Complex Analysis I

Fall 2014

These are notes for the graduate course Math 5283 (Complex Analysis I) taught by Dr. Anthony Kable at the Oklahoma State University (Fall 2014). The notes are taken by Pan Yan (pyan@okstate.edu), who is responsible for any mistakes. If you notice any mistakes or have any comments, please let me know.

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1 Complex Numbers (08/18)

The complex field $\mathbb{C}$ is $\mathbb{R}^2$ with a pair of operations that makes it a field:

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2), (x_1, y_1) \cdot (x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1).$$

The point is that we have two elements: $1 = (1, 0), i = (0, 1)$, and

$$i^2 = (0, 1) \cdot (0, 1) = (-1, 0) = -1.$$

Any complex number can be expressed uniquely as $x + yi$ where $x, y \in \mathbb{R}$. If $z = x + yi$ with $x, y \in \mathbb{R}$, then

$$\text{Re}(z) = x, \text{Im}(z) = y, |z| = \sqrt{x^2 + y^2}, \overline{z} = x - yi.$$

Conjugates are automorphism of the field $\mathbb{C}$ since

$$\overline{z + w} = \overline{z} + \overline{w}, \overline{zw} = \overline{z} \cdot \overline{w}.$$

Notice that $z\overline{z} = |z|^2$.

We use the metric $d(z, w) = |z - w|$ on $\mathbb{C}$ to make it a metric space. Note this is the standard d_2 metric on $\mathbb{R}^2$. So we already know that $\mathbb{C}$ is a complete metric space.

Example 1.1. Describe the set $\{z \in \mathbb{C} : |z| + |z - 1| = 2\}$. It is an ellipse with foci at 0 and 1, major axis the real axis, and one vertex at 3/2.

If we switch to polar coordinates, then every non-zero complex number gets a unique representation in the form $(r\cos(\theta), r\sin(\theta))$, where $r > 0$ and $\theta \in [0, 2\pi)$. By definition, $e^{it} = \cos(t) + i\sin(t)$. So the polar representation can be expressed as $z = re^{i\theta}$ with $r > 0$ and $\theta \in [0, 2\pi)$. This reveals some information about multiplication:

$$r_1e^{i\theta_1} \cdot r_2e^{i\theta_2} = r_1r_2e^{i(\theta_1+\theta_2)}.$$

This tells us that the map $M_w : \mathbb{C} \to \mathbb{C}$ given by $M_w(z) = wz$ consists of the composition of a homothety with factor $|w|$ and a rotation anticlockwise by the angle θ such that $w = |w|e^{i\theta}$. Note the map $S : \mathbb{C} \to \mathbb{C}$ given by $S(z) = \overline{z}$ is a reflection.

Example 1.2. Determine all solutions to the equation $z^4 = i$ with $z \in \mathbb{C}$. Write the solutions in the form $x + iy$. Use polar representation, say $z = re^{i\theta}$, then $z^4 = r^4e^{4i\theta} = i$. This tells us that $r^4 = 1$ and $r = 1$. Thus $e^{4i\theta} = i = e^{\frac{\pi}{2}i}$. Thus,

$$4\theta = \frac{\pi}{2} + 2k\pi, k \in \mathbb{Z},$$

$$\theta = \frac{\pi}{8} + \frac{1}{2}k\pi, k \in \mathbb{Z}.$$

The θ in the polar representation is an *argument* of the complex number. If $z = re^{i\theta}$ then $\arg(z) = \theta$. Note that arg is not a function — it has many values. In Complex Made Simple (CMS), the principle argument is defined to be the unique argument that lies in $(-\pi, \pi]$. It is denoted by $\text{Arg}(z)$.

2 Differentiability and the Cauchy-Riemann Equations (08/20)

We consider functions $f : \mathbb{C} \to \mathbb{C}$. These can be regarded as functions from $\mathbb{R}^2$ to $\mathbb{R}^2$.

Definition 2.1. Let $V \subset \mathbb{C}$ be an open set, $f : V \to \mathbb{C}$, and $p \in V$. Then f is *complex differentiable* at p if the limit

$$\lim_{h \to 0} \frac{f(p+h) - f(p)}{h}$$

exists.

If the limit exists, then it will be some complex number w. Then we will have

$$f(z) = f(p) + w(z-p) + E(z) \tag{2.1}$$

where $E(p) = 0$, and $\lim_{z \to p} \frac{E(z)}{z-p} = 0$. Conversely, if we have an expression like (2.1) then f is differentiable at p and its derivative at p is w.

Recall that a linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ can be represented by a 2×2 matrix $[T]$. The column of $[T]$ are $T(e_1)$ and $T(e_2)$ and it has the property that $T(v) = [T]v$. What if we think of $\mathbb{R}^2$ as $\mathbb{C}$ and consider the linear map $M_w : \mathbb{C} \to \mathbb{C}$ given by $M_w(z) = wz$? What is $[M_w]$?

Let $w = a + bi$. In $\mathbb{C}$ the standard basis is $1, i$. Then

$$M_w(1) = w \times 1 = w = a + bi,$$

$$M_w(i) = w \times i = (a+bi)i = -b + ai,$$

and so

$$[M_w] = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}.$$

Among all linear maps $T : \mathbb{R}^2 \to \mathbb{R}^2$, those that are complex linear are precisely the ones whose matrix has the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$.

Say we have an open set $V \subset \mathbb{C}$, $p \in V$, $f : V \to \mathbb{C}$. Let $f = u + iv$, where $u : V \to \mathbb{R}$ and $v : V \to \mathbb{R}$. Say that f is real differentiable at p. Then

$$f(x+yi) = f(p) + T(\begin{bmatrix} x \\ y \end{bmatrix} - p) + E(\begin{bmatrix} x \\ y \end{bmatrix}) \tag{2.2}$$

where $E(p) = 0$ and E is sublinear at p. We know that $u_x(p), u_y(p), v_x(p), v_y(p)$ all exist. Moreover,

$$[T] = \begin{bmatrix} u_x(p) & u_y(p) \\ v_x(p) & v_y(p) \end{bmatrix}$$

is the Jacobian matrix.

Note that (2.2) would say that f is (complex) differentiable at p if T is actually complex linear. We just found out that this happens when the matrix of T has the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ and, in that case, it is multiplication by $a+bi$. This means that f is complex differentiable at p if

$$u_x(p) = v_y(p) \tag{2.3}$$

and

$$u_y(p) = -v_x(p) \tag{2.4}$$

When this happens, $f'(p) = u_x(p) + v_x(p)i$. Equations (2.3) and (2.4) are called *Cauchy-Riemann equations* (C-R equations) . We have the exact equivalence:

$$\Big(\text{real differentiable at } p \text{ and satisfy CR equations}\Big) \Leftrightarrow \Big(\text{complex differentiable at } p\Big)$$

Recall from Advanced Calculus: If $f = u+iv : V \to \mathbb{C}, p \in V$, u_x, u_y, v_x, v_y all exist on V, and are continuous at p, then f is real differentiable at p. In particular, if $u, v : V \to \mathbb{R}$ are C^1 and satisfy CR equations, then $u+iv$ is complex differentiable at every point of V.

3 Power Series (08/22)

Definition 3.1. A *power series* is a series of the form

$$\sum_{n=0}^{\infty} c_n(z-z_0)^n.$$

Here $c_0, c_1, c_2, \cdots \in \mathbb{C}$, $z, z_0 \in \mathbb{C}$, and we have the convention that $(z-z_0)^0$ means 1.

Theorem 3.2. *Given a power series $\sum_{n=0}^{\infty} c_n(z-z_0)^n$ we define*

$$R = sup\{r \geq 0 : (c_n r^n) \text{ is bounded}\}.$$

The power series is uniformly absolutely convergent on any compact subset of the disk $D(z_o, R)$. (Note that $R \in [0,\infty) \cup \{\infty\}$, $D(z_o, R) = \{z \in \mathbb{C} : |z-z_0| < R\}$.)

Proof. Let the compact set be $K \subset D(z_0, R)$. Then there is a number $0 < r < R$ such that $K \subset D(z_0, r)$ (compact: each open cover has a finite subcover, pick the largest one), unless $R = 0$. Note that $R = 0$ is trivial, and so we shall assume that $R > 0$ and choose such a r. It suffices to show that the series converges uniformly absolutely on $D(z_0, r)$. There is a number ρ such that $r < \rho < R$ and $(c_n \rho^n)$ is bounded. Let M be a bound for $(c_n \rho^n)$. Thus $|c_n \rho^n| \leq M$ for all $n \geq 0$. Equivalently, $|c_n| \leq M\rho^{-n}$ for all $n \geq 0$. Suppose $z \in D(z_0, r)$. Then

$$|c_n(z-z_0)^n| \leq |c_n r^n| = |c_n \rho^n (\frac{r}{\rho})^n| \leq M(\frac{r}{\rho})^n.$$

Note $0 < \frac{r}{\rho} < 1$. The (geometric) series $\sum_{n=0}^{\infty} M(\frac{r}{\rho})^n$ converges. By the Direct Comparison Test, $\sum_{n=0}^{\infty} |c_n(z-z_0)^n|$ converges uniformly on $D(z_0, r)$. Thus $\sum_{n=0}^{\infty} c_n(z-z_0)^n$ converges uniformly absolutely on $D(z_0, r)$. □

We know that the sum of a power series on $D(z_0, R)$ is continuous (Uniform Limit Theorem: the uniform limit of any sequence of continuous functions is continuous). In fact, a power series is differentiable on $D(z_0, R)$. Say that $f(z) = \sum_{n=0}^{\infty} c_n(z-z_0)^n$ with $z \in D(z_0, R)$. Suppose that $w \in D(z_0, R)$. Then

$$f(w) = \sum_{n=0}^{\infty} c_n(w-z_0)^n$$

and so

$$\begin{aligned}
f(z) - f(w) &= \sum_{n=0}^{\infty} c_n \left[(z-z_0)^n - (w-z_0)^n\right] \\
&= \sum_{n=0}^{\infty} c_n \left[(z-z_0) - (w-z_0)\right] \left[(z-z_0)^{n-1} + (z-z_0)^{n-2}(w-z_0) + \cdots + (w-z_0)^{n-1}\right] \\
&= (z-w) \sum_{n=0}^{\infty} c_n \sum_{j=0}^{n-1} (z-z_0)^{n-1-j}(w-z_0)^j.
\end{aligned}$$

If $z \neq w$ then we get

$$\frac{f(z)-f(w)}{z-w} = \sum_{n=0}^{\infty} c_n \sum_{j=0}^{n-1} (z-z_0)^{n-1-j}(w-z_0)^j.$$

Let

$$\Phi(z,w) = \sum_{n=0}^{\infty} c_n \sum_{j=0}^{n-1} (z-z_0)^{n-1-j}(w-z_0)^j.$$

If we set $z = w$ in Φ we get

$$\Phi(z,z) = \sum_{n=0}^{\infty} c_n n(z-z_0)^{n-1} = \sum_{n=1}^{\infty} c_n n(z-z_0)^{n-1}.$$

To conclude that f is differentiable and that $f'(z) = \sum_{n=1}^{\infty} c_n n(z-z_0)^{n-1}$, all we need is that $\Phi : D(z_0, R) \times D(z_0, R) \to \mathbb{C}$ is continuous. We can show that this is so by using the Weierstrass M-test. We can do this by comparison with a series $\sum_{n=1}^{\infty} n\{\frac{r}{p}\}^{n-1}$ as before.

Corollary 3.3. *Say that $f(z) = \sum_{n=0}^{\infty} c_n(z-z_0)^n$ is a power series with positive radius of convergence R. Then f is infinitely differentiable on $D(z_0, R)$, we have*

$$c_n = \frac{f^{(n)}(z_0)}{n!}$$

for $n \geq 0$, and f uniquely determines (c_n).

Definition 3.4. Let $\sum_{n=0}^{\infty} a_n$, $\sum_{n=0}^{\infty} b_n$ be two complex series. The *Cauchy product* is defined as

$$\sum_{n=0}^{\infty} c_n = \sum_{n=0}^{\infty} a_n \sum_{n=0}^{\infty} b_n$$

where $c_n = \sum_{j=0}^{n} a_j b_{n-j}$.

4 Sin, Cos and Exp (08/25)

For $z \in \mathbb{C}$, we define $\exp(z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$. Note that $|\frac{1}{(n+1)!} z^{(n+1)}| = \frac{|z|}{n+1} |\frac{1}{n!} z^n|$ and so the sequence $(|\frac{1}{n!} z^n|)$ is decreasing once $n + 1 > |z|$. It follows that $(|\frac{1}{n!} z^n|)$ is always bounded. Thus $R = \infty$ for this power series and so we get an infinitely differentiable function $\exp : \mathbb{C} \to \mathbb{C}$. We have

$$\begin{aligned}\exp'(z) &= \sum_{n=1}^{\infty} \frac{n}{n!} z^{n-1} \\ &= \sum_{n=1}^{\infty} \frac{1}{(n-1)!} z^{n-1} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} z^n \\ &= \exp(z).\end{aligned}$$

Fix $w \in \mathbb{C}$ and consider $g : \mathbb{C} \to \mathbb{C}$ given by $g(z) = \exp(-z) \cdot \exp(z + w)$. By the Product and Chain Rules, g is infinitely differentiable. Also

$$g'(z) = -\exp(-z) \cdot \exp(z + w) + \exp(-z) \cdot \exp(z + w) = 0$$

for all $z \in \mathbb{C}$. Two possibilities to show that $g'(z) = 0 \Rightarrow g$ is constant: One is to use the Mean Value Inequality ($\|\phi(x) - \phi(y)\| \leq \sup_{p \in [x,y]} \|\mathrm{D}\phi(p)\| \cdot \|x - y\|$) from Advanced Calculus, since $g'(z)$ is just a special case of the Fréchet derivative. Second possibility is to show that g is given by a power series with center 0. (Note: $\exp(z + w) = \sum_{n=0}^{\infty} \frac{1}{n!} (z + w)^n = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{j=0}^{n} \binom{n}{j} z^j w^{n-j} = \cdots$) and then use the Cauchy product and the corollary. We deduce that g is constant. So $g(z) = g(0)$ for all z. That is, $\exp(-z) \cdot \exp(z + w) = \exp(-0) \cdot \exp(0 + w) = \exp(w)$ for all $z, w \in \mathbb{C}$. Set $w = 0$, we get $\exp(-z) \cdot \exp(z) = 1$, and so $\exp(-z) = \exp(z)^{-1}$. Now multiply the identity $\exp(-z) \cdot \exp(z + w) = \exp(w)$ by $\exp(z)$ to obtain

$$\exp(z + w) = \exp(z)\exp(w), \forall z, w \in \mathbb{C}.$$

Remark 4.1. *This implies that $\exp(z) \neq 0$ for all $z \in \mathbb{C}$.*

For $z \in \mathbb{C}$,

$$\begin{aligned}
\exp(iz) &= \sum_{n=0}^{\infty} \frac{1}{n!}(iz)^n \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} i^n z^n \\
&= \sum_{m=0}^{\infty} \frac{1}{(2m)!} i^{2m} z^{2m} + \sum_{m=0}^{\infty} \frac{1}{(2m+1)!} i^{2m+1} z^{2m+1} \\
&= \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} z^{2m} + i \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} z^{2m+1} \\
&\stackrel{\text{def}}{=} \cos(z) + i\sin(z).
\end{aligned}$$

The series imply that cos is even, sin is odd, and so

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}, \ \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$$

for all $z \in \mathbb{C}$. Note that if $y \in \mathbb{R}$ then

$$\cos(iy) = \frac{e^{i(iy)} + e^{-i(iy)}}{2} = \frac{e^{-y} + e^{y}}{2} = \cosh(y),$$

$$\sin(iy) = \frac{e^{i(iy)} - e^{-i(iy)}}{2i} = \frac{e^{-y} - e^{y}}{2i} = -i\frac{e^{-y} - e^{y}}{2} = i\frac{e^{y} - e^{-y}}{2} = i\sinh(y).$$

Previously, we define $e^{i\theta} = \cos(\theta) + i\sin(\theta)$. Now this is true by definition of $\cos, \sin, \exp$. This justifies the polar representation that we used before. Note if $z = re^{i\theta}$ with $r > 0$, then $z = e^{\ln(r)} \cdot e^{i\theta} = e^{\ln(r) + i\theta}$. This tells us that $\exp(\mathbb{C}) = \mathbb{C} - \{0\}$.

Remark 4.2. *For $z = x + iy \in \mathbb{C}$, $|\exp(z)| = \exp(x)$.*

5 Preliminary Results on Holomorphic Functions I (08/27)

Definition 5.1. If X is a metric space then a *curve* in X is a map $\gamma : [a, b] \to X$ for some $a \le b$ that is continuous. The interval $[a, b]$ is the parameter interval and $\gamma([a, b])$ is denoted γ^* and called the trace of γ.

Definition 5.2. A curve $\gamma : [a, b] \to \mathbb{C}$ is *smooth* (better "*piecewise smooth*") if there is a partition $a = t_0 < t_1 < \cdots < t_n = b$ such that
(1) $\gamma|_{(t_{j-1},t_j)}$ is C^1 for $1 \le j \le n$;
(2) The one-sided limits of γ' exist at each $t_j (0 \le j \le n)$.

Definition 5.3. Let $V \subset \mathbb{C}$ be open and $\gamma : [a, b] \to V$ a piecewise smooth curve. Let $f : V \to \mathbb{C}$ be a continuous function. Then we define

$$\int_\gamma f(z)\, dz = \int_a^b f(\gamma(t))\gamma'(t)\, dt.$$

This is the *path* or *contour integral* of f over γ.

Remark 5.4. *Note that $f(\gamma(t))\gamma'(t)$ is Riemann integrable because it is bounded and the set of discontinuities is at worse a finite set (hence a zero set). Also, if $a = t_0 < t_1 < \cdots < t_n = b$ is a suitable partition for γ then*

$$\int_a^b f(\gamma(t))\gamma'(t)\, dt = \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} f(\gamma(t))\gamma'(t)\, dt.$$

Lemma 5.5 (Cauchy's Theorem for Derivatives)**.** *Let $V \subset \mathbb{C}$ be an open set, $f : V \to \mathbb{C}$ continuous and suppose that there is an $F : V \to \mathbb{C}$ such that $F' = f$. Let γ be a piecewise smooth curve in V. Then*

$$\int_\gamma f(z)\, dz = F(\gamma(b)) - F(\gamma(a))$$

where $[a, b]$ is the parameter interval for γ. In particular, $\int_\gamma f(z)\, dz = 0$ if γ is closed (means $\gamma(a) = \gamma(b)$).

Proof. By the Chain Rule for vector functions, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(F(\gamma(t))) = F'(\gamma(t)) \cdot \gamma'(t) = f(\gamma(t)) \cdot \gamma'(t)$$

for all t where the derivative γ' exists. Let $a = t_0 < t_1 < \cdots < t_n = b$ be a suitable partition. Then

$$\begin{aligned}\int_\gamma f(z)\,dz &= \sum_{j=1}^{n}\int_{t_{j-1}}^{t_j} f(\gamma(t))\gamma'(t)\,dt\\ &= \sum_{j=1}^{n}\int_{t_{j-1}}^{t_j} \frac{\mathrm{d}}{\mathrm{dt}}(F(\gamma(t)))\,dt\\ &= \sum_{j=1}^{n}[F(\gamma(t_j)) - F(\gamma(t_{j-1}))]\\ &= F(\gamma(b)) - F(\gamma(a)).\end{aligned}$$

□

Definition 5.6. If $\gamma : [a, b] \to \mathbb{C}$ is a piecewise smooth curve then the *length* of γ is

$$L(\gamma) = \int_a^b |\gamma'(t)|\,dt.$$

Lemma 5.7 (ML Inequality)**.** *Let $V \subset \mathbb{C}$ be an open set, $f : V \to \mathbb{C}$ be continuous, and suppose that $|f(z)| \leq M$ for all $z \in \gamma^*$, where $\gamma : [a, b] \to V$ is a piecewise smooth curve. Then*

$$\left|\int_\gamma f(z)\,dz\right| \leq ML(\gamma).$$

Proof. We have

$$\begin{aligned}\left|\int_\gamma f(z)\,dz\right| &= \left|\int_a^b f(\gamma(t))\gamma'(t)\,dz\right|\\ &\leq \int_a^b |f(\gamma(t))|\left|\gamma'(t)\right|\,dz\\ &\leq \int_a^b M\left|\gamma'(t)\right|\,dz\\ &= M\int_a^b \left|\gamma'(t)\right|\,dz\\ &= ML(\gamma).\end{aligned}$$

□

6 Preliminary Results on Holomorphic Functions II (08/29)

Definition 6.1. Let $S \subset \mathbb{C}$. Define $A_1(S)$ to be the free abelian group on the set of piecewise smooth curves in S. That is to say,

$$A_1(S) = \{\sum_{j=1}^{d} n_j\gamma_j | n_j \in \mathbb{Z}, \gamma_j \text{ is a piecewise smooth curve in } S\}.$$

If $f \in C(S)$ then we have defined $\int_\gamma f(z)\,dz \in \mathbb{C}$ for each piecewise smooth path in S. If $\Gamma = \sum_{j=1}^{d} n_j\gamma_j$ then we can define $\int_\Gamma f(z)\,dz = \sum_{j=1}^{d} n_j \int_{\gamma_j} f(z)\,dz \in \mathbb{C}$.

Definition 6.2. $B_1(S)$ is defined as

$$B_1(S) = \{\Gamma \in A_1(S) | \int_\Gamma f(z)\,dz = 0 \text{ for all } f \in C(S)\}.$$

Remark 6.3. *This is a subgroup of* $A_1(S)$*.*

Definition 6.4. The *chains* in S are elements of $A_1(S)/B_1(S)$.

Remark 6.5. *Two chains* $\Gamma_1 = \Gamma_2$ *if* $\int_{\Gamma_1} f(z)\,dz = \int_{\Gamma_2} f(z)\,dz$ *for all* $f \in C(S)$*.*

Example 6.6. Define

$$\begin{aligned}
&\gamma : [0, 2\pi] \to \mathbb{C}, \gamma(t) = \exp(it),\\
&\gamma_1 : [0, \pi] \to \mathbb{C}, \gamma(t) = \exp(it),\\
&\gamma_2 : [\pi, 2\pi] \to \mathbb{C}, \gamma(t) = \exp(it).
\end{aligned}$$

We have $\gamma = \gamma_1 \dot{+} \gamma_2$ in the group of chains in $\mathbb{C}$. To verify, let $f \in C(\mathbb{C})$. Then

$$\begin{aligned}
\int_\gamma f(z)\,dz &= \int_0^{2\pi} f(\exp(it))i\exp(it)\,dt\\
&= \int_0^{\pi} f(\exp(it))i\exp(it)\,dt + \int_\pi^{2\pi} f(\exp(it))i\exp(it)\,dt\\
&= \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz\\
&= \int_{\gamma_1 \dot{+} \gamma_2} f(z)\,dz.
\end{aligned}$$

Thus $\gamma = \gamma_1 \dot{+} \gamma_2$ in the group of chains in $\mathbb{C}$.

There is a map $\partial : A_1(S) \to A_0(S)$, where $A_0(S)$ is the free group on S. It satisfies $\partial(\gamma) = \gamma(b) - \gamma(a)$ if $[a, b]$ is the parameter integral of γ.

The critical thing we need is that $\partial(\gamma) = 0$ if $\gamma \in B_1(S)$. If so then we can define the $\partial(\gamma)$ for γ a chain.

Definition 6.7. A *cycle* is a chain in the kernel of ∂.

In the example, $\partial(\gamma) = 1-1 = 0, \partial(\gamma_1 \dot{+} \gamma_2) = \partial(\gamma_1) + \partial(\gamma_2) = [(-1)-(1)]+[1-(-1)] = 0$.

If $z, w \in \mathbb{C}$, then $[z, w]$ denotes the curve

$$\gamma : [0,1] \to \mathbb{C}$$

such that $\gamma(t) = (1-t)z + tw$.

Example 6.8. Show that $[w, z] = \dot{-}[z, w]$.

Solution. $[w, z]$ is the curve

$$\gamma_1 : [0,1] \to \mathbb{C} \text{ such that } \gamma_1(t) = (1-t)w + tz.$$

$\dot{-}[z, w]$ is the curve

$$\gamma_2 : [0,1] \to \mathbb{C} \text{ such that } \gamma_2(t) = -[(1-t)z + tw].$$

Let $f \in C(\mathbb{C})$, then

$$\begin{aligned}
\int_{\gamma_1} f(z)\, dz &= \int_0^1 f((1-t)w + tz) \cdot \gamma_1'(t)\, dt \\
&= \int_0^1 f((1-t)w + tz) \cdot (z-w)\, dt \\
&= \int_1^0 f(uw + (1-u)z) \cdot (z-w) \cdot (-1)\, du \text{ (change of variable)} \\
&= -\int_0^1 f((1-t)z + tw) \cdot (w-z)\, dt \\
&= \int_{\gamma_2} f(z)\, dz.
\end{aligned}$$

Thus $[w, z] = \dot{-}[z, w]$.

Definition 6.9. If $a, b, c \in \mathbb{C}$ then define $[a, b, c]$ to be the *triangle* with vertices a, b, c.

By definition, $\partial[a, b, c] = [a, b] \dot{+} [b, c] \dot{+} [c, a]$.

7 Preliminary Results on Holomorphic Functions III (09/03)

Theorem 7.1 (Cauchy-Goursat). *Let $V \subset \mathbb{C}$ be an open set, and $f : V \to \mathbb{C}$ a (complex) differentiable function. Suppose that $T \subset V$ is a triangle. Then $\int_{\partial T} f(z)\, dz = 0$.*

Proof. Suppose not. Take a triangle $T \subset V$ such that $\int_{\partial T} f(z)\,dz \neq 0$. Let $\eta = |\int_{\partial T} f(z)\,dz| \neq 0$. Call T T_0 and note that $|\int_{\partial T_0} f(z)\,dz| = \eta$. Write $T_0 = T_0^1 \cup T_0^2 \cup T_0^3 \cup T_0^4$ and note that

$$\left|\int_{\partial T_0} f(z)\,dz\right| \leq \left|\int_{\partial T_0^1} f(z)\,dz\right| + \left|\int_{\partial T_0^2} f(z)\,dz\right| + \left|\int_{\partial T_0^3} f(z)\,dz\right| + \left|\int_{\partial T_0^4} f(z)\,dz\right|.$$

Thus there is at least one $j \in \{1,2,3,4\}$ such that

$$\left|\int_{\partial T_0^j} f(z)\,dz\right| \geq \frac{1}{4}\eta.$$

Call T_o^j with this property T_1. We have

$$\left|\int_{\partial T_1} f(z)\,dz\right| \geq \frac{1}{4}\eta.$$

We repeat this process to construct a sequence $(T_n)_{n=0}^{\infty}$ of triangles such that

(1) $T_0 \supset T_1 \supset T_2 \cdots$
(2) $\operatorname{diam}(T_n) = 2^{-n}\operatorname{diam}(T_0)$
(3) $L(\partial T_n) = 2^{-n}L(\partial T_0)$
(4) $\left|\int_{\partial T_n} f(z)\,dz\right| \geq 4^{-n}\eta.$

From a general fact about complete metric space, we know that $\cap_{n=0}^{\infty} T_n = \{p\}$ for some $p \in V$. Now f is differentiable at p and so $f(z) = f(p) + f'(p)(z-p) + E(z)$. Note that $\frac{d}{dz}(f(p)z) = f(p)$ and $\frac{d}{dz}(\frac{f'(p)(z-p)^2}{2}) = f'(p)(z-p)$ and so $\int_{\partial T_n} f(z)\,dz = \int_{\partial T_n} E(z)\,dz$.

Let $\epsilon > 0$. Then there is some $\delta > 0$ such that $|E(z)| \leq \epsilon|z-p|$ for all z such that $|z-p| < \delta$. Choose an n such that $T_n \subset D(p,\delta)$. Then

$$\begin{aligned}
\left|\int_{\partial T_n} f(z)\,dz\right| &= \left|\int_{\partial T_n} E(z)\,dz\right| \\
&\leq \int_{\partial T_n} |E(z)|\,\,dz \\
&\leq \epsilon \cdot \operatorname{diam}(T_n) \cdot L(\partial T_n) \\
&= \epsilon \cdot 2^{-n} \cdot \operatorname{diam}(T_0) \cdot 2^{-n} \cdot L(\partial T_n) \\
&= \epsilon \cdot \operatorname{diam}(T_0) \cdot L(\partial T_n) \cdot 4^{-n}.
\end{aligned}$$

We conclude that for any $\epsilon > 0$ there is an n such that

$$\eta \cdot 4^{-n} \leq \epsilon \cdot \operatorname{diam}(T_0) \cdot L(\partial T_0) \cdot 4^{-n}.$$

Thus

$$\eta \leq \epsilon \cdot \operatorname{diam}(T_0) \cdot L(\partial T_0)$$

for all $\epsilon > 0$. This implies $\eta = 0$, a contradiction. Thus no such triangle exists. $\square$

Proposition 7.2. *Let $V \subset \mathbb{C}$ be a convex open set and $f : V \to \mathbb{C}$ be a continuous function such that $\int_{\partial T} f(z)\,dz = 0$ for any triangle $T \subset V$. Then there is a differentiable function $F : V \to \mathbb{C}$ such that $F' = f$.*

Proof. Let $z_0 \in V$. Define $F(z) = \int_{[z_0,z]} f(w)\,dw$. Note $[z_0, z] \subset V$ and so this is well defined. For small enough $h \in \mathbb{C}$, $z + h \in V$ and so

$$F(z+h) = \int_{[z_0,z+h]} f(w)\,dw$$

and

$$\begin{aligned}
F(z+h) - F(z) &= \int_{[z_0,z+h]\dot{+}[z,z_0]} f(w)\,dw \\
&= \int_{[z_0,z+h]\dot{+}[z+h,z]\dot{+}[z,z_0]\dot{-}[z+h,z]} f(w)\,dw \\
&= \int_{\dot{-}[z+h,z]} f(w)\,dw \\
&= \int_{[z,z+h]} f(w)\,dw.
\end{aligned}$$

Thus

$$F(z+h) - F(z) - f(z)h = \int_{[z,z+h]} (f(w) - f(z))\,dw.$$

Hence

$$\frac{F(z+h) - F(z)}{h} - f(z) = \frac{1}{h} \int_{[z,z+h]} (f(w) - f(z))\,dw$$

for all small enough $h \in \mathbb{C} - \{0\}$. It suffices to show

$$\lim_{h \to 0} \frac{1}{h} \int_{[z,z+h]} (f(w) - f(z))\,dw = 0$$

Let $\epsilon > 0$. Choose $\delta > 0$ such that if $0 < |h| < \delta$ then $z + h \in V$ and $|f(w) - f(z)| < \epsilon$ for all $w \in [z, z+h]$. If $|h| < \delta$ then

$$\left| \frac{1}{h} \int_{[z,z+h]} (f(w) - f(z))\,dw \right| \le \frac{1}{|h|} \cdot \epsilon \cdot |h| = \epsilon.$$

That confirms that $\lim_{h \to 0} \frac{1}{h} \int_{[z,z+h]} (f(w) - f(z))\,dw = 0$. $\square$

8 Preliminary Results on Holomorphic Functions IV (09/05)

Theorem 8.1 (Cauchy's Theorem for Convex Sets)**.** *Let $V \subset \mathbb{C}$ be convex and open and $f : V \to \mathbb{C}$ be differentiable on V. Let γ be a closed path in V. Then $\int_\gamma f(z)\,dz = 0$.*

Proof. By Cauchy-Goursat Theorem, if T is a triangle, $T \subset V$, then $\int_{\partial T} f(z)\,dz = 0$. Since V is convex, the fact that $\int_{\partial T} f(z)\,dz = 0$ implies that there is some $F : V \to \mathbb{C}$ such that $F' = f$. By Cauchy's Theorem for Derivatives, if $\gamma : [a,b] \to V$ then $\int_\gamma f(z)\,dz = F(\gamma(b)) - F(\gamma(a)) = 0$ because $\gamma(a) = \gamma(b)$. □

Theorem 8.2 (Morera's Theorem)**.** *Suppose $V \subset \mathbb{C}$ is open, $f : V \to \mathbb{C}$ is continuous. If $\int_{\partial T} f(z)dz = 0$ for every triangle $T \subset V$, then $f \in H(V)$.*

Theorem 8.3 (Cauchy's Integral Formula)**.** *Let $V \subset \mathbb{C}$ be open, $r > 0$, $z_0 \in V$ and $\overline{D(z_0, r)} \subset V$ and also let $\gamma : [0, 2\pi] \to V$ be the path $\gamma(t) = z_0 + re^{it}$. Suppose that $f : V \to \mathbb{C}$ is complex differentiable and $z \in D(z_0, r)$. Then*

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w - z}\,dw.$$

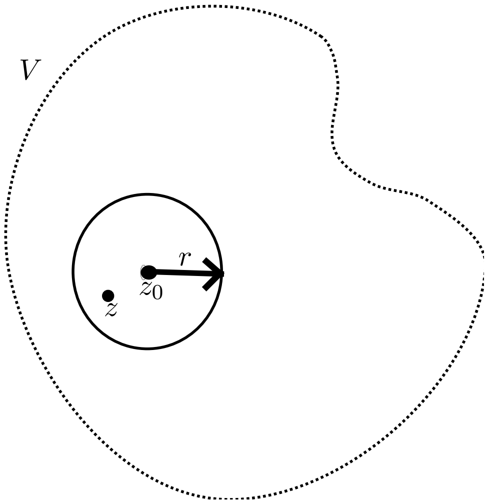

Figure 1:

Proof. Choose $\rho > 0$ so small that $\overline{D(z, \rho)} \subset D(z_0, r)$ (see Figure 2). Let $\gamma_\rho : [0, 2\pi] \to D(z_0, r)$ be $\gamma_\rho(t) = z + \rho e^{it}$. Let $g : V - \{z\} \to \mathbb{C}$ be $g(w) = \frac{f(w)}{w-z}$. Then g is differentiable on $V - \{z\}$. Note that $\sigma_1 \dot{+} \sigma_2 \dot{+} \sigma_3 = \gamma \dot{-} \gamma_\rho$ (see Figure 3). This tells us that

$$\int_\gamma g(w)\,dw - \int_{\gamma_\rho} g(w)\,dw = \int_{\sigma_1} g(w)\,dw + \int_{\sigma_2} g(w)\,dw + \int_{\sigma_3} g(w)\,dw.$$

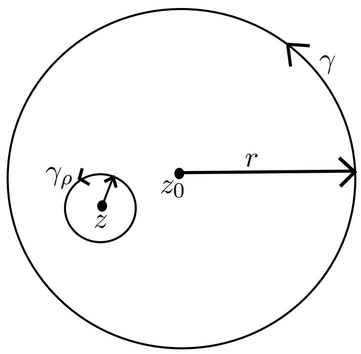

Figure 2:

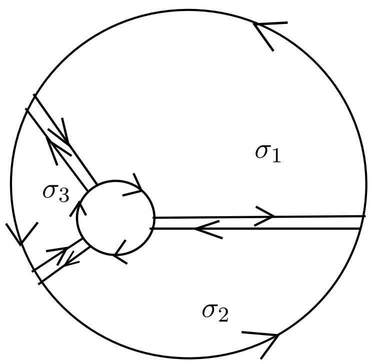

Figure 3:

We know that $\int_{\sigma_1} g(w)\,dw = 0$ by Cauchy's Theorem for Convex Sets (See Figure 4). Similarly, $\int_{\sigma_2} g(w)\,dw = \int_{\sigma_3} g(w)\,dw = 0$, and so $\int_\gamma g(w)\,dw = \int_{\gamma_\rho} g(w)\,dw$. Now,

$$\begin{aligned}\int_{\gamma_\rho} g(w)\,dw &= \int_{\gamma_\rho} \frac{f(w)}{w-z}\,dw \\ &= \int_0^{2\pi} \frac{f(z+\rho e^{it})}{\rho e^{it}} i\rho e^{it}\,dt \\ &= i\int_0^{2\pi} f(z+\rho e^{it})\,dt.\end{aligned}$$

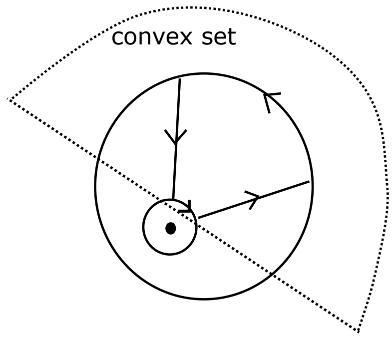

Figure 4:

This means that

$$
\begin{aligned}
\int_\gamma \frac{f(w)}{w-z}\,dw &= \lim_{\rho\to 0^+} i\int_0^{2\pi} f(z+\rho e^{it})\,dt \\
&= i\int_0^{2\pi} f(z)\,dt \\
&= 2\pi i f(z)
\end{aligned}
$$

because f is continuous at z and so $f(z+\rho e^{it})$ uniformly converges in t to $f(z)$ as $\rho \to 0^+$. $\square$

Remark 8.4. *Cauchy's Integral Formula implies that every (complex) differentiable function can be represented on small disks by a power series.*

$$
\begin{aligned}
\frac{1}{w-z} &= \frac{1}{(w-z_0)-(z-z_0)} \\
&= \frac{\frac{1}{w-z_0}}{1-\frac{z-z_0}{w-z_0}} \\
&= \frac{1}{w-z_0}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{w-z_0}\right)^n \ \left(\textit{provided } \left|\frac{z-z_0}{w-z_0}\right|<1\right) \\
&= \sum_{n=0}^{\infty}\frac{1}{(w-z_0)^{n+1}}(z-z_0)^n \ \left(\textit{provided } \left|\frac{z-z_0}{w-z_0}\right|<1\right).
\end{aligned}
$$

This tells us that

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\,dw \\ &= \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-z_0)^n\int_\gamma \frac{f(w)}{(w-z_0)^{n+1}}\,dw \\ &= \sum_{n=0}^{\infty} c_n(z-z_0)^n \end{aligned}$$

with

$$c_n = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-z_0)^{n+1}}\,dw.$$

Note then

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_\gamma \frac{f(w)}{(w-z_0)^{n+1}}\,dw.$$

9 Elementary Results on Holomorphic Functions I (09/08)

Theorem 9.1 (Cauchy's Integral Formula for Derivatives)**.** *Let $V \subset \mathbb{C}$ be open, $z_0 \in V$, $r > 0$ such that $\overline{D(z_0, r)} \subset V$, $f : V \to \mathbb{C}$ be differentiable. Let $\gamma : [0, 2\pi] \to V$ be $\gamma(t) = z_0 + re^{it}$. Then*

$$f^{(m)}(z) = \frac{m!}{2\pi i} \int_\gamma \frac{f(w)}{(w-z)^{m+1}}\, dw$$

for all $m \geq 0$ and $z \in D(z_0, r)$.

Proof. Choose $\rho > 0$ so small that $\overline{D(z, \rho)} \subset D(z_0, r)$. Let $\gamma_\rho(t) = z + \rho e^{it}$. By the argument in the proof of Cauchy's Integral Formula we get

$$\begin{aligned}\frac{m!}{2\pi i} \int_\gamma \frac{f(w)}{(w-z)^{m+1}}\, dw &= \frac{m!}{2\pi i} \int_{\gamma_\rho} \frac{f(w)}{(w-z)^{m+1}}\, dw \\ &= f^{(m)}(z)\end{aligned}$$

by the fact that z is in the center of the circle γ_ρ^*. □

Theorem 9.2 (Cauchy's Estimates)**.** *Let $V \subset \mathbb{C}$ be open, $f \in H(V)$, $z \in V$, $r > 0$ such that $\overline{D(z, r)} \subset V$. Suppose that $|f(w)| \leq M$ for all $w \in \partial D(z, r)$. Then*

$$\left| f^{(m)}(z) \right| \leq \frac{Mm!}{r^m}$$

for all $m \geq 0$.

Proof. Let $\gamma : [0, 2\pi] \to V$ be $\gamma(t) = z + re^{it}$. Then

$$f^{(m)}(z) = \frac{m!}{2\pi i} \int_\gamma \frac{f(w)}{(w-z)^{m+1}}\, dw.$$

We assume that $|f(z)| \leq M$ for all $w \in \gamma^*$. Also, $|w - z| = r$ for all $w \in \gamma^*$. Thus $\left| \frac{f(w)}{(w-z)^{m+1}} \right| \leq \frac{M}{r^{m+1}}$ for all $w \in \gamma^*$. The ML-inequality says that

$$\begin{aligned}\left| f^{(m)}(z) \right| &\leq \frac{m!}{2\pi} \frac{M}{r^{m+1}} L(\gamma) \\ &= \frac{m!}{2\pi} \frac{M}{r^{m+1}} 2\pi r \\ &= \frac{Mm!}{r^m}.\end{aligned}$$

□

Notation. $H(V)$ is the set of all functions holomorphic on V.

Definition 9.3. The elements of $H(\mathbb{C})$ are called *entire functions*.

Remark 9.4. *If f is an entire function, and $z_0 \in \mathbb{C}$ then*

$$f(z) = \sum_{m=0}^{\infty} \frac{f^{(m)}(z_0)}{m!}(z - z_0)^m$$

for all $z \in \mathbb{C}$.

Theorem 9.5 (Liouville's Theorem)**.** *A bounded entire function is constant.*

Proof. Let $f \in H(\mathbb{C})$ be an entire function such that $|f(z)| \leq M$ for all $z \in \mathbb{C}$. Then

$$f(z) = \sum_{m=0}^{\infty} \frac{f^{(m)}(0)}{m!} z^m$$

for all $z \in \mathbb{C}$. Fix $m \geq 1$. The Cauchy's Estimate for $\overline{D(0,r)}$ says that

$$\left| f^{(m)}(0) \right| \leq \frac{Mm!}{r^m}.$$

Note that this estimate is valid for $r > 0$. Let $r \to \infty$ then $\left|f^{(m)}(0)\right| \leq 0$. Thus $\left|f^{(m)}(0)\right| = 0$ for all $m \geq 1$. Thus $f(z) = f(0)$ for all $z_0 \in \mathbb{C}$. That is, f is constant. $\square$

Theorem 9.6 (Fundamental Theorem of Algebra)**.** *Let $P \in \mathbb{C}[z]$ be a polynomial with complex coefficients. If P is non-constant, then P has a root in $\mathbb{C}$.*

Analysis: Say that $P(z) = a_n z^n + \cdots + a_0$ with $a_n \neq 0$. Then the term $a_n z^n$ dominates when $|z|$ is large. For example,

$$\begin{aligned}
|P(z)| &\geq |a_n z^n| - \sum_{j=0}^{n-1} |a_j z^j| \\
&= |z^n| \cdot \left(|a_n| - \sum_{j=0}^{n-1} \frac{|a_j|}{|z^{n-j}|} \right) \\
&\geq |z^n| \cdot \left(|a_n| - \sum_{j=0}^{n-1} \frac{|a_j|}{|z|} \right) \quad \text{(if } |z| > 1\text{)}.
\end{aligned}$$

Thus $P(z)$ has no roots in the set where $|a_n| - \frac{\sum_{j=0}^{n-1} a_j}{|z|} > 0$ and $|z| > 1$. This is equivalent to $|z| > \max\{1, \frac{\sum_{j=0}^{n-1} a_j}{|a_n|}\}$. Also, it follows that $\lim_{|z| \to \infty} \frac{1}{|P(z)|} = 0$.

Proof of Theorem 9.6. Suppose not. Then $\frac{1}{P} \in H(\mathbb{C})$ and $\frac{1}{P}$ is bounded because $\lim_{|z|\to\infty} \frac{1}{|P(z)|} = 0$. Thus $\frac{1}{P}$ is constant by Liouville's Theorem, contrary to our assumption. $\square$

10 Elementary Results on Holomorphic Functions II (09/10)

Proposition 10.1. *Let $\Omega \subset \mathbb{C}$ be a connected open set, $f \in H(\Omega)$, $z_0 \in \Omega$ and suppose that $f^{(m)}(z_0) = 0$ for all $m \geq 0$. Then f is constant (in fact $f \equiv 0$).*

Proof. Let

$$\begin{aligned} C &= \{z \in \Omega | f^{(m)}(z) = 0 \text{ for all } m \geq 0\} \\ &= \bigcap_{m=0}^{\infty} \{z \in \Omega | f^{(m)}(z) = 0\}. \end{aligned}$$

Since each $f^{(m)}$ is continuous, C is closed. Now $z_0 \in C$ and so $C \neq \emptyset$. We claim that C is also open. Let $w \in C$ and $r > 0$ such that $D(w, r) \subset \Omega$. For $z \in D(w, r)$ we have

$$f(z) = \sum_{m=0}^{\infty} \frac{f^{(m)}(w)}{m!}(z - w)^m = 0.$$

That is, $f|_D(w, r) \equiv 0$. Since $D(w, r)$ is open, this implies that $f^{(m)} \equiv 0$ on $D(w, r)$ for all $m \geq 0$, and so $D(w, r) \subset C$. This confirms that C is open. Thus $C = \Omega$ (Ω is connected), as required. $\square$

Definition 10.2. Let $\Omega \subset \mathbb{C}$ be a connected open set, $f \in H(\Omega)$, $z_0 \in \Omega$ such that $f(z_0) = 0$. Assume f is not constant. Then the *order* of z_0 as a zero of f is the least N such that $f^{(N)}(z_0) \neq 0$. A zero of order one is called a *simple zero*.

Theorem 10.3. *Let $V \subset \mathbb{C}$ be open, $f \in H(V)$, $z_0 \in V$, and f has a zero of order n at z_0. Then there is a holomorphic function $g \in H(V)$ such that $g(z_0) \neq 0$ and $f(z) = (z - z_0)^n g(z)$ for all $z \in V$.*

Proof. Let Ω be the connected component of V that contains z_0. Choose $r > 0$ so such that $D(z_0, r) \subset \Omega$ and write $f(w) = \sum_{m=0}^{\infty} c_m (w - z_0)^m$ for $w \in D(z_0, r)$. Note that $c_0 = c_1 = \cdots = c_{n-1} = 0$, $c_n \neq 0$, and so

$$f(w) = \sum_{m=n}^{\infty} c_m (w - z_0)^m$$

for $w \in D(z_0, r)$. Define $g : V \to \mathbb{C}$ by

$$g(z) = \begin{cases} \frac{f(z)}{(z-z_0)^n}, & \text{if } z \neq z_0, \\ c_n, & \text{if } z = z_0. \end{cases}$$

Note that $g|_{V-\{z_0\}} \in H(V-\{z_0\})$. Also, if $w \in D(z_0, r)-\{z_0\}$, then $g(w) = \sum_{m=n}^{\infty} c_m (z - z_0)^{m-n}$. This equation is also true if $w = z_0$. This tells us that $g|_{D(z_0,r)} \in H(D(z_0, r))$. Since $V - \{z_0\}$ and $D(z_0, r)$ are open, $g \in H(V)$ as required. Note $g(z_0) = c_n \neq 0$ by assumption. $\square$

Corollary 10.4. *Let $V \subset \mathbb{C}$ be open and connected, $f \in H(V)$ non-constant, $z_0 \in V$ a zero of f. Then there is a disk $D(z_0, r) \subset V$ such that $f(z) \neq 0$ for all $z \in D(z_0, r) - \{z_0\}$.*

Proof. Let n be the order of z_0 as a zero of f. Write $f(z) = (z - z_0)^n g(z)$ with $g \in H(V)$, $g(z_0) \neq 0$. g is continuous and $g(z_0) \neq 0$, so there is a disk $D(z_0, r) \subset V$ such that $g(z) \neq 0$ for all $z \in D(z_0, r)$. The identity $f(z) = (z - z_0)^n g(z)$ now implies that $f(z) \neq 0$ for all $z \in D(z_0, r) - \{z_0\}$. $\square$

11 Elementary Results on Holomorphic Functions III (09/15)

If $V \subset \mathbb{C}$ is open and $D'(z, r) = D(z, r) - \{z\} \subset V$ then a function $f \in H(V)$ is said to have an isolated singularity at z. It is enough to study $H(D'(z, r))$.

Theorem 11.1 (Riemann's Removable Singularity Theorem)**.** *If $f \in H(D'(z, r))$ and f is bounded then there is some $g \in H(D(z, r))$ such that $g|_{D'(z,r)} = f$.*

Proof. Let $h : D(z, r) \to \mathbb{C}$ be defined by

$$h(w) = \begin{cases} (w - z)^2 f(w), & \text{if } w \in D'(z, r) \\ 0, & \text{if } w = z. \end{cases}$$

We claim that $h \in H(D(z, r))$. Certainly h is differentiable at every point of $D'(z, r)$. Now if $w \in D'(z, r)$, then

$$\frac{h(w) - h(z)}{w - z} = \frac{(w - z)^2 f(w)}{w - z} = (w - z)f(w).$$

Thus $|\frac{h(w)-h(z)}{w-z}| = |(w - z)f(w)| \leq M|w - z|$ where M is a bound for f. It follows that

$$\lim_{w \to z} \frac{h(w) - h(z)}{w - z} = 0.$$

Thus h is also differentiable at z and $h'(z) = 0$. So $h \in H(D(z, r))$ and h has a zero of order at least two at z. Let the order be N (if h is constantly zero then so is f and the conclusion is easy). There is a $\varphi \in H(D(z, r))$ such that

$$h(w) = (w - z)^N \varphi(w), \forall w \in D(z, r).$$

This implies that $f(w) = (w - z)^{N-2}\varphi(w), \forall w \in D(z, r)$. We take $g \in H(D(z, r))$ to be $g(w) = (w - z)^{N-2}\varphi(w)$. $\square$

Next, what if $f \in H(D'(z, r))$ is not bounded?

Definition 11.2. The function f has a *pole* at z if $\lim_{w \to z} |f(w)| = \infty$. It has an *essential singularity* if $\lim_{w \to z} |f(w)|$ does not exist (is not finite and is not ∞).

Example 11.3. (i) $f : D'(0,1) \to \mathbb{C}$ given by $f(z) = \frac{1}{z}$ has a pole at 0.
(ii) $g : D'(0,1) \to \mathbb{C}$ given by $g(z) = \exp(\frac{1}{z})$ has an essential singularity at 0. (Note that $g(D'(0,1)) = \mathbb{C} - \{0\}$ for any $r > 0$.)

Proposition 11.4. *Let $f \in H(D'(z,r))$ and suppose that f has a pole at z. Then there is some $N \geq 1$ and $\varphi \in H(D(z,r))$ such that $f(w) = \frac{\varphi(w)}{(w-z)^N}$, $\forall w \in D'(z,r), \varphi(z) \neq 0$.*

Proof. Since $\lim_{w\to z} |f(w)| = \infty$, there is some $0 < \rho \leq r$ such that $|f(w)| \geq 1$ for all $w \in D'(z,\rho)$. In particular, f has no zeros in $D'(z,\rho)$ and so $g : D'(z,\rho) \to \mathbb{C}$ defined by $g(w) = \frac{1}{f(w)}$ is holomorphic. Also, $|g(w)| \leq 1$, $\forall w \in D'(z,\rho)$. It follows that g extends to a holomorphic function on $D(z,\rho)$. Note that g is not constant (for f is not) and $\lim_{w\to z} |g(w)| = 0$. Thus g has a zero at z. Let N be the order of this zero. Then $\exists$ $\psi \in H(D(z,\rho))$ such that $g(w) = (w-z)^N \psi(w)$ and $\psi(z) \neq 0$. Thus there is a $0 < \sigma \leq \rho$ such that $\psi(w) \neq 0$ for all $w \in D(z,\sigma)$. On $D'(z,\sigma)$ we have

$$f(w) = \frac{1}{(w-z)^N \psi(w)} = \frac{\varphi(w)}{(w-z)^N}$$

if we let $\varphi = \frac{1}{\psi}$. Note that $\varphi(z) \neq 0$. Note that if we define $\varphi : D(z,r) \to \mathbb{C}$ by letting it be a function we already have on $D(z,r)$ and $\varphi(w) = (w-z)^N f(w)$ on $D(z,r) - \overline{D(z,\frac{\sigma}{2})}$ then it follows that $\varphi \in H(D(z,r))$. ☐

Theorem 11.5 (Identity Principle)**.** *If $F, G \in H(V)$, and $F(w) = G(w)$ for $w \in S \subset V$ where S has a limit point in V. Then $F = G$ on the connected component of V containing this limit point.*

Remark 11.6. *If $g \in H(V)$ and g is not a polynomial then $h : \mathbb{C} - \{0\} \to \mathbb{C}$ given by $h(z) = g(\frac{1}{z})$ has an essential singularity at 0.*

Remark 11.7. *If f has a pole at z, then $\frac{1}{f}$ has a removable singularity at z. This is because $\exists r$ such that $f \in H(D'(z,r))$ since $\lim_{w\to z} |f(w)| = \infty$, there is some $0 \leq \rho \leq r$ such that $|f(w)| \geq 1$ for $w \in D'(z,\rho)$. f has no zeroes in $D'(z,\rho)$ and so $\frac{1}{f} \in H(D'(z,\rho))$. Also, $|\frac{1}{f(w)}| \leq 1$ for $w \in D'(z,\rho)$ and so z is a removable singularity by Riemann's Removable Singularity Theorem. In fact, $\frac{1}{f(z)} = 0$.*

Example 11.8. Suppose that $f \in H(D'(z,r))$. Can f' have a simple pole at 0?

A simple pole means "poles of order 1". g has a pole of order 1 at 0 if the
(i) $g(z) = \frac{h(z)}{z}$ with $h \in H(D(0,1))$, $h(0) \neq 0$.
(ii) $g(z) = \frac{A}{z} + k(z)$ with $k \in H(D(0,1))$.
(iii) $\lim_{z\to 0} zg(z)$ exists and is nonzero.
(iv) $g(z) = \frac{c_{-1}}{z} + c_0 + c_1 z + \cdots$.

If f' has a pole of order 1 then $f'(z) = \frac{A}{z} + k(z)$, $k \in H(D(0,1))$, $A \neq 0$. Take $\gamma(t) = \frac{1}{2}e^{it}, 0 \leq t \leq 2\pi$. Then

$$\begin{aligned}\int_\gamma f'(z)\, dz &= \int_\gamma \frac{A}{z}\, dz + \int_\gamma k(z)\, dz \\ &= \int_0^{2\pi} \frac{A}{\frac{1}{2}e^{it}} \frac{1}{2} i e^{it}\, dt + \int_\gamma k(z)\, dz \\ &= 2\pi i A + \int_\gamma k(z)\, dz.\end{aligned}$$

By Cauchy's Theorem for Derivatives, we have $\int_\gamma f'(z)\, dz = 0$. By Cauchy's Theorem for Convex Sets, we have $\int_\gamma k(z)\, dz = 0$. Hence $0 = 2\pi i A + 0$, and so $A = 0$. Contradiction.

Example 11.9. Suppose that $f \in H(D'(z,r))$ has an essential singularity at z. Is it true that $\frac{1}{f}$ also has an essential singularity at z?

f has an essential singularity at z means: (i) $f \in H(D'(z,r))$; (ii) $\lim_{w\to z} |f(w)|$ does not exist. If $f \in H(D'(z,r))$, then $\frac{1}{f} \in H(D'(z,r)\backslash Z(f))$. $\frac{1}{f}$ does not necessarily have an essential singularity at z because $\frac{1}{f}$ may not have an isolated singularity.

For example, $f(z) = \exp(\frac{1}{z}) - 1$, $f(\frac{1}{2\pi i k}) = 0$ for $k \in \mathbb{Z}\backslash\{0\}$, so $\frac{1}{f} \in H(D(0,1)\backslash\{\{0\} \cup \{\frac{1}{2\pi i k} : k \in \mathbb{Z}\backslash\{0\}\}\})$. 0 is not an isolated singularity.

Another example is $f(z) = \exp(\frac{1}{z})$. $\frac{1}{f(z)} = \exp(\frac{-1}{z})$ does not have essential singularity at 0.

12 Elementary Results on Holomorphic Functions IV (09/17)

Parseval's Formula

Let $V \subset \mathbb{C}$ be open, $f \in H(V)$, $z_0 \in V$, $r > 0$ such that $\overline{D(z_0,r)} \subset V$. Our aim is to calculate $\int_0^{2\pi} |f(z_0 + re^{it})|^2\, dt$.

Note: (i) $|f|^2$ is continuous on the set $\{z : |z - z_0| = r\}$ so the integral exists. (ii) We have a power series $f(z) = \sum_{n=0}^\infty c_n(z-z_0)^n$ that converges uniformly absolutely to f on $\overline{D(z_0,r)}$.

We have

$$
\begin{aligned}
\left|f(z_0+re^{it})\right|^2 &= f(z_0+re^{it})\overline{f(z_0+re^{it})} \\
&= \sum_{n=0}^{\infty} c_n r^n e^{int} \overline{\left(\sum_{m=0}^{\infty} c_m r^m e^{imt}\right)} \\
&= \sum_{n=0}^{\infty} c_n r^n e^{int} \left(\sum_{m=0}^{\infty} \overline{c_m} r^m e^{-imt}\right) \\
&= \sum_{l=0}^{\infty}\sum_{j=0}^{l} c_j r^j e^{ijt} \cdot \overline{c_{l-j}} r^{l-j} e^{-i(l-j)t} \\
&= \sum_{l=0}^{\infty} r^l e^{-ilt} \sum_{j=0}^{l} c_j \overline{c_{l-j}} e^{2ijt} \\
&= \sum_{l=0}^{\infty} r^l \sum_{j=0}^{l} c_j \overline{c_{l-j}} e^{(2j-l)it}
\end{aligned}
$$

and this converges absolutely and uniformly in t. So

$$
\begin{aligned}
\int_0^{2\pi} |f(z_0+re^{it})|^2\,dt &= \sum_{l=0}^{\infty} r^l \sum_{j=0}^{l} c_j \overline{c_{l-j}} \int_0^{2\pi} e^{(2j-l)it\,dt} \\
&= \sum_{k=0}^{\infty} |c_k|^2 r^{2k} 2\pi.
\end{aligned}
$$

Hence $\frac{1}{2\pi}\int_0^{2\pi} |f(z_0+re^{it})|^2\,dt = \sum_{k=0}^{\infty} |c_k|^2 r^{2k}$.

Note: $\frac{1}{2\pi}\int_0^{2\pi} |f(z_0+re^{it})|^2\,dt$ is the average value of $|f|^2$ over the circle $\{z : |z-z_0| = r\}$ since

$$
\frac{1}{2\pi r}\int_0^{2\pi} |f(z_0+re^{it})|^2\,ds = \frac{1}{2\pi r}\int_0^{2\pi} |f(z_0+re^{it})|^2 r\,dt = \frac{1}{2\pi}\int_0^{2\pi} |f(z_0+re^{it})|^2\,dt.
$$

From

$$
\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} |f(z_0+re^{it})|^2\,dt &= \sum_{k=0}^{\infty} |c_k|^2 r^{2k} \\
&= |c_0|^2 + \sum_{k=1}^{\infty} |c_k|^2 r^{2k} \\
&= |f(z_0)|^2 + \sum_{k=1}^{\infty} |c_k|^2 r^{2k}
\end{aligned}
$$

we conclude that $|f(z_0)|^2 \leq \frac{1}{2\pi}\int_0^{2\pi} |f(z_0+re^{it})|^2\,dt$ with equality if and only if $c_k = 0$ for $k \geq 1$. This is equivalent to $|f(z_0)|^2 \leq \frac{1}{2\pi}\int_0^{2\pi} |f(z_0+re^{it})|^2\,dt$ with equality if and only if f is constant on $\overline{D(z_0, r)}$.

Theorem 12.1 (Maximum Modulus Principle)**.** *Let $V \subset \mathbb{C}$ be a connected open set and $f \in H(V)$ be non-constant. Then $|f|$ does not achieve a maximum at any point in V.*

Proof. Let $z_0 \in V$ and $r > 0$ be such that $\overline{D(z_0, r)} \subset V$. Since f is not constant in V and V is connected, f is not constant on $\overline{D(z_0, r)}$. Thus

$$|f(z_0)|^2 < \frac{1}{2\pi}\int_0^{2\pi} |f(z_0+re^{it})|^2\,dt.$$

Thus there is some point w such that $w - z_0 = r$ and $|f(w)|^2 > |f(z_0)|^2$. This implies that $|f(w)| > |f(z_0)|$. Since $r > 0$ can be taken as small as desire, it follows that z_0 is not a local maximum point for $|f|$. □

Remark 12.2. *(i) If $V \subset \mathbb{C}$ is a connected open set and $f \in H(V) \cap C(\overline{V})$ is non-constant, then $|f(z)| \leq \max\{|f(w)| : w \in \partial V\}$ for all $z \in V$.*

Example 12.3. Let $f(z) = \exp(z)$. Let $V = \{z = x + iy | x > 0\}$. Notice that $|f| = 1$ on ∂V, and $|f(z)| = \exp(x)$. So the modulus $|f|$ increases along the x-axis.

Example 12.4. Let $f \in H(D(0,1))$ and $f(D(0,1)) \subset D(0,1)$. Suppose that $f \in C(\overline{D(0,1)})$. Suppose also that $f(0) = 0$. Let $g(z) = \frac{f(z)}{z}$ for $z \in D'(0,1)$. Note that g has a removable singularity at 0. So $g \in H(D(0,1))$. We have $|g(z)| = \frac{|f(z)|}{|z|} \leq \frac{1}{|z|} = 1$ when $|z| = 1$. By Maximum Modulus Principle, $|g(z)| \leq 1$ for all $z \in D(0,1)$. It follows that $|f(z)| \leq |z|$ for any $z \in D(0,1)$.

13 Logarithms, Winding Numbers and Cauchy's Theorem I (09/19)

Theorem 13.1. *Let $V \subset \mathbb{C}$ be an open set and $f \in H(V)$. Then there exists $F \in H(V)$ such that $F' = f$ if and only if $\int_\gamma f(z)\,dz = 0$ for all closed curve γ in V.*

Proof. ($\Rightarrow$) We know that if $\exists\ F \in H(V)$ with $F' = f$ then $\int_\gamma f(z)\,dz = 0$ for all closed curves in V. (This is Cauchy's Theorem for Derivatives.)

($\Leftarrow$) Suppose that $\int_\gamma f(z)\,dz = 0$ for all closed curves γ in V. We may write V as a union of its connected components. It suffices to verify that $\exists F \in H(\Omega)$ such that $F' = f$ on Ω, for each connected component Ω of V. Note that the connected components are all open sets – this is important. Now say that Ω is a connected component of V. Then Ω is path connected. Fix $z_0 \in \Omega$ and define a function $F : \Omega \to \mathbb{C}$ by $F(z) = \int_{\gamma_z} f(w)\,dw$ where γ_z is any smooth curve from z_0 to z in Ω. Note that the hypothesis implies that $F(z)$ does not depend on the choice of γ_z. Once again using the hypothesis, we have $F(z+h) - F(z) = \int_{[z,z+h]} f(w)\,dw$ once h is small enough that it lies in a disk contained in Ω and centered at z. This implies that

$$\lim_{h\to 0} \frac{F(z+h) - F(z)}{h} = f(z).$$

Thus F works on Ω. $\square$

Remark 13.2. *$\frac{1}{z} \in H(D'(0,1))$, $\int_\gamma \frac{1}{z}\,dz$ can be non-zero. For example, $\int_\gamma \frac{1}{z}\,dz = 2\pi i$ where $\gamma(t) = \frac{1}{2}e^{it}$, $t \in [0, 2\pi]$.*

Now we turn to logarithms.

Definition 13.3. Let $V \subset \mathbb{C}$ be an open set. A function $f \in H(V)$ is said to be a *branch of logarithm* in V if $\exp(f(z)) = z$ for all $z \in V$.

Remark 13.4. *(i) If there is a branch of the logarithm in V then $0 \notin V$. This is because if it were then $\exp(f(0)) = 0$ which is impossible.*

(ii) A branch of the logarithm in V is always one-to-one. This is because if f is a branch of the logarithm in V, $z, w \in V$, and $f(z) = f(w)$, then $z = \exp(f(z)) = \exp(f(w)) = w$.

(iii) If f is a branch of the logarithm in V, $z \in V$, and $z = re^{i\theta}$ is a polar representation of z then

$$f(z) = \ln(r) + i\theta + 2\pi i k$$

for some $k \in \mathbb{Z}$. This is because $\exp(f(z)) = z$ and $\exp(\ln(r) + i\theta) = re^{i\theta} = z$. From Assignment 1, this implies that

$$f(z) = \ln(r) + i\theta + 2\pi i k$$

for some $k \in \mathbb{Z}$.

Lemma 13.5. *Suppose that $V \subset \mathbb{C}$, $f \in C(V)$, and $\exp(f(z)) = z$ for all z. Then f is a branch of the logarithm in V.*

Proof. We have to show that f is differentiable. If $z \in V$ and $h \neq 0$ is small enough, then $\frac{h}{f(z+h)-f(z)}$ is defined. (This is because f must be one-to-one by a similar argument to one above). Also,

$$\frac{h}{f(z+h)-f(z)} = \frac{(z+h)-z}{f(z+h)-f(z)} = \frac{\exp(f(z+h)) - \exp(f(z))}{f(z+h)-f(z)}.$$

Since f is continuous, $\lim_{h\to 0}(f(z+h)-f(z)) = 0$. This implies that

$$\begin{aligned} \lim_{h\to 0} \frac{h}{f(z+h)-f(z)} &= \lim_{h\to 0} \frac{\exp(f(z+h)) - \exp(f(z))}{f(z+h)-f(z)} \\ &= \exp(f(z)) \quad \text{(since exp is differentiable)} \\ &= z. \end{aligned}$$

As before, $0 \notin V$, and so

$$\lim_{h\to 0} \frac{f(z+h)-f(z)}{h} = \frac{1}{z}.$$

Thus, f is differentiable. □

Theorem 13.6. *Let $V \subset \mathbb{C}$ be open with $0 \notin V$. Then there is a branch of the logarithm in V if and only if $f(z) = \frac{1}{z}$ has an antiderivative in V.*

Proof. ($\Rightarrow$) We just saw that the equation $\exp(F(z)) = z$, $\forall z \in V$ implies that $F'(z) = \frac{1}{z}$ for all $z \in V$.

($\Leftarrow$) Conversely, suppose that $F \in H(V)$ and $F'(z) = \frac{1}{z}$ for all $z \in V$. Let $G(z) = z\exp(-F(z))$. Then $G \in H(V)$ and

$$\begin{aligned} G'(z) &= \exp(-F(z)) + z\exp(-F(z)) \cdot (-F'(z)) \\ &= \exp(-F(z)) + z\exp(-F(z)) \cdot (-\frac{1}{z}) \\ &= 0. \end{aligned}$$

Thus G is constant on each connected component of V. That is, for each component Ω of V, $\exists\, c_\Omega$ such that

$$\exp(F(z)) = c_\Omega z.$$

Note that $c_\Omega \neq 0$ because $\exp(\mathbb{C})$ does not contain 0. Choose c_Ω such that $\exp(c_\Omega) = c_\Omega$ and define $L : V \to \mathbb{C}$ by $L(z) = F(z) - c_\Omega$. Then $L \in H(V)$ and $\exp(L(z)) = z$. So $L(z)$ works. □

Remark 13.7. *$V \subset \mathbb{C}$ is an open set. $\exists\, f \in H(V)$ such that $e^{f(z)} = z$ if and only if*
(i) $0 \notin V$;
(ii) $\int_\gamma \frac{1}{z}\, dz = 0$ for all closed curve γ in V.

Remark 13.8. *$V \subset \mathbb{C}$ is an open set, $f \in H(V)$. $g \in H(V)$ is a branch of logarithm of f if $e^{g(z)} = f(z)$ for all $z \in V$. Such g exists if and only if*
(i) $Z(f) \cap V \neq \emptyset$;
(ii) $\int_\gamma \frac{f'(z)}{f(z)}\,dz = 0$ for all closed curve γ in V.

Remark 13.9. *We will see that if $g \in H(V)$ where*
(i) V is simple connected,
(ii) g does not vanish in V,
then g has a logarithm in V. That is to say, there exists $f \in H(V)$ such that $e^{f(z)} = g(z)$. Then g has a m-th root $h(z) = \exp\left(\frac{f(z)}{m}\right) \in H(V)$ such that $g(z) = h(z)^m$.

14 Logarithms, Winding Numbers and Cauchy's Theorem II (09/22)

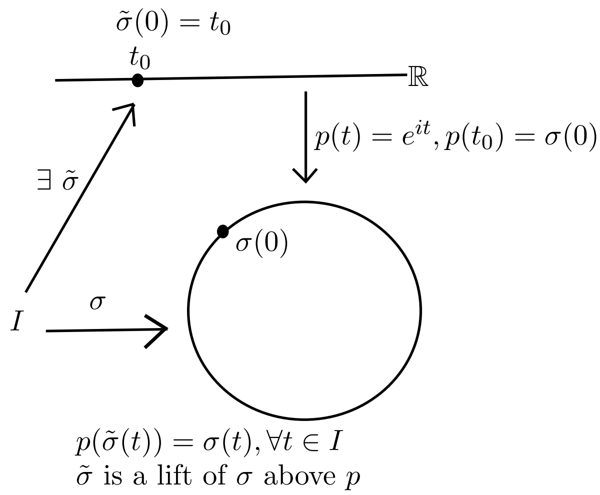

Figure 5:

Let $I = [0, 1]$. Given a metric space X then define

$$\pi(X) = \text{set of all continuous maps } \sigma : I \to X,$$

$$\Omega(X) = \text{set of all continuous loops } \sigma : I \to X, \sigma(0) = \sigma(1).$$

So $\Omega(X) \subset \pi(X)$, and $\pi(X)$ is a metric space with

$$d_\pi(\sigma, \omega) = \sup_{t \in I} d_X(\sigma(t), \omega(t)).$$

$\Omega(X)$ is a subspace of $\pi(X)$.

We want to study $\pi(S)$ and $\Omega(S)$, where $S = \{z \in \mathbb{C} : |z| = 1\}$.

Let $S_+ = \{z \in S : \mathrm{Re}(z) > 0\}$. Define $\varphi : S_+ \to \mathbb{R}$ by $\varphi(z) = \arctan(\frac{\mathrm{Im}(z)}{\mathrm{Re}(z)})$. From basis trigonometry, we have $\varphi(1) = 0$ and $p(\varphi(z)) = z$ for all $z \in S_+$. φ is continuous and if $\psi : S_+ \to \mathbb{R}$ such that $\psi(1) = 0$ and $p(\psi(z)) = z$ for any $z \in S_+$, then $\varphi = \psi$. Here is a non-elementary way to prove this. Let $k(z) = \psi(z) - \varphi(z)$. Then $\exp(ik(z)) = \exp(i(\psi(z) - \varphi(z))) = z \cdot z^{-1} = 1$. So $ik(z) \in 2\pi i\mathbb{Z}$. That is, $k(z) \in 2\pi\mathbb{Z}$. $k(z)$ is continuous and $k(1) = 0$ and so $k(S_+)$ is a connected subset of $2\pi\mathbb{Z}$ that contains 0. Thus $k(S_+) = \{0\}$ and so $\varphi = \psi$.

Remark 14.1. *$\varphi : S_+ \to \mathbb{R}$ is similar to "inverse functions" like* $\arcsin, \arccos, \cdots$. *It is an inverse once the domain is restricted.*

More formally, let $z_0 \in S$, let $S_+(z_0) = z_0 S_+$ be the half-circle centered at z_0, choose $t_o \in \mathbb{R}$ such that $p(t_0) = z_0$. Then $\psi : S_+(z_0) \to \mathbb{R}$ given by $\psi = T_{t_0} \circ \varphi \circ R_{z_0^{-1}}$ is an inverse map for p such that $\psi(z_0) = t_0$ and it is unique among continuous maps with these properties. Here $R_w : S \to S$ is $R_w(z) = wz$, $T_\tau : \mathbb{R} \to \mathbb{R}$ is $T_\tau(t) = t + \tau$. Check the properties of ψ: it is continuous because T_{to}, φ, $R_{z_0^{-1}}$ are all continuous.

$$\psi(z_0) = T_{t_0}(\varphi(R_{z_0^{-1}}(z_0))) = T_{t_0}(\varphi(1)) = T_{t_0}(0) = t_0 + 0 = t_0$$

and

$$\begin{aligned} p(\psi(z)) &= p(T_{t_0}(\varphi(R_{z_0^{-1}}(z)))) \\ &= p(t_0 + \varphi(R_{z_0^{-1}}(z))) \\ &= p(t_0) \cdot p(\varphi(R_{z_0^{-1}}(z))) \\ &= z_0 \cdot R_{z_0^{-1}}(z) \\ &= z_0 \cdot z_0^{-1} \cdot z \\ &= z. \end{aligned}$$

Theorem 14.2 (Path Lifting for the Circle)**.** *Let $\sigma \in \pi(S)$ and choose $t_0 \in \mathbb{R}$ such that $p(t_0) = \sigma(0)$. Then there is one and only one $\tilde{\sigma} = \pi(\mathbb{R})$ such that $\tilde{\sigma}(0) = t_0$ and $p(\tilde{\sigma}(t)) = \sigma(t)$.*

Proof. Let $U = \{zS_+ | z \in S\}$ be an open cover of S. Then $\xi = \{\sigma^{-1}(u) | u \in U\}$ is an open cover of I. Choose a Lebesgue number $\lambda > 0$ for ξ. This means that if $[s_1, s_2] \subset I$

and $s_2 - s_1 < \lambda$ then $[s_1, s_2]$ is entirely inside one element of ξ. Thus $\sigma([s_1, s_2])$ is entirely inside one half-circle in S. Partition I into

$$0 = s_0 < s_1 < s_2 < \cdots < s_{n-1} < s_n = 1$$

such that $s_j - s_{j-1} < \lambda$ for all j. Then $\sigma([s_0, s_1]), \sigma([s_1, s_2]), \cdots, \sigma([s_{n-1}, s_n])$ are all inside half-circles in S. We can now use the inverse maps ψ previously constructed to write down $\tilde{\sigma}$ on each interval. Finally, $\tilde{\sigma}$ is unique by connectedness argument. $\square$

All this says that every path $\sigma : I \to S$ has the form $\sigma(t) = \exp(i\theta(t))$ where $\theta : I \to \mathbb{R}$ is continuous. Also θ is uniquely determined once $\theta(0)$ is chosen.

If σ is a loop ($\sigma(0) = \sigma(1)$) then $\theta(1)$ and $\theta(0)$ differ by a multiple of 2π.

$$\exp(i\theta(1)) = \sigma(1) = \sigma(0) = \exp(i\theta(0))$$

implies that $\theta(1) - \theta(0) \in 2\pi\mathbb{Z}$.

Definition 14.3. We define the *index* of σ by

$$\operatorname{ind}(\sigma) = \frac{1}{2\pi}(\theta(1) - \theta(0)).$$

Remark 14.4. *Note that although θ is not unique, once you have one θ, the others are $\theta_k = \theta + 2\pi k$. Then*

$$\theta_k(1) - \theta_k(0) = \theta(1) - \theta(0).$$

So index is well defined.

Index is easy to compute for explicit map.

Example 14.5. Say $\sigma(t) = \exp(4\pi i t)$. For this loop, $\theta(t) = 4\pi t$ and so $\operatorname{ind}(\sigma) = \frac{1}{2\pi}(\theta(1) - \theta(0)) = \frac{1}{2\pi}(4\pi - 0) = 2$.

15 Logarithms, Winding Numbers and Cauchy's Theorem III (09/24)

Recall from last time: If $\sigma : I \to S$ is continuous then σ can be expressed as $\sigma(t) = \exp(i\theta(t))$ where $\theta : I \to \mathbb{R}$ is continuous. θ is uniquely determined by $\theta(0)$. If σ is smooth, so is θ. The correspondence $\sigma \rightsquigarrow \theta$ is "continuous". Given $\sigma > 0$, there is a $\delta > 0$ such that if $d_{\pi(S)}(\sigma, w) < \delta$ and $\tilde{\sigma}, \tilde{w}$ are lifts of σ, w, then $d_{\pi(\mathbb{R})}(\tilde{\sigma} - \tilde{\sigma}(0), \tilde{w} - \tilde{w}(0)) < \sigma$.

Definition 15.1. We define $\operatorname{ind} : \Omega(S) \to \mathbb{Z}$ by

$$\operatorname{ind}(\sigma) = \frac{1}{2\pi}\left(\theta(1) - \theta(0)\right)$$

where $\sigma(t) = \exp(i\theta(t))$, θ continuous.

In fact, ind is continuous (even uniformly continuous). There is $\delta > 0$ such that if $\sigma, w \in \Omega(S)$ and $d_{\Omega(S)}(\sigma, w) < \delta$ then $\operatorname{ind}(\sigma) = \operatorname{ind}(w)$.

Definition 15.2. If $\gamma : I \to \mathbb{C} - \{0\}$ then we define $\sigma : I \to S$ by $\sigma(t) = \frac{\gamma(t)}{|\gamma(t)|}$. We define $\operatorname{Ind}(\gamma, 0) = \operatorname{ind}(\sigma)$.

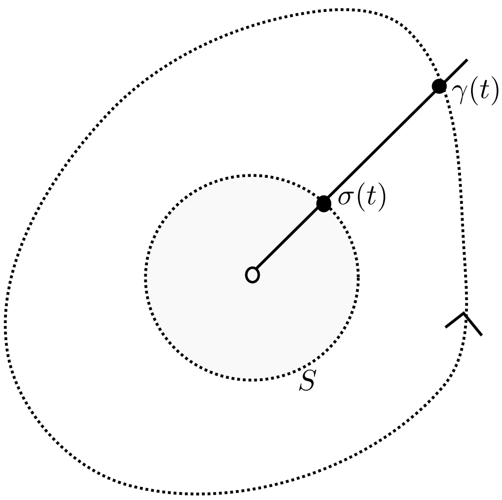

Figure 6:

The map $\gamma \mapsto \sigma$ is continuous. In fact,

$$\begin{aligned}
\left|\frac{\gamma_1(t)}{|\gamma_1(t)|} - \frac{\gamma_2(t)}{|\gamma_2(t)|}\right| &= \left|\frac{|\gamma_2(t)|\gamma_1(t) - |\gamma_1(t)|\gamma_2(t)}{|\gamma_1(t)||\gamma_2(t)|}\right| \\
&= \left|\frac{|\gamma_2(t)|(\gamma_1(t) - \gamma_2(t)) + |\gamma_2(t)|\gamma_2(t) - |\gamma_1(t)|\gamma_2(t)}{|\gamma_1(t)||\gamma_2(t)|}\right| \\
&\leq \frac{|\gamma_1(t) - \gamma_2(t)|}{|\gamma_1(t)|} + \frac{|\gamma_2(t)| - |\gamma_1(t)|}{|\gamma_1(t)|} \\
&\leq \frac{2|\gamma_1(t) - \gamma_2(t)|}{|\gamma_1(t)|}.
\end{aligned}$$

Given γ_1, One can choose $\eta > 0$ such that $|\gamma_1(t)| \geq \eta \ \forall t$ and then

$$\left|\frac{\gamma_1(t)}{|\gamma_1(t)|} - \frac{\gamma_2(t)}{|\gamma_2(t)|}\right| \leq \frac{2}{\eta} \cdot |\gamma_1(t) - \gamma_2(t)|.$$

This shows that $\gamma \to \frac{\gamma}{|\gamma|}$ is continuous (not uniformly so). This tells us that $\gamma \mapsto \operatorname{Ind}(\gamma, 0)$ from $\Omega(\mathbb{C} - \{0\}) \to \mathbb{Z}$ is continuous. Finally, if $a \in \mathbb{C}$ and $a \notin \gamma^*$, then we define $\operatorname{Ind}(\gamma, a) = \operatorname{Ind}(\gamma - a, 0)$.

If $K \subset \mathbb{C}$ is a compact set then $\mathbb{C}-K$ always has a unique unbounded component. Here is the reason. Choose $R > 0$ such that $K \subset \overline{D(0,R)}$. Then $\mathbb{C} - \overline{D(0,R)} = \{w \in \mathbb{C} | |w| > R\} \subset \mathbb{C} - K$ and is connected. The component that meets this set is unbounded because it contains the set. Every other component does not meet this set and so is contained in $\overline{D(0,R)}$ and hence is bounded.

Theorem 15.3. *Let $\gamma : I \to \mathbb{C}$ be a loop. Then the function $a \mapsto \mathrm{Ind}(\gamma, a)$ from $\mathbb{C}-\gamma^* \to \mathbb{Z}$ is constant on each component of $\mathbb{C} - \gamma^*$ and zero on the unbounded component.*

Proof. The function $a \mapsto \mathrm{Ind}(\gamma, a)$ is continuous, since it is the composition of some continuous functions:

$$\begin{array}{cccc} \mathbb{C} - \gamma^* \to & \Omega(\mathbb{C} - \{0\}) \to & \Omega(S) \to & \mathbb{Z} \\ a \mapsto & \gamma - a \quad \mapsto & \dfrac{\gamma - a}{|\gamma - a|} \mapsto & \mathrm{Ind}(\gamma, a) \end{array}$$

Thus if w is a component of $\mathbb{C}-\gamma^*$ then the image of w under $a \mapsto \mathrm{Ind}(\gamma, a)$ is a connected subset of $\mathbb{Z}$. This must be a singleton. We may choose $R > 0$ such that the unbounded component of $\mathbb{C} - \gamma^*$ contains $\{w||w| > R\}$. Choose $a \in \mathbb{C}$ with $|a| > R$. Consider the map $[1, \infty) \to \mathbb{Z}$ given by

$$s \mapsto \mathrm{Ind}(\gamma, sa) = \mathrm{ind}(\frac{\gamma - sa}{|\gamma - sa|}).$$

As $s \to \infty$, $\frac{\gamma - sa}{|\gamma - sa|} \to -\mathrm{sgn}(a) = -\frac{a}{|a|}$ uniformly. The index thus converges to ind(constant map) $=$ 0. Since $\mathrm{Ind}(\gamma, sa)$ is also constant, it equals 0. $\square$

Theorem 15.4. *If γ is a smooth loop in $\mathbb{C}$ and $a \notin \gamma^*$ then*

$$\mathrm{Ind}(\gamma, a) = \frac{1}{2\pi i} \int_\gamma \frac{1}{z - a}\, dz.$$

Proof. Let θ be a lift of $\frac{\gamma - a}{|\gamma - a|}$ and write $\rho = |\gamma - a|$. Then

$$\gamma(t) - a = |\gamma(t) - a| \cdot \frac{\gamma(t) - a}{|\gamma(t) - a|} = \rho(t) \cdot \exp(i\theta(t)).$$

So

$$\begin{aligned} \gamma'(t) &= \rho'(t) \cdot \exp(i\theta(t)) + \rho(t) \cdot i \exp(i\theta(t)) \cdot \theta'(t) \\ &= \frac{\rho'(t)}{\rho(t)} \rho(t) \exp(i\theta(t)) + i\theta'(t)\rho(t) \exp(i\theta(t)) \\ &= \frac{\rho'(t)}{\rho(t)} (\gamma(t) - a) + i\theta'(t)(\gamma(t) - a) \end{aligned}$$

and thus

$$\frac{\gamma'(t)}{\gamma(t)-a} = \frac{\rho'(t)}{\rho(t)} + i\theta'(t).$$

Hence,

$$\begin{aligned}
\frac{1}{2\pi i}\int_\gamma \frac{1}{z-a}\,dz &= \frac{1}{2\pi i}\int_0^1 \frac{\gamma'(t)}{\gamma(t)-a}\,dt \\
&= \frac{1}{2\pi i}\int_0^1 \left(\frac{\rho'(t)}{\rho(t)} + i\theta'(t)\right)\,dt \\
&= \frac{1}{2\pi i}\left(\ln(\rho(1)) - \ln(\rho(0)) + i(\theta(1)-\theta(0))\right) \\
&= \frac{1}{2\pi}(\theta(1)-\theta(0)) \\
&= \operatorname{Ind}(\gamma, a).
\end{aligned}$$

□

16 Logarithms, Winding Numbers and Cauchy's Theorem IV (09/26)

Lemma 16.1. *Let $V_1, V_2 \in \mathbb{C}$ be open, Γ a formal sum of smooth paths in V_2, $F : V_1 \times V_2 \to \mathbb{C}$ is (jointly) continuous and holomorphic in the first variable. Define $f : V_1 \to \mathbb{C}$ by $f(z) = \int_\Gamma F(z,w)dw$. Then $f \in H(V_1)$.*

Proof. First, f is continuous on V_1 because F is continuous and Γ^* is compact. Let $T \subset V_1$ be a triangle, then

$$\begin{aligned}
\int_{\partial T} f(z)\,dz &= \int_{\partial T}\left(\int_\Gamma F(z,w)dw\right)\,dz \\
&= \int_\Gamma \left(\int_{\partial T} F(z,w)dz\right)\,dw \text{ (by Fubini's Theorem for Riemann integrals)} \\
&= \int_\Gamma 0\,dw \text{ (by Cauchy-Goursat Theorem)} \\
&= 0.
\end{aligned}$$

By Morera's Theorem, $f \in H(V_1)$. □

Theorem 16.2 (Cauchy's Integral Formula, Homology Version)**.** *Let $V \subset \mathbb{C}$ be open, $\Gamma \subset V$ a formal sum of closed curve, and suppose* $\operatorname{Ind}(\Gamma, a) = 0$ *for all $a \in \mathbb{C}\backslash V$. Let $f \in H(V)$, then*

$$\frac{1}{2\pi i}\int_\Gamma \frac{f(w)}{w-z}\,dw = \operatorname{Ind}(\Gamma, z)f(z)$$

for all $z \in V\backslash\Gamma^*$.

Proof. Let $\Omega = \{w \in \mathbb{C}\backslash\Gamma^* | \mathrm{Ind}(\Gamma, w) = 0\}$. Then Ω is open (it is a union of connected components of $\mathbb{C}\backslash\Gamma^*$) and $\Omega \supset \mathbb{C}\backslash V$. Thus $\mathbb{C} = V \cup \Omega$ and also $\Gamma^* \cap \Omega = \emptyset$. Consider $\varphi : V \times V \to \mathbb{C}$ given by

$$\varphi(z, w) = \begin{cases} \frac{f(z)-f(w)}{z-w} \text{ if } z \neq w, \\ f'(z) \text{ if } z = w. \end{cases}$$

Recall that we showed that φ can be written locally as a sum of a power series in z, w. It follows that φ is continuous and holomorphic in each variable separately. We define $\Phi_1 : V \to \mathbb{C}$ by

$$\Phi_1(z) = \frac{1}{2\pi i} \int_\Gamma \varphi(z, w)\, dw.$$

By Lemma 16.1, $\Phi_1 \in H(V)$. We define $\Phi_2 : \Omega \to \mathbb{C}$ by

$$\Phi_2 = -\frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{w - z}\, dw.$$

By Lemma 16.1, $\Phi_2 \in H(V)$.

Next, I want to check that if $z \in V \cap \Omega$, then $\Phi_1(z) = \Phi_2(z)$. Let $z \in V \cap \Omega$, then $z \notin \Gamma^*$, $\mathrm{Ind}(\Gamma, z) = 0$, and so

$$\frac{1}{2\pi i} \int_\Gamma \frac{1}{w - z}\, dz = \mathrm{Ind}(\Gamma, z) = 0.$$

Thus

$$\begin{aligned} \Phi_1(z) &= \frac{1}{2\pi i} \int_\Gamma \varphi(z, w)\, dw \\ &= \frac{1}{2\pi i} \int_\Gamma \frac{f(z) - f(w)}{z - w}\, dw \\ &= \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z - w}\, dw - \frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{z - w}\, dw \\ &= \frac{f(z)}{2\pi i} \int_\Gamma \frac{1}{z - w}\, dw + \Phi_2(z) \\ &= -f(z)\mathrm{Ind}(\Gamma, z) + \Phi_2(z) \\ &= \Phi_2(z). \end{aligned}$$

It follows that $\Phi : \mathbb{C} \to \mathbb{C}$ given by

$$\Phi(z) = \begin{cases} \Phi_1(z), & \text{if } z \in V \\ \Phi_2(z), & \text{if } z \in \Omega \end{cases}$$

is holomorphic. That is, Ω is entire. We know that the (unique) unbounded component of $\mathbb{C}\backslash\Gamma^*$ is contained in Ω. Thus

$$\Phi(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{w - z}\, dz$$

for all z with $|z|$ large enough. It follows that $\lim_{|z|\to\infty} \Phi(z) = 0$. Liouville's Theorem says Φ is constant, and then $\Phi \equiv 0$ because of the limit. So $\Phi_1(z) = 0$ for all z. This says that $\frac{1}{2\pi i}\int_\Gamma \frac{f(z)-f(w)}{z-w}\,dw = 0$ for all $z \notin \Gamma^*$, and this can be rearranged to give the conclusion. $\square$

Remark 16.3. *This proof is due to John Dixon (A brief Proof of Cauchy's Integral Theorem, 1971, Proceedings of the American Mathematical Society.).*

17 Logarithms, Winding Numbers and Cauchy's Theorem V (09/29)

Theorem 17.1 (Cauchy's Theorem, Homology Version)**.** *Let $V \subset \mathbb{C}$ be open, $\Gamma \subset V$ be a formal sum of closed curves. Suppose that* $\operatorname{Ind}(\Gamma, a) = 0$ *for all $a \in \mathbb{C}\backslash V$. Let $f \in H(V)$, then*

$$\int_\Gamma f(w)\,dw = 0.$$

Proof. Choose $z_0 \in V\backslash\Gamma^*$. Let $g(z) = (z - z_0)f(z)$. Apply Cauchy's Integral Formula, Homology Version to g, we get

$$\frac{1}{2\pi i}\int_\Gamma \frac{g(w)}{w - z_0}\,dw = \operatorname{Ind}(\Gamma, z_0)g(z_0) = 0.$$

On the other hand, $\frac{1}{2\pi i}\int_\Gamma \frac{g(w)}{w-z_0}\,dw = \frac{1}{2\pi i}\int_\Gamma f(w)\,dw$ and so $\int_\Gamma f(w)\,dw = 0$. $\square$

Definition 17.2. Let $0 \le r < R \le \infty$. We define, for $a \in \mathbb{C}$,

$$A(a, r, R) = \{z \in \mathbb{C} | r < |z - a| < R\}$$

and call this the *annulus* centered at a with radii r, R.

Remark 17.3. $D'(a, R) = A(a, 0, R)$, $\mathbb{C} - \{a\} = A(a, 0, \infty)$.

Theorem 17.4. *Let $f \in H(A(a, r, R))$. Then there are functions $f_1 \in H(D(a, R))$ and $f_2 \in H(A(a, r, \infty))$ such that $f = f_1 + f_2$ on the common domain and* $\lim_{z\to\infty} f_2(z) = 0$*. Moreover, f_1, f_2 are unique.*

Proof. We may assume, after translating, that $a = 0$. For $r < \rho < R$, let $\gamma_\rho : [0, 2\pi] \to A(0, r, R)$ be $\gamma_\rho(t) = \rho e^{it}$. Let $r < r_1 < R_1 < R$ (see Figure 7), note that $\operatorname{Ind}(\gamma_{R_1} \dot{-} \gamma_{r_1}, c) = 0$ for all $c \notin A(0, r, R)$. Thus we may apply Cauchy's Integral Formula, Homology Version to $\gamma_{R_1} \dot{-} \gamma_{r_1}$. If we do so, then we get

$$\operatorname{Ind}(\gamma_{R_1} \dot{-} \gamma_{r_1}, z)f(z) = \frac{1}{2\pi i}\int_{\gamma_{R_1} \dot{-} \gamma_{r_1}} \frac{f(w)}{w - z}\,dw$$

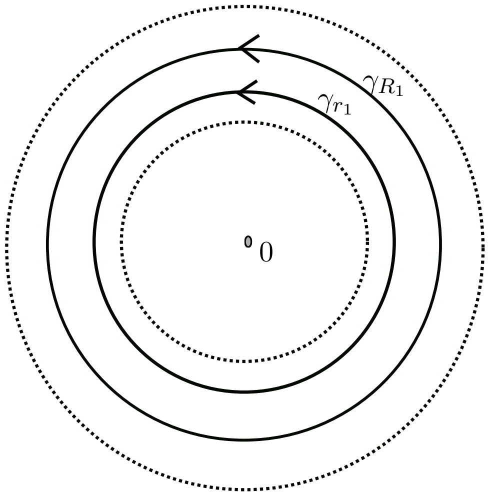

Figure 7:

for all $z \notin \gamma_{R_1}^* \cup \gamma_{r_1}^*$. In particular, if $r < r_1 < |z| < R_1 < R$, then

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_{R_1}} \frac{f(w)}{w-z}\,dw - \frac{1}{2\pi i}\int_{\gamma_{r_1}} \frac{f(w)}{w-z}\,dw.$$

If $R_1 < R_2 < R$ and $r < r_1 < |z| < R_1 < R$ then

$$\frac{1}{2\pi i}\int_{\gamma_{R_1}} \frac{f(w)}{w-z}\,dw = \frac{1}{2\pi i}\int_{\gamma_{R_2}} \frac{f(w)}{w-z}\,dw$$

again by Cauchy's Integral Formula, Homology Version. Similarly if $r < r_1 < r_2$ then

$$\frac{1}{2\pi i}\int_{\gamma_{r_1}} \frac{f(w)}{w-z}\,dw = \frac{1}{2\pi i}\int_{\gamma_{r_2}} \frac{f(w)}{w-z}\,dw.$$

We now define $f_1 : D(0,R) \to \mathbb{C}$ by

$$f_1(z) = \frac{1}{2\pi i}\int_{\gamma_{R_1}} \frac{f(w)}{w-z}\,dw$$

where R_1 is any number such that $|z| < R_1 < R$. We also define $f_2 : A(0,r,\infty) \to \mathbb{C}$ by

$$f_2(z) = -\frac{1}{2\pi i}\int_{\gamma_{r_1}} \frac{f(w)}{w-z}\,dw$$

where r_1 is any number such that $r < r_1 < |z|$. These functions are well defined and holomorphic by Lemma 16.1. We already observed that $f = f_1 + f_2$. We have $\lim_{z\to\infty} \frac{f(w)}{z-w} = 0$ uniformly for $w \in \gamma_{r_1}^*$. Thus $\lim_{z\to\infty} f_2(z) = 0$.

Finally, suppose $\tilde{f}_1, \tilde{f}_2$ have all the same properties. Then $f_1(z)+f_2(z) = \tilde{f}_1(z)+\tilde{f}_2(z)$ for all $z \in A(0,r,R)$. So $f_1(z) - \tilde{f}_1(z) = \tilde{f}_2(z) - f_2(z)$ for all $z \in A(0,r,R)$. It follows that $F : \mathbb{C} \to \mathbb{C}$ given by

$$F(z) = \begin{cases} f_1(z) - \tilde{f}_1(z), & \text{if } z \in D(0,r) \\ \tilde{f}_2(z) - f_2(z), & \text{if } z \in A(0,r,\infty) \end{cases}$$

is entire. Moreover, $\lim_{z\to\infty} F(z) = 0$. Thus, $F \equiv 0$ by Liouville's Theorem, and so $f_1(z) = \tilde{f}_1(z)$ and $f_2(z) = \tilde{f}_2(z)$. $\square$

Theorem 17.5. *Let $f \in H(A(a,r,R))$. Then there is a Laurent series*

$$\sum_{n\in\mathbb{Z}} c_n(z-a)^n$$

that converges uniformly absolutely on compact subsets of $A(a,r,R)$ to f. Moreover,

$$c_n = \frac{1}{2\pi i}\int_{\gamma_\rho} \frac{f(w)}{(w-a)^{n+1}}\, dw$$

for all $n \in \mathbb{Z}, r < \rho < R$.

Proof. Once again, we may assume $a = 0$. Write $f = f_1 + f_2$ as in Theorem 17.4. Define $g : D'(0, \frac{1}{r}) \to \mathbb{C}$ by $g(z) = f_2(\frac{1}{z})$. Then $\lim_{z\to 0} g(z) = 0$ because $\lim_{z\to\infty} f_2(z) = 0$, and so g extends to a holomorphic function on $D(0, \frac{1}{r})$ with $g(0) = 0$ by Riemann's Removable Singularity Theorem. We know that $f_1(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$ with uniformly absolute convergence on compact subsets. Thus

$$f(z) = f_1(z) + g(\frac{1}{z}) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n (\frac{1}{z})^n = \sum_{n\in\mathbb{Z}} c_n z^n$$

with uniformly absolute convergence on compact subsets. To get the formula for c_n, note that we are justified in integrating term by term. $\square$

18 Logarithms, Winding Numbers and Cauchy's Theorem VI (10/1)

Definition 18.1. Let $V \subset \mathbb{C}$ be open. If γ_0, γ_1 are loops in V then γ_0, γ_1 are said to be *homotopic* in V, denoted as $\gamma_0 \underset{V}{\sim} \gamma_1$, if there is a path from γ_0 to γ_1 in $\Omega(V)$, where $\Omega(V)$ is the space of all loops in V with the sup-metric.

Example 18.2. Let $V \subset \mathbb{C}$, $\gamma_0(t) = \sin(\pi t)$, $t \in [0,1]$ and $\gamma_1(t) = 0, t \in [0,1]$. Define $\tau : I \to \Omega(\mathbb{C})$ from γ_0 to γ_1:

$$\tau(s)(t) = (1-s)\sin(\pi t).$$

Now τ is continuous as a function of two variables, and

$$\tau(0)(t) = \sin(\pi t) = \gamma_0(t),$$

$$\tau(1)(t) = 0 = \gamma_1(t),$$

$$\tau(s)(0) = (1-s)\sin(\pi 0) = (1-s)\sin(\pi 1) = \tau(s)(1).$$

Hence γ_0 and γ_1 are homotopic in $V = \mathbb{C}$.

Remark 18.3. *If X is path connected then there is a path through loops from a loop based anywhere to a loop based at $p \in X$.*

Suppose $V \subset \mathbb{C}$ open, γ_0, γ_1 are loops in V, $\gamma_0 \underset{V}{\sim} \gamma_1$, $a \in \mathbb{C}\backslash V$. Then $\mathrm{Ind}(\gamma_0, a) = \mathrm{Ind}(\gamma_1, a)$. We know that $\gamma \mapsto \mathrm{Ind}(\gamma, a)$ is continuous. Since $\gamma_0 \underset{V}{\sim} \gamma_1$ there is a path σ from γ_0 to γ_1. The function $s \mapsto \mathrm{Ind}(\sigma(s), a)$ is constant. So it is constant.

Remark 18.4. *This implies that $\gamma_0 \underset{V}{\sim} \gamma_1$ then $\mathrm{Ind}(\gamma_1 \dot{-} \gamma_0, a) = 0$ for all $a \in \mathbb{C}\backslash V$. Thus the Cauchy's Theorem, Homology Version applies to the formal sum of paths $\Gamma = \gamma_1 \dot{-} \gamma_0$. Thus $\int_{\gamma_0} f(z)\, dz = \int_{\gamma_1} f(z)\, dz$ for any $f \in H(V)$.*

Example 18.5. Let $V = \mathbb{C} - \{-1, 1\}$, γ be the Barnes' double circuit as in Figure 8. In this case, $\mathrm{Ind}(\gamma, 1) = 0$, $\mathrm{Ind}(\gamma, -1) = 0$, $\gamma \underset{V}{\sim}$ constant path.

Definition 18.6. Let $f \in H(D'(z_0, r))$. We know that $f(z) = \sum_{n\in\mathbb{Z}} c_n (z - z_0)^n$ with uniformly absolute convergence on compact subsets of $D'(z_0, r)$. We define the *residue* of f at z_0 to be

$$\mathrm{Res}(f, z_0) = c_{-1}.$$

Let $\gamma(t) = z_0 + \rho e^{it}$, $t \in [0, 2\pi]$, $0 < \rho < r$. Then

$$\begin{aligned}
\int_\gamma f(z)\, dz &= \sum_{n\in\mathbb{Z}} c_n \int_\gamma (z - z_0)^n\, dz \\
&= 2\pi i \cdot c_{-1} \\
&= 2\pi i \cdot \mathrm{Res}(f, z_0).
\end{aligned}$$

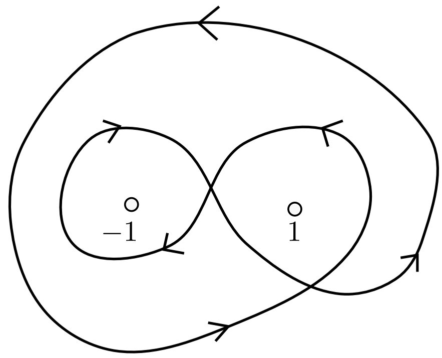

Figure 8:

19 Logarithms, Winding Numbers and Cauchy's Theorem VII (10/3)

Theorem 19.1 (Residue Theorem)**.** *Let $V \subset \mathbb{C}$ be an open set. Suppose that $S \subset V$ is closed in V and every point of S is isolated in S. Let $\Gamma \subset V \backslash S$ be a formal sum of closed curve such that* $\operatorname{Ind}(\Gamma, a) = 0$ *for every $a \in \mathbb{C} \backslash V$. Let $f \in H(V - S)$. Then the set* $\{p \in S | \operatorname{Ind}(\Gamma, p) \neq 0\}$ *is finite and*

$$\frac{1}{2\pi i} \int_{\Gamma} f(z)\, dz = \sum_{p \in S} \operatorname{Ind}(\Gamma, p) \operatorname{Res}(f, p).$$

Example 19.2. Let $\lambda > 0$ and $f(z) = \frac{\exp(i\lambda z)}{z^2+1}$. Then $f \in H(\mathbb{C} - \{\pm i\})$. $\Gamma = \gamma_1 \dot{+} \gamma_2$ where $\gamma_1(t) = t$ for $t \in [-R, R]$, $R > 1$, $\gamma_2(t) = Re^{it}$ for $t \in [0, \pi]$. (To apply Residue Theorem, we should parametrize Γ using a single interval so that Γ is a closed curve. It makes no difference.) Let $S = \{\pm i\}$. Note that $\Gamma \subset \mathbb{C} \backslash \{\pm i\}$ provided $R > 1$. To use the Residue Theorem, we need $\operatorname{Ind}(\Gamma, i)$ and $\operatorname{Ind}(\Gamma, -i)$. We know $\operatorname{Ind}(\Gamma, -i) = 0$ because $-i$ is in the unbounded component of $\mathbb{C} \backslash \Gamma^*$. It is a fact that $\operatorname{Ind}(\Gamma, i) = 1$. One way would be to show that $\Gamma \underset{\mathbb{C}-S}{\sim} \sigma$ where σ is a small circle centered at i. Then $\operatorname{Ind}(\sigma, i) = 1$ by computation, so $\mathrm{Ind}(\Gamma, i) = 1$. Next, we need $\mathrm{Res}(f, i)$.

$$f(z) = \frac{\exp(i\lambda z)}{(z-i)(z+i)} = \frac{\exp(i\lambda z)}{z+i} \cdot \frac{1}{z-i} = \left(\frac{e^{-\lambda}}{2i} + \cdots\right) \cdot \frac{1}{z-i}.$$

Thus $\mathrm{Res}(f, i) = \frac{e^{-\lambda}}{2i}$. By Residue Theorem,

$$\frac{1}{2\pi i} \int_\Gamma f(z)\, dz = \mathrm{Ind}(\Gamma, i) \cdot \mathrm{Res}(f, i) + \mathrm{Ind}(\Gamma, -i) \cdot \mathrm{Res}(f, -i).$$

Thus $\int_\Gamma f(z)\, dz = \pi e^{-\lambda}$.

On the other hand,

$$\begin{aligned} \int_\Gamma f(z)\, dz &= \int_{\gamma_1} f(z)\, dz + \int_{\gamma_2} f(z)\, dz \\ &= \int_{-R}^{R} \frac{\exp(i\lambda t)}{t^2+1}\, dt + \int_{\gamma_2} f(z)\, dz. \end{aligned}$$

Now

$$\begin{aligned} |f(z)| &= \left|\frac{\exp(i\lambda z)}{z^2+1}\right| = \left|\frac{\exp(i\lambda(x+iy))}{z^2+1}\right| \\ &= \left|\frac{\exp(i\lambda x - \lambda y)}{z^2+1}\right| = \left|\frac{e^{-\lambda y}}{z^2+1}\right| \\ &\le \frac{1}{|z^2+1|} \text{ (for } z \text{ such that } y \ge 0). \end{aligned}$$

Thus

$$\begin{aligned} \left|\int_{\gamma_2} f(z)\, dz\right| &\le \left(\max_{z \in \gamma_2^*} \frac{1}{|z^2+1|}\right) \cdot \pi R \quad \text{(ML-inequality)} \\ &= \frac{1}{R^2-1} \cdot \pi R. \end{aligned}$$

So far, we have $\pi e^{-\lambda} = \int_{-R}^{R} \frac{\exp(i\lambda t)}{t^2+1}\, dt + F(R)$ where $|F(R)| \le \frac{\pi R}{R^2-1}$. Let $R \to \infty$ in this equation, we get

$$\pi e^{-\lambda} = \lim_{R\to\infty} \int_{-R}^{R} \frac{\exp(i\lambda t)}{t^2+1}\, dt$$

because $\lim_{R\to\infty} F(R) = 0$.

$$\int_{-\infty}^{\infty} \frac{\exp(i\lambda t)}{t^2+1}\, dt = \lim_{R_1\to\infty, R_2\to\infty} \int_{R_1}^{R_2} \frac{\exp(i\lambda t)}{t^2+1}\, dt$$

provided this exists, it equals $\lim_{R\to\infty}\int_{-R}^{R}\frac{\exp(i\lambda t)}{t^2+1}\,dt$. But $\int_{-R}^{R}$ might exist even when the other doesn't. By comparison with $\int_{-\infty}^{\infty}\frac{1}{1+t^2}\,dt$, we know that $\int_{-\infty}^{\infty}\frac{\exp(i\lambda t)}{t^2+1}\,dt$ converges. Thus

$$\int_{-\infty}^{\infty}\frac{\exp(i\lambda t)}{t^2+1}\,dt = \lim_{R\to\infty}\int_{-R}^{R}\frac{\exp(i\lambda t)}{t^2+1}\,dt = \pi e^{-\lambda}.$$

Note that $\int_{-\infty}^{\infty}\frac{\sin(\lambda t)}{t^2+1}\,dt = 0$ because $\frac{\sin(\lambda t)}{t^2+1}$ is odd. Thus, $\int_{-\infty}^{\infty}\frac{\cos(\lambda t)}{t^2+1}\,dt = \pi e^{-\lambda}$, for $\lambda > 0$. Finally, $\int_{-\infty}^{\infty}\frac{\cos(-\lambda t)}{t^2+1}\,dt = \int_{-\infty}^{\infty}\frac{\cos(\lambda t)}{t^2+1}\,dt$ and $\int_{-\infty}^{\infty}\frac{1}{t^2+1}\,dt = \pi$, so we can remove the restriction on λ.

20 Counting Zeroes and the Open Mapping Theorem I (10/6)

Let $V \subset \mathbb{C}$ be open and $f \in H(V)$ is not constant on any component of V. Say $z_0 \in Z(f)$, we can write $f(z) = (z - z_0)^n g(z)$ where $g(z_0) \neq 0$, $g \in H(V)$. Then $n = \operatorname{ord}(f, z_0)$ is the order of zero at z_0. Then

$$f'(z) = n(z - z_0)^{n-1} g(z) + (z - z_0)^n g'(z)$$

and so

$$\frac{f'(z)}{f(z)} = \frac{n}{z - z_0} + \frac{g'(z)}{g(z)}.$$

This is valid as long as we stay close enough to z_0 that g is non-zero, except for z_0. We conclude that $\frac{f'}{f} \in H(V - Z(f))$ has a simple pole at z_0 with residue $\operatorname{Res}(\frac{f'}{f}, z_0) = n = \operatorname{ord}(f, z_0)$.

Theorem 20.1 (The Argument Principle)**.** *Let $V \subset \mathbb{C}$ be open, $f \in H(V)$, and f is not constant on any component of V. Let $q \in \mathbb{C}$ and define $S = \{z \in V | f(z) = q\}$. Suppose that $\gamma \subset V \backslash S$ is a closed curve such that* $\operatorname{Ind}(\gamma, a) = 0$ *for all $a \in \mathbb{C} \backslash V$ and* $\operatorname{Ind}(\gamma, a) = 0$ *or 1 for $a \in \mathbb{C} \backslash \gamma^*$. Let $\Omega = \{z \in V \backslash \gamma^* | \operatorname{Ind}(\gamma, z) = 1\}$. Define $\tilde{\gamma}$ in $\mathbb{C} - \{q\}$ to be $\tilde{\gamma} = f \circ \gamma$. Then the number of solutions to the equations $f(w) = q$ with $w \in \Omega$, counted with multiplicity, is equal to* $\operatorname{Ind}(\tilde{\gamma}, q)$.

Proof.

$$\begin{aligned}
\operatorname{Ind}(\tilde{\gamma}, q) &= \frac{1}{2\pi i} \int_{\tilde{\gamma}} \frac{1}{\zeta - q}\, d\zeta = \frac{1}{2\pi i} \int_0^1 \frac{1}{\tilde{\gamma}(t) - q} \tilde{\gamma}'(t)\, dt \\
&= \frac{1}{2\pi i} \int_0^1 \frac{1}{f(\gamma(t)) - q} \cdot f'(\gamma(t)) \cdot \gamma'(t)\, dt \\
&= \frac{1}{2\pi i} \int_0^1 \frac{f'(\gamma(t))}{f(\gamma(t)) - q} \cdot \gamma'(t)\, dt \\
&= \frac{1}{2\pi i} \int_0^1 \frac{f'(z)}{f(z) - q}\, dz \\
&= \frac{1}{2\pi i} \int_0^1 \frac{(f(z) - q)'}{f(z) - q}\, dz \\
&= \sum_{w \in Z(f(z) - q)} \operatorname{Ind}(\gamma, w) \cdot \operatorname{Res}\left(\frac{(f(z) - q)'}{f(z) - q}, w\right) \\
&= \sum_{w \in Z(f(z) - q) \cap \Omega} \operatorname{Res}\left(\frac{(f(z) - q)'}{f(z) - q}, w\right) \\
&= \sum_{w \in Z(f(z) - q) \cap \Omega} \operatorname{ord}\left(f(z) - q, w\right) \\
&= \text{number of solutions to } f(w) = q \text{ in } \Omega \text{ counted with multiplicity.}
\end{aligned}$$

□

Remark 20.2. *"Counted with multiplicity" means* $\sum_{w\in\Omega} ord(f(w)-q,w)$.

Example 20.3 (Basic Case)**.** $V = D(0,1)$, $f \in H(V)$ is $f(z) = z^k$ $(k \in \mathbb{N})$, $\gamma(t) = \frac{1}{2}e^{it}, t \in [0,2\pi]$. Here $\Omega = D(0,\frac{1}{2})$, $\tilde{\gamma}(t) = (\frac{1}{2}e^{it})^k = \frac{1}{2^k}e^{ikt}$. So $\mathrm{Ind}(\tilde{\gamma},0) = k$. f has a single zero in Ω, but the order of the zero is k. So the Argument Principle works in this case. We know that $q \mapsto \mathrm{Ind}(\tilde{\gamma},q)$ is constant on components of $\mathbb{C}\backslash\tilde{\gamma}$. Thus $\mathrm{Ind}(\tilde{\gamma},q) = k$ for all $q \in D(0,\frac{1}{2^k})$. Thus $z^k = q$ has k solutions counted with multiplicity, in $\Omega = D(0,\frac{1}{2})$. Except for $q = 0$, these k solutions are all distinct.

Definition 20.4. A function $\varphi : X \to Y$ between two metric spaces is said to be an *open map* if $\varphi(U)$ is open whenever U is open.

Theorem 20.5 (Open Mapping Theorem)**.** *Let $V \subset \mathbb{C}$ be a connected open set and $f \in H(V)$ be non-constant. Then $f : V \to \mathbb{C}$ is an open map.*

Proof. Let $U \subset V$ be a nonempty open set. Let $q \in f(U)$. We have to show that q is an interior point of $f(U)$. Choose $p \in U$ such that $f(p) = q$. Choose $r > 0$ such that $\overline{D(p,r)} \subset U$ and f takes the value q on $\overline{D(p,r)}$ only at p. Let γ be $\partial D(p,r)$. Then the Argument Principle applies. It tells us that the number of solutions to $f(w) = q$ in $D(p,r)$ is equal to $\mathrm{Ind}(\tilde{\gamma},q)$ where $\tilde{\gamma} = f \circ \gamma$. Since $f(p) = q$, $\mathrm{Ind}(\tilde{\gamma},q) \neq 0$. Let W be the component of $\mathbb{C}\backslash\tilde{\gamma}^*$ that contains q. Then $\mathrm{Ind}(\tilde{\gamma},s) \neq 0$ for all $s \in W$. By the Argument Principle again, the equation $f(w) = s$ has solutions in $D(p,r)$. That is, $W \subset f(D(p,r)) \subset f(U)$. Also, W is an open set. Thus, $q \in \mathrm{int}(f(U))$, as required. □

Remark 20.6. *Open Mapping Theorem distinguishes complex differentiability and real differentiability. For example, on the real line, the function $f(x) = x^2$ is not an open map, since it maps an open integral $(-1,1)$ to a non-open set $[0,1)$.*

21 Counting Zeroes and the Open Mapping Theorem II (10/8)

Say that γ_0 and γ_1 are two loops in $\mathbb{C} - \{0\}$. Define γ_s for $s \in [0,1]$ by

$$\gamma_s(t) = (1-s)\gamma_0(t) + s\gamma_1(t).$$

This is a homotopy from γ_0 to γ_1 in $\mathbb{C}$. This is a homotopy in $\mathbb{C} - \{0\}$ provided $\gamma_s(t) \neq 0$ for all s,t. If $\gamma_s(t) = 0$ for some s,t then 0 is in the segment of $[\gamma_0(t),\gamma_1(t)]$. So γ_s is a homotopy from γ_0 to γ_1 in $\mathbb{C} - \{0\}$ if and only if $|\gamma_1(t) - \gamma_0(t)| < |\gamma_1(t)| + |\gamma_0(t)|$ for all t. If $|\gamma_1(t) - \gamma_0(t)| < |\gamma_1(t)| + |\gamma_0(t)|$ for all t, then $\mathrm{Ind}(\gamma_0,0) = \mathrm{Ind}(\gamma_1,0)$.

Theorem 21.1 (Rouché's Theorem)**.** *Let $V \subset \mathbb{C}$ be a connected open set. Let γ be a closed curve in V such that* $\mathrm{Ind}(\gamma,p) = 0$ *for all $p \in \mathbb{C}\backslash V$ and* $\mathrm{Ind}(\gamma,p) = 0$ *or 1 for* *all $p \in \mathbb{C}\backslash\gamma^*$. Let $\Omega = \{z \in \mathbb{C}\backslash\gamma^* : \mathrm{Ind}(\gamma, z) = 1\}$. Let $f, g \in H(V)$ and suppose that $|f(z) - g(z)| < |f(z)| + |g(z)|$ for all $z \in \gamma^*$. Then f and g has same number of zeroes counted with multiplicity in Ω.*

Proof. If f and g are both nonconstant, then define $\tilde{\gamma}_f = f \circ \gamma$, $\tilde{\gamma}_g = g \circ \gamma$. We have

$$|\tilde{\gamma}_f(t) - \tilde{\gamma}_g(t)| < |\tilde{\gamma}_f(t)| + |\tilde{\gamma}_g(t)|$$

for all t by the hypothesis on f, g. So $\mathrm{Ind}(\tilde{\gamma}_f, 0) = \mathrm{Ind}(\tilde{\gamma}_g, 0)$ by preliminary discussion. By the Argument Principle, both of these count zeroes in Ω with multiplicity.

If f or g (or both) are constant then the claim follows by considering cases. □

Example 21.2. Let $f(z) = z^n(z-2) - 1$. Take $g(z) = z^n(z-2)$. Take γ to be $\partial D(0, 1)$. Then $f(z) - g(z) = -1$ and so to apply Rouché's Theorem we need

$$1 < |z^n(z-2) - 1| + |z^n(z-2)|$$

for all z with $|z| = 1$. This is equivalent to

$$1 < |z^n(z-2) - 1| + |z - 2|$$

for all z with $|z| = 1$. Now $1 < |z - 2|$ for all z with $|z| = 1$ except $z = 1$. Note that the other summand does not vanish at $z = 1$ and so we have the inequality. This implies that f and g have the same number of zeroes counted with multiplicity, in $\Omega = D(0, 1)$. For g this number is n. This means there are n zeroes of f in $D(0, 1)$. By the Intermediate Value Theorem, f has a zero slightly larger than 2.

Example 21.3. Let $\lambda \in \mathbb{C} - \{0\}$ and $f(z) = z\sin(z) - \lambda\cos(z)$ and $g(z) = z\sin(z)$. I claim that for a fixed λ, $|f(z) - g(z)| \le |g(z)|$ on a rectangle provided that T and m are large enough. This inequality is $|\lambda| \cdot |\cos(z)| < |z| \cdot |\sin(z)|$ for any z in the rectangle (see Figure 9). In general,

$$|\sin(z)|^2 = \sin^2(x) + \sinh^2(y), |\cos(z)|^2 = \cos^2(x) + \sinh^2(y),$$

where $z = x + iy$.

On side ①, we need $|\lambda|\cdot|\sinh(y)| < |z|\sqrt{1 + \sinh^2(y)}$. So $|\lambda|\cdot|\sinh(y)| < \frac{(2m+1)\pi}{2}\sqrt{1 + \sinh^2(y)}$ is sufficient. All that is necessary if $|\lambda| \le \frac{(2m+1)\pi}{2}$. This can be done by making m big enough. Side ③ is similar.

On side ②, we need $|\lambda|\cdot\sqrt{\cos^2(x) + \sinh^2(T)} < |z|\sqrt{1 + \sinh^2(T)}$. So $|\lambda|\cdot\sqrt{\cos^2(x) + \sinh^2(T)} < T \cdot \sqrt{\sin^2(x) + \sinh^2(T)}$ is enough. $|\lambda| \cdot \sqrt{1 + \sinh^2(T)} < T \cdot \sinh^2(T)$ is enough. We can do this by making T large. ④ is similar.

In the box, g has a double zero at 0, and a simple zero at $\pm\pi, \pm 2\pi, \cdots, \pm m\pi$. This is $2m + 2$ zeroes inside the rectangle in total. Thus f also has this many once T and m are large enough. The equation

$$z\sin(z) = \lambda\cos(z)$$

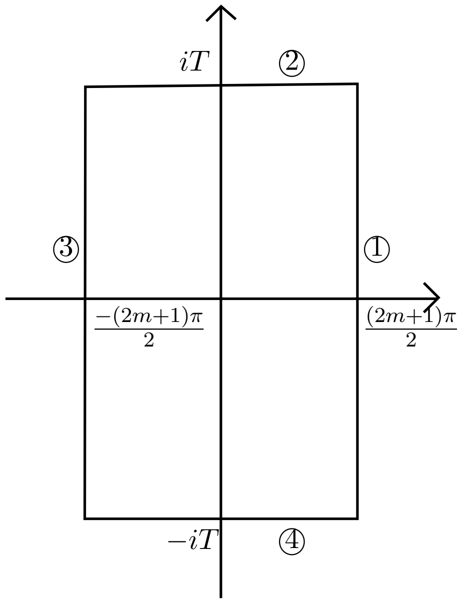

Figure 9:

is equivalent to

$$\tan(z) = \frac{\lambda}{z}.$$

This equation has $2m+2$ real solutions in the rectangle, so all solutions are real if $\lambda > 0$. If $\lambda < 0$, then there are only $2m$ real solutions, so there must be 2 non-real solutions. They are purely imaginary.

Theorem 21.4 (Hurwitz's Theorem)**.** *Suppose that D is an open set, $f_n \in H(D)$, $f_n \to f$ uniformly on compact subsets of D, $\overline{D(z,r)} \subset D$, and f has no zero on $\partial D(z,r)$. Then there exists N such that f_n and f have the same number of zeroes in $D(z,r)$ for all $n > N$.*

Proof. Let $\epsilon = \min_{w \in \partial D(z,r)} |f(w)|$, which exists and is non-zero since $|f|$ is continuous and $\partial D(z,r)$ is compact. Since $f_n \to f$ uniformly on compact subsets of D, there exists N such that for $w \in \partial D(z,r)$,

$$|f_n(w) - f(w)| < \epsilon \leq |f(w)| \leq |f(w)| + |g(w)|$$

for all $n > N$. Then Rouché's Theorem implies that f_n and f have the same number of zeroes in $D(z,r)$ for all $n > N$. $\square$

22 Product Formula for Sine Function I (10/13)

The objective of this chapter is to prove the following two formulas

$$\sin(\pi z) = \pi z \prod_{n=1}^{\infty}(1 - \frac{z^2}{n^2}) \quad \forall z \in \mathbb{C}, \tag{22.1}$$

$$\pi \cot(\pi z) = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z-n} + \frac{1}{z+n} \right) \quad \forall z \in \mathbb{C}\backslash\mathbb{Z}. \tag{22.2}$$

(22.2) is basically a partial function decomposition of $\pi \cot(\pi z)$. The function has simple poles at every integer and the residue at each pole is 1 because $\pi \cot(\pi z) = \frac{(\sin(\pi z)')}{\sin(\pi z)}$ (recall $\frac{f'}{f}$ has simple poles at zeroes of f with residue $\mathrm{ord}(f,p)$). First aim is to prove (22.2).

Formula (22.2) comes from using Cauchy's Integral Formula. The idea is to come up with a function with $\pi \cot(\pi z)$ as a residue and the terms in the sum as residues. Then show that as some parameter $\to \infty$, the integral itself goes to zero.

The first thing I need is an estimate for the size of $\cot(\pi z)$ away from its poles.

Lemma 22.1. *There is a constant M such that $|\cot(\pi z)| \leq M$ if $\mathrm{Im}(z) \geq 1$ or $Re(z) \in \frac{1}{2} + \mathbb{Z}$.*

Proof. Let $z = x + iy$. Then

$$|\cot(\pi z)|^2 = \frac{|\cos(\pi z)|^2}{|\sin(\pi z)|^2} = \frac{\cos^2(\pi x) + \sinh^2(\pi y)}{\sin^2(\pi x) + \sinh^2(\pi y)}.$$

If $|y| \geq 1$ then

$$|\cot(\pi z)|^2 \leq \frac{1 + \sinh^2(\pi y)}{\sinh^2(\pi y)} = 1 + \frac{1}{\sinh^2(\pi y)} \leq 1 + \frac{1}{\sinh^2(\pi)}.$$

This shows that $|\cot(\pi z)|$ is bounded on the set $|y| \geq 1$.

Now say $x \in \frac{1}{2} + \mathbb{Z}$. Then

$$|\cot(\pi z)|^2 = \frac{\sinh^2(\pi y)}{1 + \sinh^2(\pi y)} \leq 1.$$

This shows $|\cot(\pi z)|$ is bounded on the set $x \in \frac{1}{2} + \mathbb{Z}$. □

Let $N \geq 1$ be an integer. I will use the contour Q_N (see Figure 10). Fix $z \in \mathbb{C}\backslash\mathbb{Z}$. Assume that N is large enough that is inside the contour Q_N. Let $f : \mathbb{C}\backslash(\mathbb{Z} \cup \{z\}) \to \mathbb{C}$ be

$$f(w) = \frac{\pi \cot(\pi w)}{w - z}.$$

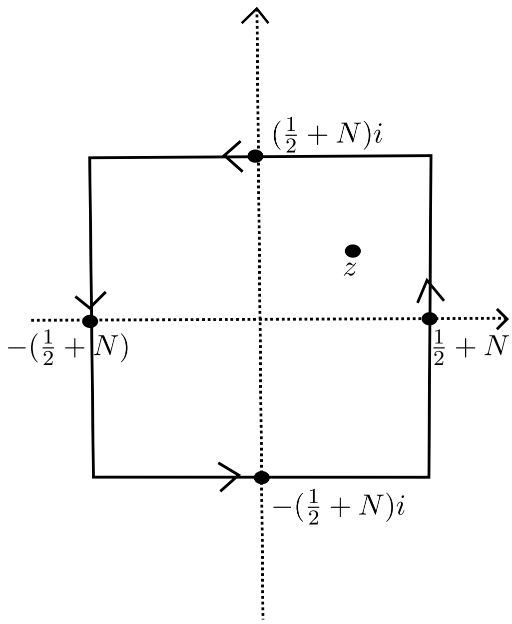

Figure 10:

The function f has simple pole at every point in $\mathbb{Z} \cup \{z\}$. The residues are

$$\operatorname{Res}(f,n) = \frac{1}{n-z}, \forall n \in \mathbb{Z},$$

$$\operatorname{Res}(f,z) = \pi \cot(\pi z).$$

Thus

$$\frac{1}{2\pi i}\int_{Q_N} f(w)\,dw = \pi\cot(\pi z) + \sum_{n=-N}^{N} \frac{1}{n-z}$$

by the Residue Theorem. If we could show $\lim_{N\to\infty} \frac{1}{2\pi i}\int_{Q_N} f(w)\,dw = 0$ then we get

$$\pi\cot(\pi z) = \lim_{N\to\infty} \sum_{n=-N}^{N} \frac{1}{z-n}$$

which is equivalent to what we wanted.

The required limit is true, but relies essentially on symmetry of Q_N.

To prove

$$\pi\cot(\pi z) = \lim_{N\to\infty} \sum_{n=-N}^{N} \frac{1}{z-n},$$

the only thing that remained was to show

$$\lim_{N\to\infty}\int_{Q_N}\frac{\pi\cot(\pi w)}{w-z}\,dw=0.$$

The key is to note that $w\mapsto\frac{\pi\cot(\pi w)}{w}$ is even. Thus

$$\int_{Q_N}\frac{\pi\cot(\pi w)}{w}\,dw=0$$

because contributions from opposite sides cancel. (Note: This is why the sum $\sum_{n=-N}^{N}\frac{1}{z-n}$ has to be symmetric.) Thus

$$\int_{Q_N}\frac{\pi\cot(\pi w)}{w-z}\,dw=\int_{Q_N}\left(\frac{\pi\cot(\pi w)}{w-z}-\frac{\pi\cot(\pi w)}{w}\right)dw=\int_{Q_N}\frac{\pi z\cot(\pi w)}{w(w-z)}\,dw.$$

We know that $|\pi\cot(\pi w)|\le M$ on Q_N^* by Lemma 22.1. So

$$\begin{aligned}\left|\int_{Q_N}\frac{\pi z\cot(\pi w)}{w(w-z)}\,dw\right|&\le\frac{M\cdot|z|}{(N+\frac{1}{2})(N+\frac{1}{2}-|z|)}\cdot 4(2N+1)\\&\quad(\text{provided }|z|<N+\frac{1}{2})\\&\quad\to 0\ (\text{as }N\to\infty,\ \text{as required}).\end{aligned}$$

So

$$\begin{aligned}\pi\cot(\pi z)&=\lim_{N\to\infty}\sum_{n=-N}^{N}\frac{1}{z-n}\\&=\frac{1}{z}+\sum_{n=1}^{\infty}\left(\frac{1}{z-n}+\frac{1}{z+n}\right)\end{aligned}$$

for $z\in\mathbb{C}\backslash\mathbb{Z}$.

Remark 22.2. *If we set $z=ia$ in the formula, where $a\in\mathbb{R}\backslash\{0\}$, then we get*

$$\begin{aligned}\pi\cot(\pi ia)&=\frac{1}{ia}+\sum_{n=1}^{\infty}\left(\frac{1}{ia-n}+\frac{1}{ia+n}\right),\\ \Rightarrow\pi\frac{\cos(\pi ia)}{\sin(\pi ia)}&=\frac{1}{ia}+\sum_{n=1}^{\infty}\frac{2ia}{-a^2-n^2},\\ \Rightarrow\pi\frac{\cosh(\pi a)}{i\sinh(\pi a)}&=\frac{1}{ia}+\sum_{n=1}^{\infty}\frac{2ia}{-(a^2+n^2)},\\ \Rightarrow\pi\frac{\cosh(\pi a)}{\sinh(\pi a)}&=\frac{1}{a}+\sum_{n=1}^{\infty}\frac{2a}{a^2+n^2},\\ \Rightarrow\sum_{n=1}^{\infty}\frac{1}{n^2+a^2}&=\frac{1}{2a}\left(\pi\coth(\pi a)-\frac{1}{a}\right).\end{aligned}$$

For example,

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1} = \frac{1}{2}\left(\pi \coth(\pi) - 1\right).$$

Also, you can solve the Basel problem by letting $a \to 0$. You could also differentiate with respect to a (needs justification). Then as $a \to 0$, you could evaluate $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

23 Product Formula for Sine Function II (10/15)

The next task is to show that $p(z) = \pi z \prod_{n=1}^{\infty}(1 - \frac{z^2}{n^2})$ is a holomorphic function.

Lemma 23.1. *Let $w_1, \cdots, w_n \in \mathbb{C}$. Then*

$$\left|1 - \prod_{j=1}^{n}(1+w_j)\right| \leq \exp\left(\sum_{j=1}^{n}|w_j|\right) - 1.$$

Proof. Well,

$$\begin{aligned}
\left|1 - \prod_{j=1}^{n}(1+w_j)\right| &= \left|\sum_{j=1}^{n} w_j + \sum_{1\leq j_1<j_2\leq n} w_{j_1}w_{j_2} + \sum_{1\leq j_1<j_2<j_3\leq n} w_{j_1}w_{j_2}w_{j_3} + \cdots + w_1w_2\cdots w_n\right| \\
&\leq \sum_{j=1}^{n}|w_j| + \sum_{1\leq j_1<j_2\leq n}|w_{j_1}w_{j_2}| + \sum_{1\leq j_1<j_2<j_3\leq n}|w_{j_1}w_{j_2}w_{j_3}| + \cdots + |w_1w_2\cdots w_n| \\
&\leq \sum_{j=1}^{n}|w_j| + \frac{1}{2}\left(\sum_{j=1}^{n}|w_j|\right)^2 + \frac{1}{3!}\left(\sum_{j=1}^{n}|w_j|\right)^3 + \cdots + \frac{1}{n!}\left(\sum_{j=1}^{n}|w_j|\right)^n \\
&\text{(by multinomial theorem)} \\
&\leq \exp\left(\sum_{j=1}^{n}|w_j|\right) - 1.
\end{aligned}$$

□

Next, we want to think about convergence of a product

$$P = \prod_{j=1}^{\infty}(1+w_j).$$

We make the partial products

$$P_N = \prod_{j=1}^{N}(1+w_j).$$

We say P converges if $P_N \to P$.

Lemma 23.2. *Suppose that* (w_j) *is a sequence of complex numbers and* $\sum_{j=1}^{\infty}|w_j|$ *converges. Then* $\prod_{j=1}^{\infty}(1+w_j)$ *converges and it doesn't converge to 0 unless one of the factors is zero.*

Proof. Let $P_N = \prod_{j=1}^{N}(1+w_j)$. First, we show that (P_N) is bounded. Well,

$$\begin{aligned} |1-P_N| &= \left|1-\prod_{j=1}^{N}(1+w_j)\right| \\ &\leq \exp\left(\sum_{j=1}^{N}|w_j|\right)-1 \\ &\leq \exp\left(\sum_{j=1}^{\infty}|w_j|\right)-1 \end{aligned}$$

and so

$$|P_N| \leq 1+|1-P_N| \leq \exp\left(\sum_{j=1}^{\infty}|w_j|\right) = B.$$

Say $N_1 < N_2$. Then

$$|P_{N_1}-P_{N_2}| = |P_{N_1}|\cdot|1-\prod_{j=N_1+1}^{N_2}(1+w_j)| \leq B\cdot\left(\exp\left(\sum_{j=N_1+1}^{N_2}|w_j|\right)-1\right)$$

by Lemma 23.1. Let $\epsilon > 0$, then we can choose $\eta > 0$ such that $B(\exp(\eta)-1) < \epsilon$. We can choose N such that if $N \leq N_1 < N_2$ then $\sum_{j=N_1+1}^{N_2}|w_j| < \eta$ because $\sum_{j=1}^{\infty}|w_j|$ converges. If $N \leq N_1 < N_2$ then

$$|P_{N_1}-P_{N_2}| \leq B\cdot|\exp(\eta-1)| < \epsilon.$$

Thus (P_N) is Cauchy and so converges to some P.

Lastly, we have to show that $P \neq 0$ provided that $w_j \neq -1$ for all j. If $M \geq 1$ is an integer then

$$\left|1-\prod_{j=M}^{\infty}(1+w_j)\right| \leq \exp\left(\sum_{j=M}^{\infty}|w_j|\right)-1.$$

By making M sufficiently large, we can make

$$\exp\left(\sum_{j=M}^{\infty}|w_j|\right)-1 < \frac{1}{2}.$$

Thus for M this large,

$$\left|1 - \prod_{j=M}^{\infty}(1+w_j)\right| < \frac{1}{2}$$

and so $\prod_{j=M}^{\infty}(1+w_j) \neq 0$. Also $\prod_{j=1}^{M-1}(1+w_j) \neq 0$ and so

$$P = \prod_{j=1}^{M-1}(1+w_j) \cdot \prod_{j=M}^{\infty}(1+w_j) \neq 0$$

as required. $\square$

Remark 23.3. *This result applies to products of functions uniformly.*

If $\prod_{j=1}^{\infty}(1+g_j(z))$ *is such that* $\sum_{j=1}^{\infty}|g_j(z)|$ *converges uniformly on some set then the product also converges uniformly and the zeroes are the obvious ones.*

24 Product Formula for Sine Function III (10/17)

The series $\sum_{n=1}^{\infty}|\frac{z^2}{n^2}|$ converges uniformly on all compact subsets of $\mathbb{C}$. Thus

$$p(z) = \pi z \prod_{n=1}^{\infty}(1 - \frac{z^2}{n^2})$$

converges uniformly on compact subsets of $\mathbb{C}$ and is zero only at integers. The partial products

$$p_N(z) = \pi z \prod_{n=1}^{N}(1 - \frac{z^2}{n^2})$$

are polynomials. So they are all holomorphic. It follows that p is holomorphic on $\mathbb{C}$. Moreover,

$$p_N' \to p'$$

uniformly on compact subsets of $\mathbb{C}$. It follows that on compact subsets of $\mathbb{C}\backslash\mathbb{Z}$,

$$\frac{p_N'}{p_N} \xrightarrow{\text{uniformly}} \frac{p'}{p}.$$

In general, $\frac{f'}{f}$ is called the logarithmic derivative of f, denoted as D. That is, $D(f) = \frac{f'}{f}$.

Main thing about D:

$$D(fg) = D(f) + D(g).$$

So

$$
\begin{aligned}
D(p_N) &= D(\pi z \prod_{n=1}^{N} (1 - \frac{z^2}{n^2})) \\
&= D(\pi) + D(z) + \sum_{n=1}^{N} D(1 - \frac{z^2}{n^2}) \\
&= 0 + \frac{1}{z} + \sum_{n=1}^{N} \frac{-\frac{2z}{n^2}}{1 - \frac{z^2}{n^2}} \\
&= \frac{1}{z} + \sum_{n=1}^{N} \frac{-2z}{n^2 - z^2} \\
&= \frac{1}{z} + \sum_{n=1}^{N} \frac{2z}{z^2 - n^2} \\
&= \frac{1}{z} + \sum_{n=1}^{N} \left(\frac{1}{z-n} + \frac{1}{z+n} \right) \\
&= \sum_{n=-N}^{N} \frac{1}{z-n}
\end{aligned}
$$

for $z \in \mathbb{C} \backslash \mathbb{Z}$. So

$$
\begin{aligned}
D(p) = \lim_{N \to \infty} D(p_N) &= \lim_{N \to \infty} \sum_{n=-N}^{N} \frac{1}{z-n} \\
&= \pi \cot(\pi z) \\
&= D(\sin(\pi z)).
\end{aligned}
$$

Now we know that $D(p) = D(\sin(\pi z))$. Thus

$$
D\left(\frac{\sin(\pi z)}{p}\right) = D(\sin(\pi z)) - D(p) = 0
$$

on $\mathbb{C} \backslash \mathbb{Z}$. So

$$
\frac{d}{dz}\left(\frac{\sin(\pi z)}{p}\right) = 0
$$

and (since $\mathbb{C} \backslash \mathbb{Z}$ is connected) $\frac{\sin(\pi z)}{p}$ is constant. There is some $c \in \mathbb{C}$ such that

$$
\sin(\pi z) = c\pi z \prod_{n=1}^{\infty} (1 - \frac{z^2}{n^2}).
$$

So

$$\frac{\sin(\pi z)}{z} = c\pi \prod_{n=1}^{\infty} (1 - \frac{z^2}{n^2}).$$

So

$$\lim_{z \to 0} \frac{\sin(\pi z)}{z} = c\pi \lim_{z \to 0} \prod_{n=1}^{\infty} (1 - \frac{z^2}{n^2}),$$

hence

$$\pi = c\pi,$$

and so

$$c = 1.$$

Thus

$$\sin(\pi z) = \pi z \prod_{n=1}^{\infty} (1 - \frac{z^2}{n^2})$$

for all $z \in \mathbb{C}$.

Example 24.1. For example, if we set $z = \frac{1}{2}$, we will get

$$\begin{aligned}
\sin(\frac{\pi}{2}) &= \frac{\pi}{2} \prod_{n=1}^{\infty} (1 - \frac{1}{4n^2}), \\
\Rightarrow \prod_{n=1}^{\infty} (1 - \frac{1}{4n^2}) &= \frac{2}{\pi}, \\
\Rightarrow \prod_{n=1}^{\infty} \frac{4n^2 - 1}{4n^2} &= \frac{2}{\pi}, \\
\Rightarrow \prod_{n=1}^{\infty} \frac{(2n-1)(2n+1)}{(2n)^2} &= \frac{2}{\pi}, \\
\Rightarrow \prod_{n=1}^{\infty} \frac{(2n)^2}{(2n-1)(2n+1)} &= \frac{\pi}{2}.
\end{aligned}$$

25 Inverses of Holomorphic Functions I (10/20)

Theorem 25.1. *Let $V \subset \mathbb{C}$ be an open set and $f \in H(V)$ be a one-to-one function. Then $f'(z) \neq 0$ for all $z \in V$.*

Proof. Suppose that $p \in V$ and $f'(p) = 0$. Note that f' is not identically zero since f is one-to-one. So p is an isolated zero of f'. Choose $r > 0$ so that $\overline{D(p,r)} \subset V$ and the only zero of f' on $D(p,r)$ is at p. Let $\gamma = \partial D(p,r)$ and $\tilde{\gamma} = f \circ \gamma$. We know that $\text{Ind}(\tilde{\gamma}, f(p))$ is equal to the number of zeroes of $f - f(p)$ in $D(p,r)$. By hypothesis, $f'(p) = 0$ and so $\text{Ind}(\tilde{\gamma}, f(p)) \geq 2$. Thus $\text{Ind}(\tilde{\gamma}, q) \geq 2$ for all q close enough to $f(p)$. Choose q close to $f(p)$ and consider the solutions to $f(z) = q$ with $z \in D(p,r)$. These solutions are all simple zeroes of $f - q$. Thus there must be at least two of them. That is, there is $z_1, z_2 \in D(p,r)$ with $f(z_1) = q = f(z_2)$. This contradicts f being one-to-one. So $f'(z) \neq 0$ at all points $z \in V$. $\square$

Remark 25.2. *(i) The implication cannot be reversed. An example would be $f(z) = \exp(z)$.*

(ii) If f is one-to-one, then all solutions to $f(z) = q$ are simple.

Theorem 25.3 (Inverse Function Theorem for Holomorphic Functions)**.** *Let $V \subset \mathbb{C}$ be an open set, $f \in H(V)$ be one-to-one, $W = f(V)$ and $g : W \to V$ the inverse function. Then $g \in H(W)$.*

Proof. Differentiability is a local property, so it suffices to show that if $q \in W$ then g is differentiable on some open set containing q. Take $q \in W$ and choose $p \in V$ such that $f(p) = q$ and $r > 0$ such that $\overline{D(p,r)} \subset V$. Let $U = f(D(p,r))$. By Open Mapping Theorem, U is open. Also, it contains q. Let $u \in U$ and consider the integral

$$\frac{1}{2\pi i} \int_{\partial D(p,r)} \frac{wf'(w)}{f(w) - u} \, dw = J.$$

First, note that J makes sense since $u \notin f(\partial D(p,r))$. Second, the integrand has one singularity in $D(p,r)$. This occurs at whichever point $a \in D(p,r)$ satisfies $f(a) = u$. We know that $f - u$ has a simple zero at a. Thus $\frac{f'(w)}{f(w)-u} = \frac{(f(w)-u)'}{f(w)-u}$ has a simple pole at a with residue equal to 1. (Slightly imprecisely - since a might be 0) we can say that $\frac{wf'(w)}{f(w)-u}$ has a simple pole at a with residue a. Thus by Residue Theorem,

$$J = \frac{1}{2\pi i} \int_{\partial D(p,r)} \frac{wf'(w)}{f(w) - u} \, dw = a \cdot 1 = a.$$

Note $a = g(u)$. That is,

$$g(u) = \frac{1}{2\pi i} \int_{\partial D(p,r)} \frac{wf'(w)}{f(w) - u} \, dw$$

for all $u \in U$. By the lemma about integral being holomorphic, it follows that $g|_U$ is holomorphic. $\square$

Lemma 25.4. *Let $V \subset \mathbb{C}$ be a convex open set. Let $f \in H(V)$. Then if $z_1 \neq z_2$, $z_1, z_2 \in V$, $\frac{f(z_1)-f(z_2)}{z_1-z_2}$ lies in the closed convex hull of $f'(V)$.*

Remark 25.5. *If $A \subset \mathbb{C}$ is a set then there is a minimum closed convex set that contains A. It is called the closed convex hull of A. One way to get it is to consider the intersection of all closed convex sets that contain A.*

Proof of Lemma 25.4. We know that

$$\begin{aligned} f(z_2) - f(z_1) &= \int_{[z_1,z_2]} f'(w)\,dw \quad \text{(note } [z_1, z_2] \subset V \text{ because } V \text{ is convex)} \\ &= \int_0^1 f'((1-t)z_1 + tz_2)(z_2 - z_1)dt \end{aligned}$$

and so

$$\begin{aligned} \frac{f(z_2) - f(z_1)}{z_2 - z_1} &= \int_0^1 f'((1-t)z_1 + tz_2)dt \\ &= \lim_{n\to\infty} \sum_{j=1}^{n} f'((1-\frac{j}{n})z_1 + \frac{j}{n}z_2) \cdot \frac{1}{n} \\ &\stackrel{\text{def}}{=} \lim_{n\to\infty} S_n. \end{aligned}$$

But $\sum_{j=1}^{n} \frac{1}{n} = 1$ and so S_n lies in the closed convex hull of $f'(V)$ for all n. Then $\lim_{n\to\infty} S_n$ also lies in the closed convex hull of $f'(V)$ because it is closed. □

26 Inverses of Holomorphic Functions II (10/22)

Theorem 26.1. *Let $V \subset \mathbb{C}$ be a convex open set, $f \in H(V)$ a non-constant holomorphic function, K a closed convex set such that $0 \notin int(K)$ and $f'(z) \in K$ for all $z \in V$. Then f is one-to-one.*

Proof. We must first consider functions with constant derivative. If the derivative is constant then $f(z) = az + b$ with $a \neq 0$. This is indeed one-to-one.

Now suppose f is not a linear function. Fix $z_1 \in V$ and consider $g_{z_1} : V\backslash\{z_1\} \to \mathbb{C}$ defined by

$$g_{z_1}(z) = \frac{f(z) - f(z_1)}{z - z_1}.$$

Then $g_{z_1} \in H(V\backslash\{z_1\})$ and g_{z_1} is not constant. Thus $g_{z_1}(V\backslash\{z_1\})$ is an open set by Open Mapping Theorem. Thus

$$\Omega = \{\frac{f(z_2) - f(z_1)}{z_2 - z_1} | z_1, z_2 \in V, z_1 \neq z_2\} = \bigcup_{z_1 \in V} g_{z_1}(V\backslash\{z_1\})$$

is an open set. By our previous result using convexity, $\Omega \subset K$ and so $\Omega \subset \text{int}(K)$. Thus $0 \notin \Omega$ and so f is one-to-one. □

Remark 26.2. *A typical example would be $f \in H(D(0,1))$ such that $Re(f'(z)) \geq 0$.*

Theorem 26.3. *Let $V \subset \mathbb{C}$ be open and $f \in H(V)$, $z_0 \in V$, and $f'(z_0) \neq 0$. Then there is an open set $U \subset V$ containing z_0 such that $f|_U$ is one-to-one.*

Proof. Note that f is not constant. Choose $\rho > 0$ such that $0 \notin \overline{D(f'(z_0), \rho)}$. Choose $r > 0$ such that $f'(D(z_0, r)) \subset \overline{D(f'(z_0), \rho)}$. By Theorem 26.1 with $V = D(z_0, r), K = \overline{D(f'(z_0), \rho)}$, we can conclude that $f|_{D(z_0,r)}$ is one-to-one. $\square$

Theorem 26.4. *Let $V \subset \mathbb{C}$ be open, $f \in H(V)$, $z_0 \in V$ and suppose z_0 is an m-fold zero of f. Then there is an open set $U \subset V$ containing z_0 and a $g \in H(U)$ such that $f(z) = (g(z))^m$ for all $z \in U$. Moreover, g has a simple zero at z_0.*

Proof. We can write

$$f(z) = (z - z_0)^m h(z)$$

where $h \in H(V)$ and $h(z_0) \neq 0$. By choosing a small enough disk centered at z_0 for U we may choose

(1) U is simple-connected.

(2) $h|_U$ does not vanish.

(1), (2) imply that h has a logarithm on U. Thus h has an m-th root. We get $k \in H(U)$ such that $k^m = h$. Then $g(z) = (z - z_0)k(z)$ works. $\square$

27 Conformal Mappings I (10/27)

Definition 27.1. A *Riemann sphere* is a set $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ where $\infty \notin \mathbb{C}$.

We want to make $\mathbb{C}_\infty$ into a metric space. We consider the *stereographic projection*

$$S(z) = (\frac{2x}{1+|z|^2}, \frac{2y}{1+|z|^2}, \frac{|z|^2-1}{1+|z|^2}) \in S^2.$$

If we extend $S : \mathbb{C} \to S^2 - \{(0,0,1)\}$ to $S : \mathbb{C}_\infty \to S^2$ by

$$S(\infty) = (0,0,1),$$

then S becomes one-to-one and onto.

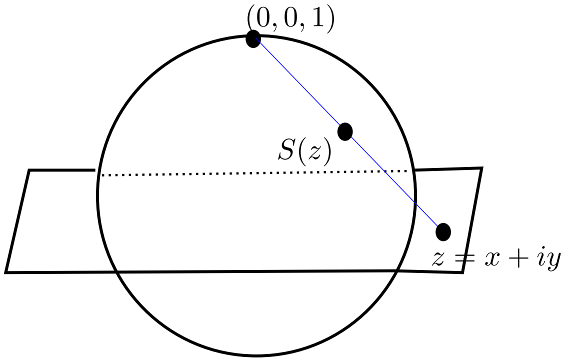

Figure 11: Stereographic Projection

We can make $\mathbb{C}_\infty$ into a metric space by saying $d_\infty(p,q) = \|S(p) - S(q)\|_2$. This is automatically a metric on $\mathbb{C}_\infty$.

Note that $d_\infty|_\mathbb{C}$ is not comparable to the usual distance. However, if $B \subset \mathbb{C}$ is bounded then $d_\infty|_B$ is comparable to the usual metric. This means that convergence of sequence in $\mathbb{C}$ is unchanged provided the sequence is bounded. Explicitly, $(z_n) \subset \mathbb{C}_\infty$ and $z \in \mathbb{C}_\infty$ then $z_n \to z$ in $\mathbb{C}_\infty$ iff (1) $z_n \in \mathbb{C}$ for all sufficiently large n and (2) $z_n \to z$ in the usual sense once ∞ terms are discarded. Also, $z_n \to \infty$ if $|z_n| \to \infty$ with the convention that $|\infty| = \infty$.

The function $k : \mathbb{C}_\infty \to \mathbb{C}_\infty$ given by $k(z) = \frac{1}{z}$ if $z \in \mathbb{C}\backslash\{0\}$, $k(0) = \infty, k(\infty) = 0$ is continuous, bijective, and its own inverse. One way to see this is to compute that the corresponding map of S^2 is $(u_1, u_2, u_3) \mapsto (u_1, -u_2, -u_3)$. This is continuous, so k is.

Definition 27.2. Let $V \subset \mathbb{C}_\infty$ be an open subset of $\mathbb{C}_\infty$. We say $h : V \to \mathbb{C}_\infty$ is *holomorphic* if
(i) h is continuous.
(ii) The function $h(z), h(\frac{1}{z}), \frac{1}{h(z)}, \frac{1}{h(\frac{1}{z})}$ are holomorphic (in the usual sense) on the set where their argument lies in $V \cap \mathbb{C}$ and they are finite.

Example 27.3. Define $\varphi : \mathbb{C}_\infty \to \mathbb{C}_\infty$ by $\varphi(z) = \frac{z+2}{z+1}$ if $z \notin \{-1, \infty\}$, $\varphi(-1) = \infty$, $\varphi(\infty) = 1$. Then $\varphi : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is holomorphic. To check the first condition, it is easy to see that φ is continuous for most points. φ is continuous at -1 because $\lim_{z\to -1} \frac{z+2}{z+1} = \infty$, φ is continuous at ∞ because $\lim_{z\to\infty} \frac{z+2}{z+1} = 1$. To check the second condition, I need
(1) $\varphi(z) = \frac{z+2}{z+1}$ is holomorphic on $\mathbb{C}\backslash\{-1\}$,
(2) $\varphi(\frac{1}{z}) = \frac{1+2z}{1+z}$ is holomorphic on $\mathbb{C}\backslash\{0, -1\}$,
(3) $\frac{1}{\varphi(z)} = \frac{z+1}{z+2}$ is holomorphic on $\mathbb{C}\backslash\{-2\}$,
(4) $\frac{1}{\varphi(\frac{1}{z})} = \frac{1+z}{1+2z}$ is holomorphic on $\mathbb{C}\backslash\{0, -\frac{1}{2}\}$.
This confirms that $\varphi : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is holomorphic.

Suppose that $V \subset \mathbb{C}$ is an open set and $f : V \to \mathbb{C}$ is holomorphic. What does this mean more concretely? Say that V is connected. Then there are two possibilities. One is f is constantly ∞. If this doesn't happen, then the set of points at which f takes the value ∞ is relatively closed and discrete in V. So there exists $S \subset V$ that is closed in V, every point of S is isolated in S, $f|_{V\backslash S} \in H(V\backslash S)$, and f has a pole at each point of S. This class of functions is denoted by $M(V)$ and they are called *meromorphic functions* on V.

28 Conformal Mappings II (10/29)

Lemma 28.1. *Let $V \subset \mathbb{C}_\infty$ be open and connected, $S \subset V$ be closed and nonempty, and C be a component of $V\backslash S$. Then $cl_V(C) \cap S \neq \emptyset$.*

Proof. Suppose $\mathrm{cl}_V(C) \cap S = \emptyset$. Then $\mathrm{cl}_V(C) \subset V\backslash S$ is a connected set and $C \subset \mathrm{cl}_V(C)$ and so C is closed in V. Since disks in $\mathbb{C}_\infty$ are connected, C is also open in $V\backslash S$ and hence in V. Since V is connected, either $C = V$ or $C = \emptyset$. Components are never empty and so $C = V$. But then $S = \emptyset$, contrary to our assumption. ☐

Proposition 28.2. *Let $V \subset \mathbb{C}_\infty$ be open and connected, $h : V \to \mathbb{C}_\infty$ a holomorphic function and $S_p = \{z \in V | h(z) = p\}$. If h is not constant, then every point of S_p is isolated in S_p. Also S_p is closed in V.*

Proof. $S_p = h^{-1}(\{p\})$ is a closed set in V because h is continuous. By replacing h by $h - p$ or $\frac{1}{h}$, we may assume that we are considering S_0. We know that S_∞ is closed in V. Suppose that S_0 has a limit point in S_0. Let C be the component of $V\backslash S_\infty$ that contains this limit point. Both $h(z)$ and $h(\frac{1}{z})$ are holomorphic where they are defined on C. From the Identity Principle, h is constantly zero on C. Thus h is identically zero on $\mathrm{cl}_V(C)$ because h is continuous. If $S_\infty \neq \emptyset$, then Lemma 28.1 implies that $\mathrm{cl}_V(C) \cap S_\infty \neq \emptyset$ and so h takes the value 0 and ∞ simultaneously at some point, which it can't. Now, we know that $C = V$ and $h \equiv 0$ on V. It follows that if h is not constant then every point of S_0 is isolated in S_0. ☐

Proposition 28.3. *If $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is holomorphic, then f is either constant or onto.*

Proof. Let $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a holomorphic function and assume that f omits a value. We may assume that it omits ∞. (If it omits $q \neq \infty$ then $\frac{1}{f(z)-q}$ omits ∞). Thus $f : \mathbb{C}_\infty \to \mathbb{C}$. Now $f(\mathbb{C}_\infty)$ is compact and so bounded. This means $f|_\mathbb{C}$ is entire and bounded. Thus $f|_\mathbb{C}$ is constant by Liouville's Theorem. Since f is continuous, f is constant. □

Theorem 28.4. *If $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is holomorphic, then there are polynomials $P, Q \in \mathbb{C}[z]$ such that $f(z) = \frac{P(z)}{Q(z)}, \forall z \in \mathbb{C}$.*

Proof. If f is constant, then this is clear. Assume f is not constant. Then S_0 and S_∞ are closed subsets of $\mathbb{C}_\infty$ in which every point is isolated. Thus they are both finite sets. At each point of $S_0 \cap \mathbb{C}$, f has a zero of some order. We can choose a polynomial $\tilde{P}$ having zeroes of the same order at each of these points. Similarly, we may find a polynomial Q such that $\frac{1}{Q}$ has poles at each point of $S_\infty \cap \mathbb{C}$ with the same order as the pole of f at this point. The function $\frac{Qf}{\tilde{P}} : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is holomorphic which does not take the zero or ∞ on $\mathbb{C}$. Since two values are missing on $\mathbb{C}$, $\frac{Qf}{\tilde{P}}$ is not onto, thus it is constant. So $f = \frac{c\tilde{P}}{Q} = \frac{P}{Q}$. □

By definition, $\mathrm{Aut}(\mathbb{C}_\infty)$ is the set of all holomorphic functions $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ that are both one-to-one and onto. Note that if $f \in \mathrm{Aut}(\mathbb{C}_\infty)$ then $f^{-1} : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is automatically holomorphic.

Definition 28.5. A continuous function $\varphi : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is said to be a *linear-fractional transformation* if there are $a, b, c, d \in \mathbb{C}$ such that $ad - bc \neq 0$ and

$$\varphi(z) = \frac{az + b}{cz + d}$$

for all $z \in \mathbb{C}$.

Theorem 28.6. *$Aut(\mathbb{C}_\infty)$ consists precisely of the linear-fractional transformations.*

Proof. Suppose $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is an automorphism which is non-constant, so

$$f(z) = \frac{P(z)}{Q(z)}$$

for some polynomials P, Q which may be assumed relatively prime. Thus $S_0 = Z(P)$ and $S_\infty = Z(Q)$ both have one point. Moreover, $p \in S_0$ and $q \in S_\infty$ are simple zeroes and poles. Thus P, Q are linear. Thus $f(z) = \frac{az+b}{cz+d}$ and $ad - bc \neq 0$ because f is not constant. □

29 Conformal Mappings III (11/1)

Proposition 29.1. *Let $\varphi \in Aut(\mathbb{C})$ then $\varphi(z) = az + b$ for some $a \in \mathbb{C}^*, b \in \mathbb{C}$.*

Proof. Let $\varphi \in \mathrm{Aut}(\mathbb{C})$. Then $\varphi^{-1} : \mathbb{C} \to \mathbb{C}$ is holomorphic, and in particular, continuous. If $K \subset \mathbb{C}$ is a compact set, then $\varphi^{-1}(K)$ is also compact. This implies that $\lim_{z\to\infty} \varphi(z) = \infty$. If we define $\varphi(\infty) = \infty$, then we extend φ to an element of $\mathrm{Aut}(\mathbb{C}_\infty)$. Thus there exists A, B, C, D with $AD - BC \neq 0$ and $\varphi(z) = \frac{Az+B}{Cz+D}, \forall z \in \mathbb{C}$. Since $\varphi(\infty) = \infty, C = 0$. Thus $D \neq 0$ and so $\varphi(z) = \frac{A}{D}z + \frac{B}{D}, \forall z \in \mathbb{C}$. Finally, $\frac{A}{D} \neq 0$ because φ is one-to-one. $\square$

Definition 29.2. Let $U, V \subset \mathbb{C}$ be open sets. A function $f : U \to V$ that is holomorphic, one-to-one and onto is said to be *biholomorphic*. A biholomorphic map is also called a *conformal mapping* between U and V.

Definition 29.3. A *conformal map* $f : U \to V$ is defined to be a map that is holomorphic and has non-vanishing derivative.

Example 29.4. $\exp : \mathbb{C} \to \mathbb{C}$ is a conformal map.

Remark 29.5. *Lots of people use "conformal map" as a synonym for "conformal equivalence". If the conformal map is from one open set to another open set, then it actually means conformal equivalence.*

We want to consider

$$\mathcal{M} = \{\varphi : \mathbb{C}_\infty \to \mathbb{C}_\infty | \varphi \text{ is fractional linear}\}.$$

Note that $\mathcal{M}$ is a group which acts on $\mathbb{C}_\infty$. The action is effective (namely, if $\varphi(z) = z, \forall z \in \mathbb{C}_\infty$, then $\varphi = e$.)

The form $\varphi(z) = \frac{az+b}{cz+d}$ with $ad - bc \neq 0$ suggests considering the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. This is an invertible matrix by the condition $ad - bc \neq 0$. The group of all invertible 2×2 complex matrices is usually denoted by $\mathrm{GL}_2(\mathbb{C})$. One can check that the map $\mathrm{GL}_2(\mathbb{C}) \to \mathcal{M}$ given by $A \mapsto \varphi_A$ is a homomorphism. That is,

$$\varphi_{AB} = \varphi_A \cdot \varphi_B.$$

This homomorphism is onto.

The next question is when is $\varphi_A = \mathrm{id}$? If so then $\frac{az+b}{cz+d} = z$ for all z. So $cz^2 + (d-a)z - b = 0$ for all $z \in \mathbb{C}$. Thus $c = 0, b = 0$ and $a = d$. Thus $\varphi_A = \mathrm{id}$ if and only if $A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = aI_2$ for some $a \in \mathbb{C}^*$. It follows that $\mathcal{M} \cong \mathrm{GL}_2(\mathbb{C})/N$ where $N = Z(\mathrm{GL}_2(\mathbb{C})) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \in \mathbb{C}^* \right\}$.

Let $\mathcal{C}$ be the collection of all lines and circles in $\mathbb{C}$ (in $\mathbb{C}_\infty$, lines should include ∞). Note that $X \in \mathcal{C}$ has an equation of the form

$$A(x^2 + y^2) + Bx + Cy + D = 0.$$

This may be rewritten as

$$A|z|^2 + \mathrm{Re}(\zeta z) + D = 0$$

where $\zeta = B - Ci$. I want to verify that if $X \in \mathcal{C}$ and $\varphi \in \mathcal{M}$ then $\varphi(X) \in \mathcal{C}$. To achieve this, I am going to find a generating set for $\mathcal{M}$. If $c = 0$, then

$$\frac{az+b}{cz+d} = \frac{a}{d}z + \frac{b}{d}.$$

If $c \neq 0$, then

$$\begin{aligned}
\frac{az+b}{cz+d} &= \frac{\frac{a}{c}(cz+d) - \frac{ad}{c} + b}{cz+d} \\
&= \frac{a}{c} + \frac{b - \frac{ad}{c}}{cz+d} \\
&= \frac{a}{c} + \frac{-\Delta}{c(cz+d)} \quad (\Delta = ad - bc) \\
&= \frac{a}{c} + \frac{-\Delta/c}{cz+d}.
\end{aligned}$$

These two expressions show that $\mathcal{M}$ is generated by

$$B = \{T_w | w \in \mathbb{C}\} \cup \{M_w | w \in \mathbb{C}^\times\} \cup \{J\}$$

where

$$\begin{aligned}
T_w(z) &= z + w, \\
M_w(z) &= wz, \\
J(z) &= \frac{1}{z}.
\end{aligned}$$

What this means is that if $\varphi \in \mathcal{M}$ then φ may be expressed as a composition of elements of B.

If

$$A|z|^2 + \mathrm{Re}(\zeta z) + D = 0$$

is the equation of an element of $\mathcal{C}$ and $w = J(z) = \frac{1}{z}$ then $z = \frac{1}{w}$ and so the equation of the image under J is

$$\begin{aligned}
&A|\frac{1}{w}|^2 + \mathrm{Re}(\zeta\frac{1}{w}) + D = 0, \\
\Rightarrow &\frac{A}{|w|^2} + \mathrm{Re}(\frac{\zeta\overline{w}}{|w|^2}) + D = 0, \\
\Rightarrow &A + \mathrm{Re}(\zeta\overline{w}) + D|w|^2 = 0, \\
\Rightarrow &A + \mathrm{Re}(\overline{\xi}w) + D|w|^2 = 0,
\end{aligned}$$

which is another element of $\mathcal{C}$. To verify the corresponding thing for T_w and M_w, for T_w use $A(x^2+y^2) + Bx + Cy + D = 0$ form, for M_w use $A|z|^2 + \mathrm{Re}(\zeta z) + D = 0$ form.

30 Conformal Mappings IV (11/3)

Now let's consider the action of $\mathcal{M}$ on $\mathbb{C}_\infty$.

Definition 30.1. If G acts on X then G is *m-transitive* if given any two lists $p_1, \cdots, p_m$ and $q_1, \cdots, q_m$ of distinct points, there is some $g \in G$ such that $gp_j = q_j$ for $1 \le j \le m$. The action is *sharply m-transitive* if the g is unique.

Theorem 30.2. *The action of $\mathcal{M}$ on $\mathbb{C}_\infty$ is sharply 3-transitive.*

Proof. Given distinct points p_1, p_2, p_3, we can show that there is a $\varphi \in \mathcal{M}$ such that $\varphi(p_1) = 0, \varphi(p_2) = 1, \varphi(p_3) = \infty$. This will establish 3-transitivity. Build φ up by composition. Start with φ_1 which is the identity if $p_3 = \infty$ and $\varphi_1(z) = \frac{1}{z-p_3}$ if not. Then $\varphi_1(p_3) = \infty$ and we let $q_1 = \varphi_1(p_1)$ and $q_2 = \varphi_1(p_2)$. Note that q_1 and q_2 are distinct and not ∞. Let φ_2 be $\varphi_2(z) = z - q_1$. Finally, take φ_3 to be $\varphi_3(z) = (q_2 - q_1)^{-1}z$, which is possible since $q_1 \ne q_2$. Take $\varphi = \varphi_3 \circ \varphi_2 \circ \varphi_1$, then it works.

To show that φ is unique, we need only that if $\psi \in \mathcal{M}$ and $\psi(0) = 0, \psi(1) = 1, \psi(\infty) = \infty$ then $\psi = \text{id}$. Let $\psi(z) = \frac{az+b}{cz+d}$ with $ad - bc \ne 0$. Since $\psi(\infty) = \infty$, $c = 0$ and the map may be written as $\psi(z) = Az + B$. $\psi(0) = B = 0$ and then $\psi(1) = A = 1$. So $\psi(z) = z$. □

Recall from geometry that given three distinct points p_1, p_2, p_3, there is a unique element of $\mathcal{C}$ through all three. (If three points are co-linear, there is a unique line; if three points are not co-linear, there is a unique circle.) We conclude that $\mathcal{M}$ acts transitively on $\mathcal{C}$. Take $C_1, C_2 \in \mathcal{C}$, choose three distinct points p_1, p_2, p_3 on C_1 and q_1, q_2, q_3 on C_2. Choose $\varphi \in \mathcal{M}$ such that $\varphi(p_j) = q_j$ $(1 \le j \le 3)$. We must have $\varphi(C_1) = C_2$.

31 Conformal Mappings V (11/5)

$\pi^+ = \{x + iy | y > 0\}$ is conformally equivalent to $\mathbb{D}$ by a fractional linear transformation. First, we construct $\psi : \pi^+ \to \mathbb{D}$.

$$\begin{aligned} z \in \pi^+ &\Leftrightarrow |z - i| < |z + i| \\ &\Leftrightarrow \left|\frac{z-i}{z+i}\right| < 1 \\ &\Leftrightarrow \frac{z-i}{z+i} \in \mathbb{D}. \end{aligned}$$

So $\psi(z) = \frac{z-i}{z+i}$ is a conformal equivalence from π^+ to $\mathbb{D}$. The matrix of ψ is $\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}$ whose inverse is $\frac{1}{2i}\begin{bmatrix} i & i \\ -1 & 1 \end{bmatrix} \sim \begin{bmatrix} i & i \\ -1 & 1 \end{bmatrix}$. So

$$\psi^{-1}(z) = \frac{iz + i}{-z + 1} = i \cdot \frac{1+z}{1-z} = \psi_c(z).$$

So $\psi_c(z) : \mathbb{D} \to \pi^+$ is a conformal equivalence, called the *Cayley transform*.

Next, we want to study the disk $\mathbb{D}$ in great detail. The first question is what is $\mathrm{Aut}(\mathbb{D})$?

Theorem 31.1 (Schwarz Lemma)**.** *Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic with $f(0) = 0$. Then $|f(z)| \le |z|$ for all $z \in \mathbb{D}$ and $|f'(0)| \le 1$. Moreover, if $|f(w)| = |w|$ for some $w \in \mathbb{D} - \{0\}$ or $|f'(0)| = 1$ then $f(z) = cz$ for some $c \in \mathbb{C}$ with $|c| = 1$.*

Proof. Let $g : \mathbb{D} \to \mathbb{C}$ be

$$g(z) = \begin{cases} \frac{f(z)}{z}, & \text{if } z \neq 0 \\ f'(0), & \text{if } z = 0. \end{cases}$$

By Riemann's Removable Singularity Theorem, g is holomorphic. On $|z| = r$, $|g(z)| \le \frac{1}{r}$ for $0 < r < 1$. This means that $|g(z)| \le \frac{1}{r}$ for all z with $|z| \le r$ by Maximum Modulus Theorem. It follows that $|g(z)| \le 1$ for all $z \in \mathbb{D}$. This gives the first two conclusions.

For the "moreover", g would achieve a maximum modulus in $\mathbb{D}$ and hence be constant. □

We want to use Schwarz Lemma to determine $\mathrm{Aut}(\mathbb{D})$.

Proposition 31.2. *If $\varphi \in Aut(\mathbb{D})$ such that $\varphi(0) = 0$ then $\varphi(z) = cz$ for some $c \in \mathbb{C}$ with $|c| = 1$.*

Proof. By the Schwarz Lemma, $|\varphi'(0)| \le 1$. Let $\psi : \mathbb{D} \to \mathbb{D}$ be φ^{-1}. By the Schwarz Lemma, $|\varphi^{-1}(0)| \le 1$. By the Chain Rule,

$$|\varphi'(0)||\psi^{-1}(0)| \le 1.$$

Thus $|\varphi'(0)| = |\psi'(0)| = 1$ and the conclusion follows from the Schwarz Lemma. □

The next step is to show that $\mathrm{Aut}(\mathbb{D})$ acts transitively on $\mathbb{D}$.

For $\alpha \in \mathbb{D}$ let φ_α be the map

$$\varphi_\alpha(z) = \frac{\alpha - z}{1 - \overline{\alpha} z}.$$

Note that φ_α is holomorphic on $\mathbb{C} \backslash \{\frac{1}{\overline{\alpha}}\}$. In particular, φ_α is holomorphic on the disk $\mathbb{D}$ because $|\frac{1}{\overline{\alpha}}| = \frac{1}{|\alpha|} > 1$.

Next,

$$\begin{aligned} 1 - |\varphi_\alpha(z)|^2 &= 1 - \frac{\alpha - z}{1 - \overline{\alpha} z} \cdot \frac{\overline{\alpha} - \overline{z}}{1 - \alpha \overline{z}} \\ &= \frac{(1 - \overline{\alpha} z)(1 - \alpha \overline{z}) - (\alpha - z)(\overline{\alpha} - \overline{z})}{|1 - \overline{\alpha} z|^2} \\ &= \frac{1 - |\alpha|^2 - |z|^2 + |\alpha|^2 \cdot |z|^2}{|1 - \overline{\alpha} z|^2} \\ &= \frac{(1 - |\alpha|^2)(1 - |z|^2)}{|1 - \overline{\alpha} z|^2}. \end{aligned}$$

Notice that if $z \in \mathbb{D}$ then it follows that $1 - |\varphi_\alpha(z)|^2 > 0$ and so $\varphi_\alpha(z) \in \mathbb{D}$. Thus $\varphi_\alpha : \mathbb{D} \to \mathbb{D}$. Note that $\varphi_\alpha(0) = \alpha$ and $\varphi_\alpha(\alpha) = 0$.

Next, we calculate $\varphi_\alpha \circ \varphi_\alpha$. Of course, you could do this directly, but there are other ways. Here is the one.

$$\varphi_\alpha(z) = \frac{\alpha - z}{1 - \overline{\alpha}z}.$$

So

$$\begin{aligned}
\varphi'_\alpha(z) &= \frac{-(1 - \overline{\alpha}z) - (\alpha - z)(-\overline{\alpha})}{(1 - \overline{\alpha}z)^2} \\
&= \frac{|\alpha|^2 - 1}{(1 - \overline{\alpha}z)^2}.
\end{aligned}$$

In particular, $\varphi'_\alpha(0) = |\alpha|^2 - 1$ and $\varphi'_\alpha(\alpha) = \frac{|\alpha|^2-1}{(1-|\alpha|^2)^2} = \frac{1}{|\alpha|^2-1}$. Thus

$$\begin{aligned}
(\varphi_\alpha \circ \varphi_\alpha)'(0) &= \varphi'_\alpha(\varphi_\alpha(0)) \cdot \varphi'_\alpha(0) \\
&= \varphi'_\alpha(\alpha) \cdot \varphi'_\alpha(0) \\
&= \frac{1}{|\alpha|^2 - 1} \cdot (|\alpha|^2 - 1) \\
&= 1.
\end{aligned}$$

Also,

$$\begin{aligned}
(\varphi_\alpha \circ \varphi_\alpha)(0) &= \varphi_\alpha(\varphi_\alpha(0)) \\
&= \varphi_\alpha(\alpha) \\
&= 0.
\end{aligned}$$

By the Schwarz Lemma, $(\varphi_\alpha \circ \varphi_\alpha)(z) = cz$ for some $c \in \mathbb{C}$ with $|c| = 1$, and actually $c = 1$ because $(\varphi_\alpha \circ \varphi_\alpha)'(0) = 1$. Thus $\varphi_\alpha^{-1} = \varphi_\alpha$. In particular, $\varphi_\alpha \in \mathrm{Aut}(\mathbb{D})$. This shows that $\mathrm{Aut}(\mathbb{D})$ acts transitively: If $\alpha, \beta \in \mathbb{D}$ then $(\varphi_\alpha \circ \varphi_\beta)(\beta) = \alpha$.

Theorem 31.3. *If $\psi \in Aut(\mathbb{D})$ then ψ can be expressed uniquely in either of the forms*

$$\psi = M_c \circ \varphi_\alpha, \tag{31.1}$$

$$\psi = \varphi_\beta \circ M_a, \tag{31.2}$$

where $\alpha, \beta \in \mathbb{D}, a, c \in \mathbb{C}$ with $|a| = |c| = 1$, and M_a, M_c are the maps given by multiplication by a and c respectively.

Proof. Let $\psi \in \mathrm{Aut}(\mathbb{D})$. Let $\beta = \psi(0)$. Consider $\varphi_\beta \circ \psi \in \mathrm{Aut}(\mathbb{D})$. We have $(\varphi_\beta \circ \psi)(0) = \varphi_\beta(\beta) = 0$ and so $\varphi_\beta \circ \psi = M_a$ for some $a \in \mathbb{C}$ with $|a| = 1$. Thus $\psi = \varphi_\beta \circ M_a$. This gives 31.2. If $\psi = \varphi_\beta \circ M_a$ then $\psi(0) = \varphi_\beta(M_a(0)) = \varphi_\beta(0) = \beta$ and so β is uniquely determined as $\psi(0)$. Then $M_a = \varphi_\beta \circ \psi$ is also uniquely determined. Thus $a = (\varphi_\beta \circ \psi)'(0)$ is uniquely determined. To get 31.1, apply 31.2 to ψ^{-1}. $\square$

This tells us that as a set

$$\mathrm{Aut}(\mathbb{D}) \leftrightarrow \mathbb{D} \times \{w \in \mathbb{C} | |w| = 1\}.$$

Unfortunately, it is "hard" to compute the group operation in this representation:

$$(M_{c_1} \circ \varphi_{\alpha_1}) \circ (M_{c_2} \circ \varphi_{\alpha_2}) = \text{horrible expression}.$$

32 Conformal Mappings VI (11/10)

Given $w, z \in \mathbb{D}$ define $d(w,z) = |\varphi_w(z)|$. (Recall $\varphi_\alpha(u) = \frac{\alpha - u}{1 - \overline{\alpha} u}$.) Actually, this is a metric on $\mathbb{D}$.

$$d(w,z) = \left| \frac{w-z}{1-\overline{w}z} \right|.$$

This shows that $d(w,z) \geq 0$ and $d(w,z) = 0$ only when $z = w$.

$$d(z,w) = \left| \frac{z-w}{1-\overline{z}w} \right| = \frac{|z-w|}{|1-\overline{z}w|} = \frac{|w-z|}{|1-\overline{w}z|} = d(w,z).$$

The only thing that remains to check is the triangle inequality, which is postponed. d is called the *pseudohyperbolic metric*.

Let $U \subset \mathbb{C}$ be an open set, $\gamma : [0,1] \to U$ be a piecewise smooth curve. One can define

$$\mathcal{L}(\gamma) = \int_0^1 |\gamma'(t)| \cdot \rho(\gamma(t))\, dt$$

where $\rho : U \to [0, \infty)$ (is a density function).

Theorem 32.1 (Pick's Theorem, Schwarz-Pick Theorem, Invariant Schwarz Lemma)**.** *Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic function. Then*
(i) $d(f(w), f(z)) \leq d(w,z), \forall w, z \in \mathbb{D}$;
(ii) $\frac{|f'(z)|}{1-|f(z)|^2} \leq \frac{1}{1-|z|^2}, \forall z \in \mathbb{D}$.
If (i) is an equality for any $z \neq w$ or (ii) is an equality for any $z \in \mathbb{D}$, then $f \in Aut(\mathbb{D})$. Conversely, (i) and (ii) are equalities if $f \in Aut(\mathbb{D})$.

Proof. Let $z \in \mathbb{D}$. Define $\alpha = f(z)$ and $g = \varphi_\alpha \circ f \circ \varphi_z$. Note that $g : \mathbb{D} \to \mathbb{D}$ is holomorphic, $g(0) = 0$, and $g \in \mathrm{Aut}(\mathbb{D})$ if and only if $f \in \mathrm{Aut}(\mathbb{D})$. From Schwarz Lemma we know that $|g(w)| \leq |w|$ for all $w \in \mathbb{D}$. Then

$$|\varphi_\alpha(f(\varphi_z(w)))| \leq |w|$$

for all $w \in \mathbb{D}$ and so

$$|\varphi_\alpha(f(\varphi_z(\varphi_z(u))))| \leq |\varphi_z(u)|$$

for all $u \in \mathbb{D}$ which implies that $\varphi_\alpha(f(u))| \le |\varphi_z(u)|$ for all $u \in \mathbb{D}$ ($\varphi_z^{-1} = \varphi_z$.) Recall that $\alpha = f(z)$ and so this inequality is

$$|\varphi_{f(z)}(f(u))| \le |\varphi_z(u)|$$

for all $u \in \mathbb{D}$. That is,

$$d(f(z), f(w)) \le d(z, u).$$

This establishes (i).

If $f \in \mathrm{Aut}(\mathbb{D})$ then so is g and so we get $|g(w)| = |w|$. If you start from this point, you get $d(f(z), f(w)) = d(z, w)$ if $f \in \mathrm{Aut}(\mathbb{D})$. If $d(f(z), f(w)) = d(z, w)$ for $z \ne w$ then $|g(w)| = |w|$ for $w \ne 0$. So $g \in \mathrm{Aut}(\mathbb{D})$ and then $f \in \mathrm{Aut}(\mathbb{D})$ too.

To get (ii), we use $|g'(0)| \le 1$ with equality if and only if $g \in \mathrm{Aut}(\mathbb{D})$. The chain rule tells us that

$$\begin{aligned} g'(0) &= \varphi'_\alpha(\alpha) \cdot f'(z) \cdot \varphi'_z(0) \\ &= \frac{1}{1 - |\alpha|^2} \cdot f'(z) \cdot (1 - |z|^2) \\ &= \frac{1}{1 - |f(z)|^2} \cdot f'(z) \cdot (1 - |z|^2) \end{aligned}$$

and so

$$\frac{|f'(z)|}{1 - |f(z)|^2} \cdot (1 - |z|^2) = |g'(0)| \le 1.$$

This gives

$$\frac{|f'(z)|}{1 - |f(z)|^2} \le \frac{1}{1 - |z|^2}.$$

The case of equality follows as before. □

Remark 32.2. *Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic but not an automorphism. Suppose that $f(p) = p, f(q) = q$ for $p \ne q$ in $\mathbb{D}$. Then*

$$d(f(p), f(q)) < d(p, q)$$

by Schwarz-Pick Lemma, which is equivalent to $d(p, q) < d(p, q)$, a contradiction. Thus such f have at most one fixed point in $\mathbb{D}$.

We will have to verify that d satisfies the triangle inequality. We want

$$d(u, z) \le d(u, w) + d(w, z), \forall u, w, z \in \mathbb{D}.$$

This is equivalent to

$$d(\varphi_w(u), \varphi_w(z)) \le d(\varphi_w(u), \varphi_w(w)) + d(\varphi_w(w), \varphi_w(z))$$

or

$$d(\varphi_w(u),\varphi_w(z)) \le d(\varphi_w(u),0) + d(0,\varphi_w(z)).$$

It is actually enough to show that

$$d(z,w) \le d(z,0) + d(0,w), \forall z,w \in \mathbb{D}.$$

Now $d(z,0) = |z|, d(0,w) = |w|$, so we need to show that

$$d(z,w) \le d(|z|,-|w|).$$

From here, the triangle inequality will follow.

On $\mathbb{D}$ we had defined $d_p(w,z) = |\varphi_w(z)|$. We had reduced the triangle inequality for d_p to the inequality $d_p(w,z) \le |w| + |z|$. Previously,

$$1 - |\varphi_\alpha(z)|^2 = \frac{(1-|\alpha|^2)(1-|z|^2)}{|1-\overline{\alpha}z|^2}.$$

Thus

$$\begin{aligned} 1 - |\varphi_w(z)|^2 &= \frac{(1-|w|^2)(1-|z|^2)}{|1-\overline{w}z|^2} \\ &\ge \frac{(1-|w|^2)(1-|z|^2)}{(1+|w||z|)^2} \\ &= \frac{(1-|-|w||^2)(1-||z||^2)}{(1-(-|w|)(|z|))^2} \\ &= 1 - \left|\varphi_{-|w|}(|z|)\right|^2 \end{aligned}$$

and so $|\varphi_w(z)| \le |\varphi_{-|w|}(|z|)|$. Thus $d_p(w,z) \le d_p(-|w|,|z|)$ for all $w,z \in \mathbb{D}$. This says

$$d_p(w,z) \le \left|\frac{-|w|-|z|}{1-(-|w|)\cdot|z|}\right| = \frac{|w|+|z|}{1+|w||z|} \le |w| + |z|.$$

This verifies that $(\mathbb{D}, d_p)$ is a metric space.

$$d_H(z,w) = \frac{1}{2}\ln\left(\frac{1+|\varphi_z(w)|}{1-|\varphi_z(w)|}\right)$$

is the hyperbolic metric.

Next, I will show that $(\mathbb{D}, d_p)$ is a complete metric space.

First, let $d_E(z,w) = |z-w|$. Note that $d_p(z,w) = \frac{|z-w|}{|1-\overline{z}w|} > \frac{1}{2}d_E(z,w)$. Second, say that $(z_n) \subset \mathbb{D}$ is a d_p-Cauchy sequence. Then $(z_n) \subset \mathbb{D}$ is also d_E-Cauchy. Thus $z_n \xrightarrow{d_E} p$ for some point $p \in \overline{\mathbb{D}}$. Third, I want to show that $p \notin \partial\mathbb{D}$. Suppose $p \in \partial\mathbb{D}$, then $p = e^{i\theta}$ for some θ. If I replace (z_n) by $(e^{-i\theta}z_n)$ then this is still d_p-Cauchy and now $e^{-i\theta}z_n \to 1$. I will simply assume that $p = 1$. Since (z_n) is d_p-Cauchy, I can find N such that if $n > m \geq N$ then $d_p(z_n, z_m) < \frac{1}{2}$. That is,

$$\frac{|z_n - z_m|}{|1 - \overline{z}_n z_m|} < \frac{1}{2}$$

for all $n > m \geq N$. Let $n \to \infty$. We know that $z_n \overset{d_E}{\to} 1$ and so

$$\lim_{n\to\infty} \frac{|z_n - z_m|}{|1 - \overline{z}_n z_m|} = \frac{|1 - z_m|}{|1 - z_m|} = 1.$$

But

$$\lim_{n\to\infty} \frac{|z_n - z_m|}{|1 - \overline{z}_n z_m|} < \frac{1}{2}$$

and this is a contradiction. Thus $p \notin \partial\mathbb{D}$. That is, $p \in \mathbb{D}$. Finally, we want to show that $z_n \overset{d_p}{\to} p$.

$$\lim_{n\to\infty} d_p(z_n, p) = \lim_{n\to\infty} \frac{|z_n - p|}{|1 - \overline{z}_n p|} = 0$$

because $\lim_{n\to\infty} |z_n - p| = 0$ and $\lim_{n\to\infty} |1 - \overline{z}_n p| = |1 - \overline{p}p| = 1 - |p|^2 > 0$. This verifies that $z_n \overset{d_p}{\to} p$. So $(\mathbb{D}, d_p)$ is complete.

33 Normal Families and the Riemann Mapping Theorem I (11/12)

Say (X, d_X) is a compact metric space. Say (Y, d_Y) is a complete metric space (usually $\mathbb{R}$ or $\mathbb{C}$ in the consideration of this course). Then you can make the space

$$C(X,Y) = \{f : X \to Y | f \text{ is continuous}\}$$

into a metric space. The metric is

$$\rho(f,g) = \max_{x \in X} d_Y(f(x), g(x)).$$

Convergence in this metric is the same as uniform convergence on X. That is, $(f_n) \subset C(X,Y)$ converges to $f \in C(X,Y)$ in the ρ-metric if and only if $f_n \to f$ uniformly on X.

Remark 33.1. *$(C(X,Y), \rho)$ is complete.*

Can we characterize compact subsets of $(C(X,Y), \rho)$?

Definition 33.2. A subset of a metric space is *precompact* if its closure is compact.

Lemma 33.3. *A set is precompact if and only if every sequence in the set has a convergent subsequence. (The convergent subsequence does not have to converge to a point in this set.)*

Can we characterize precompact subsets of $(C(X,Y), \rho)$? Yes – that is what the Arzela-Ascoli Theorem does.

Remark 33.4. *In compact metric space, point-wise continuity is the same as uniform continuality. But we may consider point-wise continuity so that it can be applied to non-compact metric space.*

34 Normal Families and the Riemann Mapping Theorem II (11/14)

Definition 34.1. A set $\mathcal{F} \subset C(X,Y)$ is *pointwise precompact* if $\{f(p) | f \in \mathcal{F}\}$ is a precompact subset of Y for all $p \in X$.

Definition 34.2. A set $\mathcal{F} \subset C(X,Y)$ is *equicontinuous* if for all $p \in X$ and all $\epsilon > 0$ there is a $\delta > 0$ such that if $x \in X$ and $d_X(x,p) < \delta$ then $d_Y(f(x), f(p)) < \epsilon$ for all $f \in \mathcal{F}$.

Theorem 34.3 (Arzela-Ascoli Theorem)**.** *Let X be a compact metric space and Y a complete metric space. A set $\mathcal{F} \subset C(X,Y)$ is precompact if and only if it is pointwise precompact and equicontinuous.*

Here is the sketch to show that $\mathcal{F}$ is totally bounded.

Assume that $\mathcal{F}$ is pointwise precompact and equicontinuous. I want to show that $\mathcal{F}$ is totally bounded. Let $\epsilon > 0$. For each $p \in X$ choose $\delta_p > 0$ such that δ_p works for $\frac{\epsilon}{3}$ in the definition of equicontinuous. $\{B_X(p, \delta_p) | p \in X\}$ is an open cover and so we choose a finite subcover $\{B_X(p, \delta_p) | p \in A\}$ where $A \subset X$ is finite. Consider the product $y = \prod_{p \in A} y_p$, where $y_p = \text{cl}\{f(p) | f \in \mathcal{F}\} \subset Y$. By assumption, y_p is compact for all $p \in A$. So $\prod_{p \in A} y_p$ with the d_∞-metric is also compact. So $\prod_{y \in A} y_p$ is totally bounded. Choose a finite set $G_1 \subset \prod_{p \in A} y_p$ such that $B_y(\gamma, \frac{\epsilon}{6}), \gamma \in G_1$ cover y. There is a map $\mathcal{F} \to \prod_{p \in A} y_p$ given by sending a function to the tuple of its values at each $p \in A$. Let $G \subset G_1$ be the set of γ such that $B_y(\gamma, \frac{\epsilon}{6})$ intersects the image of this map. Choose f_γ that maps into $B_y(\gamma, \frac{\epsilon}{6})$ for each $\gamma \in G$. We claim that $\{B_{C(X,Y)}(F_\gamma, \epsilon) | \gamma \in G\}$ covers $\mathcal{F}$. If $f \in \mathcal{F}$, then f maps into $\prod_{p \in A} y_p$ and its image lands inside some $B_y(\gamma, \frac{\epsilon}{6})$ for $\gamma \in G$. Thus $d_Y(f(p), f_\gamma(p)) < \frac{\epsilon}{3}$ for all $p \in A$. Also if $x \in X$ then $x \in B_X(p, \delta_p)$ for some $p \in A$ and so $d_Y(f(x), f(p)) < \frac{\epsilon}{3}$ and $d_Y(f_\gamma(x), f_\gamma(p)) < \frac{\epsilon}{3}$. Thus $d_Y(f(x), f_\gamma(x)) < \epsilon$ for all $x \in X$.

35 Normal Families and the Riemann Mapping Theorem III (11/17)

We have proved that if X is compact, Y is complete, then $\mathcal{F} \subset C(X, Y)$ is precompact if and only if it is pointwise precompact and equicontinuous.

What if X is not necessarily compact?

One reasonable type of convergence in $C(X, Y)$ where X need not be compact is uniform convergence on compacta (u.c.c.).

Definition 35.1. A sequence $(f_n) \subset C(X, Y)$ *converges uniformly on compacta* to $f \in C(X, Y)$ if $(f_n|_K)$ converges uniformly to $f|_K$ for all compact $K \subset X$.

Definition 35.2. X is said to have a *compact exhaustion* if there is a sequence (K_n) of compact subsets of X such that
(i) $K_n \subset \text{int}(K_{n+1})$ for all $n \geq 1$;
(ii) $X = \cup_{n=1}^\infty K_n$.

Note that if X has a compact exhaustion, then $X = \cup_{n=1}^\infty \text{int}(K_n)$. Further, if $K \subset X$ is any compact subset then $K \subset \text{int}(K_n)$ for some n. This follows because $\{\text{int}(K_n) | n \geq 1\}$ is an open cover of X.

Any open set $V \subset \mathbb{R}^n$ has a compact exhaustion. One way to see this is to define

$$K_n = \{x \in V | \text{dist}(x, \partial V) \geq \frac{1}{n} \text{ and } \|x\| \leq n\}.$$

With this definition, K_n is automatically
- a subset of V,
- closed

• bounded.
So K_n is compact (closed, bounded $\Rightarrow$ compact). Say that $x \in K_n$. Then $\|x\| \leq n$ and $\text{dist}(x, \partial V) \geq \frac{1}{n}$. We need to find $\epsilon > 0$ such that $B(x, \epsilon) \subset K_{n+1}$. Since V is open, we can choose $\epsilon > 0$ small enough that $B(x, \epsilon) \subset V$. If $y \in B(x, \epsilon)$, then

$$\begin{aligned} \|y\| &= \|x + (y - x)\| \\ &\leq \|x\| + \|y - x\| \\ &\leq n + \epsilon \end{aligned}$$

and so $\epsilon < 1$ will ensure that $\|y\| \leq n + 1$. Then

$$\begin{aligned} \text{disc}(y, \partial V) &\geq \text{dist}(x, \partial V) - \|x - y\| \\ &\geq \frac{1}{n} - \epsilon \end{aligned}$$

and

$$\text{disc}(y, \partial V) \geq \frac{1}{n+1}$$

provided that $\epsilon < \frac{1}{n(n+1)}$. Combining these, we find $\epsilon > 0$ such that $B(x, \epsilon) \subset K_{n+1}$. Thus $x \in \text{int}(K_{n+1})$. (Another way to see it is to use the inverse limit, and define partial order $K_n < K_{n+1}$ if $K_n \subset \text{int}(K_{n+1})$.)

Let X be a metric space with a compact exhaustion (K_n). Suppose that you have a family of functions (g_n) where $g_n \in C(K_n, Y)$ for each n. Suppose that $g_{n+1}|_{K_n} = g_n$ for all $n \geq 1$. Then there is a function $g \in C(X, Y)$ such that $g|_{K_n} = g_n$.

We can certainly define $g : X \to Y$ by $g(x) = g_n(x)$ for any n such that $x \in K_n$. Why is this continuous? Let $\epsilon > 0$. Suppose that $x \in K_n$. Then $x \in \text{int}(K_{n+1})$ and so there is $\delta_1 > 0$ such that $B(x, \delta_1) \subset K_{n+1}$. Since g_{n+1} is continuous there is $\delta_2 > 0$ such that if $d_X(x, y) < \delta_2$ then $d_Y(g_{n+1}(x), g_{n+1}(y)) < \epsilon$. Let $\delta = \min\{\delta_1, \delta_2\}$. If $d_X(x, y) < \delta$ then $y \in K_{n+1}$ and

$$d_Y(g(x), g(y)) = d_Y(g_{n+1}(x), g_{n+1}(y)) < \epsilon.$$

(The condition $K_n \subset \text{int}(K_{n+1})$ is very critical here.)

36 Normal Families and the Riemann Mapping Theorem IV (11/19)

Next, we want to consider all the space $C(K_n, Y)$ at once. One natural way is to consider their product:

$$\prod_{n=1}^{\infty} C(K_n, Y).$$

Theorem 36.1. *Suppose that (Z_m, d_m) is a sequence of metric space. Let*

$$Z = \prod_{m=1}^{\infty} Z_m.$$

Define $d : Z \times Z \to \mathbb{R}$ by

$$d(z,w) = \sum_{m=1}^{\infty} 2^{-m} \cdot \frac{d_m(z(m), w(m))}{1 + d_m(z(m), w(m))}$$

Then (Z, d) is a metric space. Moreover, a sequence (z_n) in Z converges to $z \in Z$ if and only if $z_n(m) \to z(m)$ for all m. Finally, if (z_m, d_m) is compact for all $m \geq 1$, then (Z, d) is compact.

Proof. We give the main ingredients for the proof.

To verify the triangle inequality, observe that $t \mapsto \frac{t}{1+t}$ is increasing on $[0, \infty)$. If $z, w, u \in Z$, then

$$\begin{aligned}
\frac{d_m(z(m), u(m))}{1 + d_m(z(m), u(m))} \leq & \frac{d_m(z(m), w(m)) + d_m(w(m), u(m))}{1 + d_m(z(m), w(m)) + d_m(w(m), u(m))} \\
= & \frac{d_m(z(m), w(m))}{1 + d_m(z(m), w(m)) + d_m(w(m), u(m))} \\
& + \frac{d_m(w(m), u(m))}{1 + d_m(z(m), w(m)) + d_m(w(m), u(m))} \\
\leq & \frac{d_m(z(m), w(m))}{1 + d_m(z(m), w(m))} + \frac{d_m(w(m), u(m))}{1 + d_m(w(m), u(m))}.
\end{aligned}$$

A sequence (b_n) in (Z_m, d_m) converges to $b \in Z_m$ with respect to the metric d_m if and only if it converges to $b \in Z_m$ with respect to the metric $\frac{d_m}{1+d_m}$. The reason is that $\frac{d_m}{1+d_m} \leq dm$ and $d_m \leq 2 \cdot \frac{d_m}{1+d_m}$ when $d_m \leq 1$. This suffices to show that the convergence is the same.

The next claim is that a sequence (z_n) in Z converges to $z \in Z$ if and only if $z_n(m) \to z(m)$ for all $m \geq 1$.

$$\begin{aligned}
z_1 &= (z_1(1), z_1(2), z_1(3), z_1(4), \cdots) \\
z_2 &= (z_2(1), z_2(2), z_2(3), z_2(4), \cdots) \\
&\vdots \\
z_n &= (z_n(1), z_n(2), z_n(3), z_n(4), \cdots)
\end{aligned}$$

Say $z_n(m) \to z(m)$ for all m. Let $\epsilon > 0$. Choose N such that

$$\sum_{m=N+1}^{\infty} 2^{-m} < \frac{\epsilon}{2}.$$

Then

$$\begin{aligned} d(z_n, z) &= \sum_{m=1}^{\infty} 2^{-m} \frac{d_m(z_n(m), z(m))}{1 + d_m(z_n(m), z(m))} \\ &< \sum_{m=1}^{N} 2^{-m} \frac{d_m(z_n(m), z(m))}{1 + d_m(z_n(m), z(m))} + \frac{\epsilon}{2}. \end{aligned}$$

I can choose L such that $n \geq L$. Then

$$\frac{d_m(z_n(m), z(m))}{1 + d_m(z_n(m), z(m))} < \frac{\epsilon}{2}$$

for all $1 \leq m \leq N$. If $n \geq L$, $d(z_n, z) < \epsilon$. Thus $z_n \to z$ in Z.

The inverse is also true. $\square$

If all (Z_m, d_m) are compact, then (Z, d) is compact. Take a sequence (z_n) in (Z, d) with (Z_m, d_m) are compact. Find infinite set $J_1 \subset N$ such that $(z_n(1))_{n \in J_1}$ converges to $w(1)$. From $(z_n(2))_{n \in J_2}$, choose infinite $J_2 \subset J_1$ such that $(z_n(2))_{n \in J_2}$ converges to $w(2)$. Continue the process. Now make a subsequence $(z_n)_{n \in J}$. J contains the smallest element of J_1, the smallest element of J_2 which is larger than that, the smallest element of J_3 which is larger than both, and so on.

Theorem 36.2 (Arzela-Ascoli Theorem, version 2)**.** *Let X be a metric space that has a compact exhaustion. Let Y be a complete metric space. There is a metric on $C(X, Y)$ such that convergence in this metric is the same as uniform convergence on compact subsets. With this metric, $\mathcal{F} \subset C(X, Y)$ is precompact if and only if it is pointwise precompact and equicontinuous.*

Proof. Choose a compact exhaustion (K_m). Let ρ_m be the metric on $C(K_m, Y)$ that corresponds to uniform convergence. Consider

$$Z = \prod_{m=1}^{\infty} C(K_m, Y)$$

with the metric

$$d(f, g) = \sum_{m=1}^{\infty} 2^{-m} \frac{\rho_m(f, g)}{1 + \rho_m(f, g)}.$$

Note that there is a one-to-one function $\iota : C(X, Y) \to \prod_{m=1}^{\infty} C(K_m, Y)$ given by

$$\iota(f) = (f|_{K_m})_{m \geq 1}.$$

We define the metric on $C(X, Y)$ to be $v(f, g) = d(\iota(f), \iota(g))$ (the pull-back metric). This metric on $C(X, Y)$ has the required property.

Say $\mathcal{F} \subset C(X, Y)$ is pointwise precompact and equicontinuous. Let $\mathcal{F}_m$ be the closure of the image of $\mathcal{F}$ under ι at the m-th coordinate. That is, $\mathcal{F}_m \subset C(K_m, Y)$ is the closure of the set $\{f|_{K_m} : f \in \mathcal{F}\}$. $\{f|_{K_m} : f \in \mathcal{F}\}$ is pointwise precompact and equicontinuous and so $\mathcal{F}_m$ is compact by the previous Arzela-Ascoli Theorem. Now $\iota(\mathcal{F}) \subset \prod_{m=1}^{\infty} \mathcal{F}_m$ which shows that $\iota(\mathcal{F})$ is precompact. Last thing is to be careful about the image of ι. $\square$

37 Normal Families and the Riemann Mapping Theorem V (11/21)

Definition 37.1. Let $V \subset \mathbb{C}$ be open and $\mathcal{F} \subset H(V)$. Then $\mathcal{F}$ is a *normal family* if for every compact subset $K \subset V$ there is a constant M such that $|f(z)| \leq M$ for all $z \in K$ and $f \in \mathcal{F}$.

Remark 37.2. *To check that a family is normal, it suffices to show that if $\overline{D}(w, r) \subset V$ then there is some M such that $|f(z)| \leq M$ for all $z \in \overline{D}(w, r)$ and $f \in \mathcal{F}$.*

Theorem 37.3 (Montel's Theorem)**.** *Let $V \subset \mathbb{C}$ be an open set and $\mathcal{F} \subset H(V)$ be a normal family. Suppose that (f_n) is a sequence in $\mathcal{F}$. Then (f_n) has a subsequence that converges uniformly on compact subsets of V.*

Proof. We have to verify that $\mathcal{F}$ is precompact in the topology of uniform convergence on compact subsets. To do this, we want to apply the Arzela-Ascoli Theorem (Version 2). Well, $H(V) \subset C(V, \mathbb{C})$, V has a compact exhaustion, and $\mathbb{C}$ is a complete metric space. It remains to verify that $\mathcal{F}$ is pointwise precompact and equicontinuous. The precompact subsets of $\mathbb{C}$ are the bounded subsets. If $z \in V$ then $\{f(z) | f \in \mathcal{F}\}$ is bounded because $\{z\}$ is compact. So $\mathcal{F}$ is pointwise precompact. Let $z \in V$. I wish to show that $\mathcal{F}$ is equicontinuous at z. Choose $r > 0$ such that $\overline{D}(z, 2r) \subset V$. Suppose that $w \in D(z, r)$. Then

$$\begin{aligned}
|f(z) - f(w)| &= \left| \frac{1}{2\pi i} \int_{\partial D(z,2r)} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{2\pi i} \int_{\partial D(z,2r)} \frac{f(\zeta)}{\zeta - w}\, d\zeta \right| \\
&= \left| \frac{1}{2\pi i} \int_{\partial D(z,2r)} f(\zeta) \frac{z - w}{(\zeta - z)(\zeta - w)}\, d\zeta \right| \\
&\leq \frac{1}{2\pi} \int_{\partial D(z,2r)} |f(\zeta)| \frac{|z - w|}{|\zeta - z| \cdot |\zeta - w|} |d\zeta|.
\end{aligned}$$

Choose M such that $|f(\zeta)| \leq M$, $\forall f \in \mathcal{F}$, $\forall \zeta \in \overline{D}(z, 2r)$. So

$$\begin{aligned}
|f(z) - f(w)| &\leq \frac{1}{2\pi} M \cdot \frac{|z - w|}{2r \cdot r} \cdot 2\pi(2r) \\
&= \frac{M}{r} \cdot |z - w|
\end{aligned}$$

for all $f \in \mathcal{F}, w \in D(z,r)$. Let $\epsilon > 0$, choose $\delta > 0$ such that $\delta < r$ and $\frac{M}{r}\delta < \epsilon$. Of $f \in \mathcal{F}$ and $|z-w| < \delta$ then

$$|f(z) - f(w)| \leq \frac{M}{r} \cdot |z - w| < \epsilon.$$

Thus f is equicontinuous at z. Since z was arbitrary, $\mathcal{F}$ is equicontinuous. By Arzela-Ascoli Theorem, it follows that $\mathcal{F}$ is precompact in the topology of uniform convergence on compact subsets. $\square$

Here are a few remarks:

1. In the topology of uniform convergence on compact subsets, $H(V)$ is a closed subset of $C(V, \mathbb{C})$. To verify this, we have to show that if $(f_n) \subset H(V)$ and $f_n \to f$ in this topology, then $f \in H(V)$. We already know this. This means that the limit of the subsequence in Montel's Theorem is always a holomorphic function.

2. The map $D : H(V) \to H(V)$ given by $D(f) = f'$ is continuous for the uniform convergence on compact subsets topology. We need to show that if $f_n \to f$ in $H(V)$ then $f_n' \to f'$ in $H(V)$. We also already know this.

3. The conditions in Montel's Theorem are actually "if and only if" conditions. That is, if $\mathcal{F}$ is precompact, then $\mathcal{F}$ has to be normal.

4. If $\mathcal{F} \subset H(V)$ is a normal family then $\{|f'(z)| : f \in \mathcal{F}\}$ is bounded for each $z \in U$. In fact, the sup is achieved by some function $g \in \mathrm{cl}(\mathcal{F})$ (because the closure of $\mathcal{F}$ is compact if $\mathcal{F}$ is a normal family and apply the extreme value theorem).

5. If $f_n \to f$ in $H(V)$ and each f_n is one-to-one then f is either one-to-one or constant. Why? Suppose f is not constant, but there are $p, q \in V$ such that $f(p) = f(q)$ but $p \neq q$. There are some $r_p > 0$ and some $r_q > 0$ such that $\overline{D}(p, r_p), \overline{D}(q, r_q) \subset V$, f does not take the value $f(p) = f(q)$ on $\partial D(p, r_p), \partial D(q, r_q)$ and $\overline{D}(p, r_p) \cap \overline{D}(q, r_q) = \emptyset$. Now $f_n \to f$ uniformly on $\overline{D}(p, r_p) \cup \overline{D}(q, r_q)$. By Hurwitz's Theorem, when n is large enough, f_n takes the value $f(p)$ in $D(p, r_p)$ and in $D(q, r_q)$. But then we get $z \in D(p, r_p)$ and $w \in D(q, r_q)$ with $f_n(z) = f(p) = f_n(w)$. This contradicts f_n being one-to-one.

38 Normal Families and the Riemann Mapping Theorem VI (11/24)

Theorem 38.1 (Riemann Mapping Theorem). *Let $V \subset \mathbb{C}$ be a non-empty, open, proper, simply connected, connected set. Take $z_0 \in V$. There is one and only one conformal equivalence $f : V \to \mathbb{D}$ such that $f(z_0) = 0$ and $f'(z_0) > 0$.*

Step 1: Uniqueness.

Suppose that $f, g : V \to \mathbb{D}$ such that $f(z_0) = g(z_0) = 0$ and $f'(z_0) > 0, g'(z_0) > 0$. The function $\varphi = f \circ g^{-1} : \mathbb{D} \to \mathbb{D}$ is holomorphic, one-to-one, and onto. That is, $\varphi \in \mathrm{Aut}(\mathbb{D})$. Also, $\varphi(0) = (f \circ g^{-1})(0) = f(z_0) = 0$. From our knowledge of $\mathrm{Aut}(\mathbb{D})$, we know that $\varphi(w) = e^{i\theta}w$ for some θ. Now $\varphi'(0) = \frac{f'(z_0)}{g'(z_0)}$ by chain rule. Thus $\varphi'(0) > 0$. Well, $\varphi'(0) = e^{i\theta} > 0$ and so $e^{i\theta} = 1$ and $\varphi = \mathrm{id}_{\mathbb{D}}$. This says that $f = g$.

Example 38.2. Let $S = \{x + iy | -1 < x < 1\}$ and $f : S \to \mathbb{D}$ be the Riemann map (the conformal equivalence) such that $f(0) = 0$ and $f'(0) > 0$. Show that $f(-z) = -f(z)$ for all $z \in S$.

One way is to consider $g : S \to \mathbb{D}$ defined by $g(z) = -f(-z)$. Note that g is one-to-one and onto, holomorphic, $g(0) = -f(-0) = -f(0) = 0$, $g'(0) = f'(0) > 0$. Thus $g = f$ by the uniqueness in the Riemann Mapping Theorem and so $f(-z) = -f(z)$.

Step 2.

If we could find $f : V \to \mathbb{D}$ such that $f(z_0) = 0$ then we could arrange to have $f'(z_0) > 0$. We know that $f'(z_0) \neq 0$ because f is one-to-one. We may choose θ such that $e^{i\theta} f'(z_0) > 0$. Then $g : V \to \mathbb{D}$ defined by $g(z) = e^{i\theta} f(z)$ would satisfy $g'(z_0) > 0$ and $g(z_0) = 0$. This means we only have to worry about $f(z_0) = 0$.

We know that if V is simply connected, then a non-vanishing holomorphic function on V has a holomorphic square-root.

Theorem 38.3 (Riemann Mapping Theorem, restated)**.** *Let $V \subset \mathbb{C}$ be a non-empty, open, connected, proper, and such that every non-vanishing holomorphic function on V has a holomorphic square root. Let $z_0 \in V$. Then there is a unique conformal equivalence $f : V \to \mathbb{D}$ such that $f(z_0) = 0$ and $f'(z_0) > 0$.*

Note that 38.3 implies 38.1.

Note that once we know this, we know that the square-root condition implies that V is simply connected.

Step 3.

We need to show there are one-to-one holomorphic functions $h : V \to \mathbb{D}$ such that $h(z_0) = 0$. Choose $p \in \mathbb{C} \backslash V$. Then $z \mapsto z - p$ is a non-vanishing holomorphic function on V. Thus there is a holomorphic function g such that $g(z)^2 = z - p$ for all $z \in V$. Choose $q \in g(V)$. We know that g is not constant. Thus $g(V)$ is open by Open Mapping Theorem. Choose $r > 0$ such that $D(q, r) \subset g(V)$. We claim that $g(V) \cap D(-q, r) = \emptyset$. Suppose not and choose $w \in g(V) \cap D(-q, r)$. Then $-w \in D(q, r) \subset g(V)$. Thus we may find $z_1, z_2 \in V$ such that $g(z_1) = w, g(z_2) = w$. This implies $z_1 - p = g(z_1)^2 = w^2 = (-w)^2 = z_2 - p$ and $z_1 = z_2$. But then $w = -w$ and so $w = 0$. So $w = 0 \notin g(V)$, a contradiction. Thus $g(V) \cap D(-q, r) = \emptyset$ and so $|g(z) + q| \geq r$ for all $z \in V$.

Also note that g is one-to-one. Thus $\tilde{g}(z) = \frac{1}{g(z)+q}$ is one-to-one and bounded. Now we may choose $a \in \mathbb{C} \backslash \{0\}$, $b \in \mathbb{C}$ such that

$$f(z) = a\tilde{g}(z) + b$$

satisfies $f(z_0) = 0$ and $f(V) \subset \mathbb{D}$. (Just choose a small enough that the diameter of $a\tilde{g}(V)$ is less than 1 and b so that $f(z_0) = 0$.)

39 Normal Families and the Riemann Mapping Theorem VII (12/1)

Step 4.

We know that $\mathcal{F} = \{f : V \to \mathbb{D} | f \in H(V), f(z_0) = 0, f \text{ one to one}\}$ is non-empty.

Next observation is that $\mathcal{F}$ is a normal family. By Montel's Theorem, $\mathcal{F}$ is precompact in the topology of uniform convergence on compact subsets of V. Thus $\mathrm{cl}(\mathcal{F})$ is a compact set. The map $g \mapsto |g'(z_0)|$ is a continuous map from $H(V)$ to $[0, \infty)$. Thus (Extreme Value Theorem implies that) this map achieves a maximum value on $\mathrm{cl}(\mathcal{F})$. That is, there is some $f \in \mathrm{cl}(\mathcal{F})$ such that $|f'(z_0)|$ is a maximum over all functions in $\mathrm{cl}(\mathcal{F})$.

Step 5.

We know that there is some $g \in \mathcal{F}$. Since g is one-to-one, $g'(z_0) \neq 0$. Thus $|g'(z_0)| > 0$ and so $|f'(z_0)| \geq |g'(z_0)| > 0$. This tells us that f is not constant. Since f is the limit of sequence of functions in $\mathcal{F}$, each of which is one-to-one, and f is not constant, it follows that f is also one-to-one. Since $g(z_0) = 0$ for all $g \in \mathcal{F}$, $f(z_0) = 0$. It is immediate (for the same reason) that $|f(z)| \leq 1$ for all $z \in V$. By the Maximum Modulus Principle, the fact V is connected, and the fact that f is not constant, it follows that $|f(z)| < 1$ for all $z \in V$. We have now shown that $f \in \mathcal{F}$. It remains to show that $\mathcal{F}$ is onto $\mathbb{D}$.

Step 6.

Say $g : V \to \mathbb{D}$ with $g(z_0) = 0$. Define $H_g : V \to [0, \infty)$ by

$$H_g(z) = \frac{|g'(z)|}{1 - |g'(z)|}.$$

Note that $H_g(z_0) = |g'(z_0)|$.

Extend H_g to all $g : V \to \mathbb{D}$ for the next lemma.

Lemma 39.1. *Let $\psi : \mathbb{D} \to \mathbb{D}$ be holomorphic. Then $H_{\psi \circ g}(z) \leq H_g(z), \forall z \in V$ with equality if and only if $\psi \in Aut(\mathbb{D})$ or $g'(z) = 0$.*

Proof. By the Schwarz-Pick Lemma, we know that

$$\frac{|\psi'(w)|}{1 - |\psi(w)|^2} \leq \frac{1}{1 - |w|^2}$$

for all $w \in \mathbb{D}$ and we have equality if and only if $\psi \in \mathrm{Aut}(\mathbb{D})$. Substituting $g(z)$ for w in this inequality, we get

$$\frac{|\psi'(g(z))|}{1 - |\psi(g(z))|^2} \leq \frac{1}{1 - |g(z)|^2}$$

with equality if and only if $\psi \in \mathrm{Aut}(\mathbb{D})$. Multiplying both sides by $|g'(z)|$ we get

$$\frac{|\psi'(g(z))g'(z)|}{1 - |\psi(g(z))|^2} \leq \frac{|g'(z)|}{1 - |g(z)|^2}$$

with equality if and only if $\psi \in \mathrm{Aut}(\mathbb{D})$ or $|g'(z)| = 0$. This says $H_{\psi\circ g}(z) \leq H_g(z)$ with equality if and only if $\psi \in \mathrm{Aut}(\mathbb{D})$ or $|g'(z)| = 0$. $\square$

Now let's go back to the main proof.

Step 7.

Suppose that our maximer is not onto $\mathbb{D}$. Choose $\alpha \in \mathbb{D}$ not in the image of f. Consider $\varphi_\alpha \circ f$. This omits the value 0 and so there exists $h \in H(V)$ such that $h^2 = \varphi_\alpha \circ f$. If $h(z_0) = \beta$, then define $F = \varphi_\beta \circ h$. Note that F is holomorphic, $F(z_0) = \varphi_\beta(h(z_0)) = \varphi_\beta(\beta) = 0$, F is one-to-one because f is, so $\varphi_\alpha \circ f$ is, so h is, so $F = \varphi_\beta \circ h$ is. This implies that $F'(z_0) \neq 0$. Since $F(z_0) = 0$,

$$\begin{aligned}
|F'(z_0)| = H_F(z_0) &= H_{\varphi_\beta^{-1}\circ F}(z_0) \text{ (because } \varphi_\beta \in \mathrm{Aut}(\mathbb{D})) \\
&= H_h(z_0) > H_{\varphi\circ h}(z_0) \text{ where } \varphi : \mathbb{D} \to \mathbb{D} \text{ is } \varphi(w) = w^2 \\
&\text{(because } \varphi \in \mathrm{Aut}(\mathbb{D}), h'(z_0) \neq 0 \text{ because } h \text{ is one-to-one)} \\
&= H_{\varphi_\alpha\circ f}(z_0) \\
&= H_f(z_0) \\
&= |f'(z_0)| \ (f(z_0) = 0).
\end{aligned}$$

This contradicts the assumption that f is the maximer. Thus f is onto.

This proves the Riemann Mapping Theorem, restated version and also the Riemann Mapping Theorem.

Corollary 39.2. *Let $V \subset \mathbb{C}$ be a connected, simple connected, open set. Then $Aut(V)$ acts transitively on V.*

Proof. If $V = \emptyset$ it is trivial. If $V = \mathbb{C}$, then the claim follows from our computation of $\mathrm{Aut}(\mathbb{C})$. Otherwise, V is non-empty and proper and so there is a biholomorphic map $f : V \to \mathbb{D}$. In addition, f is one-to-one and onto. We know that $\mathrm{Aut}(\mathbb{D})$ acts transitively on $\mathbb{D}$. The claim follows. $\square$

40 Topology of Uniform Convergence on Compact Subsets (12/3)

The set of all holomorphic maps from $\mathbb{D}$ to $\mathbb{D}$ is precompact in the topology of uniform convergence on compact subsets. This implies, for example, that if $(\alpha_n) \subset \mathbb{D}$ and $\alpha_n \to 1$ then (φ_{α_n}) must have a convergent subsequence. In fact, φ_{α_n} converges to the constant sequence 1 in the topology of uniform convergence on compact subsets. Note that $\varphi_{\alpha_n}(\alpha_n) = 0$ so $(\varphi_{\alpha_n}(\alpha_n))$ is not converging to 1! To show $\varphi_{\alpha_n} \to 1$ in the topology of uniform convergence on compact sets, let $0 < r < 1$ and consider $z \in \overline{D}(0,r)$. Then

$$\begin{aligned}
|\varphi_{\alpha_n}(z) - 1| &= \left|\frac{\alpha_n - z}{1 - \overline{\alpha}_n z} - 1\right| \\
&= \left|\frac{(\alpha_n - z) - (1 - \overline{\alpha}_n z)}{1 - \overline{\alpha}_n z}\right| \\
&= \left|\frac{(\alpha_n - 1) + (\overline{\alpha}_n - 1)z}{1 - \overline{\alpha}_n z}\right| \\
&\leq \frac{|\alpha_n - 1| + |\overline{\alpha}_n - 1|r}{1 - r} \\
&= |\alpha_n - 1| \cdot \frac{1 + r}{1 - r}.
\end{aligned}$$

Now I get $\varphi_{\alpha_n}|_{\overline{D}(0,r)} \to 1$ uniformly. Thus $\varphi_{\alpha_n} \to 1$ in the topology of uniform convergence on compact sets because any compact $K \subset \mathbb{D}$ is contained in $\overline{D}(0,r)$ for some $0 < r < 1$.

The metric space $H(\mathbb{D})$ is not compact, but it is *separable* (that is, it has a countable dense subset). We will verify that

$$P = \{g | g \text{ polynomials on } \mathbb{D} \text{ with coefficients having rational real and imaginary parts}\}$$

is dense in $H(\mathbb{D})$.

Choose a compact exhaustion (K_n) of $\mathbb{D}$. Specifically, we could choose $K_n = \overline{D}(0, r_n)$ where $r_n \nearrow 1$ (increasingly approach to 1). Say $f \in H(\mathbb{D})$. Let $f(z) = \sum_{m=0}^{\infty} a_m z^m$ be the Maclaurin series for f. Since $f \in H(\mathbb{D})$, we know that this power series has radius of convergence at least 1. Also the power series converges uniformly to f on any compact subset of $\mathbb{D}$. In particular, the power series converges uniformly to f on K_n. In particular, given n, I can choose some partial sum of the power series g_n such that $|g_n(z) - f(z)| < \frac{1}{2n}$ for all $z \in K_n$. Note $g_n \in \mathbb{C}[z]$. Next, say $g_n(z) = \sum_{m=0}^{N} a_m z^m$. Then if $z \in K_n$, we have

$$\begin{aligned}
\left|\sum_{m=0}^{N} a_m z^m - \sum_{m=0}^{N} b_m z^m\right| &\leq \sum_{m=0}^{N} |a_m - b_m| \\
&\leq (N+1) \cdot \max_{0 \leq m \leq N} |a_m - b_m|.
\end{aligned}$$

I may choose $b_m \in \mathbb{Q}(i)$ such that if $h_n = \sum_{m=0}^{N} b_m z^m$ then

$$|g_n(z) - h_n(z)| < \frac{1}{2n}$$

for all $z \in K_n$. Then $h_n \in P$ and we have

$$|h_n(z) - f(z)| < \frac{1}{n}$$

for all $z \in K_n$. We hope that $h_n \to f$ in the topology of uniform convergence on compact sets.

Let $K \subset \mathbb{D}$ be compact and $\epsilon > 0$. Choose L such that $K \subset K_L$ and $\frac{1}{L} < \epsilon$. If $n \geq L$ then $K \subset K_n$ also, and so

$$|h_n(z) - f(z)| < \frac{1}{n} \leq \frac{1}{L} < \epsilon$$

for all $z \in K$. This verifies that $h_n \to f$ uniformly on K. Hence $h_n \to f$ in the topology of uniform convergence on compact sets.

$H(\mathbb{D})$ is also connected. Why – it is a vector space with the vector space operation matching with the topology and so it is actually path connected. Given $f, g \in H(\mathbb{D})$ define $\Phi : [0,1] \to H(\mathbb{D})$ by $\Phi(t) = (1-t)f + tg$. This is path from f to g in $H(\mathbb{D})$. The only thing we need to check is that Φ is continuous.

Given N, let

$$\begin{aligned} Z_N = \{f \in H(\mathbb{D}) | &f \text{ is not constant and } f \text{ has at least } N \text{ zeroes in } \mathbb{D} \\ &\text{counted with multiplicity}\}. \end{aligned}$$

Show that $Z_N \subset H(\mathbb{D})$ is open. It will suffice to show that if $g_n \to f$ then g_n is eventually in Z_N. First, choose a point $w \in \mathbb{D}$ where $f'(w) \neq 0$. We know that $g_n'(w) \to f'(w)$ and so $g_n{\prime}(w) \neq 0$ for large enough 0. Thus g_n is not constant once n is large enough. Next, let $w_1, \cdots, w_k$ be N zeroes of f counted with multiplicity. Then $w_1, \cdots, w_k$ are contained in $\overline{D}(0,r)$ for all large enough $r < 1$. Because the zeroes of f are isolated, we may choose r large enough so that $w_1, \cdots, w_k \in \overline{D}(0,r)$ and $f|_{\partial D(0,r)}$ does not vanish. Let $\epsilon = \frac{1}{2}\min_{z \in \partial D(0,r)} |f(z)| > 0$ and choose L such that if $n \leq L$ then $|g_n(z) - f(z)| < \epsilon$ for all $n \geq L$ and all $z \in \overline{D}(0,r)$. By Rouché's Theorem, g_n has at least N zeroes in $D(0,r)$ and so in $\mathbb{D}$. This is what we needed.

41 Determination of the Automorphism Groups Aut$(A(0, r, R))$ (12/5)

First, let $0 \leq r < R$. Consider a homeomorphism $f : A(0, r, R) \to A(0, r, R)$ (a continuous function that has a continuous inverse). Then f induces a permutation of the set of boundary components. To see this, let $\eta > 0$ be a small number and consider $K = \overline{A}(0, r + \eta, R - \eta)$. Then $f^{-1}(K)$ is a compact subset of $A(0, r, R)$. Thus there is a $\lambda > 0$ such that

$$A(0, r, r + \lambda) \subset A(0, r, R) \backslash f^{-1}(K).$$

Then

$$f(A(0, r, r + \lambda)) \subset A(0, r, R) \backslash K = A(0, r, r + \eta) \cup A(0, R - \eta, R)$$

and $f(A(0, r, r + \lambda))$ is connected. Thus $f(A(0, r, r + \lambda))$ belongs to one of the two sets $A(0, r, r + \eta)$ or $A(0, R - \eta, R)$. For definiteness, say $f(A(0, r, r + \lambda)) \subset A(0, r, r + \eta)$. Also, there is a $\tau > 0$ such that

$$A(0, r, r + \tau) \subset f(A(0, r, r + \lambda)) \subset A(0, r, r + \eta).$$

(To see this, repeat the argument with f^{-1}.) If this happens, then the permutation induced by f is the identity permutation. The other alternative is that there is a $\tau > 0$ such that $A(0, R - \tau, R) \subset f(A(0, r, r + \lambda)) = \subset A(0, R - \eta, R)$. If this happens, then the permutation is the one that exchanges the two boundary components.

Example 41.1. The map $f : D'(0, 1) \to D'(0, 1)$ given in polar coordinates by

$$f(re^{i\theta}) = (1 - r)e^{i\theta}, 0 < r < 1, 0 \leq \theta < 2\pi,$$

is a homeomorphism. This homeomorphism interchanges the boundary components. Notice f does not extend continuously to the closure of $D'(0, 1)$ in $\mathbb{R}^2$!

Use this to compute Aut$(A(0, r, R))$. First assume that $r > 0$. Then $f \in$ Aut$(A(0, r, R))$ is a homeomorphism and so it either interchanges or fixes the boundary components in the sense discussed above. Define $\psi : A(0, r, R) \to A(0, r, R)$ by $\psi(z) = \frac{rR}{z}$. Note if $r < |z| < R$ then

$$\frac{1}{r} > \frac{1}{|z|} > \frac{1}{R}$$

and hence

$$r < \frac{rR}{|z|} < R$$

. So ψ preserves the annulus. $\psi \circ \psi =$ id and ψ is holomorphic on $A(0, r, R)$. This automorphism interchanges the boundary components. By considering $\psi \circ f$ if necessary we may assume that f fixes the boundary components.

Assume f fixes the boundary components. This means that $\lim_{|z| \to r} |f(z)| = r$ and $\lim_{|z| \to R} |f(z)| = R$. Thus $|f|$ extends continuously to $\overline{A}(0, r, R)$ by giving it the value r on the circle $|z| = r$ and R on the circle $|z| = R$. Consider $g : A(0, r, R) \to \mathbb{C}$ giving by $g(z) = \frac{f(z)}{z}$. Note that g is holomorphic on $A(0, r, R)$ and $|g|$ extends continuously to $\overline{A}(0, r, R)$ with value 1 on both boundary components. Since g does not have a zero in $A(0, r, R)$, we conclude from Maximum Modulus Theorem and Minimum Modulus Theorem that g is constant. In fact, $g(z) = c$ with $|c| = 1$. This tells us that $f(z) = cz$ for some c with $|c| = 1$. That is, f is a rotation about 0. So

$$\mathrm{Aut}(A(0, r, R)) = \{\text{rotations}\} \cup \{\text{rotations composed with } \psi\}.$$

We still have to look at $r = 0$. What is $\mathrm{Aut}(D'(0, R))$? Suppose $f \in \mathrm{Aut}(D'(0, R))$. Note that f has a removable singularity at 0. So f extends to an element of $H(D(0, R))$. Suppose that f interchanges the boundary components. Then f extends to an element of $H(D(0, R))$ and $|f|$ extends to a continuous function on $\overline{D}(0, R)$ with value 0 on the other boundary. By Maximum Modulus Theorem, $f \equiv 0$ and this is impossible. Thus f does not switch the boundary components. This tells us that the extended f satisfies $f(0) = 0$. So $\mathrm{Aut}(D(0, R))$ and hence f is a rotation.

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