Convex Optimization

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CAMBRIDGE
UNIVERSITY PRESS

PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building, Trumpington Street, Cambridge, United Kingdom

CAMBRIDGE UNIVERSITY PRESS
The Edinburgh Building, Cambridge, CB2 2RU, UK
40 West 20th Street, New York, NY 10011-4211, USA
477 Williamstown Road, Port Melbourne, VIC 3207, Australia
Ruiz de Alarcón 13, 28014 Madrid, Spain
Dock House, The Waterfront, Cape Town 8001, South Africa

http://www.cambridge.org

First published 2004
Reprinted with corrections 2004

Printed in the United Kingdom at the University Press, Cambridge

Typeset in Computer Modern Roman using LaTeX

A catalogue record for this book is available from the British Library

Library of Congress Cataloguing-in-Publication data

Boyd, Stephen P.
Convex Optimization / Stephen Boyd & Lieven Vandenberghe
p. cm.
Includes bibliographical references and index.
ISBN 0 521 83378 7
1. Mathematical optimization. 2. Convex functions. I. Vandenberghe, Lieven. II. Title.

QA402.5.B69 2004
519.6–dc22 2003063284

For

Anna, Nicholas, and Nora

Daniël and Margriet

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Preface

This book is about *convex optimization*, a special class of mathematical optimization problems, which includes least-squares and linear programming problems. It is well known that least-squares and linear programming problems have a fairly complete theory, arise in a variety of applications, and can be solved numerically very efficiently. The basic point of this book is that the same can be said for the larger class of convex optimization problems.

While the mathematics of convex optimization has been studied for about a century, several related recent developments have stimulated new interest in the topic. The first is the recognition that interior-point methods, developed in the 1980s to solve linear programming problems, can be used to solve convex optimization problems as well. These new methods allow us to solve certain new classes of convex optimization problems, such as semidefinite programs and second-order cone programs, almost as easily as linear programs.

The second development is the discovery that convex optimization problems (beyond least-squares and linear programs) are more prevalent in practice than was previously thought. Since 1990 many applications have been discovered in areas such as automatic control systems, estimation and signal processing, communications and networks, electronic circuit design, data analysis and modeling, statistics, and finance. Convex optimization has also found wide application in combinatorial optimization and global optimization, where it is used to find bounds on the optimal value, as well as approximate solutions. We believe that many other applications of convex optimization are still waiting to be discovered.

There are great advantages to recognizing or formulating a problem as a convex optimization problem. The most basic advantage is that the problem can then be solved, very reliably and efficiently, using interior-point methods or other special methods for convex optimization. These solution methods are reliable enough to be embedded in a computer-aided design or analysis tool, or even a real-time reactive or automatic control system. There are also theoretical or conceptual advantages of formulating a problem as a convex optimization problem. The associated dual problem, for example, often has an interesting interpretation in terms of the original problem, and sometimes leads to an efficient or distributed method for solving it.

We think that convex optimization is an important enough topic that everyone who uses computational mathematics should know at least a little bit about it. In our opinion, convex optimization is a natural next topic after advanced linear algebra (topics like least-squares, singular values), and linear programming.

Goal of this book

For many general purpose optimization methods, the typical approach is to just try out the method on the problem to be solved. The full benefits of convex optimization, in contrast, only come when the problem is known ahead of time to be convex. Of course, many optimization problems are not convex, and it can be difficult to recognize the ones that are, or to reformulate a problem so that it is convex.

> *Our main goal is to help the reader develop a working knowledge of convex optimization, i.e., to develop the skills and background needed to recognize, formulate, and solve convex optimization problems.*

Developing a working knowledge of convex optimization can be mathematically demanding, especially for the reader interested primarily in applications. In our experience (mostly with graduate students in electrical engineering and computer science), the investment often pays off well, and sometimes very well.

There are several books on linear programming, and general nonlinear programming, that focus on problem formulation, modeling, and applications. Several other books cover the theory of convex optimization, or interior-point methods and their complexity analysis. This book is meant to be something in between, a book on general convex optimization that focuses on problem formulation and modeling.

We should also mention what this book is *not*. It is not a text primarily about convex analysis, or the mathematics of convex optimization; several existing texts cover these topics well. Nor is the book a survey of algorithms for convex optimization. Instead we have chosen just a few good algorithms, and describe only simple, stylized versions of them (which, however, do work well in practice). We make no attempt to cover the most recent state of the art in interior-point (or other) methods for solving convex problems. Our coverage of numerical implementation issues is also highly simplified, but we feel that it is adequate for the potential user to develop working implementations, and we do cover, in some detail, techniques for exploiting structure to improve the efficiency of the methods. We also do not cover, in more than a simplified way, the complexity theory of the algorithms we describe. We do, however, give an introduction to the important ideas of self-concordance and complexity analysis for interior-point methods.

Audience

This book is meant for the researcher, scientist, or engineer who uses mathematical optimization, or more generally, computational mathematics. This includes, naturally, those working directly in optimization and operations research, and also many others who use optimization, in fields like computer science, economics, finance, statistics, data mining, and many fields of science and engineering. Our primary focus is on the latter group, the potential *users* of convex optimization, and not the (less numerous) experts in the field of convex optimization.

The only background required of the reader is a good knowledge of advanced calculus and linear algebra. If the reader has seen basic mathematical analysis (*e.g.*, norms, convergence, elementary topology), and basic probability theory, he or she should be able to follow every argument and discussion in the book. We hope that readers who have not seen analysis and probability, however, can still get all of the essential ideas and important points. Prior exposure to numerical computing or optimization is not needed, since we develop all of the needed material from these areas in the text or appendices.

Using this book in courses

We hope that this book will be useful as the primary or alternate textbook for several types of courses. Since 1995 we have been using drafts of this book for graduate courses on linear, nonlinear, and convex optimization (with engineering applications) at Stanford and UCLA. We are able to cover most of the material, though not in detail, in a one quarter graduate course. A one semester course allows for a more leisurely pace, more applications, more detailed treatment of theory, and perhaps a short student project. A two quarter sequence allows an expanded treatment of the more basic topics such as linear and quadratic programming (which are very useful for the applications oriented student), or a more substantial student project.

This book can also be used as a reference or alternate text for a more traditional course on linear and nonlinear optimization, or a course on control systems (or other applications area), that includes some coverage of convex optimization. As the secondary text in a more theoretically oriented course on convex optimization, it can be used as a source of simple practical examples.

Acknowledgments

We have been developing the material for this book for almost a decade. Over the years we have benefited from feedback and suggestions from many people, including our own graduate students, students in our courses, and our colleagues at Stanford, UCLA, and elsewhere. Unfortunately, space limitations and shoddy record keeping do not allow us to name everyone who has contributed. However, we wish to particularly thank A. Aggarwal, V. Balakrishnan, A. Bernard, B. Bray, R. Cottle, A. d'Aspremont, J. Dahl, J. Dattorro, D. Donoho, J. Doyle, L. El Ghaoui, P. Glynn, M. Grant, A. Hansson, T. Hastie, A. Lewis, M. Lobo, Z.-Q. Luo, M. Mesbahi, W. Naylor, P. Parrilo, I. Pressman, R. Tibshirani, B. Van Roy, L. Xiao, and Y. Ye. We are grateful to J. Jalden and A. d'Aspremont, who contributed to the time-frequency analysis example in §6.5.4, and the consumer preference bounding example in §6.5.5, respectively. P. Parrilo suggested exercises 4.4 and 4.56.

We want to single out two others for special acknowledgment. Arkadi Nemirovski incited our original interest in convex optimization, and encouraged us to write this book. We also want to thank Kishan Baheti for playing a critical role in the development of this book. In 1994 he encouraged us to apply for a National Science Foundation combined research and curriculum development grant, on convex optimization with engineering applications, and this book is a direct (if delayed) consequence.

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July 2003