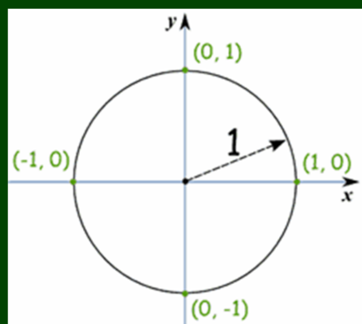
# Lesson #3:
# The Unit Circle

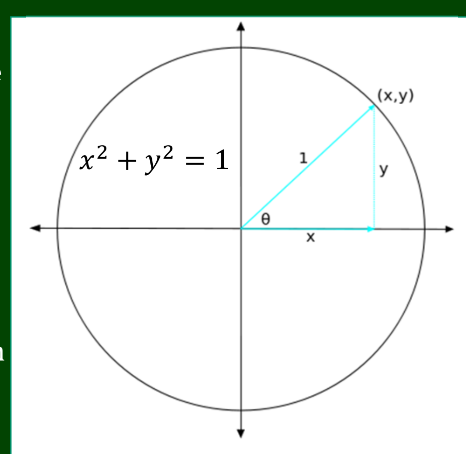



**What is the unit circle?**

- Circle with radius equal to 1
- Circle centered at the origin

# The Unit Circle

- Any point on the circle can be represented by the point $(x, y)$
- Lengths $x$ and $y$ become the lengths of the sides of the right triangle
- Hypotenuse of the right triangle is equal to 1 by the Pythagorean Theorem
- $\theta$ is the angle of elevation to point $(x, y)$



# The Unit Circle

All points on the unit circle must satisfy the equation $x^2 + y^2 = 1$. Verify that this equation is true for each of the coordinate points given below:

Ex 1) $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

$$(\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2 = 1$$

$$\frac{1}{4} + \frac{3}{4} = 1$$

$$1 = 1 \checkmark$$

Ex 2) $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

$$(\frac{\sqrt{2}}{2})^2 + (-\frac{\sqrt{2}}{2})^2 = 1$$

$$\frac{2}{4} + \frac{2}{4} = 1$$

$$\frac{4}{4} = 1$$

$$1 = 1 \checkmark$$

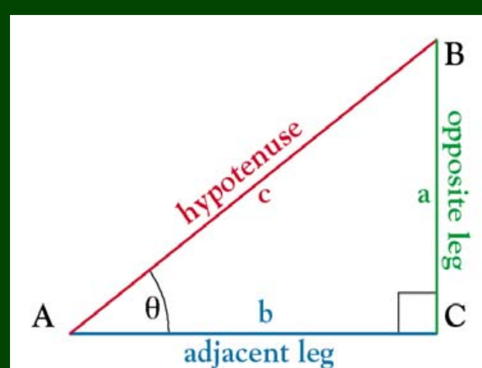
## *Recall*: Three Trigonometric Functions

Find the three basic trigonometric functions of angle $\theta$.

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{a}{c}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{b}{c}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{a}{b}$$



## Three Reciprocal Trigonometric Functions

Each basic trigonometric function has a reciprocal function.
These functions are reciprocals of the *original trig function*.

$$\textbf{reciprocal function} = \frac{\mathbf{1}}{\textbf{original function}}$$

| | |
|---|---|
| $\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$ | $\text{cosecant}\,\theta = \csc\theta = \dfrac{\text{hyp}}{\text{opp}} = \dfrac{1}{\sin\theta}$ |
| $\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$ | $\text{secant}\,\theta = \sec\theta = \dfrac{\text{hyp}}{\text{adj}} = \dfrac{1}{\cos\theta}$ |
| $\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}}$ | $\text{cotangent}\,\theta = \cot\theta = \dfrac{\text{adj}}{\text{opp}} = \dfrac{1}{\tan\theta}$ |

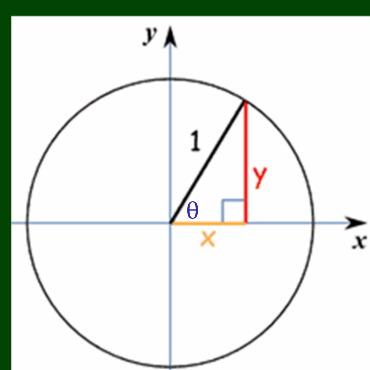
## The Unit Circle

Using the unit circle, find the three basic trigonometric functions of angle $\theta$.

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{1} = y$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{x}{1} = x$$

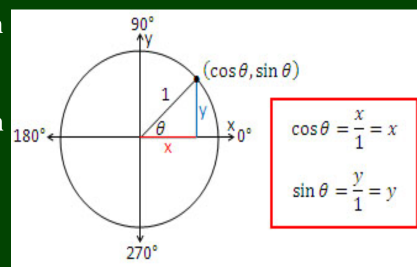
$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{y}{x} = \frac{\sin\theta}{\cos\theta}$$



# The Unit Circle

In the unit circle, since the radius is equal to 1:

- The $x-$coordinate associated with angle $\theta$ is represented as $\cos\theta$
- The $y-$coordinate associated with angle $\theta$ is represented as $\sin\theta$
- Therefore, the ordered pair $(x, y)$ can be expressed as $(\cos\theta, \sin\theta)$




*Recall*: $x^2 + y^2 = 1$ and $\cos\theta = x$ and $\sin\theta = y$

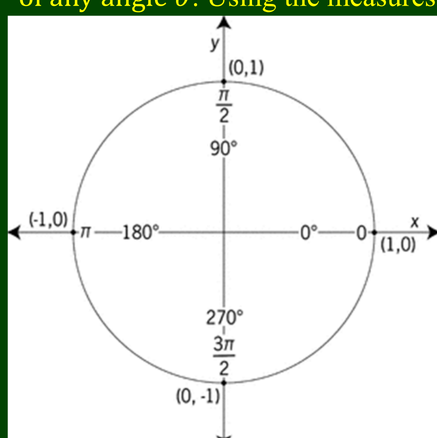
Substitute: $(\cos\theta)^2 + (\sin\theta)^2 = 1$

$$\cos^2\theta + \sin^2\theta = 1$$

**Pythagorean Identity**

# The Unit Circle

The Unit Circle can be used to evaluate trigonometric functions of any angle $\theta$. Using the measures of the quadrantal angles first:




What is the measure of angle $\theta$, in radians, at the point (1,0)?

$$\theta = 0$$

Determine the value of $\cos 0$.

$$\cos 0 = 1$$

Determine the value of $\sin 0$.

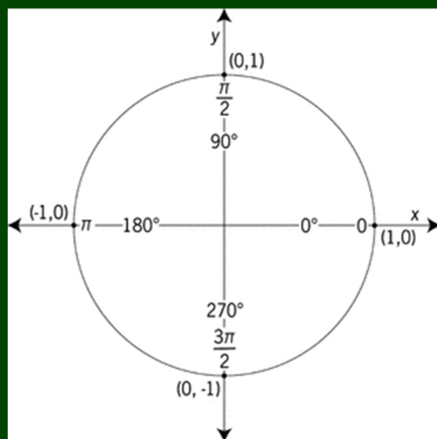
$$\sin 0 = 0$$

Determine the value of $\tan 0$.

$$\tan 0 = \frac{\sin 0}{\cos 0} = \frac{0}{1} = 0$$

# The Unit Circle

Use the Unit Circle to evaluate trigonometric functions below:




$\cos\frac{\pi}{2} = 0$ | $\cos\frac{3\pi}{2} = 0$

$\sin\frac{\pi}{2} = 1$ | $\sin\frac{3\pi}{2} = -1$

$\tan\frac{\pi}{2} =$ undefined | $\tan\frac{3\pi}{2} =$ undefined

$\cos\pi = -1$ | $\cos 2\pi = 1$

$\sin\pi = 0$ | $\sin 2\pi = 0$

$\tan\pi = 0$ | $\tan 2\pi = 0$

# Domain and Range

| **Function** | $y = \sin x$ | $y = \cos x$ | $y = \tan x$ |
|---|---|---|---|
| **Domain** | $(-\infty, \infty)$ | $(-\infty, \infty)$ | $x \neq \frac{k\pi}{2}$ where $k$ is any odd integer |
| **Range** | $[-1, 1]$ | $[-1, 1]$ | $(-\infty, \infty)$ |

**The *angles* of a trigonometric function represent the domain of the function.**

**The *ratios* of a trigonometric function represent the range of the function.**

## The Unit Circle

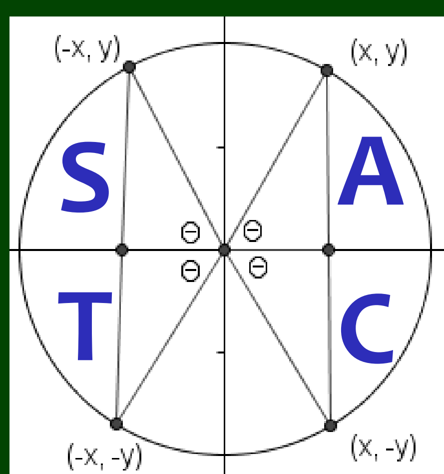
What happens as points move around the unit circle, and the angle increases from 0 to $2\pi$?

$\sin\theta$, $\cos\theta$, and $\tan\theta$ change signs

| | |
|---|---|
| $\sin\theta$ – positive<br>$\cos\theta$ – negative<br>$\tan\theta$ – negative | $\sin\theta$ – positive<br>$\cos\theta$ – positive<br>$\tan\theta$ – positive |
| $\sin\theta$ – negative<br>$\cos\theta$ – negative<br>$\tan\theta$ – positive | $\sin\theta$ – negative<br>$\cos\theta$ – positive<br>$\tan\theta$ – negative |

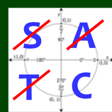
## The Unit Circle

**To help you remember the signs:**



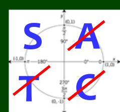
## The Unit Circle

Ex 1) If $\sin\theta < 0$ and $\cos\theta > 0$, in what quadrant does $\angle\theta$ terminate?



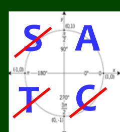

**Quadrant IV**

Ex 2) If $\tan\theta < 0$ and $\csc\theta > 0$, in what quadrant does $\angle\theta$ terminate?




**Quadrant II**

Ex 3) If $\sec\theta > 0$ and $\cot\theta > 0$, in what quadrant does $\angle\theta$ terminate?




**Quadrant I**

## The Unit Circle

Ex 4) Angle $\theta$ has a terminal ray that falls in the second quadrant. If $\sin\theta = \frac{3}{5}$, determine the value of $\cos\theta$.

**2 Methods to Solve!**

**(1) Draw a right triangle!**

**(2) Use the Pythagorean Identity $\cos^2\theta + \sin^2\theta = 1$**

$$\cos\theta = -\frac{4}{5}$$

Ex 5) Angle $\theta$ has a terminal ray that falls in the third quadrant. If $\tan\theta = \frac{8}{15}$, determine the value of $\sin\theta$.

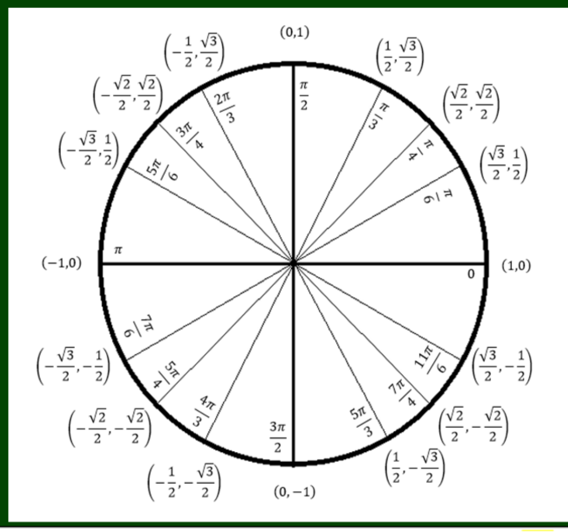
**Draw a right triangle!**

$$\sin\theta = -\frac{8}{17}$$

# The Unit Circle

**How can the unit circle be used to evaluate trigonometric functions of angles that are *not* quadrantal, i.e. angles that terminate in a quadrant?**

**Right Triangles and Reference Angles**




# *Recall*: Special Right Triangles

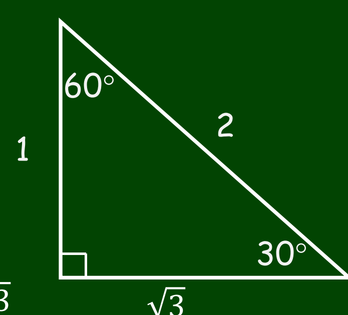
## 30 – 60 – 90 Right Triangle

Find all trigonometric ratios of each acute angle.

$\sin 30° = \frac{1}{2}$ $\quad$ $\csc 30° = 2$

$\cos 30° = \frac{\sqrt{3}}{2}$ $\quad$ $\sec 30° = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$

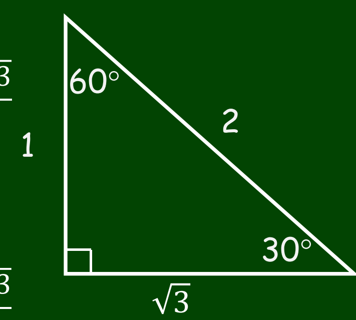
$\tan 30° = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$ $\quad$ $\cot 30° = \sqrt{3}$



## *Recall*: Special Right Triangles

### 30 – 60 – 90 Right Triangle

Find all trigonometric ratios of each acute angle.




$\sin 60° = \frac{\sqrt{3}}{2}$ $\quad \csc 60° = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$

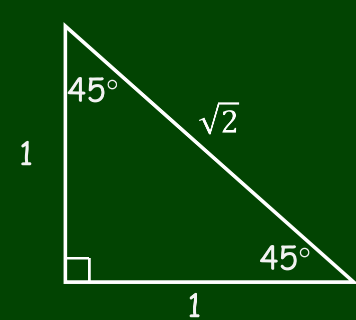
$\cos 60° = \frac{1}{2}$ $\quad \sec 60° = 2$

$\tan 60° = \sqrt{3}$ $\quad \cot 60° = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

## *Recall*: Special Right Triangles

### 45 – 45 – 90 Right Triangle

Find all trigonometric ratios of each acute angle.




$$\sin 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\cos 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\tan 45° = \frac{1}{1} = 1$$

## *Recall*: Special Right Triangles

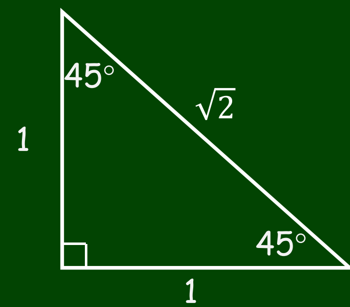
### 45 – 45 – 90 Right Triangle

Find all trigonometric ratios of each acute angle.

$$\csc 45° = \sqrt{2}$$

$$\sec 45° = \sqrt{2}$$

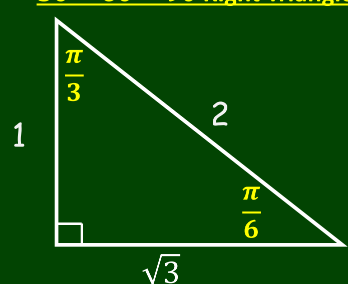
$$\cot 45° = \frac{1}{1} = 1$$



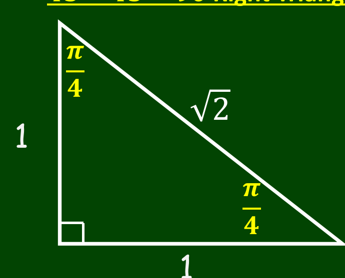

## Special Right Triangles using Radian Measure

Label the **acute** angles in each triangle using radian measure.

### 30 – 60 – 90 Right Triangle




### 45 – 45 – 90 Right Triangle




$\frac{\pi}{6} = 30°$ $\frac{\pi}{3} = 60°$ $\frac{\pi}{4} = 45°$

## *Recall*: Reference Angles

Examples: Determine the reference angle for each of the following radian angle measures.

**Recall: reference angle**
Positive, acute angle that is measured from the terminal side to the closest x-axis

Ex 1) $\theta = \frac{2\pi}{3}$

**Ref $\angle = \frac{\pi}{3}$**

Ex 2) $\theta = \frac{7\pi}{4}$

**Ref $\angle = \frac{\pi}{4}$**

Ex 3) $\theta = \frac{4\pi}{3}$

**Ref $\angle = \frac{\pi}{3}$**

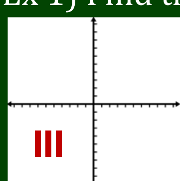
Ex 4) $\theta = -\frac{7\pi}{6}$

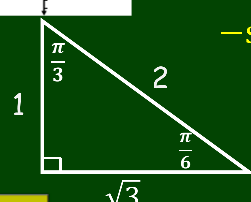
**Ref $\angle = \frac{\pi}{6}$**

## Using the Unit Circle to Evaluate Trig. Functions

Now, we are going to put together the concepts of the unit circle, special right triangles, and reference angles to evaluate trigonometric functions of angles that terminate in any quadrant.

Ex 1) Find the *exact value* of $\sin\frac{4\pi}{3}$.




$\sin\frac{4\pi}{3}$

**Ref. Angle = $\frac{\pi}{3}$**

$-\sin\frac{\pi}{3}$

$-\frac{\sqrt{3}}{2}$

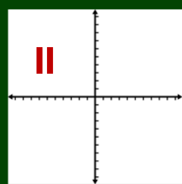
**Step 1: Sketch the angle, determine the measure of the reference angle, and the location of the angle.**

**Step 2: Determine the sign of the function (positive or negative).**

**Step 3: Evaluate the function (using special right triangles or the unit circle).**

## Using the Unit Circle to Evaluate Trig. Functions

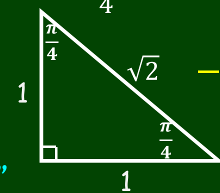
Ex 2) Find the *exact value* of $\sec\frac{3\pi}{4}$.




Ref. Angle = $\frac{\pi}{4}$

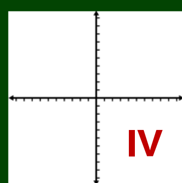
$-\sec\frac{\pi}{4}$

*think cosine & "flip"



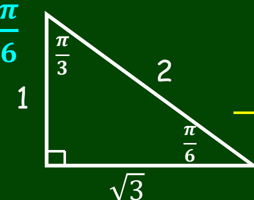

$-\frac{\sqrt{2}}{1} = \boxed{-\sqrt{2}}$

Ex 3) Find the *exact value* of $\tan\frac{11\pi}{6}$.




Ref. Angle = $\frac{\pi}{6}$

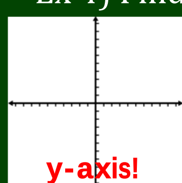
$-\tan\frac{\pi}{6}$




$-\frac{1}{\sqrt{3}} = \boxed{-\frac{\sqrt{3}}{3}}$

## Using the Unit Circle to Evaluate Trig. Functions

Ex 4) Find the *exact value* of $\sin\frac{3\pi}{2} + \cot\frac{11\pi}{4}$.



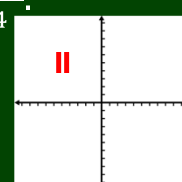

Quad. Angle = $\frac{3\pi}{2}$

*think: $y-$ value @ $\frac{3\pi}{2}$

$\sin\frac{3\pi}{2} = -1$

$\frac{11\pi}{4} - \frac{8\pi}{4} = \frac{3\pi}{4}$

coterminal!

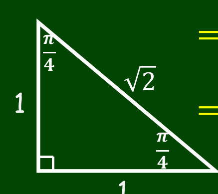



$\cot\frac{11\pi}{4}$

$= \cot\frac{3\pi}{4}$ Ref. Angle = $\frac{\pi}{4}$

$= -\cot\frac{\pi}{4} = -1$

*think tangent & "flip"



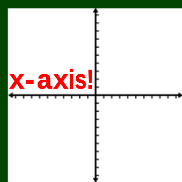

Answer:

$= -1 + (-1)$

$= \boxed{-2}$

# Using the Unit Circle to Evaluate Trig. Functions

Ex 5) Find the *exact value* of $\cos(-3\pi) + \tan\left(-\frac{5\pi}{6}\right)$.



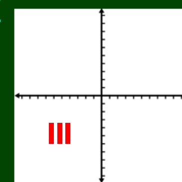

**Quad. Angle** $= -3\pi$

*think: $x-$ value @ $-3\pi$

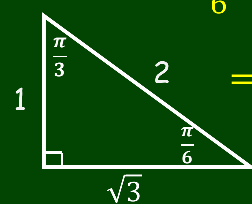
$\cos(-3\pi) = -1$

**Ref. Angle** $= \frac{\pi}{6}$

$\tan\left(-\frac{5\pi}{6}\right)$




$= +\tan\frac{\pi}{6}$




$= \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

Answer:

$= -1 + \frac{\sqrt{3}}{3}$

or

$= \frac{-3+\sqrt{3}}{3}$

# More Examples using the Unit Circle

Ex 6) For an angle β whose terminal side lies in the third quadrant, cos β = −0.96. Determine the value of sin β.

*Use the Pythagorean Identity*

$(\sin\theta)^2 + (\cos\theta)^2 = 1$

$(\sin\beta)^2 + (-.96)^2 = 1$

$(\sin\beta)^2 + (0.9216) = 1$

$(\sin\beta)^2 = 0.0784$

$\sqrt{(\sin\beta)^2} = \sqrt{0.0784}$

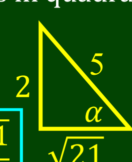
$\sin\beta = \pm 0.28$

$\sin\beta = -0.28$

Ex 7) Determine the value of $\cos\alpha$ and $\tan\alpha$ if $\csc\alpha = \frac{5}{2}$ $\frac{H}{O}$ if the terminal side of angle $\alpha$ lies in quadrant II.

Reciprocal of Sine

(1) Draw Triangle & determine quadrant
(2) Pythagorean Theorem
(3) Evaluate




$2^2 + b^2 = 5^2$

$b^2 = 21$

$b = \sqrt{21}$

$\cos\alpha = -\frac{\sqrt{21}}{5}$

$\tan\alpha = -\frac{2\sqrt{21}}{21}$

## More Examples using the Unit Circle

Ex 8) If $A(-3, 4)$ is a point on the terminal side of β, an angle in standard position, then $\cos\beta$ equals what?

(1) Plot the given point & create the triangle

(2) Determine quadrant & missing side of triangle

(3) Evaluate

$$\cos\beta = -\frac{3}{5}$$

## More Examples using the Unit Circle

Ex 9) If $P(4, 5)$ is a point on the terminal side of $x$, an angle in standard position, determine the *exact* values of all six trigonometric function of angle $x$.

(1) Plot the given point & create the triangle

(2) Determine quadrant & missing side of triangle

(3) Evaluate all six trig functions.

$$\sin x = \frac{5}{\sqrt{41}} = \frac{5\sqrt{41}}{41} \qquad \csc x = \frac{\sqrt{41}}{5}$$

$$\cos x = \frac{4}{\sqrt{41}} = \frac{4\sqrt{41}}{41} \qquad \sec x = \frac{\sqrt{41}}{4}$$

$$\tan x = \frac{5}{4} \qquad \cot x = \frac{4}{5}$$