_i$ with the aim of proving that $\varepsilon \leq e_i$.

If $\varepsilon \not\leq e_i$, fix an element w in $\varepsilon \setminus e_i$ of minimal height. Thus, $\tau \leq w$ and, for any x in T,

(16) $\quad$ if $x \in [\tau, w]$ and $x \notin e_i$, then $x = w$.

Now $w \in \varepsilon \leq e_1 + \cdots + e_n$, so $w \in e_j$ for a unique $j \neq i$. Thus, $t(j)$, the least point of e_j, satisfies $t(j) \leq w$. It follows from τ, $t(j) \leq w$ that $t(j) \leq \tau$ or $\tau \leq t(j)$; we will obtain a contradiction in both cases.

If $t(j) \leq \tau$, then $\tau \in [t(j), w]$. By $t(j)$, $w \in e_j$ and convexity of e_j, we have $\tau \in e_j$, contradicting $\tau \in e_i$.

Now assume that $\tau \leq t(j)$; we shall reach a contradiction to the normal form assumption on $b = e_1 + \cdots + e_n$ by proving that $t(j) = w = s$ for some $s \in S(i)$. By $t(j) \in [\tau, w]$, $t(j) \notin e_i$ and (16), we get $t(j) = w$. Since $t(i) \leq \tau \leq w$ but $w \notin e_i$, there is some s in $S(i)$ such that $s \leq w$. We claim that $\tau \leq s$: otherwise, by

$\tau, s \leq w$, we have $s < \tau$ and $\tau \not\leq e_i$; a contradiction. Thus, $s \in [\tau, w]$ but $s \not\leq e_i$, and (16) implies that $s = w$.

(c) If $b_t \cdot - \Sigma_{s \in S} b_s$ and $b_\tau \cdot - \Sigma_{\sigma \in \Sigma} b_\sigma$ are both normal forms of $e \in E$, then (a) implies that $t = \tau$, for each $s \in S$ there is some $\sigma \in \Sigma$ such that $\sigma \leq s$, and vice versa. Since both S and Σ are antichains of T, we find that $S = \Sigma$. Likewise, if $e_1 + \cdots + e_n$ and $\varepsilon_1 + \cdots + \varepsilon_m$ are both normal forms of $b \in B_T$, then it follows from (b) that for each i there is some j such that $\varepsilon_i \leq e_j$, and vice versa. So $\{e_1, \ldots, e_n\} = \{\varepsilon_1, \ldots, \varepsilon_m\}$, since both sets consist of pairwise disjoint elements.

(d) Assume for contradiction that $b \leq \beta$. By (b), there is some $i' \in \{1, \ldots, m\}$ such that $e_j \leq \varepsilon_{i'}$. It follows that $\varepsilon_i < e_j \leq \varepsilon_{i'}$ and thus that $i \neq i'$. But this contradicts disjointness of the ε_k and $\varepsilon_i > 0$. □

16.2. Basic facts on tree algebras

We prove several facts on tree algebras which are easily obtained from the first normal form lemma and the criterion 16.4 on construction of homomorphisms. The first three of these, in particular Proposition 16.7, are technically important. The main result, from a structural point of view, is Theorem 16.12: every tree algebra is embeddable into an interval algebra.

16.7. PROPOSITION. *For every tree T there is a tree T^* with a single root such that B_T is isomorphic to B_{T^*}.*

PROOF. Consider first the case that T has finitely many roots, say x, $t(1), \ldots, t(k)$. Let then T^* have the same underlying set as T and the partial ordering defined by

$$s \leq_{T^*} t \quad \text{iff } s \leq_T t \text{ or } s = x$$

for s and t in T^*; so x is the single root of $(T^*, \leq_{T^*})$. We apply Corollary 16.4 to T^* to get a monomorphism $g: B_{T^*} \to B_T$: let $R = T^* \setminus \{x\}$ and $f(b_t^{T^*}) = b_t^T$ for $t \in R$. Now $b_x^{T^*} = 1$ and $b_x^T = -(b_{t(1)}^T + \cdots + b_{t(k)}^T)$, thus $\{b_t^{T^*}: t \in R\}$ generates B_{T^*}, $\{b_t^T: t \in R\}$ generates B_T and g is an isomorphism.

If T has infinitely many roots, let T^* have underlying set $T \cup \{x\}$, where $x \not\in T$ and define, for s and t in T^*,

$$s \leq_{T^*} t \quad \text{iff } (s, t \in T \text{ and } s \leq_T t) \text{ or } s = x\,;$$

again x is the single root of T^*. As in the first case, there is an isomorphism from B_{T^*} onto B_T mapping $b_t^{T^*}$ onto b_t^T for each $t \in T^* \setminus \{x\}$. □

Every subset T' of a tree T is again a tree with the partial order inherited from T; a *subtree* of T. In particular for t in T, we can consider the subtree $T \restriction t$ of T defined by

$$T \restriction t = b_t = \{s \in T: t \leq s\}\,.$$

16.8. LEMMA. *For every point t of a tree T, the tree algebra $B_{T \upharpoonright t}$ is isomorphic to the relative algebra $B_T \upharpoonright b_t$ of B_T.*

PROOF. In the subtree $T \upharpoonright t$ of T, let $R = T \upharpoonright t$ and, for s in R,

$$f(b_s^{T \upharpoonright t}) = b_s^T ;$$

of course $b_s^{T \upharpoonright t} = b_s^T$ for $s \in R$. For every s in R, $b_s^T \leq b_t^T$ since $t \leq s$; thus f maps the canonical generators of $B_{T \upharpoonright t}$ into the relative algebra $B_T \upharpoonright b_t^T$. By 16.4, f extends to a monomorphism from $B_{T \upharpoonright t}$ into $B_T \upharpoonright b_t^T$ which is easily checked to be an isomorphism. □

16.9. PROPOSITION. *If T' is a subtree of T, then $B_{T'}$ embeds into B_T.*

PROOF. We consider two cases.

Case 1. All roots of T are in T', or T' has infinitely many roots. Then 16.4 guarantees the existence of a monomorphism $g: B_{T'} \to B_T$ satisfying $g(b_t^{T'}) = b_t^T$ for t in T'.

Case 2. T' has finitely many roots $t(1), \ldots, t(n)$. Then $T'(i) = T' \upharpoonright t(i)$ is a subtree of $T(i) = T \upharpoonright t(i)$ and $T(i)$, $T'(i)$ both have $t(i)$ as their single root. By the preceding lemma and Case 1,

$$\begin{aligned} B_{T'} &\cong (B_{T'} \upharpoonright b_{t(1)}^{T'}) \times \cdots \times (B_{T'} \upharpoonright b_{t(n)}^{T'}) \\ &\cong B_{T'(1)} \times \cdots \times B_{T'(n)} \end{aligned}$$

is embeddable into

$$\begin{aligned} B_{T(1)} \times \cdots \times B_{T(n)} &\cong (B_T \upharpoonright b_{t(1)}^T) \times \cdots \times (B_T \upharpoonright b_{t(n)}^T) \\ &\cong B_T \upharpoonright (b_{t(1)}^T + \cdots + b_{t(n)}^T) , \end{aligned}$$

a relative algebra of B_T. The proposition follows by noting that for any non-zero element b of a Boolean algebra B, $B \upharpoonright b$ embeds into B. To see this, assume without loss of generality that $b \neq 1$, let p be an ultrafilter of $B \upharpoonright b$ and consider the monomorphisms (respectively inclusions)

$$f: B \upharpoonright b \to (B \upharpoonright b) \times 2 \subseteq (B \upharpoonright b) \times (B \upharpoonright -b) \cong B ,$$

where $f(x) = (x, 1)$ for $x \in p$ and $f(x) = (x, 0)$ for $x \not\in p$. □

We can use the first normal form lemma to describe the atoms, and Corollary 16.4 to describe the ultrafilters of a tree algebra.

For each element x of a tree T, say $x \in U_\alpha$, an (*immediate*) *successor* of x is an element y of $U_{\alpha+1}$ satisfying $x < y$. We write

$$\text{succ}(x) = \{ y \in T : y \text{ an immediate successor of } x \} .$$

16.10. REMARK. If $t \in T$ has only finitely many successors $t(1), \ldots, t(n)$, then

$$b_t \cdot -(b_{t(1)} + \cdots + b_{t(n)}) = \{t\}$$

is an atom of B_T; by (b) and (e) in 16.3, every atom arises in this way. Thus, B_T is atomless iff every element of T has infinitely many successors. It is easily checked that, in this case, the canonical generators are dense in B_T.

Call a subset C of T an *initial chain* of T if C is a chain under the tree ordering, $t \in C$ and $s \leq t$ in T imply $s \in C$, and C is non-empty if T has only finitely many roots.

16.11. PROPOSITION. *There is a canonical bijection between ultrafilters of B_T and initial chains of T, given by*

$$\phi(p) = \{t \in T: b_t \in p\}$$

for an ultrafilter p of B_T.

PROOF. For any ultrafilter p of B_T, $\phi(p)$ is a chain in T since p has the finite intersection property; also $t \in \phi(p)$ and $s \leq t$ imply $b_t \leq b_s$, hence $s \in \phi(p)$. If T has finitely many roots $t(1), \ldots, t(n)$, then $1 = b_{t(1)} + \cdots + b_{t(n)}$ and thus $b_{t(i)} \in p$ for some i. Thus, ϕ is really a map from the set of ultrafilters of B_T into the set of initial chains of T. It is one-to-one since an ultrafilter of B_T is uniquely determined by its intersection with the set of canonical generators. Finally, ϕ is onto since for any initial chain C, there is, by 16.4, a homomorphism $f: B_T \to 2$ such that, for t in T, $f(b_t) = 1$ iff $t \in C$; thus the set $\{b \in B_T: f(b) = 1\}$ is an ultrafilter of B_T satisfying $\phi(p) = C$. □

In view of Proposition 16.9, the proof of the following theorem will be simplified by using a standard example of trees. For any ordinals α and λ, the set

$$T_{\alpha\lambda} = \bigcup_{\nu<\lambda} {}^{\nu}\alpha$$

consisting of all sequences in α with length less than λ, is a tree of height λ, under inclusion. It is not difficult to prove that every tree is isomorphic to a subtree of $T_{\alpha\lambda}$, for α and λ large enough.

16.12. THEOREM. *Every tree algebra embeds into an interval algebra. In particular, every tree algebra is retractive.*

PROOF. By 16.9 and the preceding remark, it suffices to prove the theorem for a tree of the form $T_{\alpha\lambda}$. Also since $T_{\alpha\lambda}$ is a subtree of $T_{\alpha'\lambda'}$ for $\alpha \leq \alpha'$ and $\lambda \leq \lambda'$, we may assume that λ is a limit ordinal and $\alpha = \beta + 1$ is an infinite successor ordinal.

We construct a linear order L from $T = T_{\alpha\lambda}$. The set

$$L^* = {}^{\lambda}\alpha$$

is totally ordered, by the lexicographic order, as follows: for $x \neq y$ in L^*, let ν be the least ordinal such that $x(\nu) \neq y(\nu)$; then let $x < y$ iff $x(\nu) < y(\nu)$. For each t in T, say $t: \nu \rightarrow \alpha$, the set $\{x \in L^*: t \subseteq x\}$ is convex with respect to the lexicographic order; it has a least element $x_t = t \cup ((\lambda \backslash \nu) \times \{0\})$ and a greatest element $y_t = t \cup ((\lambda \backslash \nu) \times \{\beta\})$. In particular for t the unique root of T, i.e. $t = \emptyset$, $x_\emptyset$ and $y_\emptyset$ are the least (respectively the greatest) elements of L^*. Let

$$L = L^* \backslash \{y_\emptyset\}.$$

By Corollary 16.4, there is a unique monomorphism g from B_T into Intalg L such that $g(b_t) = [x_t, y_t)$ for $t \in T$.

Retractivity of B_T follows from Rubin's theorem 15.18. □

As a consequence of Theorems 16.12 and 15.16, the class of tree algebras is quite narrow. For example, no infinite Boolean algebra satisfying the countable separation property and no uncountable free algebra can be embedded into a tree algebra.

16.3. A construction of rigid Boolean algebras

Tree algebras allow a particularly simple construction for Boolean algebras with an extraordinary property, rigidity.

16.13. DEFINITION. A Boolean algebra B is *rigid* if the identity map id_B is the only automorphism of B.

16.14. THEOREM. *Let T be a tree of height ω with a single root such that the cardinals $|\mathrm{succ}(t)|$, $t \in T$, are infinite, regular, and pairwise distinct. Then the algebra B_T is rigid.*

It can be proved that for every uncountable cardinal κ, there is a rigid Boolean algebra of size κ; cf. VAN DOUWEN, MONK and RUBIN [1980] for a brief survey of rigid algebras. No countable Boolean algebra with at least four elements is rigid (see Exercise 4), and as a rule, rigid Boolean algebras are somewhat difficult to obtain. The construction in Theorem 16.14 is a comparatively simple one, at the expense of giving an algebra with high cardinality.

16.15. PROPOSITION. *A Boolean algebra B is rigid iff there are no non-zero disjoint elements in B with isomorphic relative algebras.*

PROOF. Assume first that a and b are non-zero and disjoint in B and that $f: B \upharpoonright a \rightarrow B \upharpoonright b$ is an isomorphism. Then $g: B \rightarrow B$ defined by

$$g(x) = f(x \cdot a) + f^{-1}(x \cdot b) + x \cdot -(a + b)$$

is an automorphism of B mapping a onto b; so B is not rigid.

Conversely, let g be a non-trivial automorphism of B and pick $x \in B$ such that $x \neq g(x)$. If $x \not\leq g(x)$, then $a = x \cdot -g(x)$ and $b = g(a) = g(x) \cdot -g^2(x)$ are disjoint and non-zero, and $B \restriction a$ is isomorphic to $B \restriction b$ via the restriction of g to $B \restriction a$. If $g(x) \not\leq x$, then similar reasoning applies to $a = x \cdot g^{-1}(-x)$ and $b = g(a) = g(x) \cdot -x$. □

16.16. LEMMA. *Assume κ is a regular uncountable cardinal and T a tree satisfying*

(17) *T has a single root ,*

(18) *$\text{height}(T) \leq \omega$*

(19) $|\text{succ}(t)| \neq \kappa$ *for all $t \in T$.*

Then B_T has no partition of size κ.

PROOF. Suppose that

(20) P is a partition of B_T and $|P| = \kappa$.

Since, by the first normal form lemma, each p in P is the sum of finitely many pairwise disjoint elements of E, we may assume that

(21) $P \subseteq E$.

Write each $p \in P$ in normal form:

$$p = b_{t(p)} \cdot - \sum_{s \in S(p)} b_s .$$

For $p \neq q$ in P, $t(p) \neq t(q)$ since p and q are disjoint. Thus, the set

$$H(P) = \{t(p) \colon p \in P\}$$

has cardinality κ and there is a unique $n < \omega$ such that

(22) $|H(P) \cap U_n| = \kappa$ and, for $m < n$, $|H(P) \cap U_m| < \kappa$,

where U_n is the nth level of T. Clearly, $n > 0$ since T has a single root. We will eventually reach a contradiction by finding T', P', n' satisfying (17) through (21) with T, P, n replaced by T', P', n' but $n' < n$.

If 0_T, the single root of T, is in $H(P)$, let p^* be the unique element p of P such that $t(p) = 0_T$. Now for $p \in P \setminus \{p^*\}$, $p^* \cdot p = 0$; so there is a unique $s \in S(p^*)$ such that $p \leq b_s$. Let $s^* \in S(p^*)$ be such that the set

$$P' = \{p \in P \colon p \neq p^*, p \leq b_{s^*}, t(p) \in U_n\}$$

has cardinality κ. Then the subtree $T' = T \restriction s^*$ and the subset P' of P satisfy (17) through (21), and $|H(P') \cap U_{n'}^{T'}| = \kappa$ for some $n' < n$.

So assume that $0_T \not\in H(P)$. Then $U_1 = \mathrm{succ}(0_T)$ is infinite, for otherwise, $\{0_T\}$ would be an atom of B_T disjoint from every p in P. Therefore

$$Q = \{b_t \colon t \in U_1\}$$

is another partition of B_T. By $0_T \not\in H(P)$, each $p \in P$ is included in some $b_t \in Q$; so P refines Q and $|Q| \leq |P|$. By (19) and $|P| = \kappa$, we have $|Q| < \kappa$. So there is some $t^* \in U_1$ such that the set

$$P' = \{p \in P \colon p \leq b_{t^*}, t(p) \in U_n\}$$

has size κ. Again $T' = T \restriction t^*$ and P' satisfy (17) through (21) and $|H(P') \cap U_{n'}^{T'}| = \kappa$ for some $n' < n$. □

Proof of Theorem 16.14. Let us say that a non-zero element b of B_T has a κ-partition if its relative algebra $B_T \restriction b$ has a partition of size κ. Then

(23) for κ a regular uncountable cardinal and $t \in T$, b_t has a κ-partition iff $|\mathrm{succ}(s)| = \kappa$ for some $s \geq t$,

as follows from the preceding lemma, applied to $B_{T \restriction t} \cong B_T \restriction b_t$ (cf. 16.8).

Assume for contradiction that g is a non-trivial automorphism of B_T; the proof of Proposition 16.15 gives disjoint non-zero elements a and b of B_T such that $g(a) = b$. By Remark 16.10, the canonical generators constitute a dense subset of B_T. Pick $t \in T$ such that $b_t \leq a$ and note that also $b_t \cdot g(b_t) = 0$. Pick $s \in T$ such that $b_s \leq g(b_t)$ and $\kappa = |\mathrm{succ}(s)|$ is uncountable. Then b_s has a κ-partition and so have $g(b_t)$ (since $b_s \leq g(b_t)$) and b_t (since $g \restriction (B \restriction b_t)$ is an isomorphism from $B \restriction b_t$ onto $B \restriction g(b_t)$). (23) implies that s, the unique element of T with exactly κ immediate successors, must satisfy $s \geq t$. It follows that $b_s \leq b_t$, contradicting $b_s \leq g(b_t) \leq -b_t$. □

16.4. Closure properties of tree algebras

The closure properties of tree algebras are very similar to those of interval algebras: we will show that the class of tree algebra is closed under quotients and under products of finite families, but not under subalgebras. These results should not come as a surprise since every tree algebra is embeddable into an interval algebra, as shown in Theorem 16.12. Also, by the results 15.14 and 15.16 on interval algebras, it follows that free products of tree algebras and cartesian products of infinite families of tree algebras are generally not isomorphic to tree algebras. We investigate in some detail the relationship between tree algebras and interval algebras, proving that not every tree algebra is isomorphic to an interval algebra and not every interval algebra embeds into a tree algebra.

16.17. PROPOSITION. *The product of finitely many tree algebras is isomorphic to a tree algebra.*

PROOF. Let $B_{T(1)}, \ldots, B_{T(n)}$ be tree algebras; we may assume that the trees $T(i)$ are pairwise disjoint and, by 16.7, that each $T(i)$ has a single root $r(i)$. Let T be the tree with underlying set $T(1) \cup \cdots \cup T(n)$ and with the partial ordering

$$s \leq t \quad \text{iff } s, t \in T(i) \text{ and } s \leq_{T(i)} t, \text{ for some } i .$$

Then

$$\begin{aligned} B_T &\cong B_T \restriction b^T_{r(1)} \times \cdots \times B \restriction b^T_{r(n)} \\ &\cong B_{T(1)} \times \cdots \times B_{T(n)} \quad \text{by 16.8.} \quad \square \end{aligned}$$

16.18. PROPOSITION. *Every quotient of a tree algebra is isomorphic to a tree algebra.*

PROOF. Let T be a tree; without loss of generality, T has a single root 0_T. Let I be an ideal of B_T; we have to show that B_T/I is isomorphic to a tree algebra. Now B_T is retractive, by Theorem 16.12. Our proof proceeds by almost literally repeating that of Rubin's theorem 15.18 for retractivity of subalgebras of interval algebras.

Since T has a single root, each element b of B_T is, by the second normal form lemma, uniquely representable in normal form

$$b = e_1 + \cdots + e_n ,$$

where, for $1 \leq i \leq n$, e_i has the normal form

$$e_i = b_{t(i)} \cdot - \sum_{s \in S(i)} b_s .$$

Define

$$\operatorname{rel}(b) = (\{t(1), \ldots, t(n)\} \cup S(1) \cup \cdots \cup S(n)) \setminus \{0_T\}$$

to be the set of relevant points of b. The relevant points lying in b, i.e. $t(1), \ldots, t(n)$, are called the positive points of b; the relevant points not lying in b, i.e. the members of $S(1) \cup \cdots \cup S(n)$, are the negative points of b.

For every subset M of T containing the root 0_T of T, put

$$A_M = \langle b_t \colon t \in M \rangle .$$

A_M is isomorphic to the tree algebra B_M, as shown in the proof of 16.9 (Case 1). Moreover,

$$A_M = \{b \in B_T \colon \operatorname{rel}(b) \subseteq M\} .$$

For the non-trivial part of this assertion, note that if b is generated by $\{b_t \colon t \in M\}$, then there is a finite subset M' of M with minimal cardinality such that $\{b_t \colon t \in M'\}$ generates b; by the uniqueness of normal forms, $\operatorname{rel}(b) \subseteq M'$.

Now for our ideal I in B_T, let $M \subseteq T$ be maximal with respect to the property that I is disjoint from A_M, in the sense of Definition 15.19; so $0_T \in M$. Denote by π the canonical map from B_T onto B_T/I. As in the proof of 15.18, it suffices to prove that π maps A_M onto B_T/I – i.e. that for each $b \in B_T$ there is $c \in A_M$ such that $\pi(b) = \pi(c)$. This is proved by induction on

$$n(b) = |\text{rel}(b) \backslash M| .$$

It is trivial if $n(b) = 0$. Suppose $b \in B_T$, $n(b) \geq 1$, and for each $b' \in B_T$ with $n(b') < n(b)$ there is $c' \in A_M$ satisfying $\pi(b') = \pi(c')$. Fix $t \in \text{rel}(b) \backslash M$. Without loss of generality, t is a negative point of b; otherwise pass from b to $-b$ and note that t is a negative point of $-b$. Since I is not disjoint from $A_{M \cup \{t\}}$, pick a non-zero element d of $I \cap A_{M \cup \{t\}}$. Thus t is a relevant point of d.

If t is a positive point of d, let $b' = b + d$; then $\text{rel}(b') \subseteq (\text{rel}(b) \cup M) \backslash \{t\}$, $n(b') < n(b)$ and, by induction hypothesis, $\pi(b) = \pi(b') \in \pi[A_M]$. If t is a negative point of d, then $b' = b \cdot -d$ works. □

It follows that every at most countable Boolean algebra is isomorphic to a tree algebra. For consider a tree T with a single root, height ω and ω immediate successors to each point of T. Its tree algebra is countably infinite, atomless by 16.10 and isomorphic to the free Boolean algebra on ω generators, by 5.16. The preceding proposition (together with 9.8) then gives the assertion.

Thus, every subalgebra of a countable tree algebra is a tree algebra. For every uncountable cardinal κ, however, there is a tree T of size κ and a subalgebra of B_T not isomorphic to a tree algebra; T can even be chosen to be a linear order. We show this, for $\kappa \geq \omega_2$, by a construction due to Brenner; for every κ having cofinality greater than ω (hence for $\kappa = \omega_1$), the appropriate construction has been carried out by van Douwen.

16.19. EXAMPLE. Let $\kappa \geq \omega_2$ be a cardinal; κ is a tree under its natural well-ordering with Intalg κ as its tree algebra. Then the subalgebra B of Intalg κ consisting of the countable and cocountable sets is not isomorphic to a tree algebra.

For assume that $B \cong B_T$, for some tree T with a single root. We list several properties of B (respectively B_T) without bothering to translate them to B_T (respectively B).

(24) B is atomic.

(25) For every b in B, either $B \upharpoonright b$ or $B \upharpoonright -b$ is countable.

(26) B has no well-ordered or inversely well-ordered chain of order type $\omega_1 + \omega_1$, since the elements of B are countable or co-countable.

By (26), T has no chain of order type $\omega_1 + \omega_1$ and thus height$(T) \leq \omega_1 + \omega_1$.

(27) There are subsets P and Q of B_T such that $P \cup Q$ is a partition of unity and

(i) $|P| \geq \omega_2$; $B_T \upharpoonright p$ is countable for each $p \in P$,
(ii) Q is a set of at most ω_1 atoms of B_T,
(iii) the quotient algebra of B_T modulo the ideal generated by P has cardinality at most ω_1.

To prove (27), note that $\omega_2 \leq \kappa = |B| = |B_T| = |T|$ but $|\text{height}(T)| \leq \omega_1$. Thus, there is a least ordinal α for which U_α, the αth level of T, has size at least ω_2. Let $P = \{b_t : t \in U_\alpha\}$; the second assertion in (i) follows from (25) and $|B_T \upharpoonright (-b_t)| \geq \omega_2$, for $t \in U_\alpha$. By the minimal choice of α, the subtree $T' = \bigcup_{\beta<\alpha} U_\beta$ of T has size at most ω_1. (24) gives a set Q of atoms of B_T such that $P \cup Q$ is a partition of unity. The description of the atoms of B_T given in 16.10 shows that each $q \in Q$ is a singleton $\{s\}$, where $s \in T'$; thus (ii) holds. (iii) follows since the quotient algebra under consideration is isomorphic to $B_{T'}$.

By (27), there is a partition of unity $M \cup N$ in B such that N is a set of at most ω_1 atoms, M has size at least ω_2 and $B \upharpoonright m$ is countable for $m \in M$, and, for I the ideal of B generated by M, $|B/I| \leq \omega_1$. Note that

(28) $$\bigcup (M \cup N) = \kappa ,$$

since $M \cup N$ is a partition in B and all singletons $\{x\}$, $x \in \kappa$, belong to B.

Each $m \in M$ is the union of a finite set $\text{Int}(m)$ of pairwise disjoint at most countable intervals of κ, so the set

$$\text{Int} = \bigcup_{m \in M} \text{Int}(m)$$

consists of at least ω_2 pairwise disjoint countable intervals. We enumerate the set Int by

$$\text{Int} = \{I_\nu : \nu < \rho\} ,$$

where $\rho \geq \omega_2$ is an ordinal and $\nu < \nu' < \rho$ implies that I_ν lies left of $I_{\nu'}$. For $\alpha < \omega_2$, consider the countable subset

$$A_\alpha = \bigcup \{I_\nu : \omega \cdot \alpha \leq \nu < \omega \cdot (\alpha + 1)\}$$

of κ ($\cdot$ denotes ordinal multiplication) and let $[x_\alpha, y_\alpha)$ be the least half-open interval of κ including A_α.

Each element of N is a singleton. Thus, $|\bigcup N| \leq \omega_1$ and there is a subset J of ω_2 such that $|J| = \omega_2$ and $[x_\alpha, y_\alpha)$ is disjoint from $\bigcup N$, for $\alpha \in J$. (28) shows that $A_\alpha = [x_\alpha, y_\alpha)$ for $\alpha \in J$; in particular, $A_\alpha \in B$ for $\alpha \in J$.

We reach a final contradiction by showing that $|B/I| \geq \omega_2$ (I the ideal generated by M). For let $\alpha \neq \beta$ in J. Then A_α and A_β are disjoint elements of B, their symmetric difference is the union $A_\alpha \cup A_\beta$, and it is not covered by finitely many elements of M since each A_α intersects infinitely many elements of M. Thus, A_α and A_β are not congruent with respect to I.

Our further comparison of tree algebras with interval algebras depends on a non-trivial property of chains in tree algebras.

16.20. PROPOSITION (Brenner, Monk). *Let κ be a regular uncountable cardinal and C a chain of cardinality κ in a tree algebra B_T. Then C has a subchain of size κ which is either well-ordered or inversely well-ordered, and T has a chain of size κ.*

PROOF. We first handle three special cases, the most general form of our assertion being proved in Case 4. Without loss of generality, assume that the zero-element $0=\emptyset$ of B_T is not contained in C.

Case 1. Each c in C has the form b_t, for some $t \in T$. Then the subset $\{t \in T\colon b_t \in C\}$ of T is well-ordered and of size κ, and C is inversely well-ordered.

Case 2. Each c in C has the form

$$c = \sum_{s \in S(c)} b_s ,$$

where $S(c)$ is a finite antichain in T. There is a subset C' of C of size κ and some $n \in \omega$ such that $|S(c)| = n$ for $c \in C'$; clearly $n \geq 1$ since $0 \notin C$. We then prove the proposition by induction on n: $n=1$ is dealt with in Case 1. For $n>1$, proceed as follows. Since C' is a chain not containing 0, choose an ultrafilter p of B_T such that $C' \subseteq p$. For each c in C', let $s(c)$ be the unique element of $S(c)$ such that $b_{s(c)} \in p$. The set $\{b_{s(c)}\colon c \in C'\}$ consists of pairwise non-disjoint elements; so it is a chain in B_T and $\{s(c)\colon c \in C'\}$ is a chain in T. Now if there is some $C'' \subseteq C'$ of size κ such that $s(c) = s(c')$ for $c, c' \in C''$, then by $|S(c) \setminus \{s(c)\}| = n-1$ and the induction hypothesis, the chain

$$\left\{\sum \{b_s\colon s \in S(c)\setminus\{s(c)\}\}\colon c \in C''\right\}$$

and hence C'' have subchains of cardinality κ which are inversely well-ordered; also T has a chain of size κ. Otherwise, there is $C'' \subseteq C'$ of size κ such that $s(c) \neq s(c')$ for $c \neq c'$ in C''. Then the sets $\{b_{s(c)}\colon c \in C''\}$ and, by part (d) of the second normal form lemma 16.6, C'' are inversely well-ordered and of cardinality κ; again $\{s(c)\colon c \in C''\}$ is a chain in T of size κ.

Case 3. Each c in C is in E, i.e. it has normal form

$$c = b_{t(c)} \cdot - \sum_{s \in S(c)} b_s .$$

If there are $C' \subseteq C$ of size κ and $t \in T$ such that $t(c) = t$ for $c \in C'$, then we are finished by Case 2 – in fact, C' then has a well-ordered subchain of cardinality κ. Otherwise, there is some $C' \subseteq C$ of size κ such that $t(c) \neq t(c')$ for $c \neq c'$ in C'. By part (a) of the second normal form lemma, $c < c'$ iff $t(c') < t(c)$ for $c, c' \in C'$, thus $\{t(c)\colon c \in C'\}$ is well-ordered and C' is inversely well-ordered.

Case 4. Each c in C has normal form

$$c = \sum E(c) ,$$

where $E(c)$ is a finite set of pairwise disjoint elements of E. Let then C' be a subset of C of cardinality κ and $n \in \omega$ such that $|E(c)| = n$ for $c \in C'$; clearly $n \geq 1$. If $n=1$, we are finished by Case 3. For $n>1$, the proposition follows from

Case 3 and part (d) of the second normal form lemma in exactly the same way as Case 2 followed from Case 1. □

16.21. COROLLARY. *Let L be an uncountable subset of the real numbers with a first element. Then the interval algebra of L is not embeddable into any tree algebra.*

PROOF. Intalg L has a chain C isomorphic to L. If it could be embedded into a tree algebra, then C and hence L would have an uncountable subchain which is well-ordered or inversely well-ordered. This, however, is impossible. □

There is a trivial example of a tree algebra not isomorphic to any interval algeba: let κ be an uncountable cardinal and let T be the tree with height 1 and exactly κ roots. Then B_T is the finite–cofinite algebra over the set of roots of T and, as shown in Example 15.12, not isomorphic to an interval algebra. We now give a less trivial and atomless example.

16.22. COROLLARY. *Let κ be a regular uncountable cardinal and T a tree of cardinality κ, height less than κ and such that each element of T has infinitely many immediate successors. Then B_T is an atomless tree algebra not isomorphic to any interval algebra.*

PROOF. B_T is atomless by Remark 16.10. If it is isomorphic to some interval algebra Intalg L, then

$$|L| = |\text{Intalg } L| = |B_T| = |T| = \kappa .$$

Intalg L and hence B_T has a chain of cardinality κ, and by Proposition 16.20, T has a chain of size κ. This contradicts the assumption that height$(T) < \kappa$. □

Exercises

1. Assume that T is a tree with a single root and infinitely many successors to each point. Then the completion of B_T is isomorphic to the regular open algebra RO(T) (see the conventions preceding 14.16).

2. Let In(T) be the set of all initial chains of T, ϕ: Ult $B_T \to$ In(T) the bijection established in Proposition 16.11 and equip In(T) with the unique topology making ϕ a homeomorphism. Describe the topology of In(T) in terms of the partial order of T.

3. Show that every tree embeds into the tree $T_{\alpha\lambda} = \bigcup_{\nu<\lambda} {}^{\nu}\alpha$ (as considered in 16.12), for suitable ordinals α, λ.

4. Show that a countable Boolean algebra with at least four elements cannot be rigid.

5. If T is a tree with height ω, a single root and such that $T \cong T \upharpoonright t$, for each $t \in T$, then B_T is homogeneous.

6. (G. Brenner) Assume that T is a tree with height ω, a single root and such that each element of the nth level of T has exactly ω_n immediate successors. Show that B_T has no homogeneous factor but is *weakly homogeneous*, i.e. that for any

two non-zero elements a, b of B_T, there are a', b' in B_T such that $0 < a' \leq a$, $0 < b' \leq b$ and $(B_T) \upharpoonright a' \cong (B_T) \upharpoonright b'$.

17. Superatomic algebras

Superatomic Boolean algebras have been briefly considered in Section 10 – there an algebra A was called superatomic if every non-trivial homomorphic image of A has at least one atom. We shall not presuppose the contents of Section 10 here, however. In this section, we define an algebra A to be superatomic if every homomorphic image of A is atomic. It is quite obvious that this definition is equivalent to the original one; see Proposition 17.5.

Under Stone duality, homomorphic images of a Boolean algebra correspond to closed subspaces of its dual space and atoms correspond to isolated points. Thus, an algebra A is superatomic iff the space Ult A is scattered, i.e. iff, for every closed subspace Y of Ult A, the isolated points of Y are dense in Y. Scattered spaces are interesting in their own right to topologists. In many cases, the topological condition of being scattered is conceptually easier to deal with than the algebraic one of being superatomic. Therefore, arguments on superatomic Boolean algebras are often carried out more easily in topological form, and we will not always bother to translate their results into the language of Boolean algebras.

Our first subsection gives several equivalent descriptions of superatomicity, one of which runs as follows. The Cantor–Bendixson derivative of a topological space X is the subspace consisting of all non-isolated points of X. Iterating the process of Cantor–Bendixson derivation, one can define the αth Cantor–Bendixson derivative of X, for every ordinal α. Then a non-trivial Boolean algebra A is superatomic iff, for some ordinal α and some non-zero natural number n, the αth derivative of Ult A has exactly n elements; the ordinals α and n are called the Cantor–Bendixson invariants of A. It is the main result of the second subsection that a countable superatomic algebra is determined, up to isomorphism, by its Cantor–Bendixson invariants.

The third subsection defines the cardinal sequence of a scattered Boolean space, a somewhat more refined sequence of invariants: if α and n are the Cantor–Bendixson invariants of X, then the cardinal sequence of X is the sequence $(c_\nu)_{\nu \leq \alpha}$, where c_ν is the number of isolated points of the νth derivative of X. LaGrange's theorem 17.14 characterizes those countable sequences of cardinals which arise as cardinal sequences of scattered Boolean spaces. No similar characterization seems to be available for uncountable sequences. For example, it was an open question in topology, for some time, whether there exists a thin-tall space, i.e. a scattered Boolean space having cardinal sequence $(c_\nu)_{\nu \leq \omega_1}$, where $c_\nu = \omega$ for $\nu < \omega_1$ and $c_\nu = 1$ for $\nu = \omega_1$. Such a space does indeed exist, as proved first in OSTASZEWSKI [1976] using the additional set-theoretical assumption ($\diamondsuit$), and later in JUHASZ and WEISS [1978], using only the axioms of ZFC set theory. More information on cardinal sequences and other topics in superatomic Boolean algebras can be found in the survey chapter by ROITMAN [Ch. 19 in this Handbook].

17.1. Characterizations of superatomicity

In this subsection, we give algebraic equivalences of the property of being superatomic (Propositions 17.5 and 17.8) and translate them to topological properties of Boolean spaces.

17.1. DEFINITION. A Boolean algebra is *superatomic* if every homomorphic image of A is atomic; in particular, the trivial Boolean algebra and every finite Boolean algebra are superatomic.

A topological space is *scattered* if, for every closed subspace Y of X, the isolated points of Y are dense in Y. The duality between homomorphic images of a Boolean algebra A and closed subspaces of its dual space Ult A (respectively between atoms of a Boolean algebra Q and isolated points of Ult Q) established in Section 7, gives the following observation.

17.2. REMARK. A Boolean algebra is superatomic iff its dual space is scattered.

We can use 17.2 to check superatomicity for two well-known types of algebras. Further constructions of scattered spaces will be encountered in the proof of La Grange's theorem 17.14.

17.3. EXAMPLE (finite–cofinite algebras). Let I be an infinite set with the discrete topology and let

$$X = I \cup \{\infty\}$$

be its one-point compactification. Then X is a Boolean space whose dual algebra A is isomorphic to the finite–cofinite algebra over I. We prove that X is scattered, i.e. that A is superatomic: let Y be a closed subspace of X. If Y is finite, then it is discrete and every point of Y is isolated in Y. Otherwise, ∞ is the only non-isolated point of Y; again the isolated points of Y are dense in Y.

17.4. EXAMPLE AND NOTATION (interval algebras of well-ordered sets). Let $(X, \leq)$ be a well-ordered set with a greatest element 1_X. Then $(X, \leq)$ is a Boolean order, in the sense of Definition 15.4; by Proposition 15.6, it is a Boolean space in its order topology and

$$A = \text{Clop}\, X \cong \text{Intalg}(X \setminus \{1_X\})$$

as follows from the Claim in the proof of 15.6. We claim that X is scattered. To see this, note that a point x of X is isolated iff x is not a limit point in the ordering $(X, \leq)$ – i.e. iff x is the smallest element 0_X of X or the immediate successor of some element of X. Consequently, the isolated points of X are dense in X. Similarly, let Y be a closed subspace of X. It is known from topology that the subspace topology of Y is induced by the restriction $\leq_Y$ of the linear order $\leq$ to Y. Since $\leq_Y$ well-orders Y, the above argument shows that the isolated points of Y are dense in Y.

We have thus seen that the interval algebra of a well-ordering is superatomic. As a special case of this, the Boolean algebra defined by

$$A_{\rho k} = \text{Intalg}(\omega^{\rho} \cdot k)$$

(ordinal operations) is a superatomic algebra, for every ordinal ρ and every integer $k \geq 1$.

17.5. PROPOSITION. *Let A be a non-trivial Boolean algebra. The following are equivalent*:

(a) *A is superatomic, i.e. every homomorphic image of A is atomic*,
(b) *every subalgebra of A is atomic*,
(c) *every subalgebra of A has an atom*,
(d) *every non-trivial homomorphic image of A has an atom.*

PROOF. Trivially (b) implies (c). The other implications are shown by contraposition.

(a) implies (b): Assume that A has a non-atomic subalgebra B. By Corollary 5.10, there is a homomorphic image A' of A having B as a dense subalgebra; so A' is a non-atomic homomorphic image of A.

(c) implies (d): Assume $\pi: A \rightarrow Q$ is an epimorphism onto a non-trivial and atomless algebra Q. Choose a countable atomless subalgebra Q' of Q. By 5.16 and 9.11, Q' is isomorhic to the free Boolean algebra on ω generators, so assume $U \subseteq Q'$ freely generates Q'. For $u \in U$, pick a preimage a_u of u under π. Then the elements a_u of A, being preimages of independent elements of Q, are independent in A, the subalgebra of A generated by $\{a_u: u \in U\}$ is free and, again by 9.11, atomless.

(d) implies (a): Assume Q is a non-atomic homomorphic image of A; so Q has a non-zero element q with an atomless relative algebra $Q \restriction q$. Then $Q \restriction q$ is a non-trivial atomless homomorphic image of Q, hence of A. □

A more refined study of superatomic algebras (respectively of scattered spaces) is possible by the following process, known as the Cantor–Bendixson analysis. Given a Boolean algebra A, it consists in assigning an ideal I_α of A to each ordinal α. The ideals I_α form an increasing chain, and it turns out in Proposition 17.8 that A is superatomic iff A/I_α is trivial, for some α.

17.6. CONSTRUCTION. For every Boolean algebra A, let $I(A)$ be the ideal of A generated by the atoms of A.

For any ordinal α, we define by induction an ideal I_α of A; if I_α has been defined, put

$$A_\alpha = A/I_\alpha$$

(the *αth Cantor–Bendixson derivative of A*), and let

$$\pi_\alpha: A \rightarrow A_\alpha$$

be canonical. Define

$$I_0 = \{0\}, \qquad I_{\alpha+1} = \pi_\alpha^{-1}[I(A_\alpha)],$$

and for λ a limit ordinal,

$$I_\lambda = \bigcup_{\alpha<\lambda} I_\alpha .$$

Finally, put

$$I_\infty = \bigcup_{\alpha \in \mathrm{Ord}} I_\alpha$$

(Ord denoting the class of all ordinals). Since $I_\alpha \subseteq I_{\alpha+1}$, it follows by induction that $(I_\alpha)_{\alpha \in \mathrm{Ord}}$ is an increasing chain of ideals of A; also I_∞ is an ideal of A.

For any ordinal α, $I_\alpha = I_{\alpha+1}$ iff A_α is atomless. In this case, A_α is either trivial or infinite and it follows by induction that $I_\alpha = I_\beta$ for every $\beta \geq \alpha$, i.e. $I_\alpha = I_\infty$. There is always an ordinal α (depending on A) such that $I_\alpha = I_{\alpha+1}$: if A is trivial, let $\alpha = 0$; if A is finite and non-trivial, let $\alpha = 1$. If $|A| = \kappa$ is an infinite cardinal, then consideration of the increasing chain $(I_\alpha)_{\alpha<\kappa^+}$ of subsets of A shows that $I_\alpha = I_{\alpha+1}$, for some $\alpha < \kappa^+$.

Theorem 7.25 sets up a correspondence between ideals of a Boolean algebra A and closed subsets of Ult A such that $\mathrm{Ult}(A/I) \cong Y$, if Y is the closed set corresponding to I. Thus, the algebraic notions and assertions of 17.6 translate to topological ones as follows.

17.7. CONSTRUCTION. For any Boolean space X, let

$$\mathrm{Is}(X) = \{x \in X\colon x \text{ isolated in } X\},$$

the set of isolated points of X and

$$X' = X \setminus \mathrm{Is}(X)$$

the (first) *Cantor–Bendixson derivative* of X. Define a decreasing sequence $(X_\alpha)_{\alpha \in \mathrm{Ord}}$ of closed subspaces of X by

$$X_0 = X, \qquad X_{\alpha+1} = (X_\alpha)',$$
$$X_\lambda = \bigcap_{\alpha<\lambda} X_\alpha \quad \text{for limit } \lambda;$$

X_α is the *αth Cantor–Bendixson derivative* of X. The closed subspace

$$X_\infty = \bigcap_{\alpha \in \mathrm{Ord}} X_\alpha$$

is the *perfect kernel* of X.

If X is the dual space of a Boolean algebra A, then the ideal $I(A) = I_1$ corresponds to the subspace $X' = X_1$, I_α corresponds to X_α, and the dual space of

$A_\alpha = A/I_\alpha$ is homeomorphic to X_α. Similarly, I_∞ corresponds to X_∞ and $\mathrm{Ult}(A/I_\infty)$ is homeomorphic to X_∞.

With these notions at hand, we obtain another characterization of superatomicity. It is often easier to apply than those given in Proposition 17.5, requiring only consideration of the Cantor–Bendixson derivatives of a space instead of all closed subspaces.

17.8. PROPOSITION. *A non-trivial Boolean algebra A is superatomic iff $I_\infty = A$ iff $I_\alpha = A$ for some ordinal α. Dually, $X = \mathrm{Ult}\, A$ is scattered iff its perfect kernel is empty iff its αth Cantor–Bendixson derivative is empty, for some α.*

PROOF. Let A be a Boolean algebra and X its dual space. By the final remark of 17.6, choose α so large that $I_\alpha = I_\infty$ and hence that $X_\alpha = X_\infty$. Then $A/I_\alpha = A/I_\infty$ is atomless and its dual space X_∞ is dense in itself, i.e. it has no isolated points.

Hence, if X is scattered, then X_∞ must be empty. Conversely suppose X is not scattered. So there is a non-empty closed subspace Y of X which is dense in itself. It follows easily by induction that $Y \subseteq X_\alpha$ for each α and hence that X_∞ is non-empty. □

17.2. The Cantor–Bendixson invariants

Superatomic Boolean algebras are roughly classified by two invariants defined below, the Cantor–Bendixson invariants. These invariants will be used in Proposition 17.10 to show that $|A| = |\mathrm{Ult}\, A|$, for each infinite superatomic algebra A. The main result here is Theorem 17.11: countable superatomic algebras are characterized up to isomorphism by their Cantor–Bendixson invariants.

17.9. LEMMA AND DEFINITION. Let A be a superatomic Boolean algebra and X its dual space. There is a unique pair $(\alpha(A), n(A)) = (\alpha(X), n(X))$ such that:

(a) $\alpha(A) = -1$ and $n(A) = 0$ if A is trivial; otherwise, $\alpha(A)$ is an ordinal and $n(A)$ is a positive integer,

(b) if A is non-trivial, then $A_{\alpha(A)}$, the $\alpha(A)$th Cantor–Bendixson derivative of A, is finite with exactly $n(A)$ atoms – i.e. if X is non-empty, then $X_{\alpha(X)}$, the $\alpha(X)$th Cantor–Bendixson derivative of X, is finite with cardinality $n(X)$,

(c) $\alpha(A) < |A|^+$ if A is infinite.

$\alpha(A)$ and $n(A)$ are called the *Cantor–Bendixson invariants* of A (respectively X).

PROOF. We prove the topological version of the lemma. Let $X = \mathrm{Ult}\, A$ be non-empty. By Proposition 17.8, there is a least ordinal β satisfying $X_\beta = \emptyset$; we claim that β is a successor ordinal $\beta = \alpha + 1$ and that the pair $(\alpha, |X_\alpha|)$ works for the lemma.

Certainly $\beta > 0$ since $X_0 = X$ is non-empty. Also β cannot be a limit ordinal since otherwise X_β, being the intersection of the decreasing sequence $(X_\gamma)_{\gamma<\beta}$ of compact non-empty subsets of X, is non-empty. Thus, $\beta = \alpha + 1$ for a unique ordinal α. By the minimal choice of β, X_α is not empty but its derivative $(X_\alpha)'$ is.

Therefore each point of X_α is isolated, X_α is discrete and, by compactness, finite.

Clearly, $(\alpha, |X_\alpha|)$ is the unique pair satisfying the requirements (a) and (b). Assertion (c) has been proved in 17.6. □

It is easily checked that in Example 17.3, the one-point compactification of an infinite discrete space, we have $\alpha(X) = 1$ and $n(X) = 1$. The algebra $A_{\rho k}$ defined in Example 17.4 has invariants (ρ, k) since its dual space is the ordinal $X = \omega^\rho \cdot k + 1$, equipped with the order topology, and the νth Cantor–Bendixson derivative of X consists of those ordinals in X which are representable in the form $\omega^\nu \cdot \xi$ (ordinal operations). In particular, we find that for each ordinal ρ and each positive integer k, there exists a superatomic Boolean algebra with invariants ρ, k.

In Theorem 5.31, we proved the inequality $|A| \leq |\text{Ult } A|$ for every infinite Boolean algebra A.

17.10. PROPOSITION. *For every infinite superatomic Boolean algebra A, $|A| = |\text{Ult } A|$.*

PROOF. To prove that $|\text{Ult } A| \leq |A|$, consider the space $X = \text{Ult } A$ and note that

$$\emptyset = X_\infty = X_{\alpha(X)+1} \,.$$

Since $(X_\beta)_{\beta < \alpha(X)+1}$ is a decreasing chain of subsets of X satisfying $X_\lambda = \bigcap_{\beta<\lambda} X_\beta$ for limit λ, we can write

$$X = \bigcup_{\beta \leq \alpha(X)} (X_\beta \setminus X_{\beta+1}) \,.$$

Now $|\alpha(X)| \leq |A|$ by (c) in Definition 17.9, and for each $\beta \leq \alpha(X)$,

$$|X_\beta \setminus X_{\beta+1}| = |\text{At } A_\beta| \leq |A_\beta| \leq A|$$

because the isolated points of X_β correspond to the atoms of A_β in a one-to-one manner and A_β is a quotient of A. Thus, $|X| \leq |A| \cdot |A| = |A|$. □

17.11. THEOREM. *Countable superatomic Boolean algebra are isomorphic if they have the same Cantor–Bendixson invariants.*

The proof of 17.11 proceeds via a topological dual of Vaught's isomorphism theorem 5.15: let $\sim$ be a binary symmetric relation on the class of Boolean spaces such that

(1) if $X \sim Y$ and $X = \emptyset$, then $Y = \emptyset$;

(2) if $X \sim Y$ and $U \subseteq X$ is clopen, then there is a clopen subset V of Y such that $U \sim V$ and $X \setminus U \sim Y \setminus V$.

Then any two Boolean spaces X and Y with countable bases satisfying $X \sim Y$ are homeomorphic.

For application of this version of Vaught's theorem, let us note that the Cantor–Bendixson invariants of scattered spaces behave nicely under the formation of disjoint unions: if X is the union of two disjoint clopen subspaces U and V, then an easy induction shows that the αth derivative X_α of X is the union of its disjoint clopen subspaces $U_\alpha \subseteq U$ and $V_\alpha \subseteq V$, these derivatives being computed in the subspaces U and V. Therefore if $\alpha(U) = \alpha(V)$, then $\alpha(X) = \alpha(U)$ and $n(X) = n(U) + n(V)$; if $\alpha(U) < \alpha(V)$, then $\alpha(X) = \alpha(V)$ and $n(X) = n(V)$.

Proof of Theorem 17.11. We prove the topological dual of the theorem, i.e. that scattered Boolean spaces with countable bases and the same Cantor–Bendixson invariants are homeomorphic. By the topological version of Vaught's theorem, it suffices to show that the symmetric relation on Boolean spaces defined by

$X \sim Y$ iff X, Y are scattered and have the same Cantor–Bendixson invariants ,

satisfies (1) and (2) above.

(1) holds since $X = \emptyset$ and $X \sim Y$ imply that $\alpha(Y) = \alpha(X) = -1$ and thus that $Y = \emptyset$. For (2), suppose that $X \sim Y$ and $U \subseteq X$ is clopen. We may assume that neither U nor $X \setminus U$ is empty and that $\alpha(U) \leq \alpha(X \setminus U)$ and consider two cases.

If $\alpha(U) < \alpha(X \setminus U)$, let $\beta = \alpha(U)$ and $k = n(U)$. Since $\alpha(Y) = \alpha(X) > \beta$, Y_β is an infinite scattered space; so choose k distinct isolated points $y_1, \ldots, y_k$ in Y_β. Then $\{y_1, \ldots, y_k\}$ is a clopen subset of the closed subspace Y_β of Y; hence by 7.6(b) there is a clopen subset V of Y such that $Y_\beta \cap V = \{y_1, \ldots, y_k\}$. Clearly, V works for (2).

If $\alpha(U) = \alpha(X \setminus U)$, let $\alpha = \alpha(X)$ (so $\alpha = \alpha(U) = \alpha(X \setminus U)$) and $n = n(X)$, $k = n(U)$ and $l = n(X \setminus U)$ (so $n = k + l$). By $X \sim Y$, Y_α is a space with exactly n points; let $y_1, \ldots, y_k$ be k of these. Again any clopen subset V of Y satisfying $Y_\alpha \cap V = \{y_1, \ldots, y_k\}$ works for (2).

By this last theorem, we have a complete description of all countable superatomic Boolean algebras: the trivial Boolean algebra and each of the algebras $A_{\rho k} = \mathrm{Intalg}(\omega^\rho \cdot k)$ defined in 17.4, where $0 \leq \rho < \omega_1$, $1 \leq k < \omega$, are countable, superatomic and pairwise non-isomorphic. Conversely, if A is countably infinite and superatomic with Cantor–Bendixson invariants ρ and k, then $\rho < |A|^+ = \omega_1$ by 17.6; so by Theorem 17.11, $A \cong A_{\rho k}$.

17.3. Cardinal sequences

Uncountable superatomic algebras are far from being characterized by their Cantor–Bendixson invariants – e.g. we have seen in 17.3 that the finite–cofinite algebra over an infinite set I has $(1, 1)$ as its pair of invariants, regardless of the size of I. These algebras are somewhat better, but not completely, described by a more elaborate sequence of invariants.

17.12. DEFINITION. Let A be a non-trivial superatomic Boolean algebra. The *cardinal sequence* of A is the sequence

$$c = (c_\alpha)_{\alpha \le \rho} ,$$

where $\rho = \alpha(A)$ and c_α is the cardinality of $\mathrm{At}(A_\alpha)$, the set of atoms of the αth Cantor–Bendixson derivative of A.

Dually, for $X = \mathrm{Ult}\, A$, c_α is the cardinality of the set $\mathrm{Is}(X_\alpha)$ of isolated points of the αth Cantor–Bendixson derivative of X; we also call $(c_\alpha)_{\alpha \le \rho}$ the cardinal sequence of X.

For example, the one-point compactification of a discrete space I with cardinality $\kappa \ge \omega$ has cardinal sequence $(\kappa, 1)$.

17.13. REMARK. The cardinal sequence $(c_\alpha)_{\alpha \le \rho}$ of a non-trivial superatomic algebra A satisfies the conditions

(3) $\omega \le c_\alpha$, for $\alpha < \rho$,

(4) $1 \le c_\rho < c_\rho$; in fact, $c_\rho = n(A)$,

as follows from the very definition of the Cantor–Bendixson invariants $\alpha(A)$ and $n(A)$.

For countable A, the cardinal sequence contains no more information than the Cantor–Bendixson invariants, since $c_\alpha = \omega$ for $\alpha < \rho = \alpha(A)$. Theorem 17.14, the principal result of this subsection, states that cardinal sequences of uncountable algebras can be considerably more complicated.

17.14. THEOREM (LaGrange). *Let ρ be a countable ordinal and $c = (c_\alpha)_{\alpha \le \rho}$ a sequence of cardinals. Then c is the cardinal sequence of a scattered Boolean space iff it satisfies* (3) *and* (4) *above, plus*

(5) *if $\beta \le \alpha \le \rho$, then $c_\alpha \le c_\beta^\omega$ (cardinal exponentiation)* .

The proof of LaGrange's theorem requires two constructions of scattered Boolean spaces and computation of their cardinal sequences. We begin with a purely set-theoretical lemma half of which is applied in the proof of La Grange's theorem; the other half, or rather the method of its proof, gives the first construction required (Example 17.16). As in Example 5.28, let us call a family W of sets *almost disjoint* if each element of W is infinite but, for distinct elements w and w' of W, $w \cap w'$ is finite.

17.15. LEMMA. *Let τ and κ be cardinals and $\tau \ge \omega$. The following are equivalent:*

(a) *for each set T of size τ, there is an almost disjoint family $W \subseteq P(T)$ of size κ,*

(b) $\kappa \le \tau^\omega$.

PROOF. Assume T has size τ, $W \subseteq P(T)$ has size κ and the elements of W are almost disjoint. For every w in W, choose a countably infinite subset c_w of w.

Then $w \neq w'$ implies that $c_w \cap c_{w'}$ is finite and thus that $c_w \neq c_{w'}$. So $\kappa = |W| \leq |T|^\omega = \tau^\omega$.

For the converse, it is enough to exhibit a set T of size τ and an almost disjoint family $W \subseteq P(T)$ of size τ^ω. The set

$$T = \bigcup_{n \in \omega} {}^n\tau$$

of all finite sequences in τ has cardinality τ; it is a tree of height ω under inclusion. Every function $b: \omega \to \tau$ defines a branch

$$w_b = \{t \in T: t \subseteq b\}$$

of length ω in T; there are τ^ω such functions and the corresponding family $\{w_b: b \in {}^\omega\tau\}$ of branches consists of almost disjoint sets. □

17.16. EXAMPLE. For infinite cardinals κ, τ satisfying $\kappa \leq \tau^\omega$, we construct a scattered Boolean space

$$N = N_{\tau\kappa}$$

with cardinal sequence $(\tau, \kappa, 1)$.

To this end, let T be the tree considered in the proof of Lemma 17.15, let $B \subseteq {}^\omega\tau$ have cardinality κ and put, with the notation of 17.15,

$$W = \{w_b: b \in B\}\ ;$$

we shall define a topology on the set $T \cup B$ and then let N be the one-point compactification of $T \cup B$.

The set

$$\boldsymbol{B} = \{\{t\}: t \in T\} \cup \{m \cup \{b\}: b \in B, m \subseteq w_b \text{ cofinite in } w_b\}$$

is the base of a topology on $T \cup B$ in which $T \cup B$ is a locally compact zero-dimensional Hausdorff space. This holds since the elements of $\boldsymbol{B}$ are clopen and compact; in particular for $m \subseteq w_b$ cofinite, $m \cup \{b\}$ is homeomorphic to the one-point compactification of ω with the discrete topology. Let $N = N_{\tau\kappa} = T \cup B \cup \{\infty\}$ be the one-point compactification of $T \cup B$; it is easily seen that N is Boolean. The Cantor–Bendixson derivatives of N are $N_0 = N$, $N_1 = B \cup \{\infty\}$, $N_2 = \{\infty\}$, $N_3 = \emptyset$. Hence, N is scattered with cardinal sequence $(\tau, \kappa, 1)$.

Let us note, for application in 17.14, an additional property of the space $N = N_{\tau\kappa}$. Considering the subsets $T_n = {}^n\tau$ of T, we find that $\{T_n: n \in \omega\}$ is a partition of Is(N), each T_n has size τ, and for every neighbourhood U of a non-isolated point of N, there are arbitrarily large n such that T_n intersects U.

We turn to the second construction required in the proof of LaGrange's theorem and then to its special case 17.18.

17.17. EXAMPLE. The following procedure constructs a scattered Boolean space S out of a scattered Boolean space X, by substituting the isolated points of X by a family of scattered Boolean spaces.

Let X be scattered and Boolean and

$$\mathrm{Is}(X) = \{x_i\colon i \in I\}$$

a one-to-one enumeration of the isolated points of X. Assume that $(X_i)_{i\in I}$ is a family of scattered Boolean spaces and that $X' = X\setminus \mathrm{Is}(X)$ and the spaces X_i, $i \in I$, are pairwise disjoint. Let then

$$S = S(X, x_i/X_i)_{i\in I} ,$$

the result of substituting x_i by X_i, be the topological space defined as follows. The underlying set of $S(X, x_i/X_i)_{i\in I}$ is

$$S = X' \cup \bigcup_{i\in I} X_i .$$

A base

$$\boldsymbol{B} = \boldsymbol{B}' \cup \boldsymbol{B}''$$

for a topology on S is given by

$$\boldsymbol{B}' = \{b\colon b \text{ a clopen subset of some } X_i\} ,$$
$$\boldsymbol{B}'' = \{b_K\colon K \text{ a clopen subset of } X\} ,$$

where for clopen $K \subseteq X$

$$b_K = (X' \cap K) \cup \bigcup \{X_i\colon i \in I \text{ such that } x_i \in K\}$$

is the result of replacing each isolated point x_i which happens to be in K by X_i. Using the fact that, for clopen subsets K and K' of X,

$$b_{X\setminus K} = S\setminus b_K , \qquad b_{K\cap K'} = b_K \cap b_{K'} ,$$

one checks that $\boldsymbol{B}$ is the base of a topology on S and that every element of $\boldsymbol{B}$, in particular every X_i, is clopen in S. Thus, S is a zero-dimensional Hausdorff space.

We prove that S is compact: suppose

$$(6) \qquad S = \bigcup \boldsymbol{B}_0' \cup \bigcup_{j\in J} b_{K(j)}$$

is a cover of S consisting of elements of $\boldsymbol{B}$ – i.e. suppose that $\boldsymbol{B}_0' \subseteq \boldsymbol{B}'$ and $K(j)$ is clopen in X for $j \in J$. Since X' is closed in X, hence compact, and is covered by the sets $b_{K(j)}$, there is a finite subset J' of J such that

$$X' \subseteq \bigcup_{j\in J'} b_{K(j)} .$$

Define $K = \bigcup_{j \in J'} K(j)$; so K is clopen in X and $b_K = \bigcup_{j \in J'} b_{K(j)}$. Since $X' \subseteq K$, K is open and each x_i is isolated in X, the set

$$I' = \{i \in I: x_i \notin K\}$$

must be finite. Now $\bigcup_{i \in I'} X_i$ is a compact subspace of S, so there is a finite subset $\boldsymbol{B}_1'$ of $\boldsymbol{B}_0'$ such that

$$\bigcup_{i \in I'} X_i \subseteq \bigcup \boldsymbol{B}_1' .$$

Thus, $S = \bigcup \boldsymbol{B}_1' \cup \bigcup_{j \in J'} b_{K(j)}$ is a finite subcover of that given in (6).

Proposition 17.8 shows that the space S is scattered: for each ordinal ν greater than $\sup_{i \in I} \alpha(X_i)$, the νth derivative S_ν of S is included in the scattered space X'; so S_η is empty for η large enough.

We describe the cardinal sequence of S in the special case that all spaces X_i have the same cardinal sequence, say $(d_\alpha)_{\alpha \leq \mu}$. Denote by $(c_\alpha)_{\alpha \leq \rho}$ the cardinal sequence of X; so in particular, $c_0 = |\mathrm{Is}(X)| = |I|$. Then the cardinal sequence of S is $(f_\alpha)_{\alpha \leq \tau}$, where $\tau = \mu + \rho$ (ordinal addition) and

$$f_\alpha = c_0 \cdot d_\alpha \quad \text{for } \alpha \leq \mu \text{ (cardinal multiplication)} ,$$

$$f_\alpha = c_\gamma \quad \text{for } \alpha = \mu + \gamma,\ 0 < \gamma \leq \rho \text{ (ordinal addition)} .$$

17.18. EXAMPLE. Consider the following special case of 17.17. Let X be the one-point compactification of an infinite discrete space $\{x_i: i \in I\}$ and let for $i \in I$ X_i be a scattered Boolean space with cardinal sequence $(c_{i\alpha})_{\alpha \leq \rho(i)}$. Then the space $S = S(X, x_i/X_i)_{i \in I}$ defined in 17.17 is the one-point compactification of the disjoint union space $\bigcup_{i \in I} X_i$ as defined in Section 8.

We compute the cardinal sequence of S under the additional assumption that, for each $\alpha < \sup\{\rho(i) + 1: i \in I\}$, there are infinitely many $i \in I$ satisfying $\alpha \leq \rho(i)$. In this case, the cardinal sequence of S is $(c_\alpha)_{\alpha \leq \rho}$, where

$$\rho = \sup\{\rho(i) + 1: i \in I\} ,$$

$$c_\rho = 1 ,$$

$$c_\alpha = \sum_{i \in I} c_{i\alpha} \quad \text{for } \alpha < \rho \text{ (cardinal addition)} .$$

We shall also use, in the proof of 17.14, the following obvious fact which could be conceived as the special case of 17.17 when substituting both (isolated) points of a two-point discrete space by scattered Boolean spaces. Let X and Y be scattered Boolean spaces with cardinal sequences $(c_\alpha)_{\alpha \leq \rho}$ and $(d_\alpha)_{\alpha \leq \mu}$, respectively; without loss of generality, assume $\rho \leq \mu$. The disjoint union space U of X and Y is scattered and its cardinal sequence $(e_\alpha)_{\alpha \leq \mu}$ is computed as follows. If $\mu = \rho$, then $e_\alpha = c_\alpha + d_\alpha$ (cardinal addition) for all $\alpha \leq \mu$. If $\rho < \mu$, then $e_\alpha = c_\alpha + d_\alpha$ for $\alpha \leq \rho$ and $e_\alpha = d_\alpha$ for $\rho < \alpha \leq \mu$.

Proof of Theorem 17.14. First suppose that X is a scattered Boolean space with cardinal sequence c. Then c satisfies conditions (3) and (4), by Remark 17.13. For condition (5), it suffices to consider the case $\beta = 0$, i.e. to prove that $d_\nu \leq d_0^\omega$ if $(d_\nu)_{\nu \leq \tau}$ is the cardinal sequence of a scattered Boolean space and $\nu < \omega_1$. This implies the general case as follows: let $\beta \leq \alpha \leq \rho$ and $\alpha < \omega_1$. Let σ be the unique ordinal such that $\alpha = \beta + \sigma$ (ordinal addition). Then $Y = X_\beta$ is a scattered Boolean space, say with cardinal sequence $(d_\nu)_{\nu \leq \tau}$, and $Y_\sigma = X_\alpha$; hence $c_\alpha = d_\sigma \leq d_0^\omega = c_\beta^\omega$.

We prove $c_\alpha \leq c_0^\omega$ by induction on $\alpha < \omega_1$: $\alpha = 0$ is trivial. Let $\alpha = 1$. For each point $p \in \mathrm{Is}(X_1)$, choose a clopen subset U_p of X such that $U_p \cap X_1 = \{p\}$. Then

$$\{U_p \setminus \{p\} \colon p \in \mathrm{Is}(X_1)\}$$

is an almost disjoint family of subsets of $\mathrm{Is}(X)$ and by 17.15,

$$c_1 = |\mathrm{Is}(X_1)| \leq |\mathrm{Is}(X_0)|^\omega = c_0^\omega .$$

If $c_\alpha \leq c_0^\omega$ and $\alpha + 1 \leq \rho$, then $c_{\alpha+1} \leq c_0^\omega$ follows by applying the inequality $c_1 \leq c_0^\omega$, proved for an arbitrary scattered Boolean space, to the space X_α; this gives

$$c_{\alpha+1} \leq c_\alpha^\omega \leq (c_0^\omega)^\omega = c_0^\omega .$$

Finally, let $\alpha \leq \rho$ be a limit ordinal and assume $c_\nu \leq c_0^\omega$ for $\nu < \alpha$. The set $Z = X \setminus X_\alpha$ has cardinality at most c_0^ω, since

$$Z = \bigcup_{\nu < \alpha} \mathrm{Is}(X_\nu) ,$$

$|\mathrm{Is}(X_\nu)| = c_\nu \leq c_0^\omega$ and $|\alpha| \leq \omega$. We shall assign to each isolated point p of X_α a countable subset $\{x_{pk} \colon k \in \omega\}$ of Z having p as its single accumulation point; it then follows that

$$c_\alpha = |\mathrm{Is}(X_\alpha)| \leq |Z|^\omega \leq c_0^\omega .$$

Since α is a countable limit ordinal, we can fix a sequence $(\alpha(k))_{k \in \omega}$ of ordinals such that $\alpha(0) < \alpha(1) < \cdots$ and $\alpha = \sup\{\alpha(k) \colon k \in \omega\}$. For $p \in \mathrm{Is}(X_\alpha)$, let U_p be a clopen subset of X such that $X_\alpha \cap U_p = \{p\}$. To define the point x_{pk}, note that $X_\alpha \subseteq X_{\alpha(k)}$ and that $\mathrm{Is}(X_{\alpha(k)})$ is dense in $X_{\alpha(k)}$; so pick

$$x_{pk} \in U_p \cap \mathrm{Is}(X_{\alpha(k)}) .$$

$\{x_{pk} \colon k \in \omega\}$, being an infinite subset of the compact space X, has at least one accumulation point, say y; we claim that $y = p$. For each n, y is in $X_{\alpha(n)}$ since $X_{\alpha(n)}$ is closed and all but finitely many of the points x_{pk}, $k \in \omega$, are in $X_{\alpha(n)}$. Thus, $y \in X_\alpha$. Also, since $\{x_{pk} \colon k \in \omega\} \subseteq U_p$ and U_p is closed, $y \in U_p$. Hence, $y = p$. This finishes the proof of (5), and we have shown the necessity of conditions (3) through (5).

Conversely, assume that $c = (c_\alpha)_{\alpha \le \rho}$ satisfies (3) through (5). We construct by induction on $\alpha \le \rho$ scattered Boolean spaces Y_α with cardinal sequence $(d_\beta)_{\beta \le \alpha}$, where $d_\beta = c_\beta$ for $\beta < \alpha$ and $d_\alpha = 1$. The disjoint union of c_ρ copies of Y_ρ will then have cardinal sequence c. The construction is carried out in such a way that for $\beta < \alpha \le \rho$, Y_β is (homeomorphic to) a clopen subspace of Y_α.

For $\alpha = 0$, let Y_0 be a one-point space. If $\alpha = 1 \le \rho$, let Y_1 be the one-point compactification of a discrete space of cardinality c_0. If $\alpha \le \rho$ is a limit ordinal, let Y_α be the one-point compactification of the disjoint union space $\bigcup_{\beta<\alpha} Y_\beta$. The cardinal sequence of Y_α is as required (see its computation in Example 17.18) since $|\alpha| = \omega \le c_\beta$ for each $\beta < \alpha$. So we are left with the case that α is a successor ordinal.

If α is the successor of a successor ordinal, say $\alpha = \beta + 2$, choose $\mu \le \beta$ such that c_μ is minimal among the cardinals c_ν, $\nu \le \beta$. By $\mu \le \beta + 1$ and (5), $c_{\beta+1} \le (c_\mu)^\omega$. Let then Y_α be the disjoint union of $Y_{\beta+1}$ and S, where S results from substituting a copy of Y_β for each isolated point of $N_{c_\mu c_{\beta+1}}$. The cardinal sequence of Y_α, computed in 17.17 and the remark following 17.18, is as required.

Finally, suppose that $\alpha = \lambda + 1$, where λ is a limit ordinal. Choose $\mu < \lambda$ such that c_μ is minimal among the cardinals c_ν, $\nu < \lambda$. Fix a strictly increasing sequence $(\alpha(n))_{n\in\omega}$ with λ as its least upper bound and let Y_α be the result of substituting a copy of $Y_{\alpha(n)}$ for each isolated point of $N_{c_\mu c_\lambda}$ which happens to be in the set $T_n \subseteq \mathrm{Is}(N_{c_\mu c_\lambda})$ – see the remark at the end of 17.16. It follows as in 17.17 that the cardinal sequence of Y_α is as required. □

The countability restriction on $\rho = \alpha(X)$ in LaGrange's theorem is quite essential. Already for $\rho = \omega_1$ there are questions on the possible cardinal sequences $(c_\alpha)_{\alpha \le \rho}$ of scattered Boolean spaces which cannot be answered on the basis of ZFC set theory – see the survey chapter by ROITMAN [Ch. 19 in this Handbook].

Exercises

1. (a) Let $(A_i)_{i\in I}$ be a family of Boolean algebras. In which cases is $\bigoplus_{i\in I} A_i$ superatomic?

(b) Let A be generated by the union of finitely many subalgebras $A_1, \ldots, A_n$. In which cases is A superatomic?

2. Call a mapping r from a Boolean algebra A into the ordinals a *rank function* on A if it satisfies

(a) $b \le a$ in A implies that $r(b) \le r(a)$,

(b) If $a = b + c$ in A, where b and c are disjoint and non-zero, then either $r(b) < r(a)$ or $r(c) < r(a)$.

Show that a Boolean algebra admits a rank function iff it is superatomic.

3. Prove that if X and Y are scattered Boolean spaces, then so is their product space.

Hint. For a scattered Boolean space X and $x \in X$, let $\mathrm{rk}(x, X)$ (the Cantor–

Bendixson rank of x in X) be the unique ordinal α such that x lies in the αth Cantor–Bendixson derivative of X but not in the $(\alpha+1)$th one. Assume Z is a non-empty closed subspace of $X \times Y$. Pick (x^*, y^*) in Z such that the pair $(\mathrm{rk}(x^*, X), \mathrm{rk}(y^*, Y))$ is minimal, with respect to the lexicographic order, among the pairs $(\mathrm{rk}(x, X), \mathrm{rk}(y, Y))$, $(x, y) \in Z$. Show that (x^*, y^*) is isolated in Z.

This proves Corollary 11.16 (the free product of two countable superatomic algebras is superatomic) without using Šapirovskii's theorem 10.17.

4. This exercise gives a proof of Proposition 17.10 ($|A| = |\mathrm{Ult}\, A|$ for every infinite superatomic algebra A) which does not use topological duality.

Assume that $|A| = \kappa$ but $|\mathrm{Ult}\, A| > \kappa$. Call $a \in A$ *large* if $|\mathrm{Ult}(A \restriction a)| > \kappa$ and prove that every large element is the sum of two disjoint large elements; conclude that A is not superatomic.

CHAPTER 7

Metamathematics

Sabine KOPPELBERG
Freie Universität Berlin

Contents

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Introduction

We present in this chapter two metamathematical results on Boolean algebras: the first order theory Th($\boldsymbol{BA}$) of Boolean algebras is decidable (Tarski); the first order theory Th($\boldsymbol{BP}$) of Boolean pairs, i.e. of pairs (A, B), where A is a Boolean algebra and B a subalgebra of A, is undecidable (M. Rubin).

Both results have been applied in proving decidability or undecidability of further theories. For example, assume that $\boldsymbol{K}$ is a class of L-structures and that each member of $\boldsymbol{K}$ is representable as the structure of global sections of a sheaf of L-structures over a Boolean space (cf. Section 8). Then in many cases the theory of $\boldsymbol{K}$ is decidable if the sheaf representations are well-behaved and the theory of the stalks is decidable. A result of this type is proved in the survey chapter by WEESE [Ch. 23 in this Handbook]. On the other hand, McKenzie has characterized, in universal algebra, the finitely generated modular varieties of finite type with a decidable theory; see BURRIS and MCKENZIE [1981]. His proof uses, in the undecidable cases, Rubin's result in an essential way.

We will assume a basic knowledge of model theory as presented, for example, in CHANG and KEISLER [1973], and of decidability theory, several definitions being recalled in Section 18. Let us point out that an *L-theory* is any set of sentences (i.e. formulas without free variables) in the formal language L and not necessarily closed under logical inference. For $\boldsymbol{K}$ a class of L-structures, the *theory of* $\boldsymbol{K}$ is the set

$$\text{Th}(\boldsymbol{K}) = \{\phi\colon \phi \text{ an } L\text{-sentence}, A \models \phi \text{ for all } A \in \boldsymbol{K}\}.$$

An L-theory T is *decidable* if there is an effective procedure deciding, for each L-sentence ϕ, whether ϕ is formally derivable from T.

18. Decidability of the first order theory of Boolean algebras

This section is devoted to the proof of a single theorem:

18.1. THEOREM (Tarski). *The first order theory* Th($\boldsymbol{BA}$) *of Boolean algebras is decidable.*

Subsequent to Tarski, stronger results were obtained stating the decidability of the theory of Boolean algebras in several logics much more expressive than first order logic. See the chapter by WEESE [Ch. 23 in this Handbook] for a survey of these.

As a general rule, a theory is decidable if its completions can be described in a uniform way. Now a theory is complete iff any two of its models are elementarily equivalent; thus our decidability proof boils down to a characterization of elementary equivalence of Boolean algebras. We shall assign a triple of numerical invariants, the so-called elementary invariants, to each Boolean algebra and then show in 18.10 that two Boolean algebras are elementarily equivalent iff they have the same invariants. It turns out that having the same elementary invariants is a Vaught relation on the class of ω-saturated Boolean algebras; 18.10 then follows from a standard back-and-forth argument.

Similarly to the Cantor–Bendixson invariants of Section 17, the elementary invariants code information on the process of repeated division of a Boolean algebra A by particular ideals. The quotient algebras thus obtained are the elementary derivatives $A^{(i)}$ of A, and the invariants express the number of atoms of the last non-trivial derivative $A^{(k)}$ of A, as well as existence or non-existence of a non-trivial atomless factor of $A^{(k)}$.

The following notation will be used in this and, partially, in the next section. The formal language L for Boolean algebras consists of the non-logical symbols

$$L = \{+, \cdot, -, 0, 1\}.$$

We stick to the convention from Section 1 of identifying an L-structure $(A, +^A, \cdot^A, -^A, 0^A, 1^A)$ with its underlying set A, and we do not bother to distinguish the symbols of L from their interpretations $+^A$, $\cdot^A, \ldots$ in A. $\boldsymbol{BA}$ is the class of all Boolean algebras, thus $\mathrm{Th}(\boldsymbol{BA})$ consists (by the completeness theorem of first order logic) of those L-sentences formally derivable from the axioms (B1) through (B5′) of Section 1. For an L-formula ϕ, we write $\phi(x_1 \ldots x_n)$ if all free variables of ϕ are among $\{x_1 \ldots x_n\}$; similar notation $\Sigma(x_1 \ldots x_n)$ applies to sets Σ of formulas.

18.1. The elementary invariants

We define, in this subsection, the elementary invariants of a Boolean algebra, prove two technical lemmas on invariants, and show how they are definable in first order predicate logic.

In the preceding section, the Cantor–Bendixson invariants of a Boolean algebra A were defined in terms of repeated division by the Cantor–Bendixson ideal $I(A)$ generated by the atoms of A. For our decidability proof, we replace the Cantor–Bendixson ideal by a larger ideal $E(A)$ which has the advantage of being definable in first order logic.

18.2. DEFINITION. For every Boolean algebra A, $E(A)$ is the ideal of A defined by

$$E(A) = \{x \in A: x = y + z \text{ where } A \restriction y \text{ is atomless and } A \restriction z \text{ is atomic}\}.$$

By induction, we define for $i \in \omega$ an ideal $E_i = E_i(A)$ of A and then let

$$A^{(i)} = A/E_i$$

(the *ith elementary derivative* of A),

$$\pi_i: A \to A^{(i)}$$

the canonical map: put

$$E_0 = \{0\}, \qquad E_{i+1} = \pi_i^{-1}[E(A^{(i)})].$$

Thus, $(E_i)_{i\in\omega}$ is an increasing sequence of ideals of A.

Taking elementary derivatives commutes with relativization:

18.3. LEMMA. *For all $a \in A$ and $i \in \omega$,*

$$(A \upharpoonright a)^{(i)} \cong A^{(i)} \upharpoonright \pi_i(a).$$

PROOF. This is trivial for $i = 0$. Clearly,

(1) $\qquad E(A \upharpoonright a) = E(A) \cap A \upharpoonright a$

and, consequently,

(2) $\qquad (A \upharpoonright a)/E(A \upharpoonright a) \cong (A/E(A)) \upharpoonright \pi_1(a)$,

since the compositions of the obvious epimorphisms

$$A \to A \upharpoonright a \to (A \upharpoonright a)/E(A \upharpoonright a)$$

and

$$A \to A/E(A) \to (A/E(A)) \upharpoonright \pi_1(a)$$

both have, by (1), the kernel $\{x \in A: x \cdot a \in E(A)\}$. Now if the lemma holds for i, then it follows for $i+1$ from (2), applied to $A^{(i)}$. □

18.4. DEFINITION. We define the set *Inv* of elementary invariants to be the following set of triples in $\{-1\} \cup (\omega + 1)$:

$$Inv = \{(-1, 0, 0), (\omega, 0, 0)\}$$
$$\cup \{(k, l, m): k \in \omega, l \in 2, m \in (\omega + 1), l + m \neq 0\}.$$

For any Boolean algebra A, we define the triple $\mathrm{inv}(A)$ of elementary invariants of A:

$\mathrm{inv}(A) = (-1, 0, 0)$ if A is the trivial (one-element) Boolean algebra

$\mathrm{inv}(A) = (\omega, 0, 0)$ if $A^{(i)}$ is non-trivial for all $i \in \omega$

$\mathrm{inv}(A) = (k, l, m)$ if $k \in \omega$ and each of the following holds:

1. $A^{(k)}$ is non-trivial but $A^{(k+1)}$ is trivial,

2. $l=0$ if $A^{(k)}$ is atomic, and $l=1$ otherwise,

3. $m=\min(\omega, |\mathrm{At}\, A^{(k)}|)$

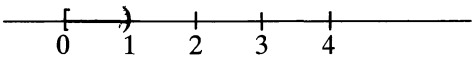
(where At B denotes the set of atoms of a Boolean algebra B).

We now prove that each triple (k, l, m) in *Inv* is actually attained.

18.5. PROPOSITION. *For each triple (k, l, m) in Inv, there is a Boolean algebra A with* $\mathrm{inv}(A)=(k, l, m)$.

PROOF. If $k=-1$, let A be the trivial algebra. For $k \geq 0$, we shall construct by induction on k a linear order X with first element such that $A = \mathrm{Intalg}\, X$ works. This is easy for $k=0$: e.g. if $l=1$ and $m=\omega$, let $A=\mathrm{Intalg}\, R$, where R is the following subset of the reals:

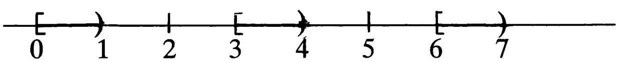
$$R=[0,1) \cup \{2,3,4,\ldots\};$$



similarly if $l=0$ or $m<\omega$, there is a clopen subset S of R such that Intalg S has invariants $(0, l, m)$.

Now assume that $0 \leq k < \omega$ and a linear order $(X, <)$ is given such that $A=\mathrm{Intalg}\, X$ has invariants (k, l, m). We construct a linear order Z such that for $B=\mathrm{Intalg}\, Z$, $B/E(B)$ is isomorphic to A; then the invariants of B are $(k+1, l, m)$. Define a subset Y of the reals by

$$Y=\bigcup_{n\in\omega}([3n, 3n+1) \cup \{3n+2\}).$$



Take pairwise disjoint copies $(Y_x, <_x)$ of $(Y, <)$ for $x \in X$ and let

$$Z=\bigcup_{x\in X} Y_x;$$

Z is totally ordered by letting, for y and y' in Z,

$$y <_Z y' \quad \text{iff for some } x \text{ in } X\text{: } y, y' \in Y_x \text{ and } y <_x y'$$
$$\text{or, for some } x \text{ and } x' \text{ in } X\text{: } y \in Y_x,\ y' \in Y_{x'} \text{ and } x < x'.$$

Recall from the proof of Proposition 15.9 that, for any ideal I of $B=\mathrm{Intalg}\, Z$, B/I is isomorphic to $\mathrm{Intalg}(Z/{\sim_I})$, where $\sim_I$ is the convex equivalence relation on Z defined by

$$y \sim_I y' \quad \text{iff } [y, y') \in I$$

for $y \leq_Z y'$ in Z. By our choice of Y above, we have for $I = E(B)$ and $y \leq_Z y'$ in Z

$$y \sim_{E(B)} y' \quad \text{iff } y, y' \in Y_x \text{ for some } x \in X\,.$$

Hence, the linear order $Z/{\sim_{E(B)}}$ is isomorphic to $(X, <)$ and $B/E(B)$ is isomorphic to $\operatorname{Intalg} X \cong A$.

Finally, we can choose by the above for every $k \in \omega$ a linear order X_k such that $\operatorname{Intalg} X_k$ has invariants $(k, 1, 0)$. Assuming that the X_k are pairwise disjoint and defining as above a linear order on $X = \bigcup_{k\in\omega} X_k$ such that each X_k is a convex subset of X, we find that $\operatorname{Intalg} X$ has invariants $(\omega, 0, 0)$. □

It is the purpose of the rest of this subsection to express the elementary invariants of a Boolean algebra in terms of first order logic. We use some obvious abbreviations in our formulas; in particular we write $x \leq y$ for $x \cdot y = x$, $x < y$ for $(x \leq y \wedge x \neq y)$ and $x \triangle y$ for $x \cdot -y + y \cdot -x$. We say that a formula $\gamma(x)$, having x as its single free variable, *defines* the subset I of an L-structure A if $I = \{a \in A\colon A \models \gamma[a]\}$.

18.6. NOTATION. Assume that A is a Boolean algebra and that $\gamma(x)$ defines an ideal I of A; for $a \in A$, denote the equivalence class of a with respect to I by a/I. There is an effective procedure assigning to each L-formula $\phi = \phi(x_1 \cdots x_n)$ another L-formula $\phi/\gamma = (\phi/\gamma)(x_1 \ldots x_n)$ such that, for any $a_1, \ldots, a_n$ in A,

(3) $$A \models (\phi/\gamma)[a_1 \ldots a_n] \quad \text{iff } A/I \models \phi[a_1/I \ldots a_n/I]:$$

simply let ϕ/γ be the result of replacing each atomic subformula $t = t'$ of ϕ, where t and t' are terms in L, by the formula $\gamma(t \triangle t')$ stating that $t/I = t'/I$. (3) follows easily by induction on the complexity of ϕ; for existential ϕ, the proof works since the canonical map from A to A/I is onto.

18.7. NOTATION. We fix several formulas in the language L and explain their interpretation in a Boolean algebra A:

at(x): $0 < x \wedge \forall y(y \leq x \rightarrow y = 0 \vee y = x)$

("x is an atom of A");

atl(x): $\neg\exists y(y \leq x \wedge \mathrm{at}(y))$

("$A \restriction x$ is atomless");

atc(x): $\forall z(z \leq x \wedge 0 < z \rightarrow \exists y(y \leq z \wedge \mathrm{at}(y)))$

("$A \restriction x$ is atomic");

$\varepsilon(x)$: $\exists y \exists z(x = y + z \wedge \mathrm{atl}(y) \wedge \mathrm{atc}(z))$.

Thus, $\varepsilon(x)$ defines the ideal $E(A)$, in every Boolean algebra A. By induction and using the notation of 18.6, we define formulas $\varepsilon_i(x)$ for $i \in \omega$ by

$$\varepsilon_0(x): \quad x = 0\,,$$

$$\varepsilon_{i+1}(x): \quad \varepsilon / \varepsilon_i\,.$$

It follows from (3) above that, in every Boolean algebra A, $\varepsilon_i(x)$ defines $E_i(A)$, i.e.

(4) $\quad A \models \varepsilon_i[a]$ iff $a \in E_i(A)$.

For k and m in ω, define the formulas:

$$\lambda_k(x): \quad \exists z[z \leq x \wedge 0 < z \wedge \mathrm{atl}(z)] / \varepsilon_k\,,$$

$$\alpha_{kn}(x): \quad \Big(\exists y_1 \cdots y_n\Big[\bigwedge_{1 \leq i \leq n} (y_i \leq x \wedge \mathrm{at}(y_i)) \wedge \bigwedge_{1 \leq i < j \leq n} y_i \neq y_j\Big]\Big) / \varepsilon_k\,.$$

By (3) above, for every Boolean algebra A and $a \in A$,

(5) $\quad A \models \lambda_k[a]$ iff $A^{(k)} \restriction \pi_k(a)$ has a non-trivial atomless relative algebra,

(6) $\quad A \models \alpha_{kn}[a]$ iff $A^{(k)} \restriction \pi_k(a)$ has at least n atoms.

For A an L-structure and $a \in A$, we say that a *realizes* a set $\Sigma(x)$ of L-formulas and write $A \models \Sigma[a]$ if $A \models \sigma[a]$, for each formula $\sigma(x)$ in $\Sigma(x)$.

18.8. NOTATION AND LEMMA. For each triple (k, l, m) in *Inv*, there are a set $\Sigma_{klm}(x)$ of L-formulas and a theory T_{klm} in L such that, for every Boolean algebra A and every a in A,

$$A \models \Sigma_{klm}[a] \quad \text{iff } \mathrm{inv}(A \restriction a) = (k, l, m)\,,$$

$$A \models T_{klm} \qquad \text{iff } \mathrm{inv}(A) = (k, l, m)\,.$$

PROOF. Given $\Sigma_{klm}(x)$ with the desired property, we simply let

$$T_{klm} = \Sigma_{klm}(1)\,,$$

i.e. T_{klm} results from $\Sigma_{klm}(x)$ by substituting in each $\sigma(x) \in \Sigma_{klm}(x)$ the constant symbol 1 of L for the free variable x.

If $k = -1$, let $\Sigma_{klm}(x)$ be the set $\{x = 0\}$.

If $k = \omega$, let $\Sigma_{klm}(x)$ by $\{\neg \varepsilon_i(x) \colon i \in \omega\}$.

If $k \in \omega$, $l = 0$ and $m < \omega$, define

$$\Sigma_{klm}(x) = \{\neg \varepsilon_k(x),\ \varepsilon_{k+1}(x),\ \neg \lambda_k(x),\ \alpha_{km}(x),\ \neg \alpha_{k,m+1}(x)\}\ .$$

If $k \in \omega$, $l = 1$ and $m < \omega$, let

$$\Sigma_{klm}(x) = \{\neg \varepsilon_k(x),\ \varepsilon_{k+1}(x),\ \lambda_k(x),\ \alpha_{km}(x),\ \neg \alpha_{k,m+1}(x)\}\ .$$

If $k \in \omega$, $l = 0$ and $m = \omega$, let

$$\Sigma_{klm}(x) = \{\neg \varepsilon_k(x),\ \varepsilon_{k+1}(x),\ \neg \lambda_k(x)\} \cup \{\alpha_{kn}(x) \colon n \in \omega\}\ .$$

Finally, for $k \in \omega$, $l = 1$ and $m = \omega$, put

$$\Sigma_{klm}(x) = \{\neg \varepsilon_k(x),\ \varepsilon_{k+1}(x),\ \lambda_k(x)\} \cup \{\alpha_{kn}(x) \colon n \in \omega\}\ .$$

This choice of $\Sigma_{klm}(x)$ works for the lemma, by (4), (5), (6), and 18.3. □

Two L-structures A and B are called *elementarily equivalent* if they satisfy the same L-sentences.

18.9. COROLLARY. *Any two elementarily equivalent Boolean algebras have the same elementary invariants.*

PROOF. Immediate from 18.8. □

18.2. Elementary equivalence of Boolean algebras

The aim of this subsection is a converse to Corollary 18.9:

18.10. PROPOSITION. *Any two Boolean algebras having the same elementary invariants are elementarily equivalent.*

The proof relies essentially on the fact that the relation of having the same elementary invariants is a Vaught relation on the class of ω-saturated Boolean algebras – see Lemma 18.12. Proposition 18.10 would easily follow if we knew that

(7) every Boolean algebra is elementarily equivalent to a countable and ω-saturated one ,

much as Theorem 17.11 (a countable superatomic Boolean algebra is determined by its Cantor–Bendixson invariants) follows from the fact that the relation of having the same Cantor–Bendixson invariants is a Vaught relation on the class of superatomic Boolean algebras. A proof of (7) is outlined in Exercise 1, but it uses the results of this subsection and does not readily follow from general principles of model theory. We therefore replace (7) by the weaker assertion

(7′) every Boolean algebra is elementarily equivalent to an ω-saturated one ,

a fact well known from model theory. In the proof of 18.10 we then cannot conclude, from Vaught's theorem 5.15, isomorphism of two ω-saturated algebras but only, using a back-and-forth argument, elementary equivalence.

We recall a few definitions from model theory. Let A be an L-structure. For arbitrary $M \subseteq A$, L_M is the language

$$L_M = L \cup \{c_a : a \in M\} ,$$

where c_a is a new constant symbol for $a \in M$. We denote by A_M the L_M-structure with L-reduct A in which c_a is interpreted by a. A set $\Sigma(x)$ of L_M-formulas is said to be *finitely satisfiable* in A_M if every finite subset of $\Sigma(x)$ is realized, in A_M, by some element of A. And A is *ω-saturated* if for every finite $M \subseteq A$, every set $\Sigma(x)$ in L_M which is finitely satisfiable in A_M is realized in A_M.

18.11. REMARK. For any Boolean algebra A and $a \in A$, the invariants of A result from those of $A \upharpoonright a$ and $A \upharpoonright -a$ as follows, by 18.3: assume

$$\text{inv}(A \upharpoonright a) = (k, l, m) , \qquad \text{inv}(A \upharpoonright -a) = (k', l', m') .$$

Then

$$\text{inv}(A) = (k', l', m') \quad \text{if } k < k'$$

and

$$\text{inv}(A) = (k, \max(l, l'), m + m') \quad \text{if } k = k' ,$$

where $m + m'$ denotes cardinal addition.

18.12. LEMMA. *Assume that A and B are Boolean algebras such that* $\text{inv}(A) = \text{inv}(B)$ *and that B is ω-saturated. Then for every a in A there is some b in B satisfying*

$$\text{inv}(A \upharpoonright a) = \text{inv}(B \upharpoonright b) , \qquad \text{inv}(A \upharpoonright -a) = \text{inv}(B \upharpoonright -b) .$$

PROOF. Let $a \in A$ be given and put

$$\text{inv}(A \upharpoonright a) = (k, l, m) , \qquad \text{inv}(A \upharpoonright -a) = (k', l', m') .$$

We may assume that $k \leq k'$, otherwise replacing a by $-a$, and consider three cases.

Case 1. $k < k'$. Then $k < \omega$ and $\text{inv}(B) = (k', l', m')$, by Remark 18.11. It is therefore sufficient to find $b \in B$ such that $\text{inv}(B \upharpoonright b) = (k, l, m)$, i.e. by 18.8 such that $B \models \Sigma_{klm}[b]$.

Now $E(B^{(k)})$ is a proper ideal in $B^{(k)}$, for otherwise the invariants of B would have the form (k'', l'', m'') *with* $k'' \leq k$, which contradicts $k < k'$. Thus, $B^{(k)}$ has infinitely many atoms but is not atomic. Considering the definition of $\Sigma_{klm}(x)$ which speaks about the kth elementary derivative, we find that $\Sigma_{klm}(x)$ is finitely satisfiable in B. Hence, by ω-saturatedness of B, $\Sigma_{klm}(x)$ is realized by some element of B.

Case 2. $k = k' = \omega$. Then $\operatorname{inv}(B) = \operatorname{inv}(A) = (\omega, 0, 0)$; it is sufficient to find $b \in B$ such that $B \models \Sigma_{\omega 00}[b]$ and $B \models \Sigma_{\omega 00}[-b]$, i.e. such that $B \models \Sigma[b]$, where

$$\Sigma(x) = \{\neg \varepsilon_i(x) \colon i \in \omega\} \cup \{\neg \varepsilon_i(-x) \colon i \in \omega\} .$$

Since, in the theory of Boolean algebras, $\neg \varepsilon_i(x)$ is a consequence of $\neg \varepsilon_{i+1}(x)$ and since B is ω-saturated, it is sufficient to prove that, for every $i \in \omega$,

$$B \models \exists x(\neg \varepsilon_i(x) \wedge \neg \varepsilon_i(-x)) .$$

Now by $\operatorname{inv}(B) = (\omega, 0, 0)$, $B^{(i)} = B/E_i(B)$ is infinite; so there is some $b \in B$ such that neither b nor $-b$ is in $E_i(B)$.

Case 3. $k = k' < \omega$. Then

$$\operatorname{inv}(B) = \operatorname{inv}(A) = (k, \max(l, l'), m + m') . \tag{8}$$

We want to find $b \in B$ such that $B \models \Sigma_{klm}[b]$ and $B \models \Sigma_{kl'm'}[-b]$, i.e. such that $B \models \Sigma[b]$, where

$$\Sigma(x) = \Sigma_{klm}(x) \cup \{\sigma(-x) \colon \sigma(x) \in \Sigma_{kl'm'}(x)\} .$$

Let $\Sigma'(x)$ be any finite subset of $\Sigma(x)$. (8) implies that

(9) $A^{(k)}$ has a non-trivial atomless relative algebra iff $B^{(k)}$ has a non-trivial atomless relative algebra ,

and, for $r \in \omega$,

(10) $A^{(k)}$ has at least r atoms iff $B^{(k)}$ has at least r atoms .

Now $\Sigma'(x)$, expressing existence or non-existence of a non-trivial atomless relative algebra and existence or non-existence of a certain number of atoms in the kth elementary derivative, is realized by a in A. By (9) and (10), it is realized in B. □

ω-saturatedness of B was used in the preceding proof only for sets of L-formulas, i.e. for sets $\Sigma(x)$ not containing any parameters from a subset M of B. But parameters are used in the proof of the following simple fact.

18.13. LEMMA. *Each relative algebra of an ω-saturated Boolean algebra is ω-saturated.*

PROOF. Let A be ω-saturated, $a \in A$, M a finite subset of $A \upharpoonright a$ and suppose that $\Sigma(x)$, a set of formulas in L_M, is finitely satisfiable in $(A \upharpoonright a)_M$. Put $M' = M \cup \{a\}$, a finite subset of A. There is an obvious assignment

$$\phi(x_1 \ldots x_n) \rightarrow \phi^*(x_1 \ldots x_n)$$

from L_M-formulas to $L_{M'}$-formulas such that, for all $a_1, \ldots, a_n$ in $A \upharpoonright a$,

$$A_{M'} \models \phi^*[a_1 \ldots a_n] \quad \text{iff} \, (A \upharpoonright a)_M \models \phi[a_1 \ldots a_n] . \tag{11}$$

Let

$$\Sigma^*(x) = \{\phi^*(x) \colon \phi(x) \in \Sigma(x)\} .$$

Then by (11), the set $\Sigma^*(x) \cup \{x \leq c_a\}$ of $L_{M'}$-formulas is finitely satisfiable in $A_{M'}$; by ω-saturatedness of A, it is realized in $A_{M'}$ by some element b of A. So $b \in A \upharpoonright a$ and, by (11) again, b realizes $\Sigma(x)$ in $(A \upharpoonright a)_M$. □

Proof of Proposition 18.10. Assume A and B are Boolean algebras with the same invariants. Since every Boolean algebra is elementarily equivalent to an ω-saturated one and elementarily equivalent algebras have the same invariants, by Corollary 18.9, we may assume that A and B are ω-saturated. Note that

$$\text{if } s \in A,\ t \in B \text{ and } \mathrm{inv}(A \upharpoonright s) = \mathrm{inv}(B \upharpoonright t), \text{ then } s = 0 \text{ iff } t = 0 . \tag{12}$$

This holds since $s = 0$ implies $\mathrm{inv}(B \upharpoonright t) = \mathrm{inv}(A \upharpoonright s) = (-1, 0, 0)$ and thus $t = 0$.

Let P be the set of those isomorphisms

$$p \colon A_0 \rightarrow B_0$$

from a finite subalgebra A_0 of A onto a finite subalgebra B_0 of B which satisfy

$$\mathrm{inv}(A \upharpoonright a) = \mathrm{inv}(B \upharpoonright p(a)) \tag{13}$$

for every atom a of A_0. P is non-empty since, by $\mathrm{inv}(A) = \mathrm{inv}(B)$, the function mapping 0_A to 0_B and 1_A to 1_B is in P. The following back-and-forth property of P is the crucial fact of our proof.

Claim. If $p \in P$ and $x \in A$ (respectively $y \in B$), then there is some $q \in P$ such that $p \subseteq q$ and $x \in \mathrm{dom}\ q$ (respectively $y \in \mathrm{ran}\ q$).

For assume $p \colon A_0 \rightarrow B_0$ and $x \in A$ are given; the proof for $y \in B$ is similar. For each atom a of A_0, $B \upharpoonright p(a)$ is ω-saturated, by 18.13. So by (13) and Lemma 18.12, there is some y_a in B such that $y_a \leq p(a)$ and

$$\mathrm{inv}(A \upharpoonright (a \cdot x)) = \mathrm{inv}(B \upharpoonright y_a) , \tag{14}$$

$$\mathrm{inv}(A \upharpoonright (a \cdot -x)) = \mathrm{inv}(B \upharpoonright (p(a) \cdot -y_a)) .$$

The sets

$$X = (\{a \cdot x: a \in \mathrm{At}(A_0)\} \cup \{a \cdot -x: a \in \mathrm{At}(A_0)\}) \setminus \{0_A\},$$

$$Y = (\{y_a: a \in \mathrm{At}(A_0)\} \cup \{p(a) \cdot -y_a: a \in \mathrm{At}(A_0)\}) \setminus \{0_B\},$$

are finite partitions of A (respectively B) generating finite subalgebras A_1 (respectively B_1) of B with X (respectively Y) as their sets of atoms. Thus, there is by (12) and (14) a unique isomorphism

$$q: A_1 \to B_1$$

such that $q \in P$, q extends p, and $q(a \cdot x) = y_a$ for $a \in \mathrm{At}(A_0)$. Clearly, $x \in A_1$ since $x = \Sigma\{a \cdot x: a \in \mathrm{At}(A_0)\}$. This finishes the proof of the Claim.

The elementary equivalence of A and B now follows immediately from the fact that P is non-empty and that for every L-formula $\phi(x_1 \ldots x_n)$, $p \in P$ and $a_1, \ldots, a_n \in \mathrm{dom}\, p$,

$$(15) \qquad A \models \phi[a_1 \ldots a_n] \quad \text{iff } B \models \phi[p(a_1) \ldots p(a_n)].$$

We prove (15) by induction on the complexity of ϕ: for ϕ atomic, (15) holds since p is an isomorphism from dom p onto ran p. The cases that ϕ is a disjunction or a negation are easy. So suppose that ϕ has the form $\exists x \psi(xx_1 \ldots x_n)$ and that (15) holds for the formula $\psi(xx_1 \ldots x_n)$, every $q \in P$ and every sequence $(a, a_1, \ldots, a_n)$ in dom q. Let $p \in P$ and $a_1, \ldots, a_n \in \mathrm{dom}\, p$. If $A \models \phi[a_1 \ldots a_n]$, pick $a \in A$ such that

$$A \models \psi[aa_1 \ldots a_n].$$

By the "forth" part of the Claim, let $q \in P$ extend p such that $a \in \mathrm{dom}\, q$. Then by the inductive hypothesis on ψ,

$$B \models \psi[q(a)q(a_1) \ldots q(a_n)],$$

$$B \models \psi[q(a)p(a_1) \ldots p(a_n)] \quad \text{by } p \subseteq q,$$

$$B \models \phi[p(a_1) \ldots p(a_n)].$$

Similarly, the "back" part of the Claim shows that $B \models \phi[p(a_1) \ldots p(a_n)]$ implies $A \models \phi[a_1 \ldots a_n]$. □

18.3. The decidability proof

Having characterized elementary equivalence of Boolean algebras, we finish the proof of Tarski's theorem by describing the completions of the theory of Boolean algebras in a uniform way.

Denote by Sent(L) the set of all sentences in the language L. For any theory S in L (i.e. for any subset of Sent(L)) the set of *consequences* of S is defined by

$$\mathrm{Cn}\, S = \{\phi \in \mathrm{Sent}(L)\colon S \models \phi\} .$$

A theory T^* in L is said to be a *completion* of a theory T if $T \subseteq T^*$ and for every $\phi \in \mathrm{Sent}(L)$, either ϕ or $\neg\phi$ is in T^*, but not both. It is well known that T^* is a completion of T iff, for some model A of T,

$$T^* = \mathrm{Th}(A) = \{\phi \in \mathrm{Sent}(L)\colon A \models \phi\} .$$

18.14. NOTATION. For each triple (k, l, m) of elementary invariants, we denote by T'_{klm} the theory which results from T_{klm} (cf. 18.8) by adding (the universal closures of) the axioms (B1) through (B5′) for Boolean algebras in Section 1.

18.15. PROPOSITION. *The completions of the first order theory* Th(***BA***) *of Boolean algebras are exactly the theories* Cn T'_{klm}, *where* $(k, l, m) \in \mathit{Inv}$.

PROOF. For any triple (k, l, m) of elementary invariants, T'_{klm} and hence Cn T'_{klm} has a model by 18.5 and 18.8, say A. If B is another model of Cn T'_{klm}, then also B has the invariants (k, l, m) and thus is, by 18.10, elementarily equivalent to A. It follows that Cn $T'_{klm} = \mathrm{Th}(A)$, hence Cn T'_{klm} is a completion of Th(***BA***).

For distinct triples (k, l, m) and (k', l', m') of elementary invariants, the theories Cn T'_{klm} and Cn $T'_{k'l'm'}$ are distinct since their union is inconsistent, i.e. every $\phi \in \mathrm{Sent}(L)$ is a consequence of Cn $T'_{klm} \cup$ Cn $T'_{k'l'm'}$.

Finally, suppose T^* is a completion of Th(***BA***), i.e. $T^* = \mathrm{Th}(A)$ for some Boolean algebra A. Let (k, l, m) be the invariants of A. Then Th(A) = Cn T'_{klm}, as shown above. □

Proof of Tarski's theorem 18.1. We give an informal proof; a reader looking for a more formal one might consult RABIN [1977]. It consists in showing that both Th(***BA***) and Sent(L)\Th(***BA***) can be effectively enumerated.

The theory Th(***BA***) has (the universal closures of) (B1) through (B5′) from Section 1 as a finite, hence recursive system of axioms. So there is an effective enumeration of Th(***BA***):

$$\mathrm{Th}(\boldsymbol{BA}) = \{\psi_n\colon n \in \omega\} .$$

Next, for each triple (k, l, m) in *Inv*, the complete theory Cn T'_{klm} has T'_{klm} as a recursive system of axioms, so again there is an effective enumeration

$$\mathrm{Cn}\, T'_{klm} = \{f_{klm}(n)\colon n \in \omega\} .$$

Also *Inv* is a decidable subset of $(\{-1\} \cup (\omega + 1)) \times 2 \times (\omega + 1)$, and the assignment of the function f_{klm} to the triple (k, l, m) can be made uniformly in k, l, m – i.e. the assignment of $f_{klm}(n)$ to (k, l, m, n) is effective. By a standard diagonal procedure, there is an effective enumeration

$$S = \bigcup\{\mathrm{Cn}\, T'_{klm}: (k, l, m) \in Inv\} = \{\chi_n: n \in \omega\}\,.$$

For each sentence σ in L, $\sigma \in S$ iff (by 18.15) σ holds true in some Boolean algebra. So

(15) $\neg\sigma \in S$ iff $\sigma \notin \mathrm{Th}(\boldsymbol{BA})$.

Given the effective enumerations of $\mathrm{Th}(\boldsymbol{BA})$ and S, the following procedure decides whether an L-sentence ϕ belongs to $\mathrm{Th}(\boldsymbol{BA})$:
check whether ϕ is ψ_0 – then $\phi \in \mathrm{Th}(\boldsymbol{BA})$,
check whether $\neg\phi$ is χ_0 – then $\phi \notin \mathrm{Th}(\boldsymbol{BA})$,
check whether ϕ is ψ_1 – then $\phi \in \mathrm{Th}(\boldsymbol{BA})$,
check whether $\neg\phi$ is χ_1 – then $\phi \notin \mathrm{Th}(\boldsymbol{BA})$,
etc. Since either $\phi \in \mathrm{Th}(\boldsymbol{BA}) = \{\psi_n: n \in \omega\}$ or, by (15), $\neg\phi \in S = \{\chi_n: n \in \omega\}$, we have decided after finitely many steps whether ϕ is in $\mathrm{Th}(\boldsymbol{BA})$. □

Exercises

1. We outline a proof of assertion (7) in this section: every Boolean algebra is elementarily equivalent to a countable and ω-saturated one.
(a) For any Boolean algebras A, A', B, and B', $A \equiv A'$ and $B \equiv B'$ implies that $A \times B \equiv A' \times B'$.
(b) For A a Boolean algebra, M a finite subset of A and $(a_1, \ldots, a_n)$ a finite sequence in A, the *elementary type* realized by $(a_1, \ldots, a_n)$ in A_M is defined by

$$\mathrm{tp}(a_1 \ldots a_n, A_M) \doteq \{\phi(x_1 \ldots x_n): \phi \text{ a formula in the language } L_M,\ A_M \models \phi[a_1 \ldots a_n]\}\,.$$

Show that for M the empty set, $\mathrm{tp}(a_1 \ldots a_n, A)$ is determined by the sequence $(\mathrm{Th}(A \restriction p_e))_{e \in E}$, where E is the set of all functions $e: \{1, \ldots, n\} \to \{+1, -1\}$ and p_e is the elementary product $e(1)a_1 \cdot \ldots \cdot e(n)a_n$.
(c) Assume A is a Boolean algebra and M a finite subset of A. Show that only countably many elementary types are realized in A_M.
(d) Every Boolean algebra is elementarily embeddable into an ω-saturated algebra of the same cardinality.
2. Consider the Lindenbaum–Tarski algebra A of the theory of Boolean algebras. It is shown in Exercise 4 of Section 2 that the ultrafilters of A are in one-to-one correspondence with the completions of $\mathrm{Th}(\boldsymbol{BA})$. Use this to prove that A is superatomic and, in fact, isomorphic to the interval algebra of the ordinal $\omega^2 + 1$ (cf. the remarks following 17.11).

19. Undecidability of the first order theory of Boolean algebras with a distinguished subalgebra

We shall prove a result due to M. Rubin: the first order theory of Boolean pairs, and also several stronger theories, are undecidable. Here a *Boolean pair* is a pair (A, B), where A is a Boolean algebra and B a subalgebra of A. Boolean pairs may be considered as structures for the language

$$L_U = \{+, \cdot, -, 0, 1, U\}$$

of Boolean algebras, augmented by a unary predicate U, where U is interpreted, in the structure (A, B), by the subalgebra B.

More precisely, we show that each of the following classes of L_U-structures has an undecidable theory:

$$\boldsymbol{BP} = \{(A, B) : (A, B) \text{ a Boolean pair}\},$$

$$\boldsymbol{BP}_{\text{rc}} = \{(A, B) \in \boldsymbol{BP} : B \text{ relatively complete in } A\},$$

$$\boldsymbol{BP}_{\text{at}} = \{(A, B) \in \boldsymbol{BP} : A, B \text{ atomic and At } A \subseteq B\},$$

$$\boldsymbol{BP}^* = \{(A, B) \in \boldsymbol{BP} : A, B \text{ atomic}, B \text{ relatively complete in } A, \; A \upharpoonright a \not\subseteq B \text{ for } a \in A^+\};$$

recall that a subalgebra B of A was defined, in 8.19, to be relatively complete in A if for every element a of A there is a greatest element b of B satisfying $b \leq a$. This is, of course, equivalent to saying that for every element a of A there is a least element b of B satisfying $a \leq b$.

The undecidability of $\text{Th}(\boldsymbol{BP}_{\text{at}})$ has been applied in universal algebra, as outlined in the introduction to Chapter 7. For readers acquainted with algebraic logic, the undecidability of $\text{Th}(\boldsymbol{BP}_{\text{rc}})$ might be most interesting. For it is easily seen that the pairs in $\boldsymbol{BP}_{\text{rc}}$ are in one-to-one correspondence with monadic algebras (respectively cylindric algebras) of dimension 1; thus, also the first order theories of monadic algebras (respectively of cylindric algebras) of dimension 1 are undecidable. It was in fact the question of decidability for these theories which motivated Rubin's investigations. The same question had previously inspired Comer who found sheaf representations of structures for arbitrary first order languages, as defined in Section 8, to be a powerful method for proving decidability results. See Exercise 2 for an easy example of this method.

We describe, in the first subsection, a general technique for showing undecidability and explain how, through this technique, all of our assertions reduce to one crucial fact: the class of finite graphs is semantically embeddable into the class $\boldsymbol{BP}^*$ (Theorem 19.5). The proof of this is postponed to the second subsection. Our presentation follows that given in BURRIS and MCKENZIE [1981].

For a survey of further undecidability results on Boolean algebras, cf. WEESE [Ch. 24 in this Handbook].

19.1. The method of semantical embeddings

We begin by formally describing, in 19.2, a method for proving undecidability. Some (pretty obvious) definitions are in order.

19.1. NOTATION AND DEFINITION. Let L be any first order language.

(a) If $\phi(x_1 \ldots x_n)$ is an L-formula and B an L-structure, then ϕ^B is the n-ary relation on B defined by ϕ:

$$\phi^B = \{(b_1, \ldots, b_n) \in B^n : B \models \phi[b_1 \ldots b_n]\} .$$

(b) An equivalence relation $\sim$ on B is a *congruence relation* for an n-ary relation r on B if, for all elements $b_1, \ldots, b_n, b'_1, \ldots, b'_n$ of B, $(b_1, \ldots, b_n) \in r$ and $b_i \sim b'_i$ for $1 \le i \le n$ implies $(b'_1, \ldots, b'_n) \in r$.

(c) Assume (by abuse of notation) that $B = (B, r_i)_{i \in I}$, where r_i is an n_i-ary relation on B and that $\sim$ is a congruence relation for each r_i; denote the equivalence class of $b \in B$ by $b/\sim$. Then the *quotient structure* of B with respect to $\sim$ is the L-structure

$$(B/\sim, s_i)_{i \in I} ,$$

where, for $b_1, \ldots, b_{n_i} \in B$,

$$(b_1/\sim, \ldots, b_{n_i}/\sim) \in s_i \quad \text{iff } (b_1, \ldots, b_{n_i}) \in r_i .$$

The following definition is crucial for the method of undecidability proofs. Loosely speaking, a class $\boldsymbol{K}'$ of L'-structures is semantically embeddable into a class $\boldsymbol{K}$ of L-structures if each member of $\boldsymbol{K}'$ is obtained, in a uniform way, from a member of $\boldsymbol{K}$ by taking quotients as described in 19.1. We may assume in 19.2 that L' has no function symbols or constant symbols, otherwise replacing n-ary function symbols by $(n+1)$-ary relation symbols and constant symbols by unary relation symbols.

19.2. DEFINITION. Let L and L' be first order languages; assume that

$$L' = \{r_1, \ldots, r_k\}$$

is finite and r_i is an n_i-ary relation symbol. Let $\boldsymbol{K}'$ be a class of L'-structures and $\boldsymbol{K}$ a class of L-structures. We say that $\boldsymbol{K}'$ is *semantically embeddable* into $\boldsymbol{K}$ and write

$$\boldsymbol{K}' \rightarrow_{\text{sem}} \boldsymbol{K}$$

if there are L-formulas

$$\text{un}(x) , \quad \text{eq}(x, y) , \quad \phi_{r_1}(x_1, \ldots, x_{n_1}) , \ldots , \phi_{r_k}(x_1, \ldots, x_{n_k})$$

such that for each $A \in \boldsymbol{K}'$ there is some $B \in \boldsymbol{K}$ satisfying

(a) un^B is non-empty and, for $1 \le i \le k$, $\phi^B_{r_i}$ is an n_i-ary relation on un^B,

(b) eq^B is an equivalence relation on un^B and a congruence relation for every $\phi^B_{r_i}$,

(c) A is isomorphic to the quotient structure of $(\text{un}^B, \phi^B_{r_1}, \ldots, \phi^B_{r_k})$ with respect to eq^B.

19.3. DEFINITION. A *finite graph* is a pair (G, r), where G is a finite non-empty set and r an irreflexive and symmetric binary relation on G. It can be considered as being a structure for the language $\{r\}$ with one binary relation symbol. $\boldsymbol{G}_{\mathrm{fin}}$ is the class of all finite graphs.

The first part of the following lemma is obvious; the second one can be found in ERSHOV, LAVROV, TAIMANOV and TAICLIN [1965].

19.4. LEMMA (Ershov, Rabin). (a) *The relation* $\rightarrow_{\mathrm{sem}}$ *is transitive: if* $\boldsymbol{K}'' \rightarrow_{\mathrm{sem}} \boldsymbol{K}'$ *and* $\boldsymbol{K}' \rightarrow_{\mathrm{sem}} \boldsymbol{K}$, *then* $\boldsymbol{K}'' \rightarrow_{\mathrm{sem}} \boldsymbol{K}$.

(b) *If* $\boldsymbol{G}_{\mathrm{fin}} \rightarrow_{\mathrm{sem}} \boldsymbol{K}$, *then the first order theory* $\mathrm{Th}(\boldsymbol{K})$ *of* $\boldsymbol{K}$ *is undecidable.*

19.5. THEOREM (M. Rubin, McKenzie). $\boldsymbol{G}_{\mathrm{fin}}$ *is semantically embeddable into the class* $\boldsymbol{BP}^*$. *Hence*, $\mathrm{Th}(\boldsymbol{BP}^*)$ *is undecidable.*

This is proved in the next subsection.

19.6. COROLLARY. $\boldsymbol{G}_{\mathrm{fin}}$ *is semantically embeddable into the classes* $\boldsymbol{BP}$ *and* $\boldsymbol{BP}_{\mathrm{rc}}$. *Hence, their first order theories are undecidable.*

PROOF. Since

$$\boldsymbol{BP}^* \subseteq \boldsymbol{BP}_{\mathrm{rc}} \subseteq \boldsymbol{BP},$$

a semantical embedding of $\boldsymbol{G}_{\mathrm{fin}}$ into $\boldsymbol{BP}^*$ is also a semantical embedding of $\boldsymbol{G}_{\mathrm{fin}}$ into $\boldsymbol{BP}_{\mathrm{rc}}$ and $\boldsymbol{BP}$. □

19.7. COROLLARY. $\boldsymbol{BP}^*$ *is semantically embeddable into* $\boldsymbol{BP}_{\mathrm{at}}$. *Hence, the first order theory of* $\boldsymbol{BP}_{\mathrm{at}}$ *is undecidable.*

PROOF. We exhibit a semantical embedding of $\boldsymbol{BP}^*$ into $\boldsymbol{BP}_{\mathrm{at}}$; the undecidability of $\mathrm{Th}(\boldsymbol{BP}_{\mathrm{at}})$ then follows from 19.5 and 19.4.

Let (A^*, B^*) in $\boldsymbol{BP}^*$ be given. We may assume, e.g. by the proof of Proposition 2.6, that A^* is a subalgebra of $P(X)$ for some set X and that the atoms of A^* are the singletons $\{x\}$, where $x \in X$. Let

$$Y = X \times \omega$$

and consider the embedding

$$e: A^* \rightarrow P(Y), \quad e(a) = a \times \omega .$$

Let *Fin* be the ideal of finite subsets of Y and define subalgebras A and B of $P(Y)$ by

$$A = \langle e[A^*] \cup Fin \rangle ,$$

$$B = \langle e[B^*] \cup Fin \rangle .$$

Clearly, the pair (A, B) is in $\boldsymbol{BP}_{\mathrm{at}}$, and $A/Fin \cong A^*$, $B/Fin \cong B^*$. Moreover, the ideal Fin of A is first order definable in the L_U-structure (A, B): let ϕ_{fin} be the L_U-formula

$$\phi_{\mathrm{fin}}(x): \quad \forall z(z \leq x \rightarrow U(z)) .$$

We show that for $a \in A$,

$$(1) \qquad (A, B) \models \phi_{\mathrm{fin}}[a] \quad \text{iff } a \in Fin .$$

For (1) is equivalent to the assertion that $a \in Fin$ iff $A \restriction a \subseteq B$, and this holds since $(A^*, B^*) \in \boldsymbol{BP}^*$ and, for $x \neq 0$ in A^*, $e(x)$ is infinite.

These remarks suggest how to recover the structure (A^*, B^*) from (A, B): we have to consider, when applying Definition 19.2 to the classes $\boldsymbol{BP}^*$ and $\boldsymbol{BP}_{\mathrm{at}}$, the languages $L = L' = L_U$ and we follow the convention preceding 19.2 and consider $+$ as a ternary, 0 as a unary relation symbol of L_U, etc. Define L_U-formulas by

$$\mathrm{un}(x): \qquad x = x ,$$

$$\mathrm{eq}(x, y): \qquad \phi_{\mathrm{fin}}(x \cdot -y + y \cdot -x) ,$$

$$\phi_+(x, y, z): \quad x + y = z ,$$

$$\phi_0(x): \qquad x = 0 ,$$

$$\phi_U(x): \qquad U(x) ;$$

the formulas $\phi_\cdot(x, y, z)$, $\phi_-(x, y)$, $\phi_1(x)$ are defined similarly. Now (1) shows that

$$(\mathrm{un}^{(A,B)}, \phi_+^{(A,B)}, \ldots)/\mathrm{eq}^{(A,B)} \cong (A^*, B^*) ,$$

which concludes the proof. □

19.2. *Undecidability of* Th($\boldsymbol{BP}^*$)

We are left with proving Theorem 19.5: the class of finite graphs is semantically embeddable into the class $\boldsymbol{BP}^*$. The proof presented here is due to McKenzie.

Let a finite graph (G, r) be given; we shall assign to it a Boolean pair (A, B) coding (G, r). A and B will be algebras of sets over X, where

$$X = G \times \omega .$$

For each $g \in G$, define the "cylinder over g" to be the set

$$c_g = \{g\} \times \omega$$

and let then A be the subalgebra of $P(X)$ generated by the cylinders plus all finite subsets of X:

$$A = \langle \{c_g: g \in G\} \cup \{e \subseteq X: e \text{ finite}\} \rangle .$$

Hence, for $a \subseteq X$,

(2) $$a \in A \quad \text{iff } a \triangle \bigcup_{g \in H} c_g \text{ is finite, for some } H \subseteq G ,$$

here H is uniquely determined by a.

To define B, call a subset p of X a *candidate* (for an atom of B) if $2 \leq |c_g \cap p| \leq 3$, for each $g \in G$; by finiteness of G, also every candidate is finite and hence an element of A. Then let

$$B = \langle P \rangle ,$$

where

P is a partition of X and each $p \in P$ is a candidate ;

so B is isomorphic to the finite–cofinite algebra over P. This assumption on P is sufficient to prove that (A, B) is actually in the class $\boldsymbol{BP}^*$ and that the set G can be recovered from (A, B) in a uniform way; a more careful choice of P described below ensures that also the relation r on G is definable from (A, B).

We check that $(A, B) \in \boldsymbol{BP}^*$: clearly both A and B are atomic. For each non-zero element a of A, $A \upharpoonright a$ is not included in B since A is atomic but the atoms of A are not in B. Also B is relatively complete in A. For let $a \in A$, say, by (2), $a \triangle \bigcup_{g \in H} c_g$ is finite where $H \subseteq G$. Let $P_0 = \{p \in P: a \cap p \neq \emptyset\}$ and $b = \bigcup P_0$. Then either H is empty, so a and P_0 are finite and b is the least element of B including a. Otherwise, P_0 is cofinite in P; again b is in B and is the least element of B including a.

We proceed to define formulas un(x) and eq(x, y) in the language $L_U = \{+, \cdot, -, 0, 1, U\}$. Note that the singletons $\{x\}$, for x in X, are the atoms of A and the elements of P are the atoms of B; in particular an element of A has cardinality $n \in \omega$ if it is the sum of n distinct atoms of A. Thus, let "$|x| \leq n$" (respectively "$|x| \geq n$") abbreviate L_U-formulas stating that x is a sum of at most n distinct atoms of A (respectively that $A \upharpoonright x$ contains at least n distinct atoms). Similarly, let $\text{at}_U(x)$ be a formula in L_U expressing that x is an atom of (the subalgebra defined by) U. Put

$$\begin{aligned} \text{un}(x): \quad & \forall z[\text{at}_U(z) \rightarrow |x \cdot z| \leq 3] \\ & \wedge \forall x_1 x_2 x_3 [x = x_1 + x_2 + x_3 \rightarrow \exists z(\text{at}_U(z) \\ & \wedge (|x_1 \cdot z| \geq 2 \vee |x_2 \cdot z| \geq 2 \vee |x_3 \cdot z| \geq 2))] , \end{aligned}$$

$$\text{eq}(x, y): \quad \text{un}(x) \wedge \text{un}(y) \wedge \text{un}(x \cdot y) .$$

It is obvious that

(3) $(A, B) \models \mathrm{un}[c_g]$, for each $g \in G$.

Also, for a in A,

(4) if $(A, B) \models \mathrm{un}[a]$, then $|a \triangle c_g| < \omega$ for a unique $g \in G$.

To prove (4), assume that $a \in A$, $(A, B) \models \mathrm{un}[a]$ and, by (2), that $H \subseteq G$ and $a \triangle \bigcup_{g \in H} c_g$ is finite; we have to show that $|H| = 1$. For if $|H| \geq 2$, let g and h be distinct elements of H. Then there is $p \in P$ such that both $c_g \cap p$ and $c_h \cap p$ are included in a; so $|a \cap p| \geq 4$, contradicting $(A, B) \models \mathrm{un}[a]$. On the other hand if H is empty, then a and hence $P_0 = \{p \in P: a \cap p \neq \emptyset\}$ are finite. By $(A, B) \models \mathrm{un}[a]$, $|a \cap p| \leq 3$ for every $p \in P$. Thus, there are pairwise disjoint a_1, a_2, a_3 in A such that $a = a_1 + a_2 + a_3$ and $|a_j \cap p| \leq 1$ for all $p \in P_0$ and $j \in \{1, 2, 3\}$, which again contradicts $(A, B) \models \mathrm{un}[a]$. This finishes the proof of (4).

By (3) and (4), each $a \in A$ satisfying $(A, B) \models \mathrm{un}[a]$ codes a unique element of G. And (4) implies that, for a and b in A,

(5) $(A, B) \models \mathrm{eq}[a, b]$ iff a, b code the same element of G.

Imposing additional restrictions on the partition P allows us to recover the relation r on G from (A, B). To this end, call a candidate p *small* if $|c_g \cap p| = 2$ for every $g \in G$; if g, h are distinct elements of G, call p a *candidate for the pair* (g, h) if $|c_g \cap p| = |c_h \cap p| = 3$ but $|c_k \cap p| = 2$ for every $k \in G \setminus \{g, h\}$. We assume, for the rest of the proof, that

(6) for each $(g, h) \in r$, P contains infinitely many candidates for the pair (g, h),

(7) each $p \in P$ is either small or a candidate for some (unique) pair $(g, h) \in r$.

A partition P of X satisfying (6) and (7) is easily constructed by induction. Fig. 7.1 shows how to code the graph (G, r) with $G = \{g, h, k\}$ and $r = \{(g, h), (h, g), (g, k), (k, g)\}$.

Let ϕ_r be the L_U-formula

$$\phi_r(x): \quad \mathrm{un}(x) \wedge \mathrm{un}(y) \wedge \neg \mathrm{eq}(x, y) \wedge \forall s, t[\mathrm{eq}(x, s) \wedge \mathrm{eq}(y, t) \rightarrow \exists z(\mathrm{at}_U(z) \wedge |s \cdot z| = 3 \wedge |t \cdot z| = 3)].$$

We claim that for a and b in A,

(8) $(A, B) \models \phi_r[a, b]$ iff $(A, B) \models \mathrm{un}[a]$, $(A, B) \models \mathrm{un}[b]$, a codes $g \in G$, b codes $h \in G$, and $(g, h) \in r$.

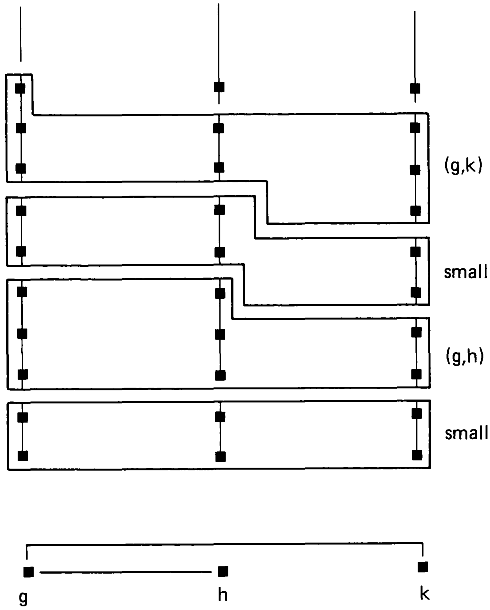


Fig. 7.1

To prove (8), assume first that the right-hand side holds, i.e. that a codes g, b codes h, and $(g, h) \in r$. Thus, $g \neq h$. Let a' and b' be in A such that $a \triangle a'$ and $b \triangle b'$ are finite. Then the set

$$P_0 = \{p \in P\colon a \cap p = a' \cap p = c_g \cap p \text{ and } b \cap p = b' \cap p = c_h \cap p\}$$

is cofinite in P, and by $(g, h) \in r$ and (6), there is some $p \in P_0$ such that $|c_g \cap p| = |c_h \cap p| = 3$. Hence, $|a' \cap p| = |b' \cap p| = 3$, and we have shown that $(A, B) \models \phi_r[a, b]$.

Conversely, assume that $(A, B) \models \phi_r[a, b]$. Then by (4) and (5), a codes some $g \in G$, b codes some $h \in G$, and $g \neq h$; we have to show that $(g, h) \in r$. By (3) and (5), the elements $a' = c_g$ and $b' = c_h$ of A satisfy

$$(A, B) \models \mathrm{un}[a'], \qquad (A, B) \models \mathrm{un}[b']$$

$$(A, B) \models \mathrm{eq}[a, a'], \qquad (A, B) \models \mathrm{eq}[b, b'].$$

Now since $(A, B) \models \phi_r[a, b]$, there exists $p \in P$ such that $|a' \cap p| = |b' \cap p| = 3$. It follows from (7) that $(g, h) \in r$, which concludes the proof of (8).

The assertions (3), (4), (5), and (8) imply that the quotient structure

$$(\mathrm{un}^{(A,B)}, \phi_r^{(A,B)}) / \mathrm{eq}^{(A,B)}$$

is isomorphic to the graph (G, r). The theorem is proved. □

Exercises

1. For a fixed natural number n, consider the following class of Boolean pairs:

$$\boldsymbol{BP}_{\mathrm{rc}n} = \{(B, A): A \text{ a relatively complete subalgebra of } B, \; B \text{ generated over } A \text{ by at most } n \text{ elements}\}.$$

(a) Find a finite axiom system for $\mathrm{Th}(\boldsymbol{BP}_{\mathrm{rc}n})$.
(b) Describe the completions of $\mathrm{Th}(\boldsymbol{BP}_{\mathrm{rc}n})$.
Hint. For $(B, A) \in \boldsymbol{BP}_{\mathrm{rc}n}$, consider the sheaf $\mathcal{S} = (S, \pi, X, (B_p)_{p \in X})$ associated with the pair (A, B) in 8.16. Show that, for $1 \leq k \leq 2^n$,

$$c_k = \{p \in X = \mathrm{Ult}\, A: |B_p| = 2^k\}$$

is a clopen subset of X, say $c_k = s(a_k)$, where $a_k \in A$ and $s: A \to \mathrm{Clop}\, X$ is the Stone isomorphism. Exercise 1 in Section 18 implies that the complete L_U-theory of the pair (B, A) is determined by the complete L-theories $\mathrm{Th}(A \upharpoonright a_k)$, $1 \leq k \leq 2^n$.

2. Conclude from Exercise 1 that $\mathrm{Th}(\boldsymbol{BP}_{\mathrm{rc}n})$ is decidable, for each $n \in \omega$.

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Sabine Koppelberg
Freie Universität Berlin

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Index of Notation, Volume 1

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Chapter 3: Topological Duality

Chapter 4: Free Constructions

Chapter 5: Infinite Operations

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