The Mathematics of Fairness

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Fairness

There are numerous examples in human activity in which individuals desire some degree of fairness.

Sometimes, mathematical analysis can enable us to analyze certain aspects of fairness.

However, we must be aware that there may be limitations in our models.

We will offer several examples to illustrate this quest for fairness.

Game Cut Short

Game Cut Short

Gwen and Dan are playing a gambling game. Each contributes \$6 to the pot.

The game consists of flipping a fair coin three times. If there are more heads than tails, then Gwen wins. Otherwise, Dan wins.

The coin is flipped for the first time. It comes up heads. Then the game is permanently interrupted.

What is a fair way for Gwen and Dan to divide up the money in the pot?

Game Cut Short

It might not be fair to assume that Gwen is going to win the game and give her all $12.

It might not be fair to assume that the money should be divided equally, since Gwen is currently ahead.

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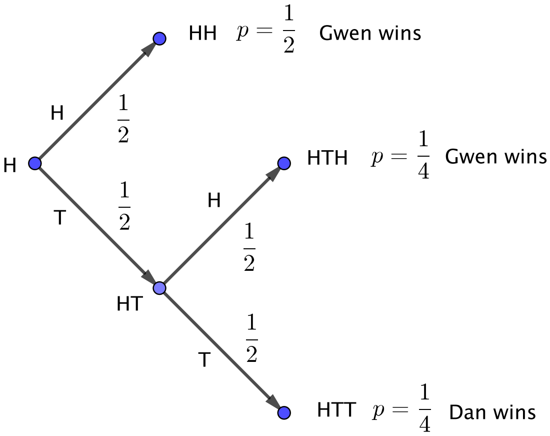
One solution is to determine the probability that Gwen will win the game and the probability that Dan will win the game.

Game Cut Short

Let's examine the possibilities of future flips. A tree diagram is helpful here.



Game Cut Short

We see that the probability that Gwen wins the game is $3/4$, and the probability that Dan wins the game is $1/4$.

So we propose to give $3/4$ of the pot (i.e., \$9) to Gwen and $1/4$ of the pot (i.e., \$3) to Dan.

This is an example of computing "expected value."

Game Cut Short

This kind of "gambling game cut short" problem lies at the origins of the probability theory. Such problems date back to the late 1400s, and was later the subject of correspondence between Pascal and Fermat. See, for example, `https://www.york.ac.uk/depts/maths/histstat/pascal.pdf`.

Apportionment

How do we fairly apportion the required number of representatives to districts or states with different size populations?

For example, the US House of Representatives (currently!) has 435 members. How do we fairly allocate them to the 50 states?

Apportionment

The source for material in this section comes from `http://www.opentextbookstore.com/mathinsociety/2.5/Apportionment.pdf`.

Apportionment

Example. The state of Rhode Island has five counties.... The Rhode Island state House of Representatives has 75 members.... The populations of the counties are as follows:

County	Population
Bristol	49,875
Kent	166,158
Newport	82,888
Providence	626,667
Washington	126,979
Total	1,052,567

How many representatives should each county get?

Apportionment

To be fair, we would expect the number of representatives to be proportional to the sizes of the populations.

Apportionment

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First, we determine the divisor: $1,052,567/75 = 14,034.22667$.

This is "population per representative."

Apportionment

Now we determine each county's quota by dividing the county's population by the divisor:

County	Population	Quota
Bristol	49,875	3.5538
Kent	166,158	11.8395
Newport	82,888	5.9061
Providence	626,667	44.6528
Washington	126,979	9.0478
Total	1,052,567	75.0000

Now what?

Apportionment

Proposal: First round down the quotas. Then add one additional representative, as necessary, to the quotas with the largest fractions.

County	Population	Quota	Initial	Final
Bristol	49,875	3.5538	3	3
Kent	166,158	11.8395	11	12
Newport	82,888	5.9061	5	6
Providence	626,667	44.6528	44	45
Washington	126,979	9.0478	9	9
Total	1,052,567	75	72	75

Apportionment

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Bristol	49,875	3.5538	3	3
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Does this seem like a fair method?

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Total	1,052,567	75	72	75

Does this seem like a fair method?

If so, then you agree with its inventor, Alexander Hamilton.

Apportionment

So why aren't we using Hamilton's method today?

"The problem is that Hamilton's method is subject to several paradoxes. Three of them happened, on separate occasions, when Hamilton's method was used to apportion the United States House of Representatives."

Apportionment

"The Alabama Paradox is named for an incident that happened during the apportionment that took place after the 1880 census. (A similar incident happened ten years earlier involving the state of Rhode Island, but the paradox is named after Alabama.) The post-1880 apportionment had been completed, using Hamilton's method and the new population numbers from the census. Then it was decided that because of the country's growing population, the House of Representatives should be made larger. That meant that the apportionment would need to be done again, still using Hamilton's method and the same 1880 census numbers, but with more representatives.

Apportionment

"The assumption was that some states would gain another representative and others would stay with the same number they already had (since there werent enough new representatives being added to give one more to every state). The paradox is that Alabama ended up losing a representative in the process, even though no populations were changed and the total number of representatives increased."

Apportionment

"The New States Paradox happened when Oklahoma became a state in 1907. Oklahoma had enough population to qualify for five representatives in Congress. Those five representatives would need to come from somewhere, though, so five states, presumably, would lose one representative each. That happened, but another thing also happened: Maine gained a representative (from New York)."

Apportionment

"The Population Paradox happened between the apportionments after the census of 1900 and of 1910. In those ten years, Virginia's population grew at an average annual rate of 1.07%, while Maine's grew at an average annual rate of 0.67%. Virginia started with more people, grew at a faster rate, grew by more people, and ended up with more people than Maine. By itself, that doesn't mean that Virginia should gain representatives or Maine shouldn't, because there are lots of other states involved. But Virginia ended up losing a representative to Maine."

Apportionment

Hamilton's method was vetoed by Washington. A new method was used from 1791 to 1842.

Find the same divisor and the same quota, and cut off the decimal parts in the same way, giving a total number of representatives that is less than the required total. Change the divisor by making it smaller, finding new (now larger) quotas with the new divisor, cutting off the decimal parts, and looking at the new total, until we find a divisor that produces the required total.

Does this seem like a fair method?

Apportionment

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Does this seem like a fair method?

If so, then you agree with its inventor, Thomas Jefferson.

Apportionment

So what's wrong with Jefferson's method?

"The Quota Rule says that the final number of representatives a state gets should be within one of that state's quota. Since we're dealing with whole numbers for our final answers, that means that each state should either go up to the next whole number above its quota, or down to the next whole number below its quota."

Jefferson's method can sometimes violate the Quota Rule.

Also, Jefferson's method tends to favor larger states. (And Virginia was the largest state.)

Apportionment

A new method was used from 1842 until 1852.

Round the quotas to the nearest whole number rather than dropping the decimal parts. If that doesn't produce the desired results at the beginning, adjust the divisor up or down until it does.

Does this seem like a fair method?

Apportionment

A new method was used from 1842 until 1852.

Round the quotas to the nearest whole number rather than dropping the decimal parts. If that doesn't produce the desired results at the beginning, adjust the divisor up or down until it does.

Does this seem like a fair method?

If so, then you agree with its inventor, Daniel Webster.

Apportionment

So what's wrong with Webster's method?

It can also violate the Quota Rule. But it appears to do so less often then Jefferson's method.

Apportionment

Hamilton's method was then used again from 1852 to 1901. Then Webster's method was readopted.

Apportionment

Why don't we just use a method that satisfies the Quota Rule and avoids the other paradoxes?

Apportionment

Why don't we just use a method that satisfies the Quota Rule and avoids the other paradoxes?

In 1980, Bilinski and Young proved that there is no such method! So in a certain sense, there is no fair method!

Apportionment

In 1941, a new apportionment method was adopted, developed by Huntington and Hill. It is also a divisor method and attempts to minimize the percent differences of how many people each representative will represent.

This is the one that is currently in use. But it is not guaranteed to satisfy the Quota Rule.

Fair Division

Fair Division — Cutting Cake

Our first example of fair division involves dividing up a continuous object such as cake (or soup).

Fair Division — Cutting Cake

Source: Steinhaus's book Mathematical Snapshots.

To divide an object like a cake into two equal parts, we can adopt the old custom of letting one partner cut and the other choose. The advantage of such a procedure is obvious: neither of the partners can object to this division. The first can secure the part due him by dividing the cake into two parts that he considers to be equally valuable; the second can secure at least his due part, by choosing the more valuable part or—if he considers them equally valuable—either part. It is presumed here that the object has the property of not losing its total value by division, i.e. that the

Fair Division — Cutting Cake

values of the parts give by addition the value of the whole, this property being admitted by both partners, even if they disagree as to the valuation of the whole object and of its parts. There exist such objects: heaps of nuts, for instance. There arises the question of how to divide fairly an object into three or more parts. The answer is given by the following rules, which may be explained here in the case of five partners, the procedure being essentially the same for any number of partners. They may be called *A*, *B*, *C*, *D*, and *E*. *A* has the right to cut from the cake an arbitrary slice; *B* is free to diminish the slice cut off by *A*, but is not compelled to do so; in turn *C* has the right (but not the duty) to diminish the (already

Fair Division — Cutting Cake

diminished or not diminished) slice, and so on. After *E* has made use of his right (or declined to do so), we see who was the last to touch the slice. Suppose it was *D*. Then *D* gets the slice, and the remainder of the cake (including the bits cut off) has to be divided fairly between *A*, *B*, *C*, and *E*. In the second round the same procedure reduces the number of partners to three, and the third round reduces it to two; the two partners divide the rest of the cake by the procedure initially explained: one cuts and the other chooses. Now let us see how every partner can secure his due part whatever his companions may do. If in the first round *A* cuts a slice that he considers to be 1/5 in value, it can happen that nobody touches it and *A* gets

Fair Division — Cutting Cake

it; in this case he is not wronged. If, however, one or more of his companions diminish this slice, the man who was the last to touch it gets it and, as it is diminished, *A* must consider that more than 4/5 of the value is left to be divided equally among 4 partners, himself being one of them. In the second round *A* has to proceed as before: If he happens to be the first again, he has to cut a slice that he considers 1/4 in value of the remainder. This policy is not sufficient; we must show how a partner has to behave when he is not the first. Suppose that *B* considers the part cut by *A* to be too great, that is to say, greater in value in *B*'s estimation than 1/5 of the whole. He has only to diminish it to the proper size; if he turns out to be the last

Fair Division — Cutting Cake

diminisher, he gets it and is not wronged. If he fails to get it, it is because somebody else has touched the slice after it had already been diminished by *B* to a size considered by *B* as 1/5. One of these subsequent diminishers thus gets a slice that *B* considers to be of smaller value than 1/5, so that *B* comes to the next round as a shareholder of a remainder that he considers of greater value than 4/5 of the whole cake, the number of partners being now 4 and *B* one of them. Now the method is clear: if you are the first of n partners in any round, you have to cut off a slice that you consider to be 1/nth in value of the part before you, whether it be the whole or the remainder of the cake; if you are not the first in the given round and you see a

Fair Division — Cutting Cake

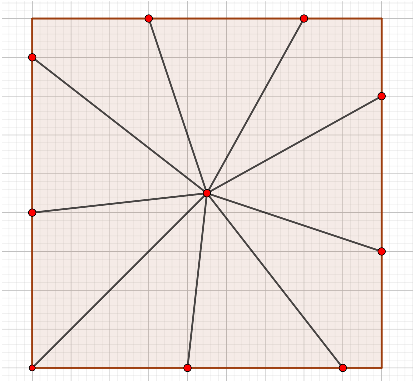
slice cut by one of your companions, a slice greater, in your estimation, than 1/nth of the part, you have to diminish it to 1/*n*th; if it has been cut so that the slice is 1/*n*th or less, in your estimation, you have to keep off. This method insures that everybody receives what he considers to be his due share.

Fair Division — Cutting Cake

Here is a related cake cutting problem. Suppose you have a square cake with evenly spread frosting on top and on the sides. How can you cut it into 9 pieces so that each peace has an equal amount of cake and each piece has an equal amount of frosting?

Fair Division — Cutting Cake

Solution: Place nine points equally spaced around the perimeter.



Now check the areas of the shapes on the top and sides of the cake.

Fair Division — Ham Sandwich Theorem

Supposed you have a ham sandwich with two slices of bread and one slice of ham. The Ham Sandwich Theorem states that with one straight cut you can divide each slice of bread as well as the slice of ham into two equal parts.

See the video `https://www.youtube.com/watch?v=YCXmUi56rao`

Fair Division — Dividing an Estate

How do you fairly divide up discrete, indivisible objects?

Fair Division — Dividing an Estate

Source: Steinhaus's book Mathematical Snapshots.

> There is another problem of division encountered in economic life: the division of indivisible objects like houses, domestic animals, pieces of furniture, cars, and works of art. If, for instance, an inheritance composed of a house, a mill, and a car has to be divided

Fair Division — Dividing an Estate

among four inheritors *A*, *B*, *C*, *D* participating in equal shares, this division is generally made by a sworn appraiser who determines the values of the objects so that the inheritors can choose the objects and, if they agree, in principle, satisfy by payments in cash the mutual claims arising from the differences in value.

This procedure has many inconveniences connected with the determination of the objective value of things by an official appraiser or by a court of justice. It is possible to make a fair division without appealing to them:

An umpire, who has to act only as a sort of automaton to keep records and make

Fair Division — Dividing an Estate

computations, summons the inheritors to write down their estimates of the objects. They are not supposed to discuss the matter among themselves but every one of them is allowed to be helped by friends and experienced persons. Thus a table of values is put down by the umpire:

	A	B	C	D
House	$ 6,000	$ 10,000	$ 7,000	$ 9,000
Mill	3,000	2,000	4,000	2,000
Car	1,500	1,200	1,000	1,000
Sum	10,500	13,200	12,000	12,000
Share	2,625	3,300	3,000	3,000
Value	1,500	10,000	4,000	0
Claim	1,125	−6,700	−1,000	3,000

Fair Division — Dividing an Estate

In the above table each person's share is got by dividing his estimate of the total by 4. In every row the greatest item appears in a frame and the corresponding object is attributed to the person whose name stands above the column. Thus *A* gets the car, *B* the house, and C the mill. The values of the objects subtracted from the shares of the persons invested with them give the claims. *A* appears with a claim of $1,125, *D* with one of $3,000, whereas *B* has a negative claim of $6,700, and *C* a negative claim of $1,000. This means that *B* and *C* have to pay money to the umpire and *A* and *D* have to get money from him:

Fair Division — Dividing an Estate

A	\$1,125	*B*	\$6,700	\$7,700
D	3,000	*C*	1,000	−4,125
	4,125		7,700	3,575 ÷ 4 = \$893.75

This computation shows that the payments will leave the umpire with a surplus of \$3,575; divided by 4 this gives \$893.75 for each inheritor. Thus

A will get the car and

\$1,125 + \$893.75 = \$2,018.75 cash

B will get the house and will have to

Fair Division — Dividing an Estate

pay

$6,700 — $893.75 = $5,806.25 off,

C will get the mill and will have to pay

$1,000 — $893.75 = $106.25 off,

D will get $3,000 + $893.75 = $3,893.75.

Thus everybody will finally get more than his due share of the inheritance, the value of the total and of the objects given to him being estimated according to his own valuation. For

Fair Division — Dividing an Estate

instance, *A* has a car and $2,018.75 in cash; as the car is worth $1,500 to him, he has a total of $3,518.75, whereas he had estimated his share at only $2,625. He has got $893.75 over his due part and the same is true of other partners. This method works with unequal shares too, and it can be modified so as to diminish the payments in cash. (How?)

Fair Division — Dividing Land

Here is another example from the same book. How can we fairly divide up a plot of land?

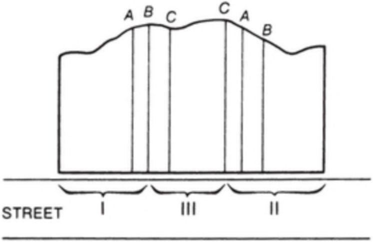
Fair Division — Dividing Land

The simplest way of dividing a plot of land is to do it on a map. Suppose that there are three joint proprietors of a garden wanting to divide it into three equal parts: they draw a sketch of the garden on transparent paper in 3 copies. Now every partner draws on his sketch two lines perpendicular to the street which, according to him, divide the garden into three equal parts. The parts do not need to be of equal area because the soil differs in quality and value, and besides, a man whose house is situated close to the garden estimates the soil near the house at a higher rate than the soil farther away—evidently the point in question is the subjective value. By superposing the transparent sketches one over the other we see

Fair Division — Dividing Land

six straight lines, which have been signed with the initials of the given names (Allan, Bertrand, Cecil). If the lines appear as shown in the above sketch **(67),** the umpire grants part I to *A*, part II to *B* and part III to *C*, thus giving to each partner more than 1/3 of the garden (in value) in the partners' own estimation. There are 8 different possibilities but there is always a division possible which gives to each person at least as much as his own estimation of 1/3 of the lot. The advantage of this method (dividing on a map) is that all shareholders are admitted simultaneously to the determination of parts; the provisional dividing lines remain unknown to those partners who did not draw them. The role of the umpire is purely automatic, and is

Fair Division — Dividing Land



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Fair Division — Dividing Land

get at least as much as his estimation of a third of the garden. In some of the 8 different cases the umpire bisects some strips without, however, violating the principle.

Voting Methods

Voting Methods

“Lexington should change how it votes by ranking candidates” — May 18, 2018, article in the Lexington Herald-Leader.

See `www.ms.uky.edu/~lee/olli18/lexington.pdf`.

Voting Methods

Let us consider an election which five candidates are running. Suppose each voter submits a rank ordered list from most favorite to least favorite—a preference ballot. We can assemble these ballots into a preference schedule.

Number of Voters	18	12	10	9	4	2
1st choice	A	B	C	D	E	E
2nd choice	D	E	B	C	B	C
3rd choice	E	D	E	E	D	D
4th choice	C	C	D	B	C	B
5th choice	B	A	A	A	A	A

How can we determine the winner in a fair way?

See The article by Joe Malkovitch `http://www.ams.org/publicoutreach/feature-column/fcarc-voting-decision`.

Voting Methods

Number of Voters	18	12	10	9	4	2
1st choice	A	B	C	D	E	E
2nd choice	D	E	B	C	B	C
3rd choice	E	D	E	E	D	D
4th choice	C	C	D	B	C	B
5th choice	B	A	A	A	A	A

Method #1: Plurality.
Select the candidate with the most first place votes.

Voting Methods

Number of Voters	18	12	10	9	4	2
1st choice	A	B	C	D	E	E
2nd choice	D	E	B	C	B	C
3rd choice	E	D	E	E	D	D
4th choice	C	C	D	B	C	B
5th choice	B	A	A	A	A	A

Method #1: Plurality.
Select the candidate with the most first place votes.

In this case the winner is A.

Voting Methods

Number of Voters	18	12	10	9	4	2
1st choice	A	B	C	D	E	E
2nd choice	D	E	B	C	B	C
3rd choice	E	D	E	E	D	D
4th choice	C	C	D	B	C	B
5th choice	B	A	A	A	A	A

Method #1: Plurality.
Select the candidate with the most first place votes.

In this case the winner is A.

But notice that A received more than 50% of the last-place votes! Is this fair?

Voting Methods

Number of Voters	18	12	10	9	4	2
1st choice	A	B	C	D	E	E
2nd choice	D	E	B	C	B	C
3rd choice	E	D	E	E	D	D
4th choice	C	C	D	B	C	B
5th choice	B	A	A	A	A	A

Method #2: Plurality with Runoff.
Count how many first place votes each candidate receives. If no candidate receives a majority, declare all candidates except those two who have gotten the largest number of first place votes as losers. Now, conduct a new election based on the preferences of the voters for these top two vote getters at this stage.

Voting Methods

Number of Voters	18	12	10	9	4	2
1st choice	A	B	C	D	E	E
2nd choice	D	E	B	C	B	C
3rd choice	E	D	E	E	D	D
4th choice	C	C	D	B	C	B
5th choice	B	A	A	A	A	A

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In this case the winner is B. Is this fair?

Voting Methods

Number of Voters	18	12	10	9	4	2
1st choice	A	B	C	D	E	E
2nd choice	D	E	B	C	B	C
3rd choice	E	D	E	E	D	D
4th choice	C	C	D	B	C	B
5th choice	B	A	A	A	A	A

Method #3: Plurality with Elimination.
If no candidate gets a majority based on first place votes, eliminate the candidate with the fewest first place votes and hold a new election based on voting only for the smaller collection of candidates. Repeat the process until some candidate receives a majority of the first place votes.

Voting Methods

Number of Voters	18	12	10	9	4	2
1st choice	A	B	C	D	E	E
2nd choice	D	E	B	C	B	C
3rd choice	E	D	E	E	D	D
4th choice	C	C	D	B	C	B
5th choice	B	A	A	A	A	A

Method #3: Plurality with Elimination.
If no candidate gets a majority based on first place votes, eliminate the candidate with the fewest first place votes and hold a new election based on voting only for the smaller collection of candidates. Repeat the process until some candidate receives a majority of the first place votes.

In this case the winner is C. Is this fair?

Voting Methods

Number of Voters	18	12	10	9	4	2
1st choice	A	B	C	D	E	E
2nd choice	D	E	B	C	B	C
3rd choice	E	D	E	E	D	D
4th choice	C	C	D	B	C	B
5th choice	B	A	A	A	A	A

Method #4: Borda Count.
For each preference ballot, the bottom candidate gets 1 point, the next one up gets 2 points, etc. The winner is the one with the highest point total.

Voting Methods

Number of Voters	18	12	10	9	4	2
1st choice	A	B	C	D	E	E
2nd choice	D	E	B	C	B	C
3rd choice	E	D	E	E	D	D
4th choice	C	C	D	B	C	B
5th choice	B	A	A	A	A	A

Method #4: Borda Count.
For each preference ballot, the bottom candidate gets 1 point, the next one up gets 2 points, etc. The winner is the one with the highest point total.

In this case the winner is D. Is this fair?

Voting Methods

Number of Voters	18	12	10	9	4	2
1st choice	A	B	C	D	E	E
2nd choice	D	E	B	C	B	C
3rd choice	E	D	E	E	D	D
4th choice	C	C	D	B	C	B
5th choice	B	A	A	A	A	A

Method #5: Pairwise Comparisons.
Consider all possible two-way races between candidates. The winner is the one that wins the most of these pairwise races.

Voting Methods

Number of Voters	18	12	10	9	4	2
1st choice	A	B	C	D	E	E
2nd choice	D	E	B	C	B	C
3rd choice	E	D	E	E	D	D
4th choice	C	C	D	B	C	B
5th choice	B	A	A	A	A	A

Method #5: Pairwise Comparisons.
Consider all possible two-way races between candidates. The winner is the one that wins the most of these pairwise races.

In this case the winner is E. In fact, E wins every two-way race! Such an individual is called a Condorcet candidate.

Is this fair?

Voting Methods

These are five examples of voting methods that are in common use. There are a number of others as well.

Voting Methods

What are some fairness criteria for voting methods?

Can we choose a voting method in advance that we can be sure is fair?

Voting Methods

The Majority Criterion. If there is a candidate who receives more than 50% of the first place folks, then that candidate should win the election.

Is this fair?

Voting Methods

Using the Borda Method can lead to a violation of the Majority Criterion.

3	2
A	B
B	C
C	A

Candidate A wins the majority of the first place votes. But candidate A receives 11 Borda points while candidate B receives 12 Borda points.

Voting Methods

The Condorcet Criterion. If there is a candidate who is favored in each pairwise head-to-head comparison with each other candidate, then that candidate should win the election.

Is this fair?

Voting Methods

Our first example shows that using the Plurality Method can lead to a violation of the Condorcet Criterion.

Voting Methods

The Monotonicity Criterion. If the winning candidate is determined, and then some ballots are changed by moving the winning candidate's name to the top of those ballots, and the election is recounted, the winner should not lose.

Is this fair?

Voting Methods

Using the Plurality with Elimination Method can lead to a violation of the Monotonicity Criterion.

5	6	6
A	C	B
B	A	A
C	B	C

B wins

5	4	2	6
A	C	B	B
B	A	C	A
C	B	A	C

Now A wins.

This can also lead to issues of "insincere voting."

Voting Methods

The Independence of Irrelevant Alternatives Criterion. If the winning candidate is determined, and then a losing candidate is removed from all ballots (e.g., is determined not to be qualified), and the election is recounted, the winner should not lose.

Is this fair?

Voting Methods

Using the any of the five methods can lead to a violation of the Independence of Irrelevant Alternatives Criterion. For example, consider using the Plurality Method.

4	3	2
A	B	C
B	C	B
C	A	A

A wins

Remove C.

4	3	2
A	B	B
B	A	A

Now B wins

Voting Methods

Arrow's Theorem asserts that there is no voting system that will always satisfy these fairness criteria.

Kenneth Arrow worked in social choice theory and won the Nobel Prize with John Hicks.

Stable Assignment

Stable Assignment

Suppose there are 10 companies and 10 interns who are interested in working at these companies. Each company prepares a ranked preference list of the interns, and each intern prepares a ranked preference list of the companies.

What is a fair way to assign interns to companies?

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This is more often referred to as the Stable Marriage Problem. See http://www.ams.org/publicoutreach/feature-column/fc-2015-03 and https://en.wikipedia.org/wiki/Stable_marriage_problem.

Stable Assignment

One measure of fairness is the notion of stability. Suppose there is an intern I and a company C for which I is not assigned to C. If I prefers C to her current company assignment, and C prefers I to its current intern assignment, then the assignment is considered to be unstable.

But if this situation never arises, then the assignment is considered to be stable.

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But if this situation never arises, then the assignment is considered to be stable.

In 1962, Gale and Shapley proved that a stable assignment is always possible, and that there is an algorithm to achieve one. Shapley, and Roth (who extended this work) won the Nobel prize in economics for this area of study.

Stable Assignment

Here is the algorithm.

Each company makes an offer to the first intern on its list of preferences. Each intern says “maybe” to the offer from the company she most prefers out of those who have made her an offer; these companies’ offers are “conditionally accepted.” She says “no” to the other offers, permanently rejecting them.

Stable Assignment

Each company with no conditionally accepted offer makes an offer to the intern it most prefers out of those who have not yet rejected it. Each intern considers any new companies who have made offers at this step and any company she had previously accepted, and conditionally accepts the offer from the company she most prefers, even if that means rejecting the company she had previously conditionally accepted.

This process repeats until every intern has conditionally accepted an offer, at which time the conditional acceptances become final and the process ends.

Stable Assignment

Why is the resulting assignment stable?

Suppose intern I is not assigned to company C. Upon completion of the algorithm, it is not possible for both I and C to prefer each other over their current assignments. If C prefers I to its current intern, it must have made an offer to I before making an offer to its current intern. If I accepted its offer, yet is not assigned to C at the end, she must have rejected C for a company she likes more, and therefore doesn't like C more than her current company. If I rejected its offer, she was already with a company she liked more than C.

Sharing Profits or Costs

Sharing Profits or Costs

Suppose there are three sizes (1, 2, 3) of planes, each requiring a certain length of runway to land and take off, $L_1 < L_2 < L_3$. (Larger planes require longer runways.)

A single runway is to be constructed. In some suitable units, let's consider some numbers.

	1	2	3
Full runway cost	200	400	500
Number of landings	50	30	20

What is a fair way to share the cost?

Sharing Profits or Costs

	1	2	3
Full runway cost	200	400	500
Number of landings	50	30	20

Solution: Share the cost of what you use.

All planes share the cost (200) of the short length they use, each paying $200/100 = 2$ units.

The medium and large planes share the additional cost $(400 - 200 = 200)$ of the medium length they use, each paying $200/50 = 4$ units.

The large planes share the additional cost $(500 - 400 = 100)$ of the long length they use, each paying $100/20 = 5$ units.

Sharing Profits or Costs

In summary,

Size	Number of Landings	Cost per Landing	Total
Small	50	2	100
Medium	30	6	180
Large	20	11	220
			500

See https://en.wikipedia.org/wiki/Airport_problem.

Sharing Profits or Costs

Now let's think about sharing profit. Suppose that there are three individuals, A, B, and C, who can work together in various subgroups to achieve some profit. We assume that the more they work together, the more they can make. Here is a table showing how much each subgroup can make by working together. If they all work together, how should the 60 be shared fairly?

Subgroup	Profit
Empty group	0
A	12
B	18
C	6
AB	48
AC	42
BC	36
ABC	60

Sharing Profits or Costs

Shapley proposed some fairness criteria.

1. If there is a individual who does not change the value of any subgroup when they are added to it, they should receive zero.
2. If there are two individuals who change the values of subgroups in exactly the same way when they are added to them, they should receive the same amounts.
3. If you have two separate profit sharing problems, and you make a new profit sharing problem by adding the values of the corresponding subgroups together, then you should add together the amounts that individuals receive.

Sharing Profits or Costs

There is a unique solution based on these axioms of fairness.

Consider all orderings of the three players. For each ordering, determine the marginal contribution that each player makes as the subgroup grows. Take the average of all of these contributions.

Sharing Profits or Costs

Ordering	A	B	C
ABC	12	36	12
ACB	12	18	30
BAC	30	18	12
BCA	24	18	18
CAB	36	18	6
CBA	24	30	6
Total	138	138	84
Average	23	23	14

These allocations are called the Shapley value. The runway cost allocations that we saw earlier are also an example of the Shapley value.

See Game Theory: A Playful Introduction, by DeVos and Kent.

Conclusion

There are many other situations in which we may wish to define what fairness means, then look for a procedure or method that is fair.

For example, there have been recent studies on fair districting methods that seek to reduce extreme gerrymandering. See `https://www.math.cmu.edu/~wes/gerrymandering.html`.