A very brief overview of complex analysis

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The shortest path between two truths in the real domain passes through the complex domain.

— J. Hadamard

1 Introduction

This chapter is a review of standard material from a first module on complex analysis. Many proofs are not included here. For further reading, see any standard text on complex analysis or the theory of complex functions (of one variable).

Complex numbers were first introduced to make sense of the solution to certain polynomial equations such as

$$x^2 + x + 1 = 0. \tag{1}$$

The standard formula for the roots of a quadratic equation gives

$$x = \frac{-1 \pm \sqrt{-3}}{2}.$$

Are we forced to discard this expression as meaningless because there is no real number whose square is -3? It turns out that if we augment the real numbers with just one extra number, Euler's "imaginary number" i, which satisfies $i^2 = -1$, then we can make sense of the solutions to a great many problems such as this. Moreover, many theorems seem more natural when stated in terms of complex numbers.

The use of complex numbers can be justified in the first instance as a powerful mathematical trick for the solution of many "real" problems. For example, the evaluation of a large class of real integrals is most easily done by imbedding the problem in the complex plane. The mathematician J. Hadamard once said "the shortest path between two truths in the real domain passes through the complex domain." Also, in electromagnetism it is computationally convenient to consider the electric and magnetic fields to be the real and imaginary parts of a single complex vector field.

More fundamentally, complex numbers appear in quantum mechanics through the Schrödinger equation,

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V(\mathbf{r},t)\psi,$$

which is arguably the most important equation in all of modern physics.

One theme that will run through this module is that the important functions that appear in applications are usually "special" from the point of view of complex analysis, regardless of whether the original problem is in the real or complex domain.

2 Complex numbers

Informally, the starting point for the theory of complex numbers is the introduction of a "number", i, such that $i^2 = -1$. The set of all complex numbers consists of all sums of real numbers and real multiples of i. More formally, we have the following.

Definition 1 *A complex number is an ordered set of real numbers* (x, y)*. The space of all complex numbers is denoted by* $\mathbf{C}$ *and has the following operations defined on it.*

$$\begin{aligned} (x_1, y_1) + (x_2, y_2) &:= (x_1 + x_2, y_1 + y_2), \\ (x_1, y_1)(x_2, y_2) &:= (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1), \\ (x, y)^{-1} &:= \left(\frac{x}{x^2+y^2}, -\frac{y}{x^2+y^2}\right), \qquad (x,y) \neq (0,0). \end{aligned}$$

Note that complex numbers of the form $(x, 0)$ satisfy $(x_1, 0) + (x_2, 0) = (x_1 + x_2, 0)$ and $(x_1, 0)(x_2, 0) = (x_1x_2, 0)$. This allows us to identify the complex number $(x, 0)$ with the real number x. In doing so, since the complex number $i := (0, 1)$ satisfies $i^2 = (-1, 0)$, we must identify i^2 with the real number -1. Every complex number (x, y) can then be written as $(x, 0) + (0, 1)(y, 0)$ and identified with the expression

$$z = x + iy.$$

We call x the real part of z and y the imaginary part of z. Although we will discard the vector notation of definition 1 as a formal trick to convince ourselves that complex numbers exist (in as much as real numbers exist), it is often useful to think of complex numbers as vectors in $\mathbf{R}^2$.

It is sometimes convenient to represent a complex in terms of polar coordinates r and θ, where $x = r\cos\theta$ and $y = r\sin\theta$. The non-negative number r is called the modulus of z and is denoted by $|z| = \sqrt{x^2+y^2}$. The angle θ is called the argument of z and is denoted by $\arg z$. The argument of z is only defined up to the addition of integer multiples of 2π. In order to get a unique value it is sometimes convenient to restrict the argument to lie in the range $-\pi < \theta \leq \pi$. This restricted argument is called the principal argument and is written $\operatorname{Arg} z$. The modulus stisfies the so-called triangle inequality $|z_1 + z_2| \leq |z_1| + |z_2|$ for any complex numbers z_1 and z_2.

Writing the complex number z in Cartesian and polar coordinates, we have

$$z = x + iy = r(\cos\theta + i\sin\theta) = r\mathrm{e}^{i\theta},$$

where we have used Euler's formula

$$\mathrm{e}^{i\theta} = \cos\theta + i\sin\theta. \qquad (2)$$

For the time being, Euler's formula can be taken to be a definition of the exponential of a pure imaginary number. This definition will be motivated later in this chapter.

Complex analysis involves the study of complex functions which in turn requires us to describe a number of special classes of subsets of the complex plane. For any $z_0 \in \mathbf{C}$ and $r > 0$, the set $D(z_0, r) := \{z \in \mathbf{C} : |z - z_0| < r\}$ is the set of all points that lie inside the circle centred at z_0 with radius r in the complex plane. This set is called the open disc centred at z_0 with radius r. A neighbourhood of a point z_0 is any open disc centred at z_0 (we usually think of $r > 0$ as small in this case.) Similarly, $\overline{D}(z_0, r) := \{z \in \mathbf{C} : |z - z_0| \leq r\}$ is called the closed disc.

A subset $U \subseteq \mathbf{C}$ is said to be open if for each $z \in U$ there is a positive number r such that $D(z, r) \subset U$. A subset $V \subseteq \mathbf{C}$ is said to be closed if $\mathbf{C} \setminus V$ is open. The only subsets of $\mathbf{C}$ that are both open and closed are the empty set $\emptyset$ and $\mathbf{C}$. A region is an open connected subset of the plane.

3 Analytic functions

In this section we define the derivative of a function $f : \mathbf{C} \to \mathbf{C}$ in an analogous manner to the way in which the derivative of a real function is defined, namely as the limit of a difference quotient. To this end we begin with a definition of limit.

Definition 2 *Let $f : \Omega \to \mathbf{C}$ be defined in a neighbourhood of $z = z_0$. The complex number l is called the limit of f as z approaches z_0 if, given $\epsilon > 0$ there exists $\delta > 0$ such that*

$$|f(z) - l| < \epsilon \quad \textit{whenever} \quad 0 < |z - z_0| < \delta.$$

In this case we write $\lim\limits_{z \to z_0} = l$.

Note ϵ and δ are necessarily real. The function f need not be defined at $z = z_0$ in order for this limit to exist.

Now we are in a position to define the derivative of a complex function.

Definition 3 *Let $f : \Omega \to \mathbf{C}$, where Ω is a domain in $\mathbf{C}$. The function f is said to be differentiable at $z \in \Omega$ if the limit*

$$\lim_{h \to 0} \frac{f(z + h) - f(z)}{h} \tag{3}$$

exists. This limit is called the derivative of f at z and is denoted by $f'(z)$.

The derivative for complex differentiable functions satifies the same product, quotient, and chain rules as the derivative for real differentiable functions. In particular, it follows that if $f(z) = z^n$, then $f'(z) = nz^{n-1}$.

The differentiability of f can be expressed simply in terms of the partial derivatives of the real and imaginary parts of f. Let $z = x + iy$ and $f(z) = u(x, y) + iv(x, y)$, where x, y, u, and v are real. Assuming f is differentiable at $z_0 = x_0 + iy_0$, we will now evaluate the limit in (3) along two different paths in the complex plane. First we take the limit along the real axis. If h is restricted to have real values then

$$\frac{f(z + h) - f(z)}{h} = \frac{u(x + h, y) - u(x, y)}{h} + i\frac{v(x + h, y) - v(x, y)}{h}.$$

Taking the limit $h \to 0$ gives

$$f'(z) = \frac{\partial u(x, y)}{\partial x} + i\frac{\partial v(x, y)}{\partial x}. \tag{4}$$

Now we will evaluate the limit in (3) along the imaginary axis. To this end, we write $h = ik$, where k is real, which gives

$$\frac{f(z+h)-f(z)}{h} = \frac{v(x,y+k)-v(x,y)}{k} - i\frac{u(x,y+k)-u(x,y)}{k}.$$

Taking the limit as $h = ik \to 0$ gives

$$f'(z) = \frac{\partial v(x,y)}{\partial y} - i\frac{\partial u(x,y)}{\partial y}. \tag{5}$$

Comparing the real and imaginary parts of the two expressions for $f'(z)$ given in equations (4) and (5) gives the so-called Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{6}$$

We have proved the following

Theorem 4 *Let $f(z) = u(x,y) + iv(x,y)$ be continuous in some neigbourhood of z. If f is differentiable at z then u and v satisfy the Cauchy-Riemann equations (6) at z. Furthermore, $f'(z)$ is given by equation (4) (or equivalently equation 5).*

We have shown that the Cauchy-Riemann equations are necessary for f to be differentiable at a point. It can be shown that they are also sufficient, provided u and v are sufficiently smooth.

Theorem 5 *Let $f(z) = u(x,y) + iv(x,y)$, $z = x + iy$, where u, v, and their first partial derivatives are continuous at (x_0, y_0). Then f is differentiable at $z_0 = x_0 + iy_0$ if and only if u and v satisfy the Cauchy-Riemann equations (6) at (x_0, y_0).*

Proof: The main idea here is to use the linear approximations of the real functions u and v. Let $h = a + ib$. Then

$$\begin{aligned} & f(z_0+h) - f(z_0) \\ = \; & [u(x_0+a, y_0+b) - u(x_0,y_0)] + i\,[v(x_0+a,y_0+b) - v(x_0,y_0)] \\ = \; & [u_x(x_0,y_0)a + u_y(x_0,y_0)b + \delta_1 a + \delta_2 b] + i\,[v_x(x_0,y_0)a + v_y(x_0,y_0)b + \delta_3 a + \delta_4 b]\,, \end{aligned}$$

where $\delta_j \to 0$, $j = 1,2,3,4$, as $(a,b) \to (0,0)$. Since the Cauchy-Riemann equations (6) at satisfied at (x_0, y_0), we have

$$f(z_0+h) - f(z_0) = (a+ib)\,[u_x(x_0,y_0) + iv_x(x_0,y_0)] + \epsilon_1 a + \epsilon_2 b, \tag{7}$$

where $\epsilon_1, \epsilon_2 \to 0$ as $(a,b) \to (0,0)$. The result follows on dividing equation (7) by $h = a+ib$ and taking the limit $h \to 0$, while noting that $|a/h| \leq 1$ and $|b/h| \leq 1$.

Exercise 6 *Verify that the Cauchy-Riemann equations hold for all $z \in \mathbf{C}$ for the function $f(z) = z^2$.*

If f is differentiable in some neighbourood of z, then f is said to be analytic at z. Let U be any subset of $\mathbf{C}$. The function f is said to be analytic on U if f is analytic at every point of U. A function is continuous on any domain on which it is analytic. A function that is analytic throughout the complex plane is called an entire function.

Exercise 7 *Show that $f(z) = |z|^2$ is only differentiable at 0 and is analytic nowhere.*

Exercise 8 *Show that $f(z) := \mathrm{e}^x(\cos y + i \sin y)$, where $z = x + iy$, is analytic for all $z \in \mathbf{C}$. Furthermore, show that f satisfies the initial value problem $f'(z) = f(z)$, $f(0) = 1$. This motivates the definition $f(z) = \exp z$ for all complex z, which gives Euler's formula (2).*

Note that the Cauchy-Riemann equations (6) can be written compactly as $\dfrac{\partial f}{\partial \bar{z}} = 0$.

4 Integration

Let γ be a curve in the complex plane given by $z(t) = x(t) + iy(t)$, where x and y are smooth real functions of the real variable t in the interval $t_1 < t < t_2$. We define the integral of f along γ by

$$\int_\gamma f(z)dz := \int_{t_1}^{t_2} f(z)\frac{dz}{dt}dt,$$

where the second integral is understood to mean the integral of the real part of the integrand plus i multiplied by the integral of the imaginary part. In terms of line integrals, this becomes

$$\int_\gamma f(z)dz = \int_\gamma \left\{(u\,dx - v\,dy) + i(v\,dx + u\,dy)\right\}.$$

We will use the symbol "$\oint_\gamma$" to denote the integral around a closed curve γ. Unless otherwise stated, we will assume that γ is traced in the positive (i.e. anti-clockwise) direction.

Theorem 9 *(The Cauchy-Goursat Theorem) Let f be analytic at all points interior to and on a closed curve γ. Then*

$$\oint f(z)dz = 0.$$

Partial Proof: We will give Cauchy's proof which requires the further assumption that f' is continuous. Goursat provided a proof without this assumption. The proof uses Green's Theorem which says that if $P(x,y)$ and $Q(x,y)$, together with their first partial derivatives with respect to x and y, are continuous then

$$\oint_\gamma (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.$$

Applying this result to the real and imaginary parts of $\oint f\,dz$ and using the Cauchy-Riemann equation proves the theorem.

Exercise 10 *Show that*

$$\oint_\gamma \frac{dz}{z} = 2\pi i,$$

where γ is any circle centred at 0.

Theorem 11 *(Cauchy's integral formula) Let f be analytic on the simple closed curve γ and on its interior. Then*

$$f(z) = \frac{1}{2\pi i}\oint_\gamma \frac{f(\zeta)}{\zeta - z}d\zeta. \tag{8}$$

One remarkable fact about analytic functions is that derivatives of all orders exist throughout the domain of analyticity. This result follows from Cauchy's integral formula. Repeated formal differentiation of equation (11) and shifting the derivative under the integral sign gives equation (9) below. We justify this result by considering the usual difference quotient for the derivative.

Theorem 12 *Let f and γ satisfy the conditions of theorem 11. Then*

$$f^{(n)}(z) = \frac{n!}{2\pi i}\oint_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}d\zeta. \tag{9}$$

Proof: Assume that equation (9) is true for some $n = 0, 1, \ldots$ Consider the difference

$$\begin{aligned} f^{(n)}(z+h) - f^{(n)}(z) &= \frac{n!}{2\pi i} \oint_\gamma f(\zeta) \left\{ \frac{1}{(\zeta - z - h)^{n+1}} - \frac{1}{(\zeta - z)^{n+1}} \right\} d\zeta \\ &= \frac{n!}{2\pi i} \cdot (n+1)h \oint_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+2}} d\zeta + O(h^2). \end{aligned}$$

Theorem 13 *(Cauchy's estimate) Let f be analytic on the closed disc $\bar{D} := \{z : |z - z_0| \leq r\}$. Then*

$$|f^{(n)}(z_0)| \leq \frac{n! M}{r^n}, \tag{10}$$

where $M := \max_{z \in \bar{D}} |f(z)|$.

Proof: From equation (9), we have

$$|f^{(n)}(z_0)| = \left| \frac{n!}{2\pi i} \oint_\gamma \frac{f(z)}{(z - z_0)^{n+1}} dz \right| \leq \frac{n!}{2\pi} \cdot 2\pi r \cdot \frac{M}{r^{n+1}} = \frac{n! M}{r^n}.$$

Theorem 14 *(Maximum Modulus Theorem) Let f be analytic in a domain D. If f has a maximum in D then f is a constant. It follows that any non-constant analytic function on some closed region $\bar{D}$ achieves its maximum on the boundary ∂D and satisfies*

$$|f(z_0)| < \sup_{z \in \partial D} |f(z)|,$$

for all $z_0 \in D$.

Theorem 15 *(Liouville's Theorem) Any bounded entire function is a constant.*

Proof: Let $M = \sup_{z \in \mathbf{C}} |f(z)|$. Since f is entire, it follows from equation (10) with $n = 1$ that

$$|f'(z)| = \frac{M}{r},$$

for all $z \in \mathbf{C}$ and all $r > 0$. Since r can be arbitrarily large, $f(z) = 0$ for all $z \in \mathbf{C}$. So f is a constant.

Corollary 16 *(The Fundamental Theorem of Algebra) Every non-constant polynomial has at least one zero in the complex plane.*

Proof: Suppose that p is a non-constant polynomial with no zeros. Then the entire function f defined by $f(z) := 1/p(z)$ is bounded since $f(z) \to 0$ as $z \to \infty$. Hence f (and therefore p) is a constant by Liouville's Theorem.

Exercise 17 *Show that there is no entire function f such that $f(z) = z^{-1} + O(z^{-2})$ as $z \to \infty$ (in all directions).*

5 Convergence of sequences and series in the complex domain

A sequence is a mapping from the set of integers greater than some given integer n_0 to the complex numbers, $n \mapsto a_n$. We represent the sequence as (a_n). Without loss of generality, we can take $n_0 = 1$.

Definition 18 *The sequence (a_n) is said to converge to the limit l if, given $\epsilon > 0$ there exists $N \in \mathbf{N}$ such that*

$$|a_n - l| < \epsilon \quad \textit{for all} \quad n > N.$$

Definition 19 *The series $\sum_{n=1}^{\infty} a_n$ is said to converge to the limit (or sum) s if the limit of the sequence of partial sums $s_n = a_1 + a_2 + \cdots + a_n$ converges to s. That is, if, given $\epsilon > 0$ there exists $N \in \mathbf{N}$ such that*

$$|s_n - s| < \epsilon \quad \textit{for all} \quad n > N.$$

Theorem 20 *If the series $\sum_{n=1}^{\infty} a_n$ converges then $a_n \to 0$.*

Proof: Let $s_n = \sum_{j=1}^{n} a_j$ and $s = \lim_{n\to\infty} s_n$. Then $a_n = s_n - s_{n-1} \to s - s = 0$.

The converse of this theorem is not true. We cannot conclude from $a_n \to 0$ alone that $\sum_{n=1}^{\infty} a_n$ converges. In particular, the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges but $a_n = 1/n \to 0$.

Theorem 21 *Let (a_n) and (A_n) be two sequences of real numbers such that $0 \leq a_n \leq A_n$, $n = 1, 2, \ldots$ If $\sum A_n$ converges then $\sum a_n$ converges.*

Corollary 22 *If the series $\sum_{n=1}^{\infty} |a_n|$ converges then the series $\sum_{n=1}^{\infty} a_n$ converges. In this case the series $\sum_{n=1}^{\infty} a_n$ is said to converge absolutely.*

Theorem 23 *(The n^{th} root test) Let (a_n) be a sequence of positive (real) numbers. Let $R := \limsup_{n\to\infty} a_n^{1/n}$.*

1. *If $R < 1$ then the series $\sum_{n=1}^{\infty} a_n$ converges.*
2. *If $R > 1$ then the series $\sum_{n=1}^{\infty} a_n$ diverges. (This includes the case in which $R = \infty$.)*

Note that this theorem says nothing about the convergence of a series for which $R = 1$.

Theorem 24 *(The ratio test) Let (a_n) be a sequence of positive (real) numbers. Let*

$$\lambda := \liminf_{n\to\infty} \frac{a_{n+1}}{a_n} \quad \textit{and} \quad \mu := \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

1. If $\mu < 1$ then the series $\sum_{n=1}^{\infty} a_n$ converges.

2. If $\lambda > 1$ then the series $\sum_{n=1}^{\infty} a_n$ diverges. (This includes the case in which $\lambda = \infty$.)

Exercise 25 *Does the series $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converge?*

6 Power series

A series of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

where z_0 and the a_n's are complex constants and z is a complex variable, is called a power series in z with centre z_0 and coefficients a_n.

Theorem 26 *For any power series*

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n, \tag{11}$$

exactly one of the following statements is true.

1. The series converges absolutely at $z = z_0$ and diverges for all other values of z.

2. There exists a finite positive number R such that the series converges absolutely whenever $|z| < R$ and diverges whenever $|z| > R$.

3. The series converges absolutely for all finite z.

The number R is called the radius of convergence of the series. In we say $R = 0$ in case 1 and $R = \infty$ in case 3.

Example 27 *From the identity*

$$1 - z^n = (1 - z)(1 + z + z^2 + \cdots + z^{n-1}),$$

$n = 1, 2, \ldots$, *it follows that*

$$\sum_{j=0}^{n-1} z^j = \frac{1 - z^n}{1 - z} \qquad z \neq 1.$$

It follows that the geometric series converges in the unit disc:

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1 - z}, \qquad |z| < 1,$$

and diverges for $|z| \geq 1$. So the radius of convergence is 1.

The radius of convergence is given by $R = 1/\alpha$, where

$$\alpha = \limsup_{n\to\infty} |a_n|^{1/n}.$$

If $\alpha = 0$, then $R = \infty$. Furthermore,

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|,$$

if this limit exists.

Exercise 28 *For each of the following series, find the radius of convergence.*

$$(a)\ \sum_{n=0}^{\infty} \frac{z^n}{2^n}, \qquad (b)\ \sum_{n=0}^{\infty} \frac{(z+2)^n}{n}, \qquad (c)\ \sum_{n=0}^{\infty} n^{2n} z^n, \qquad (d)\ \sum_{n=0}^{\infty} \frac{(z-1)^n}{(2n)!}.$$

Theorem 29 *Let Ω be any open subset of $\mathbf{C}$. A function $f : \Omega \to \mathbf{C}$ is analytic on Ω if and only if at each point $z \in \Omega$, $f(z)$ is the sum of a power series with nonzero radius of convergence.*

We see that convergent power series can be used to define an analytic function on the the interior of its domain of convergence. This provides a straightforward procedure for extending classical functions of a real variable to the complex domain. In particular, it motivates the following definition which agrees with the standard definitions when z is real.

Definition 30 *The following definitions are for all complex z.*

$$\begin{aligned}
\exp z &:= \sum_{n=0}^{\infty} \frac{z^n}{n!}, \\
\cos z &:= \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!}, \\
\sin z &:= \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!}, \\
\cosh z &:= \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!}, \\
\sinh z &:= \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!}.
\end{aligned}$$

Exercise 31 *Show that Euler's formula*

$$\exp iz = \cos z + i \sin z$$

is true for all $z \in \mathbf{C}$.

The following thoerem says that the sum of a convergent power series is an analytic function.

Theorem 32 *If the series (11) has radius of convergence $R > 0$, then*

$$f(z) := \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

is analytic in the open ball of radius R centred at z_0. Furthermore, for all positive integers k, the k^{th} derivative of f is given by the convergent series

$$f^{(k)}(z) := \sum_{n=0}^{\infty} n(n-1)\cdots(n-k+1)a_n(z-z_0)^{n-k}, \qquad |z-z_0| < R.$$

The coefficients in the power series expansion of f are given by $a_n = \frac{1}{n!}f^{(n)}(z_0)$.

Conversely, we have

Theorem 33 *(Taylor series) Let f be analytic in the disc $|z-z_0| < r$ for some $r > 0$. Then*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n, \quad \text{for} \quad |z-z_0| < r. \tag{12}$$

The series (12) is called the Taylor series for f about $z = z_0$. The Taylor series for f about 0 is called the Maclaurin series of f.

Proof: For fixed $z \in B(z_0, r)$, there is a number r such that $|z - z_0| < \rho < r$. Let γ be the circle with centre z_0 and radius ρ. From Cauchy's integral formula we have

$$f(z) = \frac{1}{2\pi i}\oint_\gamma \frac{f(\zeta)}{\zeta - z}d\zeta.$$

Now

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) - (z - z_0)} = \frac{1}{\zeta - z_0}\frac{1}{1-w}, \qquad w = \frac{z-z_0}{\zeta - z_0}.$$

Recall that

$$\frac{1}{1-w} = 1 + w + w^2 + \cdots + w^n + \frac{w^{n+1}}{1-w}.$$

$$f(z) = \frac{1}{2\pi i}\oint_\gamma f(\zeta)\left\{\frac{1}{\zeta - z_0} + \frac{z-z_0}{(\zeta-z_0)^2} + \cdots + \frac{(z-z_0)^n}{(\zeta-z_0)^{n+1}} + \frac{1}{\zeta - z}\frac{(z-z_0)^{n+1}}{(\zeta-z_0)^n}\right\}d\zeta.$$

Using equation (9), we have

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + \cdots + \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n + R_n,$$

where

$$R_n = \frac{(z-z_0)^{n+1}}{2\pi i}\oint_\gamma \frac{f(\zeta)}{(\zeta - z)(\zeta - z_0)^n}d\zeta.$$

Let $M = \max_{z\in\gamma}|f(z)|$. Then

$$|R_n| \le \frac{|z-z_0|^{n+1}}{2\pi}\cdot 2\pi \cdot \frac{M}{(\rho - |z-z_0|)\rho^n} = \frac{M|z-z_0|}{\rho - |z-z_0|}\left(\frac{|z-z_0|}{\rho}\right)^n.$$

Since $|z-z_0| < \rho$, $R_n \to 0$ as $n \to \infty$.

Exercise 34 *Find the Maclaurin series of $f(z) = z^2\cos\sqrt{z}$. (Hint: do NOT use theorem 33).*

7 Isolated Singularities

A Laurent series is a natural extension of a power series that includes negative powers of the expansion variable. Such series represent functions that are analytic on annuli.

Theorem 35 *Any function f that is analytic on the annulus $0 \le r_1 < |z - z_0| < r_2 \le \infty$ has a unique Laurent series expansion,*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n,$$

where

$$a_n = \frac{1}{2\pi i} \oint_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta$$

and γ is any circle $|z - z_0| = r$ such that $r_1 < r < r_2$. Furthermore the series converges uniformly to $f(z)$ on the annulus.

In the above, the coefficient a_{-1} is called the residue of f at z_0.

Definition 36 *A complex-valued function f is said to have an isolated singularity at $z = z_0$ if there exists $\epsilon > 0$ such that f is analytic for all z such that $0 < |z - z_0| < \epsilon$ but f is not analytic at $z = z_0$.*

Definition 37 *Let f have an isolated singularity at $z = z_0$ with Laurent series*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n.$$

1. *If $a_n = 0$ for all $n < 0$, then f has a removable singularity at $z = z_0$.*
2. *If there exists a positive integer m such that $a_{-m} \neq 0$ but $a_{-n} = 0$ for all $n > m$, then f has a pole of order m at $z = z_0$.*
3. *If there is no positive integer m such that $a_{-n} = 0$ for all $n > m$, then f has an essential singularity at $z = z_0$.*

In case 1, the singularity at $z = z_0$ can be removed by extending the definition of f to a function $\tilde{f}$ which is analytic in a neighbourhood of $z = z_0$ given by

$$\tilde{f}(z) := \begin{cases} a_0 & z = z_0, \\ f(z) & z \neq z_0. \end{cases}$$

Throughout this module, we will therefore remove any removable singularity and treat it as a regular (i.e. analytic) point.

Exercise 38 *For each of the following functions, classify the point $z = 0$ as a removable singularity (which we treat as a regular point), a pole, or an essential singularity.*

$$(a) \quad z^{-1} \sin z, \qquad (b) \quad z \sin(z^{-1}), \qquad (c) \quad z^{-2} \sin z..$$

Theorem 39 *(The Residue Theorem) Let γ be a closed contour on which a function f is analytic. Let f be analytic on the interior of γ except for a finite number of points $z_1, z_2, \ldots, z_n$. Then*

$$\frac{1}{2\pi i} \oint_\gamma f(z)\, dz = \sum_{j=1}^{n} r_j,$$

where r_j is the residue of f at z_j, $j = 1, \ldots, n$.

Corollary 40 *Let Ω be a domain bounded by a Jordan curve γ. Suppose that f is meromorphic in Ω and analytic on γ. If $Z_f(\Omega)$ and $P_f(\Omega)$ are the number of zeros and poles respectively of f in Ω (counting multiplicities) then*

$$\frac{1}{2\pi i}\oint_\gamma \frac{f'(z)}{f(z)}dz = Z_f(\Omega) - P_f(\Omega).$$

Theorem 41 *(Picard's Theorem) Let f have an isolated essential singularity at $z = a$. Then for any complex number $c \in \mathbf{C}$ with at most one possible exception, the equation $f(z) = c$ has infinitely many solutions in any neighbourhood of $z = a$.*

8 Singularities at infinity

For many purposes we wish to consider the point "$z = \infty$" to be like any other point in the complex plane. The complex plane can be mapped to a sphere without one point by stereographic projection.

Definition 42 *The function $f(z)$ is said to have a regular point, a pole, an essential singularity, or a branch point at infinity if $f(1/z)$ has the corresponding property at $z = 0$.*

Example 43 *The function $f(z) = \dfrac{z}{2z^2+1}$ has a simple zero at infinity, since $f(1/z) = \dfrac{z}{2+z^2}$ has a simple zero at $z = 0$.*

Example 44 *The function $f(z) = \exp(z)$ has an essential singularity at infinity since $f(1/z) = \exp(1/z)$ has an essential singularity at $z = 0$.*

9 Branch points

Consider the function $f(z) := \sqrt{z}$. That is, $f(z)$ satisfies $[f(z)]^2 = z$. Let us "fix a branch" of f by requiring that $f(1) = 1$. This definition agrees with the usual definition for $\sqrt{x}$, where x is a real number.

As defined, f can still have one of two values at any point in the complex plane except $z = 0$ and $z = 1$. At each point we would like to choose a value to make f as smooth as possible. If we were to write down the Taylor series expansion of the real-valued function $\sqrt{x}$ about the point $x = 1$ and then replace the real variable x with the complex variable z, we would have a power series that converges on the disc $|z - 1| < 1$ to a function whose square is z. So this gives us the unique determination of f that is analytic on this disc. Since f is analytic on the disc, we can find its power series about any other point z_1 in the open disc and that series will converge on a disc centred at z_1 with radius $|z_1|$ (since $z = 0$ is the only singularity). By continuing this process we get a unique determination of f along any curve starting from $z = 1$. However, something strange happens when we analytically continue f around the singularity at $z = 0$ and back to $z = 1$.

Let us consider the behaviour of f on the unit circle. Any point on the unit circle is given by $z = \mathrm{e}^{i\theta}$, where $\theta = 0$ corresponds to $z = 1$. At any point on the circle, $f(z) = \sqrt{z}$ must be either $\mathrm{e}^{i\theta/2}$ or $-\mathrm{e}^{i\theta/2}$. For $\theta = 0$, we have $f(\mathrm{e}^{i0}) = 1 = \mathrm{e}^{i0/2}$. Since f must be a continuous function of θ, it follows that we must take $f(\mathrm{e}^{i\theta}) = \mathrm{e}^{i\theta/2}$. However, as $\theta \to 2\pi$, we have $z \to 1$. So originally we had $f(1) = 1$ but now, after looping once around the circle, we return to find that $f(1) = \mathrm{e}^{i\pi} = -1$. Hence, although at every point of the complex plane, we can choose two determinations of f such that f is analytic in a disc centred at the point, we cannot glue these patches together to get a determination of f that is analytic on the entire complex plane with only the origin removed. The singularity at $z = 0$ is called a branch point of f.

There are two commonly used approaches of avoiding this multi-valued behaviour of branched functions such as $\sqrt{z}$. The first is to introduce a branch cut, which means that we restrict the domain of f in such a way that we cannot loop around the origin. For example, we could delete the negative real axis and only define f for $z = re^{i\theta}$, where $r > 0$ and $-\pi < \theta < \pi$ by $f(re^{i\theta}) = \sqrt{r}\,e^{i\theta/2}$. In this approach, the negative real axis is called a branch cut.

The other standard approach is to extend the domain of f from the complex plane to something called a Riemann surface. Imagine taking two copies of the cut complex plane described above. On one copy of the plane, we define $f(re^{i\theta}) = \sqrt{r}\,e^{i\theta/2}$ and on the other copy we define $f(re^{i\theta}) = -\sqrt{r}\,e^{i\theta/2}$. Now take the top of the branch cut of one of the planes and glue it to the bottom of the branch cut on the second plane. Do a similar "gluing" along the two remaining sides of the branch cuts (warning: you will need to work in more than three space dimensions to do this). Note that 0 (and ∞ if we think in terms of the complex sphere) are the same on both planes. In this way you have constructed a surface (called a Riemann surface) on which f is single-valued and analytic everywhere except 0 and ∞.

Example 45 *The logarithm defined by*

$$\log z = \int_\gamma \frac{dz}{z},$$

where γ is any path in $\mathbf{C} \setminus \{0\}$ from 1 to z, is branched at 0 and ∞. It follows that

$$\log(re^{i\theta}) = \log r + i\theta.$$

It follows that $\log z$ naturally lives on a Riemann surface with infinitely sheets. At each point $z \in \mathbf{C} \setminus \{0\}$, two determinations of $\log z$ differ by an integer multiple of $2\pi i$.

10 The Gamma function

The gamma function is defined by

$$\Gamma(z) := \int_0^\infty e^{-t} t^{z-1}\, dt, \qquad \Re(z) > 0. \tag{13}$$

Exercise 46 *Use integration by parts to show that $\Gamma(z+1) = z\Gamma(z)$. Show that for any positive integer n, $\Gamma(n+1) = n!$.*

Note that Γ can be extended to $\Re(z) \le 0$ using $\Gamma(z) = z^{-1}\Gamma(z+1)$. Γ is single-valued throughout the complex plane. It is analytic everywhere except where z is a non-positive integer, in which case Γ has a simple pole.

For $\Re(\mu), \Re(\nu) > 0$, the integral

$$B(\mu, \nu) := \int_0^1 t^{\mu-1}(1-t)^{\nu-1}\, dt,$$

where the path of integration is along the real axis, is known as the beta integral. On replacing t with $1-t$, it is straightforward to show that $B(\mu, \nu) = B(\nu, \mu)$. The beta integral has a simple expression in terms of the Γ function.

Theorem 47 *For $\Re(\mu), \Re(\nu) > 0$,*

$$B(\mu, \nu) := \int_0^1 t^{\mu-1}(1-t)^{\nu-1}\, dt = \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)}. \tag{14}$$

Proof: We will give the proof in the case for which $\Re(\mu), \Re(\nu) > 1/2$. Consider

$$\Gamma(\mu)\Gamma(\nu) = \int_0^\infty \int_0^\infty \mathrm{e}^{-s-t} s^{\mu-1} t^{\nu-1}\, ds\, dt.$$

Converting to "polar coordinates" $s = x^2 = (r\cos\theta)^2$, $t = y^2 = (r\sin\theta)^2$, gives

$$\begin{aligned}\Gamma(\mu)\Gamma(\nu) &= 4\int_0^\infty \int_0^{\pi/2} \mathrm{e}^{-r^2} (r\cos\theta)^{2\mu-1} (r\sin\theta)^{2\nu-1}\, r\, dr\, d\theta \\ &= 4\int_0^\infty \mathrm{e}^{-r^2} r^{2(\mu+\nu)-1} dr \int_0^{\pi/2} \cos^{2\mu-1}\theta\, \sin^{2\nu-1}\theta\, d\theta.\end{aligned}$$

Setting $u = r^2$ and $v = \cos^2\theta$ in the last expression gives

$$\Gamma(\mu+\nu)B(\mu,\nu).$$

11 Summary

Key words, phrases and concepts

1. Absolute convergence
2. Analytic function
3. Beta integral
4. Branch point
5. Cauchy's integral formula
6. Cauchy-Riemann equations
7. Essential singularity
8. Gamma function
9. Liouville's Theorem
10. Pole of order m
11. Removable singularity
12. Residue
13. Simple pole, simple zero
14. Zero of order m

Exercises

1. Write each of the following in the form $x + iy$.

   (a) $\dfrac{1-4i}{2+3i}$,

   (b) $(1+7i)^{-1}$,

   (c) $(1+\sqrt{3}\,i)^{29}$ (Hint: use polar form).

2. Use the Cauchy-Riemann equations to verify that $f(z) = z^3$ is analytic (i.e. differentiable) for all complex z.

3. For each of the following functions, classify the point $z = 0$ as a regular point, a pole, an essential singularity, or a branch point.

$$\text{(a)} \quad \frac{\sin(\sqrt{z})}{z}, \qquad \text{(b)} \quad \frac{\sin(z)}{1 - \mathrm{e}^z}, \qquad \text{(c)} \quad \frac{1}{\sin(z^{-1})}, \qquad \text{(d)} \quad \frac{1}{z^2 - 1}.$$

4. Find and classify all singularities of the following functions on the complex sphere (i.e., including the point at infinity).

$$\text{(a)} \quad \frac{\cos(z)}{z^2 + 1}, \qquad \text{(b)} \quad \sqrt{z(z-1)}, \qquad \text{(c)} \quad \mathrm{e}^{-z}, \qquad \text{(d)} \quad \frac{z^2 + 1}{z - 1}.$$

5. Show that every rational function has either a regular point or a pole at infinity.

6. Let $f(z) = \cos(z^6)$. Calculate $f^{(24)}(0)$ and $f^{(25)}(0)$.

7. Recall that a level set of a function f of two (real) variables (x, y) has the form $\{(x, y) : f(x, y) = C\}$, for some constant C.

   Let $z = x + iy$. Show that the level sets of $|\exp(1/z)|$ and $\arg(\exp(1/z))$ are circles and straight lines through the origin. Plot some of these level curves on a diagram and explain how it illustrates Picard's theorem 41.

8. Consider the (branched) function $f(z) = z^{1/3}$. Suppose we (locally) choose a branch such that $f(1) = 1$. If we analytically continue f along the circle $|z| = 1$ in a anti-clockwise direction, what value will we obtain for f when we return to $z = 1$? What value do we get if we trace the circle in the clockwise direction?

9. Use the Cauchy integral formula to evaluate $\displaystyle\int_C \frac{\exp z}{(z^2 - 25)(z^2 - 3)}\, dz$ where C is the circle $|z| = 4$ traversed in the clockwise direction.

10. Show that the gamma function Γ has a non-isolated singularity at infinity.

11. Show that

$$\int_0^\infty \mathrm{e}^{-t^\alpha}\, dt = \Gamma\left(1 + \frac{1}{\alpha}\right),$$

    where $\Re(\alpha) > 0$.

12. Show that

$$\int_0^\infty \mathrm{e}^{-r^2} r^{2(\mu+\nu)-1} dr = \frac{1}{2}\Gamma(\mu + \nu)$$

    and

$$\int_0^{\pi/2} \cos^{2\mu-1}\theta \, \sin^{2\nu-1}\theta \, d\theta = \frac{1}{2}B(\mu, \nu),$$

    where $\Re(\mu), \Re(\nu) > 1/2$. [Hint: in the last integral, use the substitution $v = \cos^2\theta$.]

13. Let f be an entire function and suppose that $|f(z)/z^n|$ is bounded for large z, where n is a positive integer. Show that f is a polynomial.