Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

1176

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H_∞ Ring Spectra and their Applications

Spring
Berlin Heidelberg New York Tokyo

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Mathematics Subject Classification (1980): 55N15, 55N20, 55P42, 55P47, 55Q10, 55Q35, 55Q45, 55S05, 55S12, 55S25, 55T15

ISBN 3-540-16434-0 Springer-Verlag Berlin Heidelberg New York Tokyo
ISBN 0-387-16434-0 Springer-Verlag New York Heidelberg Berlin Tokyo

Printed in Germany

Printing and binding: Beltz Offsetdruck, Hemsbach/Bergstr.
2146/3140-543210

PREFACE

This volume concerns spectra with enriched multiplicative structure. It is a truism that interesting cohomology theories are represented by ring spectra, the product on the spectrum giving rise to the cup products in the theory. Ordinary cohomology with mod p coefficients has Steenrod operations as well as cup products. These correspond to an enriched multiplicative structure on the Eilenberg-MacLane spectrum HZ_p. Atiyah has shown that the Adams operations in KU-theory are related to similar structure on its representing spectrum and tom Dieck and Quillen have considered Steenrod operations in cobordism coming from similar structure on Thom spectra. Kahn, Toda, Milgram, and others have exploited the same kind of structure on the sphere spectrum to construct and study homotopy operations, and Nishida's proof of the nilpotency of the stable stems is also based on this structure on the sphere spectrum.

In all of this work, the spectrum level structure is either implicit or treated in an ad hoc way, although Tsuchiya gave an early formulation of the appropriate notions. Our purpose is to give a thorough study of such structure and its applications. While there is much that is new here, we are also very interested in explaining how the material mentioned above, and other known results, can be rederived and, in many cases, sharpened and generalized in our context.

The starting point of our work is the existence of extended powers of spectra generalizing the extended powers

$$D_jX = E\Sigma_j \ltimes_{\Sigma_j} X^{(j)} = E\Sigma_j \times_{\Sigma_j} X^{(j)}/E\Sigma_j \times_{\Sigma_j} \{*\}$$

of based spaces X. Here Σ_j is the symmetric group on j letters, $E\Sigma_j$ is a contractible space on which Σ_j acts freely, the symbol $\ltimes$ denotes the "half smash product", and $X^{(j)}$ denotes the j-fold smash power of X. This construction and its variants play a fundamental role in homotopy theory. They appear ubiquitously in the study of torsion phenomena.

It will come as no surprise to anyone that extended powers of spectra can be constructed and shown to have all of the good properties present on the space level. However, those familiar with the details of the analysis of smash products of spectra will also not be surprised that there are onerous technical details involved. In working with spectra, the precise construction of smash products is seldom relevant, and I think most workers in the field are perfectly willing to use them without bothering to learn such details. The same attitude should be taken towards extended powers.

With this in mind, we have divided our work into two parts, of which this volume is the first. We here assume given extended powers and structured spectra and show how to exploit them. This part is meant to be accessible to anyone with a standard background in algebraic topology and some vague idea of what the stable category is. (However, we should perhaps insist right at the outset that, in stable homotopy theory, it really is essential to work in a good stable category and not merely to think in terms of cohomology theories on spaces; only in the former do we have such basic tools as cofibration sequences.) All of the technical work, or rather all of it which involves non-standard techniques, is deferred until the second volume.

We begin by summarizing the properties of extended powers of spectra and introducing the kinds of structured ring spectra we shall be studying. An H_∞ ring spectrum is a spectrum E together with suitably related maps $D_j E \to E$ for $j \geq 0$. The notion is analogous to that of an E_∞ space which I took as the starting point of my earlier work in infinite loop space theory. Indeed, H_∞ ring spectra may be viewed as analogs of infinite loop spaces, and we shall also give a notion of H_n ring spectrum such that H_n ring spectra are analogs of n-fold loop spaces. However, it is to be emphasized that this is only an analogy: the present theory is essentially independent of infinite loop space theory. The structure maps of H_∞ ring spectra give rise to homology, homotopy, and cohomology operations. However, for a complete theory of cohomology operations, we shall need the notion of an H_∞^d ring spectrum. These have structural maps $D_j \Sigma^{di} E \to \Sigma^{dji} E$ for $j \geq 0$ and all integers i.

While chapter I is prerequisite to everything else, the blocks II, III, IV-VI, and VII-IX are essentially independent of one another and can be read in any order.

In chapter II, which is primarily expository and makes no claim to originality, I give a number of rather direct applications of the elementary properties of extended powers of spectra. In particular, I reprove Nishida's nilpotency theorems, explain Jones' recent proof of the Kahn-Priddy theorem, and describe the relationship of extended powers to the Singer construction and to theorems of Lin and Gunawardena.

In chapter III, Mark Steinberger introduces homology operations for H_∞ (and for H_n) ring spectra. These are analogs of the by now familiar (Araki-Kudo, Dyer-Lashof) homology operations for iterated loop spaces. He also carries out extensive calculations of these operations in the standard examples. In particular, it turns out that the homology of HZ_p is monogenic with respect to homology operations, a fact which neatly explains many of the familiar splittings of spectra into wedges of Eilenberg-MacLane and Brown-Peterson spectra.

In chapters IV-VI, Bob Bruner introduces homotopy operations for H_∞ ring spectra and gives a thorough analysis of the behavior of the H_∞ ring structure with respect to the Adams spectral sequence and its differentials. As very special cases, he uses this theory to rederive the Hopf invariant one differentials and certain key odd primary differentials due to Toda. The essential point is the relationship between the structure maps $D_pE \to E$ and Steenrod operations in the E_2 term of the Adams spectral sequence. Only a few of the Steenrod operations survive to homotopy operations, and the attaching maps of the spectra D_pS^q naturally give rise to higher differentials on the remaining Steenrod operations. An attractive feature of Bruner's work is his systematic exploitation of a "delayed" Adams spectral sequence originally due to Milgram to keep track of these complex phenomena.

In chapters VII-IX, Jim McClure relates the notion of an H_∞^d ring spectrum to structure on the familiar kinds of spectra used to represent cohomology theories on spaces. For example, he shows that the representing spectrum KU for complex periodic K-theory is an H_∞^2 ring spectrum, that the Atiyah-Bott-Shapiro orientations give rise to an H_∞^2 ring map $MSpin^c \to KU$, and that similar conclusions hold with $d = 8$ in the real case. He then describes a general theory of cohomology operations and discusses its specialization to ordinary theory, K-theory, and cobordism. Finally, he gives a general theory of homology operations and uses the resulting new operations in complex K-theory to compute the K-theory of $QX = \operatorname{colim} \Omega^n\Sigma^nX$ as a functor of X. This is a striking generalization of work of Hodgkin and of Miller and Snaith, who treated the cases $X = S^0$ and $X = RP^n$ by different methods.

Our applications - and I have only mentioned some of the highlights - are by no means exhaustive. Indeed, our examples show that this is necessarily the case. Far from being esoteric objects, the kinds of spectra we study here abound in nature and include most of the familiar examples of ring spectra. Their internal structure is an essential part of the foundations of stable homotopy theory and should be part of the tool kit of anybody working in this area of topology.

There is a single table of contents, bibliography, and index for the volume as a whole, but each chapter has its own introduction; a reading of these will give a much better idea of what the volume really contains. References are generally by name (Lemma 3.1) within chapters and by number (II.3.1) when to results in other chapters. References to "the sequel" or to [Equiv] refer to "Equivariant stable homotopy theory", which will appear shortly in this series; it contains the construction and analysis of extended powers of spectra.

J. Peter May
Feb. 29, 1984

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