The Mathematics of Juggling
by Burkard Polster

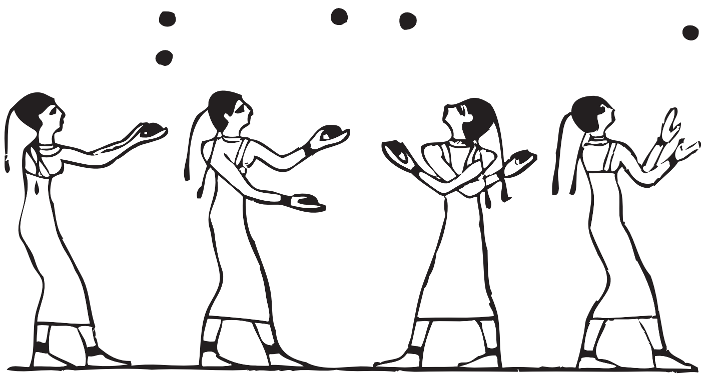
Next time you see some jugglers practising in a park, ask them whether they like mathematics. Chances are, they do. In fact, a lot of mathematically wired people would agree that juggling is "cool" and most younger mathematicians, physicists, computer scientists, engineers, etc. will at least have given juggling three balls a go at some point in their lives.

I myself also belong to this category and, although I am only speaking for myself, I am sure that many serious mathematical jugglers would agree that the satisfaction they get out of mastering a fancy juggling pattern is very similar to that of seeing a beautiful equation, or proof of a theorem click into place.

Given this fascination with juggling, it is probably not surprising that mathematical jugglers have investigated what mathematics can be found in juggling. But before we embark on a tour of the mathematics of juggling, here is a little bit of a history.

1 A mini history

The earliest historical evidence of juggling is a 4000 year old wall painting in an ancient Egyptian tomb. Here is a tracing of part of this painting showing four jugglers juggling up to three objects each.



The earliest juggling mathematician we know of is Abu Sahl al-Kuhi who lived around the 10th century. Before becoming famous as a mathematician, he juggled glass bottles in the market place of Baghdad.

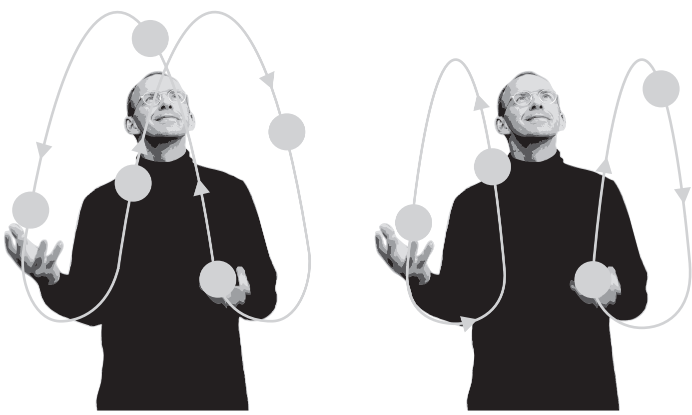
But he seems to be the exception. Until quite recently, it was mostly professional circus performers or their precursors, who engaged in juggling. There are some countries, Japan and Tonga for example, where for a long time juggling was a popular game practised by girls. But in most countries the professionals dominated until the second half of the 20th century. It was at that time that more and more people got into juggling as a hobby. This was particularly true among college students, and many juggling clubs were formed at colleges and universities around that time. Today there are tens of thousands of amateur jugglers, all around the world.

It was around 1985 that at least three groups of people independently from each other started developing and popularizing a mathematical language for noting juggling patterns: Bengt Magnusson and Bruce 'Boppo' Tiemann in Los Angeles; Paul Klimak in Santa Cruz; and Adam Chalcraft, Mike Day, and Colin Wright in Cambridge.[1]

Based on this mathematical method of describing juggling patterns, there are a number of freely available juggling simulators. If you are not a juggler yourself (and even if you are), I recommend that you download and play with some of these simulators before you read on. To get you started, I particularly recommend *Juggling Lab* by Jack Boyce, arguably the most powerful and useful simulator for mathematical juggling involving just one juggler. To find out what is possible in terms of simulating multiple jugglers manipulating chainsaws while riding unicyles in 3D, have a play with *JoePass!* by Werner Westerboer, or *Jongl* by Hermann Riebesel.

2 Juggling numbers

Let's have a close look at a juggler juggling the basic 5-ball pattern (left) and the basic 4-ball pattern (right). In the basic 5-ball pattern all balls trace a somewhat distorted infinity sign, and the same is true for all basic juggling patterns using an odd number of balls. On the other hand, all the basic even-ball juggling patterns split up into two circular patterns that are juggled independently by the left and right hands.



Our juggler juggles to a certain regular beat. For simplicity, we shall assume that balls are caught and immediately thrown again on the beats

[1]It should be noted, however, that there were a few earlier attempts at tackling juggling mathematically. In particular, Claude Shannon, the famous information theorist, wrote a paper in 1981 on the *Scientific Aspects of Jugging* [14]. However, Shannons paper only got published a decade after it was written.

(hot-potato style) and that the left and right hands take turns doing this. This means that *on every beat at most one ball gets caught and tossed again.*[2]

The higher a ball gets tossed the more beats it stays in the air. We call a throw a 1-throw, 2-throw, 3-throw, etc. if it keeps a ball exactly 1, 2, 3, etc. beats in the air. For example, in the basic n-ball pattern every ball stays n beats in the air. On some beats it may happen that no ball lands, and that therefore also no ball gets tossed. We express this by saying that a 0-throw gets performed.

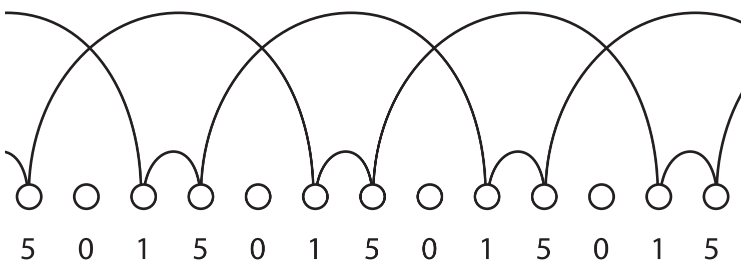
To avoid worrying about starting and stopping, let's assume that our juggler has been juggling forever and will never stop. Then recording the numbers in the different throws that a juggler makes on consecutive beats, we arrive at an infinite sequence of numbers. This sequence is ...5, 5, 5, 5, 5,... in the case of the basic 5-ball pattern. For some other pattern this sequence may turn out to be ...5, 0, 1, 5, 0, 1, 5, 0, 1,...

Note that because juggling patterns performed by real jugglers eventually repeat so will these sequences. This means that to pin down such a sequence we only need to record a small part of the sequence that when repeated gives the whole infinite sequence. So, for example, 5 and $5, 5$ and $5, 5, 5, 5$ all capture the first sequence, and $5, 0, 1$ and $1, 5, 0$ and $0, 1, 5$ and $5, 0, 1, 5, 0, 1, 5, 0, 1$ all capture the second sequence. These finite strings of numbers are called *juggling sequences* or *site swaps*.[3]

3 Juggling diagrams

This immediately raises the question: Are there finite sequences of nonnegative integers that are not juggling sequences? Well, obviously there are, since we all have physical limitations—we surely cannot juggle any pattern with juggling sequence $1, 1, 100000$. However, if we assume for the moment that there are no physical limitations, what juggling sequences can actually be juggled?

To decide whether or not a sequence can be juggled we draw its *juggling diagram*. Here is the juggling diagram of the sequence $5, 0, 1$.

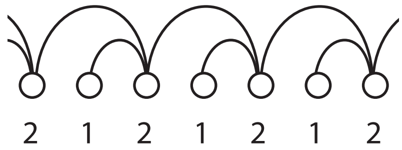


[2]There are also ways of catching and tossing multiple objects with one hand or with both hands simultaneously. It is even possible for a number of jugglers to pass props back and forth between them. They can even juggle themselves while they are doing this by moving around in intricate patterns. There are extensions to the basic mathematical model for juggling that we will be focussing on in this article that cover all these more complicated scenarios.

[3]It is common to skip the commas in a juggling sequence if none of its elements is greater then 9, but for the purpose of this article it turns out to be more convenient to leave the commas in.

The dots at the bottom stand for the beats on a virtual timeline, the arches represent the different throws. Looking at this diagram we can tell at a glance that $5, 0, 1$ can be juggled. Why? Because the diagram extends indefinitely to both sides as indicated and on every beat at most one ball lands and gets tossed again.

A sequence that cannot be juggled is $2, 1$. This becomes obvious when we draw its juggling diagram.



In order to juggle 21, a juggler would have to catch two incoming balls in one hand on every second beat. Moreover, he would need to have balls mysteriously materialize and vanish. None of this is possible within our simple juggling formulation, and therefore 2,1 cannot be juggled. In fact, everything that can go wrong, does go wrong in the case of 2,1.

Juggling Theorem 1 *To check whether a finite sequence of numbers is jugglable simply draw a large enough part of its juggling diagram to capture the periodic nature of the sequence, and then check that on every dot representing a beat: (1) either exactly one arch ends and one starts or no arches end and start; (2) all dots with no arches correspond to 0-throws.*

It should be clear that all the different juggling sequences that capture the same pattern will yield the same juggling diagram. For example, $5, 0, 1$ and $1, 5, 0$ and $0, 1, 5$ and $5, 0, 1, 5, 0, 1, 5, 0, 1$ all correspond to the same juggling diagram.

4 An algebraic juggling detector

All this is very well for short sequences consisting of small numbers, but what about checking a sequence like 1000000, 1, 1? Fancy drawing (at least) a million dots? Luckily, there is an easier algebraic method. Here is what you do.

Let's say we want to check whether

$$4, 4, 1, 3$$

is jugglable. The number of elements in the sequence is called its *period.* This means the period of $4, 4, 1, 3$ is 4. We form a second sequence by adding 0, 1, 2, 3 to the elements of the first sequence. This gives

$$4 + \mathbf{0}, 4 + \mathbf{1}, 1 + \mathbf{2}, 3 + \mathbf{3} = 4, 5, 3, 6.$$

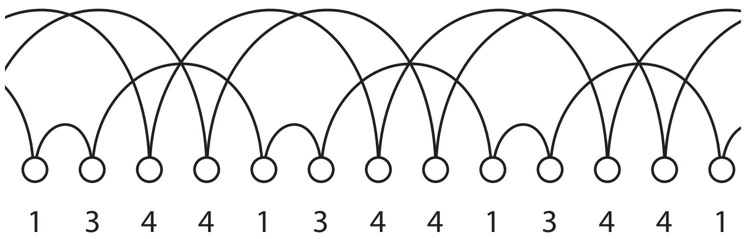
Following this we form the *test sequence* whose elements are the remainders of the elements of the second sequence when you divide them by the period 4. This gives

$$0, 1, 3, 2.$$

Now, it has been shown that

Juggling Theorem 2 *If all the numbers in a test sequence are different, then the original sequence is a juggling sequence, and otherwise it is not.*

This means that $4, 4, 1, 3$ is a juggling sequence. To double-check, here is its juggling diagram.



We already know that $2, 1$ cannot be juggled. Let's confirm this using our algebraic juggling detector. In this case, the period is 2 and the second sequence is $2+0, 1+1 = 2, 2$. This means that the test sequence is $0, 0$. Two 0s means no juggling sequence.

5 New from old

Here are a few easy consequences of this basic juggling theorem that allow us to make new juggling sequences from given ones.

Adding or subtracting the period

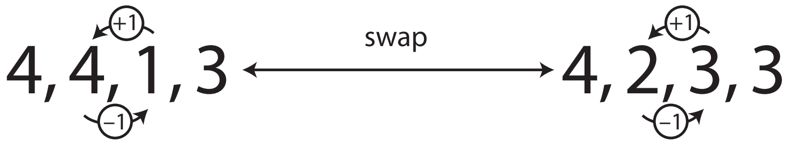
First, adding the period of a sequence to any of its elements does not change the associated test sequence. This means that starting with a juggling sequence, we can make up more juggling sequences by simply adding multiples of the period to its elements. Subtracting the period is also possible as long as the resulting sequence does not contain any negative numbers. For example, $4, 4, 1, 3$ has period 4. Therefore $4, 0, 1, 3$ and $8, 4, 5, 7$ and $0, 0, 1, 3$, etc. are all juggling sequences.

Scramblable juggling sequences

Simply permuting the elements of a juggling sequence usually results in a sequence that cannot be juggled. However, there are *scramblable* juggling sequences that stay jugglable no matter how you permute their elements. Clearly, any constant juggling sequence and any juggling sequence derived from a constant sequence using the adding and subtracting method is scramblable. In fact, it is easy to see that all scramblable juggling sequences arise in this manner. For example, starting with 5, 5, 5 we add the period 3 to the first 5 and subtract it from the second 5 to arrive at the scramblable juggling sequence 8, 2, 5.

Swap

From what we just said it should also be clear that simply swapping two adjacent elements in a juggling sequence will usually not result in another juggling sequence. However, you will always get a new juggling sequence if you swap two adjacent elements in a juggling sequence, adding 1 to the element that is moving left and subtracting 1 from the element that is moving right.[4] Why? Because at the level of the test sequence this results in a simple swap of the corresponding elements. For example, performing this *swap operation* on the second and third elements of $4, \mathbf{4}, \mathbf{1}, 3$ gives $4, 2, 3, 3$. Note also that applying the same swapping operation to the second sequence gets you back to the first sequence.



But let's not stop here and let's also swap the first and second elements of the second sequence $\mathbf{4}, \mathbf{2}, 3, 3$. This gives the constant juggling sequence $3, 3, 3, 3$. In fact, it is fairly easy to see that any juggling sequence can be turned into a constant juggling sequence by repeatedly applying this swapping operation together with the occasional cyclic permutation (which in essence leaves the sequence unchanged). Here is an example in which we transform the juggling sequence $2, 6, 4$ into the constant juggling sequence $4, 4, 4$:

$$2, \mathbf{6}, \mathbf{4} \xrightarrow{\text{swap}} 2, 5, 5 \xrightarrow{\text{cycle}} 5, \mathbf{5}, \mathbf{2} \xrightarrow{\text{swap}} \mathbf{5}, \mathbf{3}, 4 \xrightarrow{\text{swap}} 4, 4, 4$$

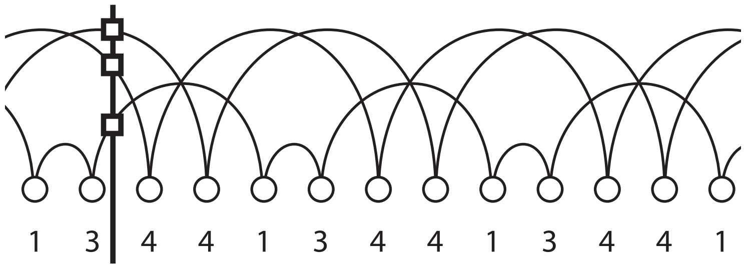
Of course, it is also possible to move in the opposite direction and transform $4, 4, 4$ into $6, 4, 2$ using the same operations.

Juggling Theorem 3 *Any juggling sequence can be transformed into a constant juggling sequence using swap operations and cyclic permutations. Conversely, any juggling sequence can be constructed from a constant juggling sequence using swap operations and cyclic permutations.*

6 How many balls?

Okay, you just verified in two different ways that the numbers in your date of birth form a juggling sequence. Of course, this means that you are destined to be a juggler, and the first thing you want to do is try and juggle your date of birth. But how many balls do you need to do this? The answer is hiding in the juggling diagram. For example, have another look at the juggling diagram of $4, 4, 1, 3$. Since all the balls are in the air in between beats the number of balls is simply the number of arches above a point in between two consecutive beat points. In the case of 4, 4, 1, 3 we find that there are three balls.

[4]Obviously, when you perform a swap like this you have to ensure that the element from which 1 is subtracted is not a 0.



However, as we already mentioned, it is not always easy to draw a juggling diagram. Luckily, there is an even easier way to determine the number of balls.

Juggling Theorem 4 *The number of balls needed for juggling a juggling sequence is the average of the numbers in the sequence.*

For example, using this theorem we confirm that we need

$$\frac{4+4+1+3}{4} = \frac{12}{4} = 3$$

balls to juggle $4, 4, 1, 3$. And we need exactly $n/1 = n$ balls to juggle the basic n-ball juggling sequence.

An ingenious proof of this fundamental result uses the swapping operation that we introduced in the previous section. It is easy to check that this operation does not change the number of balls and also does not change the average of a sequence. Consequently, after transforming a juggling sequence into a constant sequence $c, c, c, c, ..., c$ as in Juggling Theorem 3, both the average and the number of balls needed are the same for both sequences. Theorem 4 then follows once you observe that for the constant sequence both the average and the number of balls needed are equal to c.

This result also implies that a given mystery sequence cannot possibly be a juggling sequence if its average is not an integer. For example, the sequence $5, 1, 1$ is not a juggling sequence—no need for drawing a juggling diagram or doing any more calculations.

On the other hand, it is important to realize that if the average of a certain mystery sequence is an integer, *this does not identify it as a juggling sequence.*[5] For example, $3, 2, 1$ is not a juggling sequence even though the average of 3, 2, and 1 is an integer.

Having said this, it is interesting to note that

Juggling Theorem 5 *Given any sequence of non-negative integers whose average is an integer, this sequence can always be rearranged into a juggling sequence.*[6]

For example, we can rearrange our previous example $3, 2, 1$ into the juggling sequence $3, 1, 2$.

[5] This should not come as a surprise given that we already mentioned that most juggling sequences are not scramblable.

[6] The proof for this result is a corollary of a theorem about abelian groups proved by Marshall Hall; see [13] for details. This is a much deeper result than any of the other juggling theorems mentioned so far.

Magic juggling sequences

As a little application of all of the above, let's have a look at *magic juggling sequences*. In analogy to magic squares, a juggling sequence of period p is magical if it contains all the integers $1, 2, 3, ..., p$. For which periods p do magic juggling sequences exist? By the last theorem, this is the same as asking for which p is the average

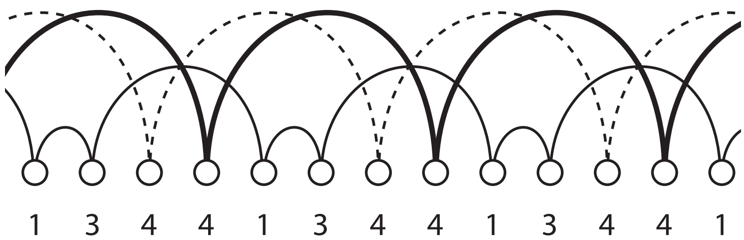
$$(1 + 2 + 3 + ... + p)/p = (p + 1)/2$$

an integer? Obviously, this is the case if and only if p is an odd number. In fact, you can use our algebraic juggling detector to prove that the sequence $1, 2, 3, ..., p$ can be juggled whenever its period p is an odd number. Usually, there are a few different magic juggling sequences for each odd period. For example, 12345, 13524, and 14253 are all the magic juggling sequences of period 5 (up to cyclic permutations).

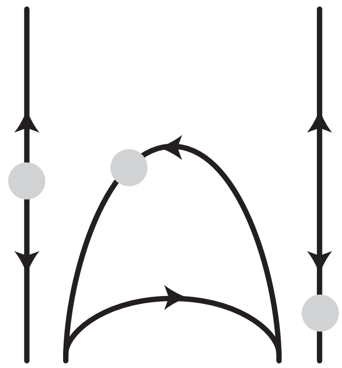
Juggling diagrams ARE useful!

Considering that it is so easy to determine algebraically whether or not a sequence of numbers is a juggling sequence, and how many balls are needed to juggle a juggling sequence, it would appear that juggling diagrams are fairly useless after all. However, juggling diagrams can tell you a lot more about the way a juggling pattern is built.

Have another look at the juggling diagram for 4, 4, 1, 3. Note that the arches in the diagram naturally combine into three continuous curves. Each curve corresponds to a distinct ball.



Highlighting the trajectories of the different balls actually tells you exactly what the different balls are doing. Having this sort of information is essential when you are actually learning to juggle a pattern. In particular, from the juggling diagram above we can see see that two of the balls are continuously tossed with 4-throws whereas the remaining ball "does" 3, 1. Since the hands take turns tossing the balls, an odd-number throw originates at one hand and lands at the other hand, whereas an even-number throw starts and ends at the same hand. This means that when you juggle 4, 4, 1, 3, one of the balls continuously moves up and down in 4s on the left side of the pattern, a second one does the same on the right side, and the third ball moves in a circle between the two hands; see the following diagram.



7 How many juggling sequences are there?

Since every non-negative integer corresponds to a juggling sequence, there are infinitely many juggling sequences. So, is there anything else to be said about the number of juggling sequences?

Associated with every juggling sequence are three parameters: (1) the number of balls b required to juggle the sequence; (2) the maximum number m contained in the sequence; and (3) the period p of the sequence.

The official world record for the most number of balls juggled is twelve and it is probably safe to say that nobody will ever be able to juggle one hundred balls. As well, there is only a limited number of different throws that anybody can control, let's be incredibly generous and say 0-, 1-, 2-, ... , 100-throws. And, it is probably also true that nobody will be able to memorize, let alone juggle, a random looking juggling sequence that consists of one million elements.

So, we have physical limitations on the three parameters b, p and m when it comes to real-life juggling. It is therefore natural to ask for the number of juggling sequences for which some of these parameters are limited, and maybe even list all such sequences to identify the ones that are interesting from a performance point of view.

It turns out that limiting only one of the parameters does not limit the number of juggling sequences (unless you go for really small limits). For example, we have already seen that there are infinitely many juggling sequences of period 1—every non-negative integer corresponds to a juggling sequence. The same is true for limiting only the maximum throw height or only the number of balls. This means that we have to restrict at least two of the three parameters in order to arrive at a finite collection of juggling sequences.

Juggling cards: limiting balls and period

Let's start by limiting the number of balls b and the period p. Then we have the following result which was first proved by Joe Buhler, David Eisenbud, Ron Graham and Colin Wright in [3].

Juggling Theorem 6 *There are exactly* $(b+1)^p$ *juggling sequences of period* p *that involve at most* b *balls.*

This immediately implies that there are exactly $(b+1)^p - b^p$ juggling sequences with period p using exactly b balls. For example, for two balls and period 3 we get the following $3^3 = 27$ juggling sequences

$$0,0,0|0,0,3|0,3,0|3,0,0|0,1,2|1,2,0|2,0,1|1,1,1$$

$$0,1,5|1,5,0|5,0,1|0,4,2|4,2,0|2,0,4|3,1,2|1,2,3|2,3,1|1,1,4|1,4,1|4,1,1$$

$$2,2,2|0,0,6|0,6,0|6,0,0|0,3,3|3,0,3|3,3,0$$

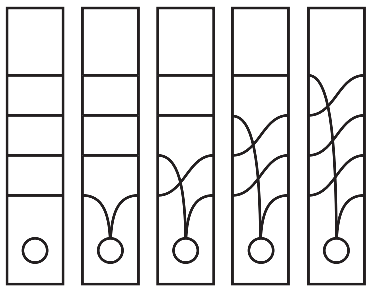
Among these 27 juggling sequences the first is a 0-ball sequence and the next seven are 1-ball sequences. This leaves us with the 19 bona fide 2-ball juggling sequences

$$0,1,5|1,5,0|5,0,1|0,4,2|4,2,0|2,0,4|3,1,2|1,2,3|2,3,1|1,1,4|1,4,1|4,1,1$$

$$2,2,2|0,0,6|0,6,0|6,0,0|0,3,3|3,0,3|3,3,0$$

One problem with such a list is that it contains the juggling sequences $1,1,4$ and $1,4,1$ and $4,1,1$, which represent the same pattern. Another problem is that $1,1,1$ gets counted as a "proper" juggling sequence of period 3 although it is really just the juggling sequence 1 repeated three times. There is a formula for b-ball juggling sequences of period p that counts all cyclic permutations of a juggling sequence as one and also only counts juggling sequences that are not repetitions of smaller juggling sequences.[7]

There is a really ingenious proof for Theorem 6 due to Richard Ehrenborg and Margaret Readdy [8] that we'd like to sketch in the following. Have a close look at the following five "cards".

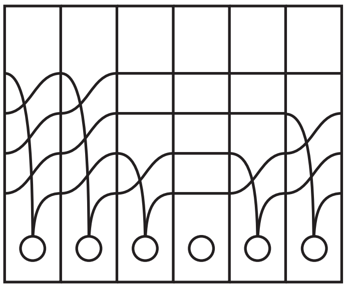


Every card contains a circle at the bottom and four curves connecting the left and right sides of the card. From a deck that contains infinitely many copies of each card we now draw a few cards which we place right next to each other. Here is a possible outcome of this experiment.

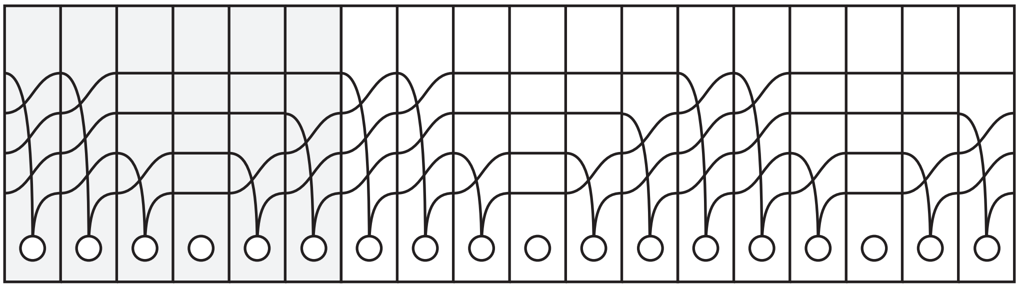
[7]This formula is

$$\frac{1}{p}\sum_{d|p}\mu(p/d)\left((b+1)^d - b^d\right).$$

Here μ is the *Möbius function*; see [13] for details.



Repeating the resulting pattern to the left and right we arrive at a slightly distorted juggling diagram of a juggling sequence.



In this case it is easy to see that the juggling sequence is 235086. This juggling sequence has period 6, the number of cards that we pulled from the deck. Furthermore, it is easy to see that the number of balls needed to juggle this sequence is 4, the number of curves connecting the left and right sides of the cards.

In fact, it is not hard to see that every possible juggling sequence requiring at most 4 balls and a certain period p can be represented using p of our cards in exactly one way. Since there are 5 different cards this gives a total of 5^p different juggling sequences of period p requiring at most 4 balls.

All this generalizes to any number of balls by suitably modifying our original set of cards. Neat!

As we have just seen, limiting the number of balls and the period results in a limited number of juggling sequences and a neat way of finding all of them using juggling cards. Since the number of balls is the average of the numbers in a juggling sequence, it is clear that by limiting the number of balls b and the period p we also limit the maximal throw height. In fact, it is easy to see that bp is the maximal possible throw height and that this throw height actually occurs in the juggling sequence $bp, 0, 0, ..., 0$.

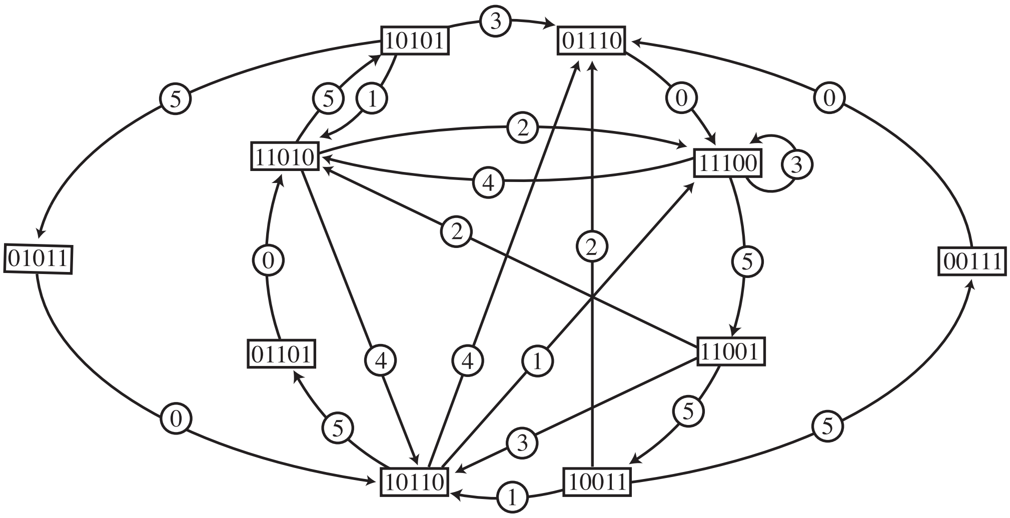
Of course, even for relatively small numbers of balls and small period their product can get quite large and therefore move a juggling sequence beyond what is physically possible. This makes the lists of juggling sequences considered in this section of only limited usefulness for the real juggler.

This also turns out to be the case if we consider the juggling sequences of a given period p and maximum throw height m. Here it is immediately clear that there will be less than $(m+1)^p$ such sequences. Furthermore, the number of balls needed for juggling such a juggling sequence will always be less than m, the number of balls necessary to juggle the sequence $m, m, m, ..., m$.

Juggling state graphs: limiting balls and throw height

As we will see, apart from some trivial exceptions, fixing the number of balls and the maximum throw height still gives an infinite class of juggling sequences. However, it is exactly pondering these limitations that has resulted in a very ingenious method for constructing useful lists of juggling sequences. The method which we are about to describe is due to Jack Boyce, who is also the author of the juggling program *Juggling Lab* recommended earlier.

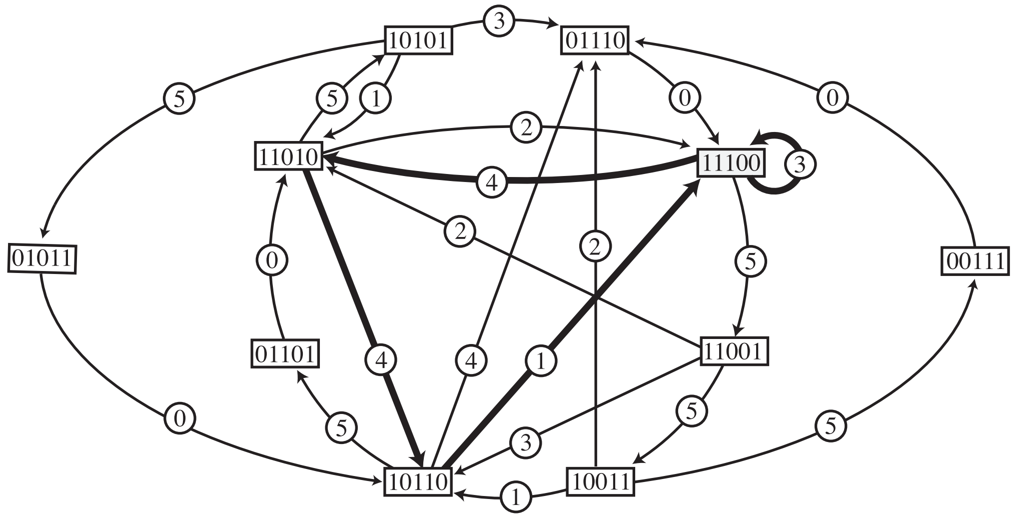
Picture yourself in four weeks time after having practised your juggling for ten hours a day. You are now able to juggle three balls proficiently, and you are able to use 0-, 1-, 2-, 3-, 4-, and 5-throws in your 3-ball patterns. Then the following *juggling state graph* is all you need to construct all the 3-ball juggling sequences that you are able to juggle.



The vertices of this graph are the numbers 11100, 11010, and so on; that is, the ten different ways of writing a five digit number with three ones and two zeros. These ten numbers/vertices are called *juggling states*. They are connected by arrows that are labeled with the numbers from 0 to 5.

To construct a juggling sequence, all you need to do is to find a closed oriented loop in the graph. Here "oriented" means that, as you travel around the loop, you are always traveling in the direction of the arrows that the loop consists of.

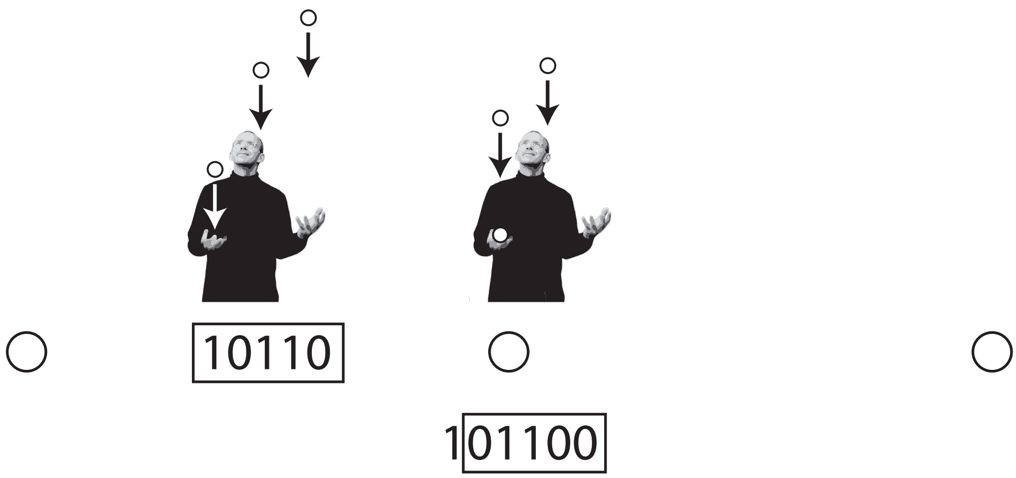
The numbers in the juggling sequence that correspond to the loop are the labels that you come across as you travel along the loop. For example, the little circular arrow attached to the juggling state 11100 and labeled with a 3 is such a loop. It corresponds to the basic 3-ball juggling sequence 3. The highlighted triangular oriented loop in the following diagram corresponds to the juggling sequence $4, 4, 1$. If we extend this oriented loop to a larger oriented loop by adding the little circular arrow, we get the juggling sequence $4, 4, 1, 3$. Note that a loop can visit the same vertex more than once. For example, the loop corresponding to $4, 4, 1, 3$ visits the vertex 11100 twice.



Why does this work? And how is this juggling state graph constructed? Imagine yourself on stage, performing a mindbogglingly complicated 3-ball juggling pattern. After a couple of minutes of doing the same pattern, your audience is starting to get bored, and you decide that it is time to change your pattern on the next beat, without hesitating. What are your options? The answer to this question comes in the form of a *juggling state*.

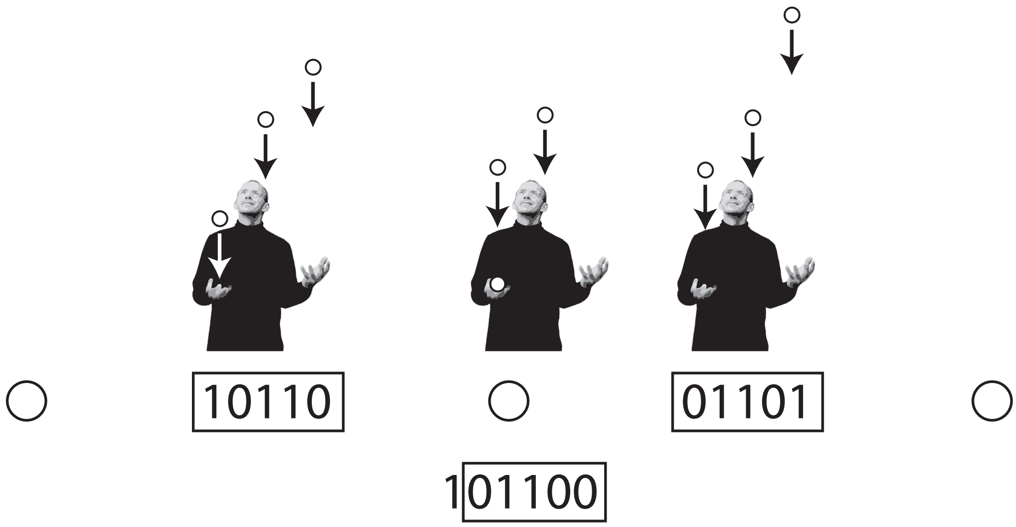
Let's freeze the juggling action in between two beats. As indicated in the diagram, the three balls are scheduled to land in 1, 3, and 4 beats from now. This means that a natural way to note down the juggling state that you are in is 10110. This just says that there is 1, no, 1, 1, and no ball expected to land on beats 1, 2, 3, 4, 5, from now on.

Let's unfreeze the action and refreeze again on the next beat. To get the new juggling state, just remove the first digit from 10110 and add a 0 at the end. This means that our new juggling state is 01100, and we now hold one ball in our hand that has to be tossed again.

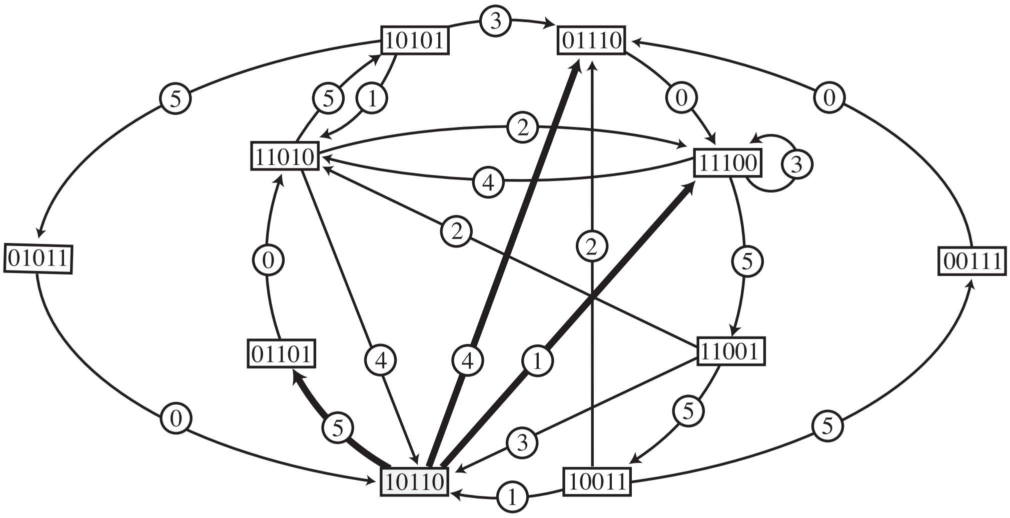


Since we hold a ball in our hand, a 0-throw is not an option. What about a 1-throw? Well, a 1-throw would land on the next beat. Since our current juggling state starts with a 0 we know that no other ball is scheduled to land on the next beat. Hence a 1-throw is definitely possible. What about a 2-throw? No, not possible, because it would collide with the other ball that is scheduled to land two beats from now. Similarly, a 3-throw is not possible. On the other hand, both 4- and 5-throws are possible because no balls are scheduled to land 4 and 5 beats from now.

You decide to perform a 5-throw. This changes your juggling state from 01100 to 01101.



If you had gone for a 1-throw, your juggling state would have changed to 11100, and if you had decided on a 4-throw, then your state would have changed to 01110. This shows the connection between your original state 10110 and the three arrows originating at it in the state graph.



Repeating the same for all possible states gives the complete juggling state graph. It should be clear that by juggling along the edges of this juggling state graph, on any beat either nothing happens or exactly one ball lands and is tossed again. This is exactly what we want to ensure. Finally, it is clear that every oriented loop will correspond to a juggling sequence.

Okay, what we have found so far is that loops in this juggling state graph correspond to 3-ball juggling sequences consisting of 0-, 1-, 2-, 3-, 4-, and 5-throws. On the other hand, it is also fairly easy to see that EVERY such juggling sequence corresponds to an oriented loop in this juggling state graph.

Of course, it is also possible to draw a juggling state graph which will allow you to construct all 245-ball juggling sequences with throws that can be up to 1056 beats in duration. Or, for any choice of nonnegative integers b and m, there is a juggling state graph from which all b-ball juggling sequences with throws up to m beats in duration can be extracted.

As we have seen, constructing such state graphs is a completely automatic task and can be performed easily by a computer. Similarly, finding loops in graphs is a routine exercise for a computer.

Finally, it is easy to see that there are infinitely many ways to find loops in all but the most trivial juggling state graphs. For example, let's have another look at the loops corresponding to the juggling sequences $4, 4, 1$ and 3. Since the two sequences both start and end in the same juggling state 11100 we can combine copies of these juggling sequences into infinitely many different juggling sequences. Here are just a few such sequences

$$4, 4, 1|4, 4, 1, 3|4, 4, 1, 3, 3|4, 4, 1, 3, 3, 3|...$$

Although there are infinitely many different loops corresponding to infinitely many different juggling sequences hiding in any juggling state graph, all these juggling sequences are made up from finitely many *prime juggling sequences*. These are the juggling sequences that correspond to loops that do not visit states more than once. The two juggling sequences $4, 4, 1$ and 3 are examples of prime juggling sequences. Identifying all cyclic permutations of a prime juggling sequence, there are a total of 26 prime 3-ball juggling sequences of maximum throw height 5.

8 Do real jugglers really care?

Well, the answer to this question is: "Some do, most don't, but they should!" Here are a couple of reasons for why most jugglers could profit from knowing a little bit about juggling sequences.

Communicate juggling patterns

Juggling sequences provide a clear and compact way to communicate juggling patterns. Just tell your friend on the other side of the planet that you are practicing $4, 4, 1, 3$ blindfolded, and with all throws going under your legs, and she will get the picture. This usually beats even watching videotapes that show a particular pattern being performed.

Find all juggling patterns

As we have seen, juggling sequences provide a comprehensive method for finding new juggling patterns. It is certainly true that the vast majority of possible juggling sequences are very boring or impossible and therefore of no interest from a performance point of view. Nevertheless quite a few new and interesting patterns have only been discovered as a result of thinking about juggling in a mathematical manner.

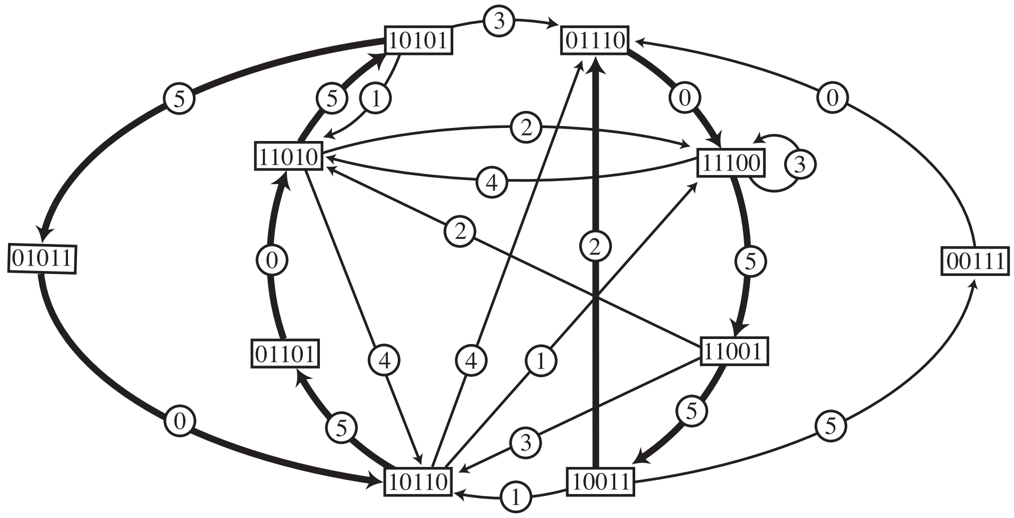
Divide and conquer

We've already seen in Section 6 how coloring in the trajectories of the balls in a juggling diagram can help to break a juggling pattern into managable subpatterns that can be practised individually.

Further, juggling sequences can help master complicated patterns, by providing examples of easier patterns that are made up of the same throws. For example, to master the basic 5-ball pattern corresponding to the juggling sequence 5 is difficult. On the other hand, it is easy to get used to the 5-throws that this pattern consists of by practicing the 2-ball sequence $5, 0, 1$, or the 4-ball sequence, $5, 5, 2$.

Create transitions

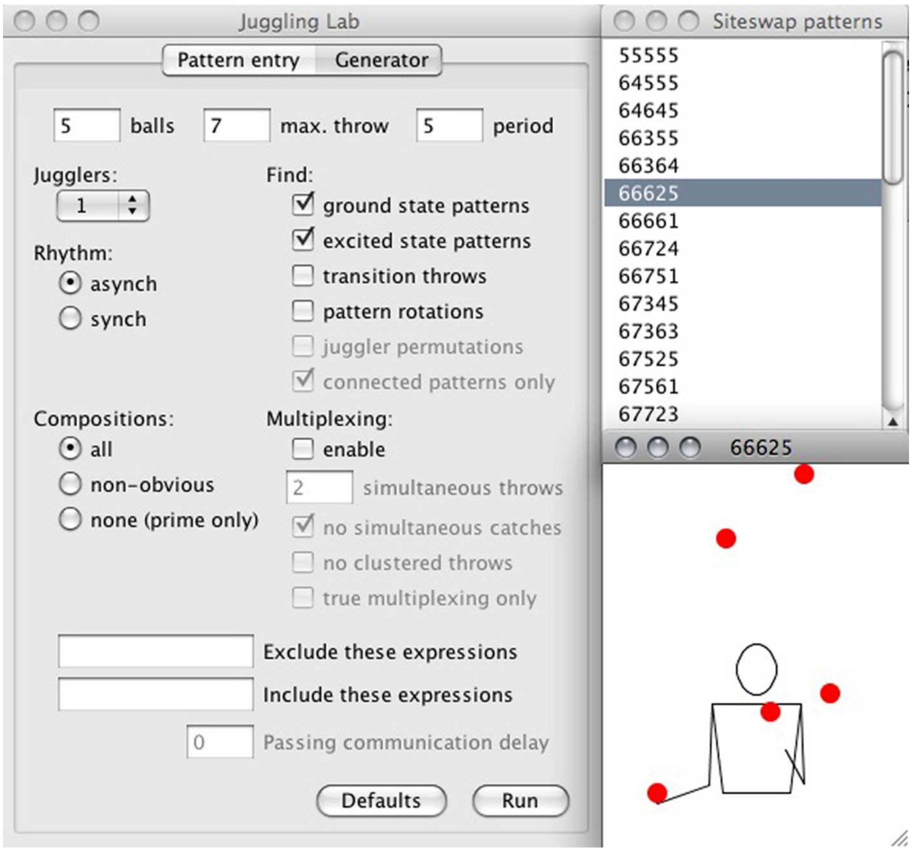
In a juggler's routine different patterns are connected seamlessly. To figure out a way to move from one juggling sequence to another one, have a look at loops that correspond to them in a juggling state graph. Basically, what you are seeking is an oriented path in the juggling state graph that connects the two loops. For example, in the following diagram I have highlighted two oriented loops. These loops are disjoint. This means that juggling a pattern corresponding to the left loop, it is not possible to go straight into a pattern corresponding to the right loop. But, as you can see, once you are in the top left juggling state 10101, you can juggle a 3-throw to get into the top right juggling state 01110, which is part of the second loop.



If the two loops that you are interested in intersect in a state, you can go from one to the other whenever you are in this state. We've already encountered an example like this. The two juggling sequences $4, 4, 1$ and 3 correspond to loops that have the state 11100 in common. Therefore you can switch from juggling one pattern to the other whenever you are in this state.

Juggling simulators

Juggling sequences form the natural language that people use to teach a computer to juggle. In fact, the first serious computer simulations of juggling were developed by some of the same people who first investigated the mathematics of juggling.

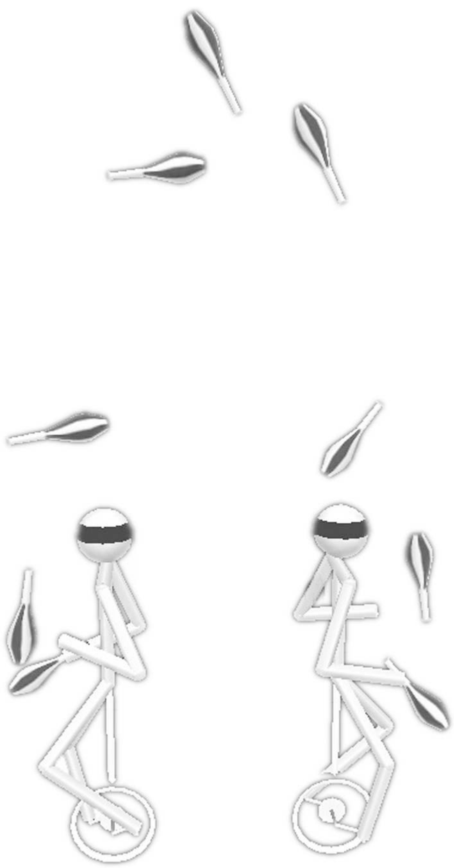


Modern juggling simulators can realistically show you just about every juggling trick ever performed by a human juggler. Using these simulators you can witness masterful juggling performed without having to leave your living room and you can slow down a complicated pattern and study it in detail before attempting it yourself.

But going way beyond what is humanly possible, a juggling simulator can show you tricks with dozens of balls woven into the most intricate patterns imaginable. The main point being that if there was a human being or robot strong and fast enough to perform a trick like this here on Earth[8] it would look very much like what the simulator shows you.

The screenshot above shows Jack Boyce's *Juggling Lab* in action. Apart from executing juggling sequences, this simulator also has a built in juggling sequence generator implementing juggling state graphs. For example, in the picture I have asked the generator to show me all juggling sequences with 5 balls, maximal throw height 7 and period 5. The beginning of the resulting list is shown in the upper right window. Clicking on the sequence 66625, for example, then brings up the little juggler in the lower right corner who juggles this sequence for us.

Here is a tracing of another screenshot. This is of Werner Westerboer's *JoePass!* conjuring up two blindfolded back-to-back jugglers on unicycles who are passing 9 clubs between them.



[8]...or on the Moon, since gravity can often be adjusted in juggling simulators.

9 Do real mathematicians really care?

Yes, they do. The mathematics of juggling is inherently beautiful and this is good enough a reason for real mathematicians to care. However, even if you are looking for some justification in the form of results that connect with other disciplines, or interesting and hard open problems, or an appealing application of mathematics to a real-world problem, the mathematics of juggling has much to offer. Here are just a few examples.

Braids

Imagine that while you are juggling some pattern in a plane in front of you, you are also jogging backwards at a constant speed, and smoke is issuing from your juggling balls. This has the effect that the balls trace their own trajectories in the air, just as some airplanes write advertisements in the sky. The following diagram shows what the set of trajectories produced by juggling the basic 3-ball pattern would look like viewed from above. As you can see, the three trajectories form the most basic braid. Braids are important mathematical objects that have made an appearance in many areas of mathematics. It can be shown that every braid can be juggled in the above sense; see [13] and [7].

Permutations

Associated with every juggling sequences are several permutations. One example is the test sequence that arises when we apply the algebraic test for jugglability that we talked about in Section 4. This is a permutation of the numbers 0, 1, 2, ... , up to $p-1$, where p is the period. Other straightforward examples include the various interpretations of juggling diagrams as permutations of the infinite set of beat points. This means that distinguished sets of juggling sequences correspond to certain sets of permutations, some of which have turned out to be of special significance in other parts of mathematics.

For example, in the investigation of juggling cards by Richard Ehrenborg and Margaret Readdy [8] the so-called *affine Weyl group* $\tilde{A}_{p-1}$ materializes as a permutation group composed of permutations associated with the juggling sequences of period p. It is then shown how juggling cards can be used to very elegantly derive some important results about this group.

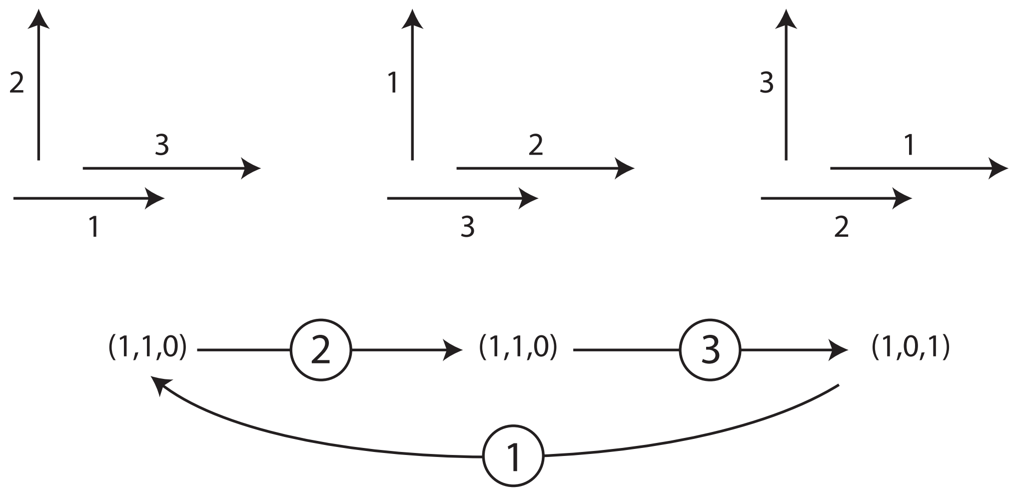
Another example, involves the original proof for the formula $(b+1)^p$ for the number juggling sequences of period p using at most b balls. This proof starts out with the possible test sequences associated with such juggling sequences. The general idea for this proof then generalized very naturally to yield some remarkable results about *posets*, which are important generalizations of permutations; see [4].

Linear algebra

Another surprising connection, this time with linear algebra, was discovered recently by Allen Knutson, Thomas Lam and David E. Speyer [10].

Given an ordered list of n vectors spanning a b-dimensional real vector space, build a basis of the vector space by including the ith vector exactly if it is not a linear combination of the 1st, . . . , $i-1$st vectors. Call this the greedy basis. Then move the 1st vector to the end of the list of vectors and construct a new greedy basis from the new list. Repeat n times, and you get n greedy bases. Interpreted as states of the b-ball juggling state graph of maximal height n, the greedy bases form an oriented loop and therefore correspond to a b-ball juggling sequence of period n all of whose throws are no larger then n.

Here is an example involving a list of three vectors in the plane, two of which are parallel. The first greedy basis contains the vectors 1 and 2, but not the vector 3. This means it corresponds to the juggling state (1,1,0), etc. Altogether we find that the juggling sequence associated with this list of vectors is $2, 3, 1$.



Note that all the numbers in juggling sequences generated in this manner are no larger than the period of the sequence. This means that, for example, the basic sequence 2 does not arise in this manner. Nonetheless, it is possible to generate a juggling sequence for any possible juggling pattern—although 2 is impossible $2, 2, 2$ is (just start with three vectors in the plane, no two of which are parallel).

This new connection of juggling sequences and linear algebra leads to important applications and some very fancy mathematics elaborated on in [10].

Infinite sequences

Recently Fan Chung and Ron Graham investigated primitive juggling sequences [5]. These are juggling sequences whose corresponding oriented loops in a juggling state graph visit their starting states only once. Counting families of primitive juggling sequences Chung and Graham came up with a number of infinite sequences that naturally show up in completely different areas of mathematics.

For example, they found the sequence of numerators of the continued fraction convergents to $\sqrt{2}$. At this stage it is not clear whether these are simply amazing coincidences or whether there are deeper connections with these other areas that are waiting to be discovered.

Other interesting open problems

Some of the most interesting outstanding problems concern the prime juggling sequences. Remember that a juggling sequence is prime if the corresponding oriented loop in a juggling state graph does not visit any state more than once. Furthermore, every juggling sequence has a cyclic permutation that is a product of prime juggling sequences. There are only finitely many prime juggling sequences in any state graph. Natural unsolved questions about prime sequences involve the number of such sequences in state graphs and the maximal length of prime sequences.

Communicating the power and beauty of mathematics

Many people leave school convinced that mathematics is a boring, difficult and inaccessible subject, practised by equally boring, difficult and inaccessible individuals. Many juggling mathematicians have found that a presentation of mathematical juggling spiced up with some fancy juggling by the presenter is a perfect way to demonstrate to people that mathematicians can be very interesting individuals whose mathematics can be a lot of fun, incredibly beautiful and very useful and accessible. This should be another good reason why mathematicians in general “should care”.

10 Concluding remarks

This concludes our tour of the mathematics of juggling. My aim was to give an accessible introduction to the most important results while skipping most of the gory details, generalizations to multihand- and multiperson juggling, models that incorporate more of the physical constraints, and so on. A fairly complete account of all aspects of mathematical juggling can be found in my 2003 book *The mathematics of juggling* [13].

If you actually feel like getting into juggling some non-trivial juggling sequences, I recommend that you first have a go at some of the juggling sequences collected in the `Lodi.jml` file in the `pattern` directory of the juggling simulator *Juggling Lab*. Also very useful is the program *JuggleMaster Java* by Yuji Konishi and Asanuma Nobuhiko, based on the original Juggle Master program by Ken Matsuoka. It comes with lots of juggling sequences and easy-to-use controls for slowing down the juggling action. This is particularly useful for figuring out how to break a juggling pattern into pieces that can be mastered individually.

The following is just a small collection of annotated references. Included in this list are a number of landmark articles, some high-profile popularizations, plus all articles of interest (to me) that have appeared since the publication of my book. Most of these articles are readily available online.

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