Lion and Man – Can Both Win?

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Abstract

This paper is concerned with continuous-time pursuit and evasion games. Typically, we have a lion and a man in a metric space: they have the same speed, and the lion wishes to catch the man while the man tries to evade capture. We are interested in questions of the following form: is it the case that exactly one of the man and the lion has a winning strategy?

As we shall see, in a compact metric space at least one of the players has a winning strategy. We show that, perhaps surprisingly, there are examples in which both players have winning strategies. We also construct a metric space in which, for the game with two lions versus one man, neither player has a winning strategy. We prove various other (positive and negative) related results, and pose some open problems.

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1 Introduction

Rado's famous 'Lion and Man' problem (see [9, pp. 114-117] or [2, pp. 45-47]) is as follows. A lion and a man (each viewed as a single point) in a closed disc have equal maximum speeds; can the lion catch the man? This has been a well known problem since at least the 1930s – it was popularised extensively by Rado and subsequently by Littlewood. The reader not familiar with this problem is urged to give it a few minutes' thought before proceeding further.

For the 'curve of pursuit' (the lion always running directly towards the man) the lion gets arbitrarily close to the man but does not ever catch him. However, the apparent 'answer' to the problem is that the lion can win by adopting a different strategy, namely that of staying on the same radius as the man. In other words, the lion moves, at top speed, in such a way that he always lies on the radius vector from the centre to the man. If we assume, as seems 'without loss of generality', that the man stays on the boundary of the circle, then it is easy to check that the lion does now catch the man in finite time. Indeed, in the time that it takes the man to run a quarter-circle at full speed the lion (say starting at the centre) performs a semicircle of half the radius of the disc, thus catching the man.

This 'answer' was well known, but in 1952 Besicovitch (see [9, pp. 114-117]) showed that it is wrong to assume that the man should stay on the boundary, and that in fact the man can survive forever. His beautiful argument goes as follows. We split time into a sequence of intervals, of lengths $t_1, t_2, t_3, \ldots$. At the ith step the man runs for time t_i in a straight line that is perpendicular to his radius vector at the start of the step. He chooses to run into the half plane that does not contain the lion (if the lion is on the radius then either direction will do). So certainly the lion does not catch the man in this time step. The man then repeats this procedure for the next time step, and so on.

Now, note that if r_i is the distance of the man from the origin at the start of the ith time step then $r_{i+1}^2 = r_i^2 + t_i^2$. Hence as long as $\sum_i t_i$ is infinite then the man is never caught, and if $\sum_i t_i^2$ is finite then the r_i are bounded, so that (multiplying by a constant if necessary) the man does not leave the arena. So taking for example $t_i = 1/i$ we have a winning strategy for the man.

We pause for a moment to reassure the reader that all of these terms like 'winning strategy' have a precise definition, and indeed there is really only one natural choice for the definitions. Thus a *lion path* is a function l from $[0, \infty)$ to the closed unit disc D such that $|l(s) - l(t)| \leq |s - t|$ for all s and t (in other words, the path is 'Lipschitz' – this corresponds to the lion having maximum speed say 1) and with $l(0) = x_0$ for some fixed x_0 in the disc (as, for definiteness, a starting point should be specified). A *man path* is defined similarly. We write L for the set of lion paths and M for the set of man paths. Then a *strategy* for the lion is a function G from M to L such that if $m, m' \in M$ agree on $[0, t]$ then also $G(m)$ and $G(m')$ agree on $[0, t]$. This 'no lookahead' rule is saying that $G(m)(t)$ depends only on the values of $m(s)$ for $0 \leq s \leq t$ (or equivalently, by continuity of m, that it depends only on the values of $m(s)$ for $0 \leq s < t$). A strategy G for the lion is a *winning strategy* if for every $m \in M$ there is a time t with $G(m)(t) = m(t)$. We make corresponding definitions for the man. Note that all of these definitions also make sense with the disc replaced by an arbitrary metric space X. We will usually suppress the dependence on the starting points, speaking for example about just 'the game on X', since for most results the actual starting points are irrelevant (as long as they are distinct, of course).

Let us briefly remark that it would be fundamentally different to use an alternative definition of 'lion strategy' (for example) by insisting on some kind of finite delay, so that the lion's position at time t depended on the man's position at times up to $t - \varepsilon$ or some such. For then one would be disallowing such natural strategies as 'aim for the man' or 'keep on the same radius as the man', and thus one would be changing the nature of the problem away from the nature that was intended (by Rado, Besicovitch and Littlewood). Indeed, it is important to remember that strategies such as 'aim for the man' do not in any sense 'look into the future', even infinitesimally, but only into the past, as the position at time t is determined by the opponent's position at all times strictly before t. (In fact, the relationship between strategies and finite delays will be considered in some detail in Section 2.)

We now ask the question that motivates the work of this paper. We have seen that the man has a winning strategy (the Besicovitch strategy); could it be that the lion also has a winning strategy?

At first sight, this seems an absurd question to ask: after all, if both players have winning strategies then let us consider a play of the game in which each is following his winning strategy, and ask who wins? But a moment's careful thought reveals that this 'proof' is in fact nonsense. For what would it mean to have a play of the game in which 'each player followed his strategy'? If the lion is using strategy G and the man is using strategy F, then we would need paths $l \in L$ and $m \in M$ such that $l = G(m)$ and $m = F(l)$. And there is no *a priori* reason why such a common fixed point should exist.

[There is also no reason why such a common fixed point, if it exists, should be unique. To see a simple example of this, suppose that the two players start on the boundary, diametrically opposite each other, and each is following the strategy 'stay on the boundary, diametrically opposite your opponent'. Then the two might stay still, might run around the boundary with a common speed, and so on.]

Now, it turns out that, in this particular case, the 'local finiteness' of the Besicovitch strategy means that it is quite simple to show that the lion cannot also have a winning strategy (see Section 2). But what would happen in a different space (in other words, with the closed disc replaced by a different metric space)? Indeed, in a different space, why should it even be true that at least one of the man and the lion has a winning strategy?

One might imagine that this is merely some 'formal nonsense', and that, once thought about from the correct viewpoint, it would become clear that exactly one of the lion and the man has a winning strategy. Surprisingly, this is not the case.

The plan of the paper is as follows. We start in Section 2 by considering the bounded-time version of the lion and man game (the man wins if he can stay alive until some fixed time T). In this case, if we considered what one might call the 'discrete' version of the game, in which the two players take turns to move, each move being a path lasting for time ϵ (for a fixed $\epsilon > 0$), then we would be in the world of finite-length games, and here of course no pathology can occur. So it is natural to seek to approximate the continuous game by the discrete version. Using this approach, we are able to show that, in a compact metric space, at least one player has a winning strategy for the bounded-time game. (Curiously, at one point the argument seems to make essential use of the Axiom of Choice.) We do not see how to extend our result to the original unbounded-time problem.

The methods of Section 2 seem to come very close to proving that it is also true that at most one player can have a winning strategy. Indeed, as we shall see, it seems that one is only an 'obvious' technical lemma away from proving this. But it turns out that this technical lemma is not true. And in fact in Section 3 we present some examples of metric spaces (even compact ones) in which both players have winning strategies – in the strongest possible sense, namely that the lion can guarantee to catch the man by a fixed time T and the man can guarantee to stay alive forever. Interestingly, the key here is to understand the nature of the required strategies; the spaces themselves are not particularly pathological.

In Section 4 we consider a related game that we call 'race to a point'. Two players, with equal top speeds, start at given points in a metric space, and race towards a given target point. The first to arrive is the winner (with the game a draw if they reach the target at the same time or if neither reaches the target). Of course, if the metric space is compact then each player has a shortest path to the target (as long as he has *some* path to the target of finite length, that is), so that either exactly one player has a winning strategy or else both players have drawing strategies. Thus the interest is in noncompact spaces. One could view race to a point as a 'simplified' version of lion and man, in the sense that the motion is towards a fixed point (as opposed to towards or away from a moving point).

We give an example of a space in which both players have winning strategies for race to a point. We also give examples to show that, perhaps more unexpectedly, there are spaces in which neither player has even a drawing strategy. We also show how this game relates to the lion and man: based on our 'race to a point' examples we give a metric space in which, for the game of two lions versus a man, neither side has a winning strategy.

Finally, in Section 5 we give a number of open problems.

There has been a considerable amount of work on the lion and man problem and related questions. For example, Croft [5] showed that if the man's path is forced to have uniformly bounded curvature then the lion can catch the man (although, strangely, the 'stay on the same radius' strategy does not achieve this). Croft also showed that in the n-dimensional Euclidean ball n

lions can catch the man while the man can escape from $n-1$ lions. For some interesting versions played in a quadrant of the plane, see Sgall [11]. There are also quantitative estimates about how long it takes the lion to get within a certain distance of the man: see for example [1].

There is also a large body of work on 'differential games', where the dynamics is modelled by a system of differential equations: see the beautiful book of Isaacs [6] for a thorough introduction, and Lewin [8] for some applications to lion and man. For results about (discrete) pursuit and evasion in general metric spaces, see Mycielski [10]. However, none of the above appear to have considered the particular questions we address here.

Finally, it is worth pointing out that our results do not seem to be related to results about determinacy of infinite games in Descriptive Set Theory (see for example Jech [7]), with the exception that our construction for the 'race to a point' game in Section 4 has perhaps some of the flavour of some constructions of infinite games in which neither player has a winning strategy.

We mention that we have an online version of this paper (see [4]), which contains some additional material. In particular, it contains a discussion of more general 'locally finite' phenomena (like the Besicovitch strategy), as well as applications of this to other games such as 'porter and student' where one player seeks to leave a region via a specified boundary and the other player wishes to catch him the instant he reaches this boundary.

2 Finite Approximation

We start by showing that, owing to the nature of the Besicovitch strategy, there cannot be a winning strategy for the lion in the original lion and man game in the closed unit disc.

Let G be a lion strategy and let F be the (winning) Besicovitch strategy defined earlier. We aim to play these two strategies against each other: that is, to find a lion path l and a man path m with $l = G(m)$ and $m = F(l)$. By definition of the Besicovitch strategy, the man's path is determined for time t_1: we may write this path as $m|_{[0,t_1]}$ (with slight abuse of notation, as we do not yet know that m exists). Now, given $m|_{[0,t_1]}$, the lion's strategy determines $l|_{[0,t_1]}$, and, in particular, determines $l(t_1)$. The Besicovitch strategy now tells the man what to do for time t_2: that is, $m|_{[0,t_1+t_2]}$ is determined, so as before the lion's path is determined up until $t_1 + t_2$.

Repeating in this way we obtain the desired pair of paths: since the man's strategy is winning we know that $l(t) \neq m(t)$ for all t. That is, the lion does not win, so the lion's strategy is not a winning strategy.

Note that, in this proof, we used the special 'discrete' nature of the man's strategy: the man committed to doing something for some positive amount of time. Many sensible strategies are not of this form: indeed, in the strategy we gave earlier for the lion of 'stay on the same radius', the lion's position continually depends on where the man is now (or, equivalently, where he was at all earlier times).

However, if one were to insist that strategies *should* be in some sense discrete, or that there was some 'delay' (the man's position at time t being allowed to depend only on the lion's position at times before $t - \varepsilon$ for some fixed $\varepsilon > 0$, and vice versa), then there should be no problem in proving that exactly one player has a winning strategy. One might view this kind of restriction as arising from a 'real world' simplification of the problem.

Based on this, let us try to approximate the continuous game by a discrete version, as follows. Before we do so, we will introduce one change to the game: in the *bounded-time lion and man game* in space X there is a fixed parameter $T > 0$, and the lion wins if he has caught the man by time T while the man wins otherwise. For the rest of this section, we shall be considering the bounded-time game.

Let X be a metric space. Fix $\varepsilon > 0$ such that T is an integer multiple of ε: say $T = n\varepsilon$. In the *discrete bounded-time game* on X the two players take turns to move, with say the lion moving first. On a turn, the player runs for time ε – or, more precisely, he chooses a path of path length at most ε, starting at his current position, and runs along it to its end. The game ends after each player has had n turns; the 'outcome' of the game is defined to be the closest distance d that occurred between the lion and man at any time.

Since this is a finite game (a game lasting for a fixed finite number of moves), it is easy to see (for example, by 'backtracking' – also known as Zermelo's theorem) that there is a $\delta = \delta(\varepsilon)$ such that for any $\delta' < \delta$ the man has a strategy that ensures d is at least δ' and for any $\delta' > \delta$ the lion has a strategy that ensures $d < \delta'$.

[We remark in passing that this would remain the case if we were considering a discrete version of the unbounded-time game, by virtue of the fact that Borel games are determined (see e.g. Jech [7]). However, it turns out that this does not seem to help with the analysis of the unbounded-time game.]

As we are interested in the relation of this discrete game to the original 'continuous' game, it is natural to consider $\varepsilon \to 0$ (of course, only through values that divide T). One would hope that $\delta \to 0$ as $\varepsilon \to 0$ corresponds to a lion win in the continuous game. More precisely, one would hope that the following four implications hold.

Implications. *For the bounded-time game on the metric space X, with δ and ε as above, we have*

A. *If $\delta \to 0$ as $\varepsilon \to 0$ in the discrete game then the lion wins the continuous game.*

B. *If $\delta \not\to 0$ as $\varepsilon \to 0$ in the discrete game then the man wins the continuous game.*

C. *If the lion wins the continuous game then $\delta \to 0$ as $\varepsilon \to 0$ in the discrete game.*

D. *If the man wins the continuous game then $\delta \not\to 0$ as $\varepsilon \to 0$ in the discrete game.*

Now, as we shall see, Implications B and C are trivial. Implication A is true, if X is compact, but seems to be not quite trivial; indeed, our proof needs the Axiom of Choice. Combining Implications A and B we see that at least one player has a winning strategy for the bounded-time game played in a compact space.

Under some extra, seemingly mild, conditions the final implication D is easy to prove. Surprisingly, however, it is false in general, as we see in Section 3.

We remark that for our purposes it does not matter who moves first in the discrete-time game. Indeed, allowing the man to move first, or equivalently forcing the lion to stay where he is for his first move, will only change the value of δ by at most ε, and so will not change whether or not $\delta \to 0$ as $\varepsilon \to 0$.

Note that a strategy for a player in the ε-discrete game (with that player moving first) gives rise naturally to a strategy for that player in the continuous game. Conversely, a strategy for a player in the continuous game gives rise naturally to a strategy for that player in the ε-discrete game (with that player moving second). Here the restrictions about who moves first are to ensure that the no-lookahead rule is not violated. Note also that these changes from discrete to continuous or vice versa change the closest distance between lion and man (in any play of the game) by at most ε.

In Lemmas 1–3 and Corollary 4 we fix a metric space X and we consider only bounded-time games.

Lemma 1. *Suppose that $\delta \not\to 0$ as $\varepsilon \to 0$. Then the man has a winning strategy in the continuous game.*

Proof. Choose $\varepsilon > 0$ such that $\delta(\varepsilon) > \varepsilon$, and let F be a man strategy for the ε-discrete game (with the man moving first) witnessing this. Then, in the continuous game, the man just follows the corresponding strategy. At all times, the lion is at most ε closer to the man than he is in the discrete game, and so does not catch the man. □

Lemma 2. *Suppose that the lion has a winning strategy for the continuous game. Then $\delta \to 0$ as $\varepsilon \to 0$.*

Proof. Let G be a winning strategy for the lion in the continuous game, and let G' be the corresponding strategy for the lion in the ε-discrete game (with the lion moving second). In any play of the discrete game, the lion (following G') must be at distance at most ε of the man (because the lion catches the man in the continuous game). Hence $\delta(\varepsilon) \leq \varepsilon$ and the result follows. □

The next lemma assumes that X is compact; this is a natural condition to impose for the whole lion-man game in general, and here it is important because it allows us to take limits of paths. We remark that, since all paths are Lipschitz, the spaces L and M, viewed as metric spaces with the supremum metric, are compact. This is by a standard Arzelà-Ascoli type argument (see for example [3, Ch. 6]) – the fact that all paths are Lipschitz guarantees equicontinuity.

To prove that a winning strategy for the lion in the discrete game lifts to a winning strategy in the continuous game we use the following lemma.

Lemma 3. *Let X be compact. Suppose that for every n there exists a lion strategy G_n in the continuous-time game such that for every man path m we have $d(G_n(m)(t), m(t)) < 1/n$ for some t. Then there exists a winning lion strategy in the continuous-time game.*

Proof. It is tempting to argue as follows. For any man path m, the compactness of L ensures that there exists a subsequence of the paths $G_n(m)$ that converges uniformly to some path l. Define $G(m) = l$, noting that since for each n there is a t with $G_n(m)(t)$ within distance $1/n$ of $m(t)$ it follows by the uniformity of the convergence (and the fact that the interval $[0, T]$ is compact) that we have $G(m)(t) = m(t)$ for some t. However, this may not yield a valid strategy: there is no reason why G should satisfy the no-lookahead rule.

Instead of this, we build up G one path at a time – or, to put it another way, we use Zorn's Lemma to construct G. Let a *partial strategy* be a function G from a subset of M to L that satisfies 'no lookahead' where it is defined – in other words, if G is defined at $m, m' \in M$, and m and m' agree on $[0, t]$, then also $G(m)$ and $G(m')$ agree on $[0, t]$. We say that a partial strategy G is *good* if for each m for which $G(m)$ is defined there is a subsequence of the paths $G_n(m)$ that converges uniformly to $G(m)$. Given two good partial strategies G_1 and G_2 with domains M_1 and M_2 respectively, we say $G_1 \leq G_2$ if $M_1 \subset M_2$ and $G_2|M_1 = G_1$.

It is obvious that every chain of good partial strategies has an upper bound, namely their union. Hence by Zorn's Lemma there is a maximal good partial strategy G, say with domain M'. We will show that this is a full strategy, i.e. that $M' = M$.

Indeed, suppose not. Fix $m \notin M'$. We aim to extend G to a good partial strategy G' on $M' \cup \{m\}$. Set t_0 be

$$t_0 = \sup\{t : \exists m' \in M',\ \forall s < t,\ m(s) = m'(s)\}.$$

There are now two cases, according to whether or not this supremum is attained. If it is attained, we have a path m' in the domain of G agreeing with m on $[0, t_0]$. Then a certain subsequence of the $G_n(m')$, say the $G_{n_i}(m')$, converges to $G(m')$. We may now choose a convergent subsequence of the $G_{n_i}(m)$, and let $G'(m)$ be the limit of that sequence.

On the other hand, if the supremum is not attained then we have a sequence $t_1, t_2, \ldots$ tending up to t_0, and paths $m_1, m_2, \ldots$ in the domain of G, such that m and m_i agree on $[0, t_i]$. For each i, we may choose n_i such that $G_{n_i}(m_i)$ is within distance $1/i$ of $G(m_i)$ (and say $n_1 < n_2 < \ldots$). We now choose a convergent subsequence of the $G_{n_i}(m)$, and let $G'(m)$ be the limit of that sequence.

It is easy to check that G' is indeed a good partial strategy. □

We remark that Lemma 3 may also be proved using limits along a (non-principal) ultrafilter. Alternatively, one may work with the space of strategies and use Tychonov's theorem. Indeed, one can check that the space of strategies is a closed subset of the space of all functions from M to L (with the obvious product topology), and hence is compact – and now any accumulation point of the sequence $G_1, G_2, \ldots$ is a winning lion strategy. But it would be interesting to know if the appeal to the Axiom of Choice (in the form of Zorn's Lemma or Tychonov's theorem) is really necessary.

By taking as the G_n the strategies corresponding to the lion strategies in the discrete game (with the lion moving first), we immediately have the following corollary.

Corollary 4. *Suppose that in the ε-discrete game we have $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$. Then the lion has a winning strategy for the continuous-time game.* □

Combining Lemma 1 and Corollary 4 we have:

Theorem 5. *In the bounded-time game played in a compact metric space, at least one of the lion and man has a winning strategy.* □

The condition that the game be played for bounded time seems crucial for the above argument: we do not know what happens if time is unbounded.

We remark that there are simple spaces showing that the man being able to escape for an arbitrarily long time does not mean that the man can escape forever. Indeed, consider a space consisting of paths from a point of path-lengths $1, 2, 3, \ldots$, with the man starting at the common point. Then the man has a winning strategy for the bounded-time game (for any T), but not for the unbounded-time game. Note that this space can easily be made compact, by 'rolling up' the paths.

We also do not know what happens if the metric space X is not compact. We suspect that it can happen that (even for the bounded-time game) neither player has a winning strategy, but we have been unable to show this. However, in Section 4 we will show that if one allows two lions (acting as a team) to pursue a man then it can indeed happen that neither side has a winning strategy.

We now turn to the last of our four implications. If the man has a winning strategy that is continuous (as a function from L to M) then the result is immediate.

Lemma 6. *Let X be a compact metric space. Suppose that the man has a continuous strategy that is winning for the bounded-time game on X. Then $\delta \not\to 0$ as $\varepsilon \to 0$.*

Proof. Let F be the continuous winning strategy. The function mapping a lion path l to $\inf_{t\in[0,T]} d(l(t), F(l)(t))$ (in other words, the closest the lion ever gets to the man when the man plays this strategy), is continuous as a function of l. As L is compact, there is a path minimising this distance. But F is a winning strategy, and so this minimal distance must be strictly greater than zero – say it is $c > 0$. Hence if the man plays the corresponding strategy in the ε-discrete game (with the man moving second) then for any $\varepsilon < c$ we have $\delta(\varepsilon) \geq c - \varepsilon$, so that δ does not tend to zero. $\square$

Now, do strategies tend to be continuous? As defined earlier, the Besicovitch strategy for the game in the closed disc is *not* continuous. This is for a reason that one feels ought to be easy to get round: that if the lion is on the same radius as the man then the man makes an arbitrary choice of which way to run. But in fact there is *no* continuous winning strategy for the man in this game.

Theorem 7. *In the lion and man game in the closed unit disc the man does not have a continuous winning strategy. Indeed, for any continuous man strategy there is a lion path catching the man by time* 1.

Proof. Suppose that F is a continuous man strategy. For each point z in the unit disc define the lion path l_z to be the constant speed path from the origin to z reaching z at time 1. Then $z \mapsto F(l_z)(1)$ is a continuous function of z from the disc to the disc. Hence by the Brouwer fixed point theorem there is some z with $F(l_z)(1) = z = l_z(1)$. In other words, the man is caught by the lion. □

One might still feel that the problems here are merely technical, arising as they do out of the arbitrary choice when the lion is on the same radius as the man. It would be natural to imagine that this can be got round by allowing multivalued strategies (thus each lion path would map to a set of man paths) – so if the lion was on the same radius as the man then we would have two paths extending our path so far, and so on. One would then hope to prove that such a set-valued function may always be chosen to be upper semi-continuous (or have some related property), which would then allow a proof as in the lemma above.

But, contrary to what the authors of this paper had thought for some time, this cannot be made to work, and indeed the whole theorem that exactly one player has a winning strategy is false – even in compact spaces. This is the content of the next section.

3 Some Examples

We start by giving an example of a compact metric space in which, for the unbounded-time game (and also for the bounded-time game), both players have winning strategies. As we shall see, the strategy for the lion is very simple. The strategy for the man, on the other hand, will rely on a curious device of 'getting out from underneath the lion'.

If X and Y are metric spaces then the l_∞ *sum* of X and Y is the metric space on $X \times Y$ in which the distance from (x, y) to (x', y') is $\max(d(x, x'), d(y, y'))$.

Theorem 8. *Let X be the l_∞ sum of the closed unit disc D and $[0,1]$. Then, in the lion and man game on X with the man starting at $(0,0)$ and the lion starting at $(0,1)$, both players have winning strategies.*

Proof. The lion has an obvious winning strategy: keep the 'disc' coordinate the same as the man and run towards him in the interval coordinate. Thus the lion catches the man in time at most 1.

For a winning strategy for the man, the first aim is to 'escape from underneath the lion'. For time $1/2$ (say), the man acts as follows. If there exists a positive time t such that the lion's disc coordinate was exactly s for all $0 \leq s \leq t$ then the man runs straight to $(-1/2, 0)$, while if there is no such positive t then the man runs straight to $(1/2, 0)$. Note that this satisfies the no-lookahead rule, because for any given time $t \leq 1/2$ the man's position at time t is determined by the lion's position at arbitrarily early times.

At time $1/2$, the man now has a different disc coordinate to the lion. He then plays the Besicovitch strategy in the disc (and does whatever he likes in the other coordinate). $\square$

The above example does not embed isometrically into Euclidean space, because the sum is taken as an l_∞ sum – and moreover the l_∞ nature of the sum was crucial to the lion having a winning strategy. It would be very interesting to know if such a construction exists in Euclidean space.

In the above example, the start positions were very important. Indeed, if the lion and man did not start with the same disc coordinate then the lion does not have a winning strategy – this is as for the earlier discussion of the Besicovitch strategy. However, we now show that by adapting the above ideas, and getting the man to use the 'escape from underneath the lion' idea not just once but repeatedly, there is a compact metric space in which, for any (distinct) starting positions, both players have winning strategies.

Theorem 9. *Let X be the closed unit ball in l_∞^2 (the l_∞ sum of $[-1,1]$ with itself). Then both players have winning strategies for the lion and man game on X, for any distinct starting positions.*

Proof. We describe the strategy when the man starts at $(1,0)$ and the lion starts at the point (x_0, y_0) – the general case is the same. The lion has an obvious winning strategy: in each cordinate, run at full speed towards the man, and when that coordinate is equal to that of the man then keep the coordinate the same as that of the man. This catches the man in time at most 2.

For the man's strategy, we will give a strategy for the man which starts with him running at full speed to one of $(0, 1)$ and $(0, -1)$ without being caught. From here he repeats the strategy, thus surviving for all time.

We split into three cases: if $y_0 < 0$ then the man runs to $(0, 1)$ and so does not get caught. If $y_0 > 0$ then the man runs to $(0, -1)$ and, again, does not get caught.

Finally we deal with the case $y_0 = 0$. Here we follow the strategy from the previous proof: if the lion's y-coordinate at time s is equal to s for all $0 < s < t$, some $t > 0$, then the man runs to $(0, -1)$, and otherwise he runs to $(0, 1)$. Again, he is not caught. $\square$

4 Race to a Point

In this section we give examples of race to a point games in which both players have a winning strategy, and also examples in which neither player has even a drawing strategy (meaning, of course, a strategy that guarantees the player either a win or a draw in each play of the game). The first of these is included for the sake of completeness, and also out of interest, because what is somehow the 'obvious' example is in fact not an example at all. The second will form the basis for an example of a lion and man game involving two lions pursuing a man.

In general, if the game is symmetric, with the two players starting at the same point, then it is clear that each player has a drawing strategy, namely 'copy the other player'. What about both players having a winning strategy?

It is natural to try to construct an example based on the idea that, in a game where one has to name a higher number than one's opponent, saying 'my opponent's number plus 1' would be a sensible thing to do, if it were allowed. So we let X consist of two points x and y at distance 1, joined by some disjoint paths of length $1 + 1/n$ for every n (the distance apart of two points on different paths is irrelevant to the argument). Both players start at x and the target is y. Then the analogue of the above would be for Player A to move as follows: if Player B runs along the path of length $1+1/n$, then Player A runs along the path of length $1+1/(n+1)$ instead, thus arriving at y before B does.

Surprisingly, this is not correct. For unfortunately Player B might 'back-track' and change his path. Or he might wait at x for a while before proceeding (and there may not even be a 'first' path that he moves onto). So the above strategy is not well-defined. In fact, there is no winning strategy, as we now show. This proposition should be contrasted with the one that follows it.

Proposition 10. *Let X be the metric space defined above. Then in the race to a point game starting at x and ending at y neither player has a winning strategy.*

Proof. Let G be a strategy for Player A. Suppose Player B's path p is just constant at the start point. Then, according to G, Player A's path $q = G(p)$ reaches the finish at some time. On this path, at some time t Player A is strictly more than distance 1 from the finish: say he is at point z, at distance $1+\varepsilon$.

Now, by the no-lookahead rule it follows that for every Player B path that stays at the start point until time t we have that Player A is at the same point z (at distance $1+\varepsilon$ from the finish) at time t. But now consider a path for Player B that waits at the start until time t and then runs straight to the finish along a path of length $1+1/n < 1+\varepsilon$. In that play of the game, B reaches the finish before A. $\square$

However, we can modify the idea of this construction, by allowing the various paths from x to y to connect to each other. The simplest way to do this is as follows.

Proposition 11. *Let X be the subset of the plane consisting of the open upper half plane with $(0,0)$ and $(1,0)$ added. Then in the race to a point game on X starting at $(1,0)$ and finishing at $(0,0)$ both players have a winning strategy.*

Proof. We give a winning strategy for (say) Player A. It is convenient to work in polar coordinates (r, θ). Suppose that Player B follows path $(r(t), \theta(t))$. Then Player 1 follows the path $(s(t), \phi(t))$ given by

$$\begin{aligned} s(t) &= (r(t) + 2(1-t))/3 \\ \phi(t) &= t + s(t) - 1 \; . \end{aligned}$$

The key point is that $s(t)$ is always smaller than $r(t)$ for $t > 0$ (because we cannot ever have $r(t) = 1 - t$). The choice of ϕ ensures that the path is Lipschitz and at the same time stays within the space. □

We now turn to our main aim in this section: a metric space (with given start points for the two players and a given finish point) in which neither player has even a drawing strategy.

Theorem 12. *There is a space X in which for the race to a point game (with specified starting positions and target) neither player has even a drawing strategy.*

Proof. The space will be the following: we will pick subsets A and B of the interval $(0, 1)$. The space X will be a subset of the complex plane: it will consist of the subset $\{e^{it} : \ 0 \leq t \leq 1\} \cup \{-e^{it} : \ 0 \leq t \leq 1\}$ of the unit circle, together with 'spokes' (radii to the origin) from each point e^{ia}, $a \in A$ and $-e^{ib}$, $b \in B$.

Player A starts at 1 and Player B at -1, and they are racing to the origin. We start by finding conditions on A and B that would ensure that neither player has a drawing strategy.

Suppose that Player A has a drawing strategy. Fix a point b in B with $b > 1/2$. Consider the Player B path going round the circle to $-e^{ib}$ and then along the spoke to the origin. Player A's strategy gives some path p, which of course must leave the unit circle at some time (as it has to reach the origin). Let t_0 be the last time at which Player A is on the boundary circle – certainly $t_0 \leq b$. Since Player A is following a strategy he follows exactly the same path p whenever Player B sets off round the circle at full speed, at least until Player B deviates from the path above. If there is some time t at which Player B is at the end of one of his spokes and Player A is not on or at the end of one of his, then the Player B path 'go around the boundary until t and then along the spoke' beats Player A's strategy (since at the time Player B leaves the circle Player A is not able to).

Now, during the time $[0, t_0]$ Player A's position varies continuously. Thus, if no such t occurs, then the function that sends t to the argument of $p(t)$ is a continuous function (even Lipschitz) from $[0, t_0] \to [0, t_0]$ mapping 0 to 0 such that every point of B is mapped to a point of A.

The same of course has to hold with the players reversed. Thus we shall be done if we can find sets A and B in $(0, 1)$ such that for no $t_0 > 0$ is there a continuous function $f : [0, t_0] \to [0, t_0]$ with $f(0) = 0$ and $f(B) \subset A$, and nor is there such a continuous function with $f(A) \subset B$.

We use a well-ordering argument to construct these sets. It will be convenient to use the term *large* to mean of the same cardinality as the reals and the term *small* to mean of strictly smaller cardinality than the reals. Readers not familiar with set theory will lose nothing if they think of these as uncountable and countable respectively (although formally that would assume the continuum hypothesis).

Consider the set S_n consisting of all continuous functions from $[0, 1/n]$ to itself that fix zero, for each natural number n, and let S denote their union (over all n). Then S has the same cardinality as the reals, and so we may well-order S as $\{f_\beta : \ \beta < c\}$, where c is the first ordinal with cardinality that of the reals.

We construct the sets A and B inductively: at any stage $\beta < c$ we will have put at most a small number of points in A and forbidden at most a small number of points from A — we will call these sets A^+ and A^- respectively. (Formally, what we mean by this is that at stage β we have a set A^+_β, and when we write for example 'place a in A^+' this means that we set $A^+_{\beta+1} = A^+_\beta \cup \{a\}$ – but we prefer to omit the subscripts for readability.) And similarly for B (we start with all of these sets empty). At the βth stage, consider the function $f = f_\beta$: say it maps $[0, 1/n] \to [0, 1/n]$. We want to pick a point b to put in B and an a to forbid from A with $f(b) = a$: that is, we want to make sure that $f(B) \not\subset A$.

First, suppose $f([0, 1/n])$ is large. Since there are only a small number of points that we have put in A so far (i.e., A^+ is small) we have that $f^{-1}(f([0, 1/n] \setminus A^+)$ is large. Since B^- is also small we can pick b

in $f^{-1}(f([0,1/n] \setminus A^+) \setminus B^-$. We put b in B^+ and $a = f(b)$ in A^-.

If $f([0,1/n])$ is not large, then, by the Intermediate Value Theorem, it must be constantly zero. Since $0 \notin A$, we just pick a point b in $[0,1/n] \setminus B^-$ and place it in B^+.

We now do the same the other way round: putting a point a' in A^+ and b' in B^- with $f(a) = b'$.

Continuing in this way we obtain the sets we require by letting $A = \bigcup A^+$ and $B = \bigcup B^+$. Indeed, suppose that we are given a continuous function $f \colon [0, t_0] \to [0, t_0]$. Pick n with $1/n < t_0$. Then f restricted to $[0, 1/n]$ belongs to S, and so there is a point $a \notin A$ and $b \in B \cap [0,1/n]$ with $f(b) = a$. $\square$

It would be interesting to know whether there is an explicit construction (meaning without use of the Axiom of Choice) of such an example. Indeed, the induction part of the above proof (which is a standard type of argument, as mentioned in the Introduction) is of a kind that does not produce Borel sets; is there an example that is Borel?

We now show how to use the construction above to give a pursuit game of two lions against a man in which neither player has a winning strategy.

The idea is to take the above space X and attach an infinite ray to the origin (out of the plane). Suppose first that we consider the usual lion and man game (with only one lion) on this space, with the lion and man starting at $(-1, 0)$ and $(1, 0)$ respectively. Then no lion strategy always catches the man, since (as above) there is a man path that reaches the origin before the corresponding lion path; if the man now runs at full speed along the infinite ray then he will not be caught. Unfortunately, it is not correct to argue that the man cannot have a winning strategy merely because he cannot guarantee to reach the origin first – it is easy to see that the man can win while staying entirely within X.

However, the addition of a second lion is enough to render this impossible. When we speak of 'two lions' they are understood to form a team: the lions win if at least one of them catches the man.

Theorem 13. *There is a metric space Y in which, for the game of two lions against a man (with specified starting positions), neither player has a winning strategy.*

Proof. We form Y from X by attaching an infinite ray at the origin and also a line of length 1 from $(-2, 0)$ to $(-1, 0)$. The lions start at $(-2, 0)$ and $(-1, 0)$ and the man starts at $(1, 0)$.

As above we see that the no lion strategy stops the man getting to the origin first (the second lion is too far away to affect this argument). Once on the ray the man, of course, wins. Hence there is no winning strategy for the lions.

To see that the man cannot have a winning strategy, note first that the man can never guarantee to reach the infinite ray (thanks to the lion that starts at $(-1, 0)$). If we then let that lion stay at the origin for ever, we can let the other lion run around the circle, eventually trapping the man on a spoke. Hence there is no winning strategy for the man. $\square$

5 Open Questions

In this section we collect the various open problems that have been mentioned in the paper, as well as giving some other ones.

The most interesting question is probably that of whether or not there exists a metric space, necessarily non-compact, in which neither lion nor man has a winning strategy in the bounded-time game.

Question 1. *Does there exist a metric space X in which for the bounded-time lion and man game neither player has a winning strategy?*

It would also be very nice to know what happens for the *unbounded-time* game in a compact (or indeed any) metric space.

Question 2. *Let X be a compact metric space. In the unbounded-time lion and man game must it be the case that at least one player has a winning strategy?*

Then there is the question about whether or not the general situation becomes nicer if we restrict to Euclidean spaces.

Question 3. *Is there a subset of a Euclidean space for which both the lion and the man have winning strategies for the bounded-time game?*

We believe that the answer to this question is no. Indeed, we believe that much stronger statements should be true, determining exactly who wins in a given subset of Euclidean space: the lion should win precisely when the subset is 'tree-like' or 'dendrite-like'. An example of such a statement would be the following (where an *arc* is an injective path).

Conjecture 4. *Let X be a subset of a Euclidean space that has an associated path length metric (in other words, any two points are joined by a path of finite length), with X compact in this metric. Then the lion has a winning strategy for the lion and man game on X from all starting positions if and only if any two points of X are joined by a unique arc.*

Perhaps related to this is the following question. Lemma 3 clearly applies to any metric space X that is isometric to the path length metric on a compact metric space – for example, it applies to the real line, since the real line is the path length metric of a suitable compact subset of the plane.

Question 5. *Is there a natural larger class of spaces to which Lemma 3 applies?*

Finally, it would be interesting to know how much of our use of the Axiom of Choice is actually necessary.

Question 6. *Is there a proof of Lemma 3 that does not use the Axiom of Choice?*

Question 7. *Is there an 'explicit' (constructive) example of a metric space in which, for race to a point, neither player has a drawing strategy? In particular, can this happen for a Borel subset of a Euclidean space?*

6 Thanks

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