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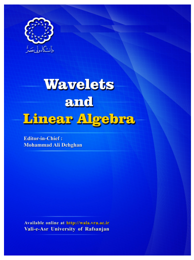

A computational wavelet method for numerical solution of stochastic Volterra-Fredholm integral equations

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Abstract

A Legendre wavelet method is presented for numerical solutions of stochastic Volterra-Fredholm integral equations. The main characteristic of the proposed method is that it reduces stochastic Volterra-Fredholm integral equations into a linear system of equations. Convergence and error analysis of the Legendre wavelets basis are investigated. The efficiency and accuracy of the proposed method was demonstrated by some non-trivial examples and comparison with the block pulse functions method.

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1. Introduction

Stochastic analysis has been an interesting research area in mathematics, fluid mechanics, geophysics, biology, chemistry, epidemiology, microelectronics, theoretical physics, economics, and finance. The behavior of dynamical systems in these areas are often dependent on a noise source and a Gaussian white noise, governed by certain probability laws, so that modeling such phenomena naturally requires the use of various stochastic differential equations or, in more complicated cases, stochastic integral equations and stochastic integro-differential equations. As in many cases analytic solutions of stochastic integral and differential equations are not available, numerical solution becomes a efficient way to challange this problem. Many research papers have been appeared on the problem of approximate the solution of stochastic integral and differential equations [1, 2, 3, 4, 7, 5, 6]. Recently, many orthogonal basis functions, such as block pulse functions, Walsh functions, Fourier series, orthogonal polynomials and wavelets, were used to estimate solution of functional equations. As a powerful tool, wavelets have found their way into many different fields of science and engineering. Wavelets permit the accurate representation of a variety of functions and operators. Moreover, wavelets establish a connection with fast numerical algorithms [8, 9]. Legendre wavelets have been widely applied in system analysis, system identification, optimal control and numerical solution of integral and differential equations[10, 11, 12, 13]. In this paper, an stochastic operational matrix for Legendre wavelets is derived. Then application of this stochastic operational matrix in solving stochastic Volterra-Fredholm integral equations is investigated. Some non-trivial examples are included to demonstrate the efficiency and accuracy of the proposed method. Also to verify the proposed method, numerical results are compared with the block pulse functions (BPFs) methed presented in [5]. This paper is organized as follows: In section 2 some basic definition and preliminaries about stochastic process and Itô integral are presented. The Legendre wavelets and their properties are introduced in Section 3. In section 4 stochastic operational matrix of the Legendre wavelets is derived. In section 5 application of this stochastic operational matrix in solving stochastic Voltera-Fredholm integral equations are described. In section 6 the efficiency of the proposed method is demonstrated by some non-trivial examples. Finally, a conclusion is given in section 7.

2. Preliminaries

In this section we review some basic definition of the stochastic calculus and the block pulse functions (BPFs).

2.1. Stochastic calculus

Definition 2.1. (Brownian motion process) A real-valued stochastic process $B(t), t \in [0, T]$ is called Brownian motion, if it satisfies the following properties
(i) The process has independent increments for $0 \leq t_0 \leq t_1 \leq ... \leq t_n \leq T$,
(ii) For all $t \geq 0$, $B(t+h) - B(t)$ has Normal distribution with mean 0 and variance h,
(iii) The function $t \to B(t)$ is continuous functions of t.

Definition 2.2. Let $\{\mathcal{N}_t\}_{t\geq 0}$ be an increasing family of σ-algebras of subsets of Ω. A process $g(t, \omega) : [0, \infty) \times \Omega \to \mathbb{R}^n$ is called $\mathcal{N}_t$-adapted if for each $t \geq 0$ the function $\omega \to g(t, \omega)$ is $\mathcal{N}_t$-measurable.

Definition 2.3. Let $\mathcal{V} = \mathcal{V}(S, T)$ be the class of functions $f(t, \omega) : [0, \infty) \times \Omega \to \times\mathbb{R}$ such that
(i) The function $(t, \omega) \to f(t, \omega)$ is $\mathcal{B}\times\mathcal{F}$-measurable, where $\mathcal{B}$ denotes the Borel algebra on $[0, \infty)$ and $\mathcal{F}$ is the σ -algebra on Ω.
(ii) f is adapted to $\mathcal{F}_t$, where $\mathcal{F}_t$ is the σ -algebra generated by the random variables $B(s)$, $s \leq t$.
(iii)$E\left(\int_S^T f^2(t, \omega)dt\right) < \infty$.

Definition 2.4. (The Itô integral) Let $f \in \mathcal{V}(S, T)$, then the Itô integral of f is defined by

$$\int_S^T f(t, \omega)dB(\omega) = \lim_{n\to\infty} \int_S^T \varphi_n(t, \omega)dB(\omega), \quad (\lim \text{in } L^2(P)),$$

where, φ_n is a sequence of elementary functions such that

$$E\left(\int_s^T (f(t, \omega) - \varphi_n(t, \omega))^2\, dt\right) \to 0, \quad as\ \ n \to \infty.$$

For more details about stochastic calculus and integration please see [2].

2.2. Block pulse functions

BPFs have been studied by many authors and applied for solving different problems. In this section we recall definition and some properties of the block pulse functions [4, 5, 14].

The m-set of BPFs are defined as

$$b_i(t) = \begin{cases} 1 & (i-1)h \ \leq t < ih \\ 0 & \ otherwise \end{cases} \tag{2.1}$$

in which $t \in [0, T)$, $i = 1, 2, ..., m$ and $h = \frac{T}{m}$. The set of BPFs are disjointed with each other in the interval $[0, T)$ and

$$b_i(t)b_j(t) = \delta_{ij}b_i(t), i, j = 1, 2, ..., m, \tag{2.2}$$

where δ_{ij} is the Kronecker delta. The set of BPFs defined in the interval $[0, T)$ are orthogonal with each other, that is

$$\int_0^T b_i(t)b_j(t)dt = h\delta_{ij}, \quad i, j = 1, 2, ..., m. \tag{2.3}$$

If $m \to \infty$ the set of BPFs is a complete basis for $L^2[0, T)$, so an arbitrary real bounded function $f(t)$, which is square integrable in the interval $[0, T)$, can be expanded into a block pulse series as

$$f(t) \simeq \sum_{i=1}^{m} f_i b_i(t), \tag{2.4}$$

where

$$f_i = \frac{1}{h}\int_0^T b_i(t)f(t)dt, \quad i = 1, 2, ..., m. \tag{2.5}$$

Rewritting Eq. (4.3) in the vector form we have

$$f(t) \simeq \sum_{i=1}^{m} f_i b_i(t) = F^T \Phi(t) = \Phi^T(t) F, \tag{2.6}$$

in which

$$\Phi(t) = [b_1(t), b_2(t), ..., b_m(t)]^T,$$
$$F = [f_1, f_2, ..., f_m]^T. \tag{2.7}$$

Morever, any two dimensional function $k(s,t) \in L^2([0,T_1] \times [0,T_2])$ can be expanded with respect to BPFs such as

$$k(s,t) = \Phi^T(t) \Pi \Phi(t), \tag{2.8}$$

where $\Phi(t)$ is the m-dimensional BPFs vectors respectively, and Π is the $m \times m$ BPFs coefficient matrix with (i, j)-th element

$$\Pi_{ij} = \frac{1}{h_1 h_2} \int_0^{T_1} \int_0^{T_2} k(s,t) b_i(t) b_j(s) dt ds, \quad i, j = 1, 2, ..., m, \tag{2.9}$$

and $h_1 = \frac{T_1}{m}$ and $h_2 = \frac{T_2}{m}$. Let $\Phi(t)$ be the BPFs vector, then we have

$$\Phi^T(t) \Phi(t) = 1, \tag{2.10}$$

and

$$\Phi(t)\Phi^T(t) = \begin{pmatrix} b_1(t) & 0 & \dots & 0 \\ 0 & b_2(t) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & b_m(t) \end{pmatrix}_{m \times m}. \tag{2.11}$$

For an m-vector F we have

$$\Phi(t)\Phi^T(t) F = \tilde{F} \Phi(t), \tag{2.12}$$

where $\tilde{F} = diag(F)$ is an $m \times m$ matrix which its diagonal arrays are elements of vector F. Also, it is easy to show that for an $m \times m$ matrix A

$$\Phi^T(t) A \Phi(t) = \hat{A}^T \Phi(t), \tag{2.13}$$

where $\hat{A} = (a_{11}, a_{22}, ..., .a_{nn})$ is an m-vector.

3. Legendre wavelets

Wavelets constitute a family of functions constructed from dilation and translation of a single function ψ called the mother wavelet. When the dilation parameter a and the translation parameter b vary continuously, we have the following family of continuous wavelets

$$\psi_{a,b}(t) = a^{-\frac{1}{2}}\psi\left(\frac{t-b}{a}\right), \;\; a, b \in \mathbb{R}, \; a \neq 0. \tag{3.1}$$

The Legendre wavelets are defined on the interval [0, 1) as

$$\psi_{mn}(t) = \begin{cases} \sqrt{m+\frac{1}{2}}\,2^{\frac{k+1}{2}} p_m\left(2^{k+1}t - (2n+1)\right) & \frac{n}{2^k} \leq t < \frac{n+1}{2^k} \\ 0 & otherwise, \end{cases} \tag{3.2}$$

where $n = 0, 1, ..., 2^k - 1$ and $m = 0, 1, \cdots, M-1$ is the degree of the Legendre polynomials for a fixed positive integer M. Here $P_m(t)$ are the well-known Legendre polynomials of degree m [10, 12]. Any square inegrable function $f(x)$ defined over [0, 1) can be expanded in terms of the extended Legendre wavelets as

$$f(x) \simeq \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{nm}\psi_{nm}(x) = C^T\Psi(x), \tag{3.3}$$

where $c_{mn} = (f(t), \psi_{mn}(t))$ and $(.,.)$ denotes the inner product on $L^2[0, 1]$. If the infinite series in (3.3) is truncated, then it can be written as

$$f(x) \simeq \sum_{n=0}^{2^k-1}\sum_{m=0}^{M-1} c_{mn}\psi_{mn}(x) = C^T\Psi(x), \tag{3.4}$$

where C and $\Psi(x)$ are $\hat{m} = 2^k M$ column vectors given by

$$C = \left[c_{00}, \ldots, c_{0(M-1)} | c_{10}, \ldots, c_{1(M-1)} |, \ldots, | c_{(2^k-1)0}, \ldots, c_{(2^k-1)(M-1)}\right]^T, \tag{3.5}$$
$$\Psi(x) = \left[\psi_{00}(x), \ldots, \psi_{0(M-1)}(x) | \psi_{10}(x), \ldots, \psi_{1(M-1)}(x) |, \ldots, | \psi_{(2^k-1)0}(x), \ldots, \psi_{(2^k-1)(M-1)}(x)\right]^T.$$

By changing indices in the vectors $\Psi(x)$ and C the series (3.4) can be rewritten as

$$f(x) \simeq \sum_{i=1}^{\hat{m}} c_i\psi_i(x) = C^T\Psi(x), \tag{3.6}$$

where

$$C = [c_1, c_2, ..., c_{\hat{m}}], \quad \Psi(x) = [\psi_1(x), \psi_2(x), ..., \psi_{\hat{m}}(x)], \tag{3.7}$$

and

$$c_i = c_{nm}, \quad \psi_i(x) = \psi_{nm}(x), \quad i = (n-1)M + m + 1. \tag{3.8}$$

Similarly, any two dimensional function $k(s, t) \in L^2\left([0, 1] \times [0, 1]\right)$ can be expanded into Legendre wavelets basis as

$$k(s, t) \approx \sum_{i=1}^{\hat{m}}\sum_{j=1}^{\hat{m}} k_{ij}\psi_i(s)\psi_j(t) = \Psi^T(s)K\Psi(t), \tag{3.9}$$

where $K = [k_{ij}]$ and $k_{ij} = \Big(\psi_i(s), \big(k(s, t), \psi_j(t)\big)\Big)$.

3.1. The Legendre wavelets and BPFs

In this section we will derive the relation between the Legendre wavelets and BPFs. It is worth mention that here we set $T = 1$ in definition of BPFs.

Theorem 3.1. *Let $\Psi(t)$ and $\Phi(t)$ be the $\hat{m}$-dimensional Legendre wavelets and BPFs vector respectively, the vector $\Psi(t)$ can be expanded by BPFs vector $\Phi(t)$ as*

$$\Psi(t) \simeq Q\Phi(t), \tag{3.10}$$

where Q is an $\hat{m} \times \hat{m}$ block matrix and

$$Q_{ij} = \psi_i\left(\frac{2j-1}{2\hat{m}}\right), i, j = 1, 2, ..., \hat{m} \tag{3.11}$$

Proof. Let $\phi_i(t), i = 1, 2, ..., \hat{m}$ be the i-th element of Legendre wavelets vector. Expanding $\phi_i(t)$ into an $\hat{m}$-term vector of BPFs, we have

$$\psi_i(t) \simeq \sum_{i=1}^{\hat{m}} Q_{ij} b_j(t) = Q_i^T \Phi(t), \quad i = 1, 2, ..., \hat{m}, \tag{3.12}$$

where Q_i is the i-th row and Q_{ij} is the (i, j)-th element of matrix Q. By using the orthogonality of BPFs we have

$$Q_{ij} = \frac{1}{h}\int_0^1 \psi_i(t) b_j(t) dt = \frac{1}{h}\int_{\frac{j-1}{\hat{m}}}^{\frac{j}{\hat{m}}} \psi_i(t) dt = \hat{m}\int_{\frac{j-1}{\hat{m}}}^{\frac{j}{\hat{m}}} \psi_i(t) dt, \tag{3.13}$$

by using mean value theorem for integrals in the last equation we can write

$$Q_{ij} = \hat{m}\left(\frac{j}{\hat{m}} - \frac{j-1}{\hat{m}}\right)\psi_i(\eta_i) = \psi_i(\eta_j), \quad \eta_j \in \left(\frac{j-1}{\hat{m}}, \frac{j}{\hat{m}}\right), \tag{3.14}$$

now by choosing $\eta_j = \frac{2j-1}{2\hat{m}}$ so we have

$$Q_{ij} = \psi_i\left(\frac{2j-1}{2\hat{m}}\right), i, j = 1, 2, ..., \hat{m}. \tag{3.15}$$

and this prove the desired result. □

The following remarks are the straight result of relations (2.12), (2.13) and Theorem 3.1.

Remark 3.2. For an $\hat{m}$-vector F we have

$$\Psi(t)\Psi^T(t)F = \tilde{F}\Psi(t), \tag{3.16}$$

in which $\tilde{F}$ is an $\hat{m} \times \hat{m}$ matrix as

$$\tilde{F} = Q\bar{F}Q^{-1}, \tag{3.17}$$

where $\bar{F} = diag\left(Q^T F\right)$.

Remark 3.3. Let A be an arbitrary $\hat{m} \times \hat{m}$ matrix, then for the Legendre wavelets vector $\Psi(t)$ we have

$$\Psi^T(t)A\Psi(t) = \hat{A}^T\Psi(t), \tag{3.18}$$

where $\hat{A}^T = UQ^{-1}$ and $U = diag(Q^T AQ)$ is a $\hat{m}$-vector.

3.2. Convergence and error analysis

In this section we investigate the convergence and error analysis of the Legendre wavelets basis.

Theorem 3.4. *Let $f(x)$ be a function defined on $[0,1)$ with bounded second derivatives, say $|f''(x)| \leq M$, and $\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{mn}\psi_{mn}(x)$ be its infinite Legendre wavelets expansion, then*

$$|c_{mn}| \leq \frac{\sqrt{12}M}{(2n)^{\frac{5}{2}}(2m-3)^2}, \tag{3.19}$$

this means the Legendre wavelets series converges uniformly to $f(x)$ and

$$f(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{nm}\psi_{nm}(x), \tag{3.20}$$

Proof. Please see [15]. □

Theorem 3.5. *Suppose $f(x)$ be a continuous function defined on $[0,1)$, with second derivatives $f''(x)$ bounded by M, then we have the following accuracy estimation*

$$\left\|e_{M,k}(t)\right\|_2 \leq \left(\frac{3M^2}{2}\sum_{n=0}^{\infty}\sum_{m=M}^{\infty}\frac{1}{n^5(2m-3)^4} + \frac{3M^2}{2}\sum_{n=2^k}^{\infty}\sum_{m=0}^{M-1}\frac{1}{n^5(2m-3)^4}\right)^{\frac{1}{2}}, \tag{3.21}$$

where

$$\left\|e_{M,k}(t)\right\|_2 = \left(\int_0^1\left(f(x) - \sum_{n=0}^{2^k-1}\sum_{m=0}^{M-1} c_{nm}\psi_{nm}(x)\right)^2 dx\right)^{\frac{1}{2}}.$$

Proof. We have:

$$\begin{aligned}\sigma_{M,k}^2 &= \int_0^1\left(f(x) - \sum_{n=0}^{2^k-1}\sum_{m=0}^{M-1} c_{nm}\psi_{nm}(x)\right)^2 dx\\ &= \int_0^1\left(\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{nm}\psi_{nm}(x) - \sum_{n=0}^{2^k-1}\sum_{m=0}^{M-1} c_{nm}\psi_{nm}(x)\right)^2 dx\\ &= \sum_{n=0}^{\infty}\sum_{m=M}^{\infty} c_{nm}^2\int_0^1\psi_{nm}^2(x)dx + \sum_{n=2^k}^{\infty}\sum_{m=0}^{M-1} c_{nm}^2\int_0^1\psi_{nm}^2(x)dx = \sum_{n=0}^{\infty}\sum_{m=M}^{\infty} c_{nm}^2 + \sum_{n=2^k}^{\infty}\sum_{m=0}^{M-1} c_{nm}^2,\end{aligned}$$

now by considering Eq. (3.19), the desired result is achieved. □

4. Stochastic operational matrix of Legendre wavelets

In this section we derive an stochastic integration operational matrix for Legendre wavelets. In this way we first remind some useful results for BPFs[4, 5].

Lemma 4.1. *[4] Let $\Phi(t)$ be the $\hat{m}$-dimensional BPFs vector defined in (2.7), then integration of this vector can be derived as*

$$\int_0^t \Phi(s)ds \simeq P\Phi(t), \tag{4.1}$$

where P is called the operational matrix of integration for BPFs and is given by

$$P = \frac{h}{2}\begin{bmatrix} 1 & 2 & 2 & \dots & 2 \\ 0 & 1 & 2 & \dots & 2 \\ 0 & 0 & 1 & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 2 \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}_{\hat{m}\times\hat{m}}. \tag{4.2}$$

Lemma 4.2. *[4] Let $\Phi(t)$ be the $\hat{m}$-dimensional BPFs vector defined in (2.7), the Itô integral of this vector can be derived as*

$$\int_0^t \Phi(s)dB(s) \simeq P_s\Phi(t), \tag{4.3}$$

where P_s is called the stochastic operational matrix of BPFs and is given by

$$P_s = \begin{bmatrix} B\left(\frac{h}{2}\right) & B(h) & B(h) & \dots & B(h) \\ 0 & B\left(\frac{3h}{2}\right) - B(h) & B(2h) - B(h) & \dots & B(2h) - B(h) \\ 0 & 0 & B\left(\frac{5h}{2}\right) - B(2h) & \dots & B(3h) - B(2h) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & B\left(\frac{(2\hat{m}-1)h}{2}\right) - B((\hat{m}-1)h) \end{bmatrix}_{\hat{m}\times\hat{m}}. \tag{4.4}$$

Now we are ready to derive a new operational matrix of stochastic integration for the Legendre wavelets basis. For this end we use BPFs and the matrix Q introduced in (3.10).

Theorem 4.3. *Suppose $\Psi(t)$ be the $\hat{m}$-dimensional Legendre wavelets vector defined in (3.7), the integral of this vector can be derived as*

$$\int_0^t \Psi(s)ds \simeq QPQ^{-1}\Psi(t) = \Lambda\Psi(t), \tag{4.5}$$

where Q is introduced in (3.10) and P is the operational matrix of integration for BPFs derived in (4.2).

Proof. Let $\Psi(t)$ be the Legendre wavelets vector, by using Theorem 3.1 and Lemma 4.1 we have

$$\int_0^t \Psi(s)ds \simeq \int_0^t Q\Phi(s)ds = Q\int_0^t \Phi(s)ds = QP\Phi(t), \tag{4.6}$$

now Theorem 3.1 give

$$\int_0^t \Psi(s)ds \simeq QP\Phi(t) = QPQ^{-1}\Psi(t) = \Lambda\Psi(t), \tag{4.7}$$

and this complete the proof. □

Theorem 4.4. *Suppose $\Psi(t)$ be the $\hat{m}$-dimensional Legendre wavelets vector defined in (3.7), the Itô integral of this vector can be derived as*

$$\int_0^t \Psi(s)dB(s) \simeq QP_sQ^{-1}\Psi(t) = \Lambda_s\Psi(t), \tag{4.8}$$

where Λ_s is called stochastic operational matrix for Legendre wavelets, Q is introduced in (3.10) and P_s is the stochastic operational matrix of integration for BPFs derived in (4.4).

Proof. Let $\Psi(t)$ be the Legendre wavelets vector, by using Theorem 3.1 and Lemma 4.2 we have

$$\int_0^t \Psi(s)dB(s) \simeq \int_0^t Q\Phi(s)dB(s) = Q\int_0^t \Phi(s)dB(s) = QP_s\Phi(t), \tag{4.9}$$

now Theorem 3.1 result

$$\int_0^t \Psi(s)dB(s) = QP_s\Phi(t) = QP_sQ^{-1}\Psi(t) = \Lambda_s\Psi(t), \tag{4.10}$$

and this complete the proof. □

5. Numerical solution of stochastic Voltera-Fredholm integral equation

In this section, we use the stochastic operational matrix of Legendre wavelets for solving stochastic Voltera-Fredholm integral equations. In this way, consider the following stochastic Voltera-Fredholm integral equation

$$X(t) = f(t) + \int_\alpha^\beta X(s)k_1(s,t)ds + \int_0^t X(s)k_2(s,t)ds + \int_0^t X(s)k_3(s,t)dB(s), \quad t \in [0,T), \tag{5.1}$$

where $X(t)$, $f(t)$ and $k_i(s,t), i = 1,2,3$ are the stochastic processes defined on the same probability space (Ω, F, P), and $X(t)$ is unknown. Also $B(t)$ is a Brownian motion process and $\int_0^t k_3(s,t)X(s)dB(s)$ are the Itô integral. For sake of simplicity and without loss of generality we set $(\alpha,\beta) = (0,1)$. Now, we approximate $X(t)$, $f(t)$ and $k_i(s,t), i = 1,2,3$ in term of $\hat{m}$-dimensional Legendre wavelets as follows

$$f(t) = F^T\Psi(t) = \Psi^T(t)F, \tag{5.2}$$

$$X(t) = X^T \Psi(t) = \Psi^T(t) X, \tag{5.3}$$

$$k_i(s,t) = \Psi^T(s) K_i \Psi(t) = \Psi^T(t) K_i^T \Psi(s), i = 1, 2, 3, \tag{5.4}$$

where X and F are Legendre wavelets coefficients vector, and $K_i, i = 1, 2, 3$ are Legendre wavelets coefficient matrices defined in Eq. (3.7) and Eq. (3.9). Substituting above approximations in Eq. (5.1), we have

$$X^T \Psi(t) = F^T \Psi(t) + X^T \left(\int_0^1 \Psi(s) \Psi^T(s) ds \right) K_1 \Psi(t)$$
$$+ \Psi^T(t) K_2^T \left(\int_0^t \Psi(s) \Psi^T(s) X ds \right) + \Psi^T(t) K_3^T \left(\int_0^t \Psi(s) \Psi^T(s) X dB(s) \right),$$

using relation $\int_0^1 \Psi(s) \Psi^T(s) ds = I_{\hat{m} \times \hat{m}}$ and Remark 3.2 we get

$$X^T \Psi(t) = F^T \Psi(t) + X^T K_1 \Psi(t) + \Psi^T(t) K_2^T \left(\int_0^t \tilde{X} \Psi(s) ds \right) + \Psi^T(t) K_3^T \left(\int_0^t \tilde{X} \Psi(s) dB_i(s) \right),$$

where $\tilde{X}$ is an $\hat{m} \times \hat{m}$ matrix. Now applying the operational matrices Λ and Λ_s for Haar wavelets derived in Eqs. (4.5) and (4.8) we have

$$X^T \Psi(t) = F^T \Psi(t) + X^T K_1 \Psi(t) + \Psi^T(t) K_2^T \tilde{X} \Lambda \Psi(t) + \Psi^T(t) K_3^T \tilde{X} \Lambda_s \Psi(t) \tag{5.5}$$

by setting $Y_2 = K_2^T \tilde{X} \Lambda,\ Y_3 = K_3^T \tilde{X} \Lambda_s$ and using Remark 3.3 we derive

$$X^T \Psi(t) - X^T K_1 \Psi(t) - \hat{Y}_2^T \Psi(t) - \hat{Y}_3^T \Psi(t) = F^T \Psi(t), \tag{5.6}$$

in which $\hat{Y}_2$ and $\hat{Y}_3$ are $\hat{m} \times \hat{m}$ matrix and they are linear function of vector X. Eq. (5.6) is hold for any $t \in [0, 1)$, so we can write

$$X^T - X^T K_1 - \hat{Y}_2^T - \hat{Y}_3^T = F^T. \tag{5.7}$$

Since $\hat{Y}_2$ and $\hat{Y}_3$ are linear function of X, Eq. (5.7) is a linear system for unknown vector X. Solving this linear system and determining X, we can approximate solution of stochastic Voltera-Fredholm integral equation (5.1) by substituting obtained vector X in Eq. (5.3).

6. Numerical examples

Here we demonstrate the efficiency and accuracy of the Legendre wavelets method (LWM) by some non-trivial examples. All algorithms are performed by Maple 13 with 20 digits precision.

Example 6.1. Consider the following stochastic Volterra-Fredholm integral equation [5]

$$X(t) = f(t) + \int_0^1 \cos(s+t) X(s) ds + \int_0^t (s+t) X(s) ds + \int_0^t e^{-3(s+t)} X(s) dB(s), \quad s, t \in [0, 1],$$

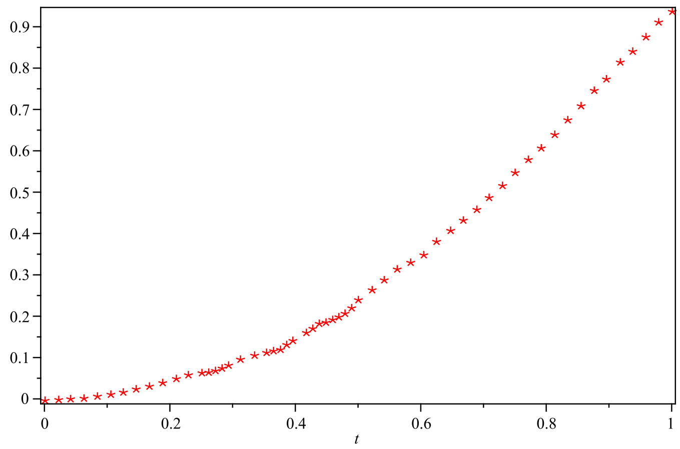

Figure 1: The approximate solution for $\hat{m} = 64$.

	$\hat{m} = 32$		$\hat{m} = 64$	
t	LWM	BPFs[5]	LWM	BPFs[5]
0.2	0.0373360485	0.0566018117	0.0125320418	0.0162899633
0.4	0.1429836416	0.1550820154	0.1140858304	0.1151902625
0.6	0.3412840467	0.3908514112	0.3128060572	0.3840300664
0.8	0.6250649268	0.6338163380	0.5873515014	0.6993271966
1.0	0.9403443751	0.9684881988	0.9055792720	1.0017286969

Table 1: Numerical results for $\hat{m} = 32$ and $\hat{m} = 64$.

in which

$$f(t) = t^2 + \sin(1+t) - 2\cos(1+t) - 2\sin(t) - \frac{7t^4}{12} + \frac{1}{40}B(t),$$

and $X(t)$ is an unknown stochastic process defined on the probability space $(\Omega, \mathcal{F}, P)$ and $B(t)$ is a Brownian motion process. The stochastic operational matrix of Legendre wavelets and the proposed method in section 5 are used for solving this stochastic Volterra-Fredholm integral equation. Fig. 6.1 presents the approximate solution computed by LWM for $\hat{m} = 64$. A comparison between the numerical results given by the LWM and BPFs method [5] are shown in Table 1.

Example 6.2. Consider the following stochastic Volterra-Fredholm integral equation[5]

$$X(t) = f(t) + \int_0^1 (s+t)X(s)ds + \int_0^t (s-t)X(s)ds + \frac{1}{125}\int_0^t \sin(s+t)X(s)dB(s), \;\; s,t \in [0,1],$$

where

$$f(t) = 2 - \cos(1) - (1+t)\sin(1) + \frac{1}{250}\sin(B(t)),$$

and $X(t)$ is an unknown stochastic process defined on the probability space $(\Omega, \mathcal{F}, P)$ and $B(t)$ is a Brownian motion process. The stochastic operational matrix of Legendre wavelets is employed for deriving numerical solution of this Volterra-Fredholm integral equation. Fig. 6.1 presents the approximate solution computed by the LWM for $\hat{m} = 64$. A comparison between the numerical results given by the LWM and BPFs method [5] are shown in Table 2.

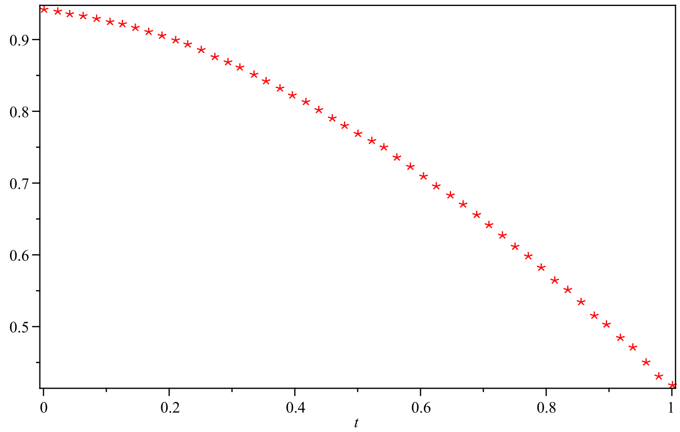

Figure 2: The approximate solution for $\hat{m} = 64$.

	$\hat{m} = 32$		$\hat{m} = 64$	
t	LWM	BPFs[5]	LWM	BPFs[5]
0.2	0.8494112576	0.9860154776	0.9024267433	0.9833522815
0.4	0.7565728441	0.9432021950	0.8208246618	0.9157653040
0.6	0.6381295249	0.8554015473	0.7124650476	0.8042753408
0.8	0.4957875008	0.7250865831	0.5758998949	0.6954537702
1.0	0.3406800668	0.5459802735	0.4191384270	0.5713651151

Table 2: Numerical results for $\hat{m} = 32$ and $\hat{m} = 64$.

7. Conclusion

A computational method based on the Legendre wavelets and their stochastic operational matrix is proposed for solving stochastic Volterra-Fredholm integral equations. Convergence and error analysis of the Legendre wavelets basis are considerd. Accuracy and efficiency of the proposed method is confirmed by some non-trivial numerical examples. Moreover, the numerical results of the proposed method is in good agreement with the BPFs method.

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