MAT166: SECTION 5.3

DOUBLE ANGLE AND HALF ANGLE FORMULAS

POWER REDUCING FORMULA

(CEC BOOK: SECTION 7.3)

Table 5-3

Double-Angle Formulas

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

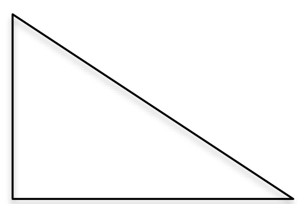
$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$= 1 - 2\sin^2\theta$$
$$= 2\cos^2\theta - 1$$

PRACTICE

Given that $\sin\theta = \frac{4}{5}$ for θ in Quadrant II, find the exact values.

■ $\sin 2\theta$

$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$= 2\left(\frac{4}{5}\right)\left(-\frac{3}{5}\right)$$
$$= -\frac{24}{25}$$



Sine is positive in Q2.

Cosine is negative in Q2.

■ $\cos 2\theta$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$= \left(-\frac{3}{5}\right)^2 - \left(\frac{4}{5}\right)^2$$
$$= \frac{9}{25} - \frac{16}{25}$$
$$= -\frac{7}{25}$$

■ $\tan 2\theta$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$
$$= 2\left(-\frac{4}{3}\right) \div \left[1 - \left(-\frac{4}{3}\right)^2\right]$$
$$= -\frac{8}{3} \div \left[1 - \frac{16}{9}\right]$$
$$= -\frac{8}{3} \cdot \left[-\frac{9}{7}\right] = \frac{24}{7}$$

ON YOUR OWN

Given that $\sec\theta = \frac{37}{12}$ for θ in Quadrant IV, find $\sin 2\theta$.

$12^2 + opp^2 = 37^2$

$opp = \sqrt{1369 - 144} = 35$

In Q4, sine is negative and cosine is positive.

$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$= 2\left(-\frac{35}{37}\right)\left(\frac{12}{37}\right)$$
$$= -\frac{840}{1369}$$

PRACTICE

Verify the identity.

■ $\cos 3x = 4\cos^3 x - 3\cos x$

$\cos(x + 2x) = 4\cos^3 x - 3\cos x$

$\cos x\cos 2x - \sin x\sin 2x = 4\cos^3 x - 3\cos x$

$\cos x\,(2\cos^2 x - 1) - \sin x\,(2\sin x\cos x) = 4\cos^3 x - 3\cos x$

$(2\cos^3 x - \cos x) - 2\cos x\,(\sin^2 x) = 4\cos^3 x - 3\cos x$

$2\cos^3 x - \cos x - 2\cos x\,(1 - \cos^2 x) = 4\cos^3 x - 3\cos x$

$2\cos^3 x - \cos x - 2\cos x + 2\cos^3 x = 4\cos^3 x - 3\cos x$

$4\cos^3 x - 3\cos x = 4\cos^3 x - 3\cos x$ QED

Table 5-4

Power-Reducing Formulas		
$\sin^2\theta = \dfrac{1 - \cos 2\theta}{2}$	$\cos^2\theta = \dfrac{1 + \cos 2\theta}{2}$	$\tan^2\theta = \dfrac{1 - \cos 2\theta}{1 + \cos 2\theta}$

The power-reducing formulas are important in integral calculus.

PRACTICE

Write the following in terms of first powers of cosine.

■ $\cos^4 x$

$$(\cos^2 x)^2 = \left(\frac{1+\cos(2x)}{2}\right)^2$$

$$(\cos^2 x)^2 = \frac{1 + 2\cos(2x) + \cos^2(2x)}{4}$$

$$(\cos^2 x)^2 = \frac{1}{4}(1 + 2\cos(2x) + \cos^2(2x))$$

$$(\cos^2 x)^2 = \frac{1}{4}\left(1 + 2\cos(2x) + \frac{1+\cos 2(2x)}{2}\right)$$

$$(\cos^2 x)^2 = \frac{1}{4}\left(1 + 2\cos(2x) + \frac{1}{2} + \frac{\cos(4x)}{2}\right)$$

$$(\cos^2 x)^2 = \frac{1}{4} + \frac{\cos(2x)}{2} + \frac{1}{8} + \frac{\cos(4x)}{8}$$

$$(\cos^2 x)^2 = \frac{3}{8} + \frac{\cos(2x)}{2} + \frac{\cos(4x)}{8}$$

Table 5-5

Half-Angle Formulas

$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}} \qquad \cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$$

$$\tan\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} = \frac{1-\cos\alpha}{\sin\alpha} = \frac{\sin\alpha}{1+\cos\alpha}$$

The sign + or − is determined by the quadrant in which the angle $\frac{\alpha}{2}$ lies.

PRACTICE

Use the half-angle formula to find the exact value.

■ $\sin 165°$

$$165° = \frac{330°}{2}$$

The angle lies in Q2 where sine is positive.

$$\sin\frac{330°}{2} = +\sqrt{\frac{1-\cos 330°}{2}}$$

$$\sin\frac{330°}{2} = +\sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}}$$

$$\sin\frac{330°}{2} = +\sqrt{\left(\frac{1-\frac{\sqrt{3}}{2}}{2}\right)\cdot\left(\frac{2}{2}\right)}$$

$$\sin\frac{330°}{2} = +\sqrt{\frac{2-\sqrt{3}}{4}}$$

$$\sin\frac{330°}{2} = +\frac{\sqrt{2-\sqrt{3}}}{2}$$

ON YOUR OWN

■ $\cos 67.5°$

The angle lies in Q1 where cosine is positive.

$$\cos\frac{135°}{2} = +\sqrt{\frac{1+\cos 135°}{2}}$$

$$\cos\frac{135°}{2} = +\sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}}$$

$$\cos\frac{135°}{2} = +\sqrt{\left(\frac{1-\frac{\sqrt{2}}{2}}{2}\right)\cdot\left(\frac{2}{2}\right)}$$

$$\cos\frac{135°}{2} = +\sqrt{\frac{2-\sqrt{2}}{4}}$$

$$\cos\frac{135°}{2} = +\frac{\sqrt{2-\sqrt{2}}}{2}$$

PRACTICE

Given that $\sin\alpha = -\frac{5}{13}$ and $\frac{3\pi}{2} < \alpha < 2\pi$, find exact values for the following.

NOTE: The sign + or – is determined by the quadrant in which the angle α lies.

$(-5)^2 + adj^2 = 13^2$

$adj = \sqrt{169 - 25} = 12$

■ $\cos\frac{\alpha}{2}$

α is in Q4, so $\frac{\alpha}{2}$ is in Q2 where cosine is negative.

$$\cos\frac{\alpha}{2} = -\sqrt{\frac{1+\cos\alpha}{2}}$$

$$\cos\frac{\alpha}{2} = -\sqrt{\frac{1+\frac{12}{13}}{2}}$$

$$\cos\frac{\alpha}{2} = -\sqrt{\left(\frac{1+\frac{12}{13}}{2}\right)\cdot\left(\frac{13}{13}\right)}$$

$$\cos\frac{\alpha}{2} = -\sqrt{\left(\frac{13+12}{26}\right)}$$

$$\cos\frac{\alpha}{2} = -\sqrt{\frac{25}{26}} = -\frac{5\sqrt{26}}{26}$$

■ $\tan\frac{\alpha}{2}$

α is in Q4 so $\frac{\alpha}{2}$ is in Q2 where tangent is negative.

$$\tan\frac{\alpha}{2} = \frac{\sin\alpha}{1+\cos\alpha}$$

$$\tan\frac{\alpha}{2} = \frac{-\frac{5}{13}}{1+\frac{12}{13}}$$

$$\tan\frac{\alpha}{2} = \left(\frac{-\frac{5}{13}}{1+\frac{12}{13}}\right)\cdot\left(\frac{13}{13}\right)$$

$$\tan\frac{\alpha}{2} = \frac{-5}{13+12} = -\frac{5}{25}$$

$$\tan\frac{\alpha}{2} = -\frac{1}{5}$$