COMPLEX ANALYSIS NOTES

HAO (BILLY) LEE

Abstract. These are notes I took in class, taught by Professor Marianna Csornyei. I claim no credit to the originality of the contents of these notes. Nor do I claim that they are without errors, nor readable.

Stein- Complex analysis
Midterm 23rd, in class.

1. Review

$D \subseteq \mathbb{C}$ is a domain (open, connected), $f : D \to \mathbb{C}$. f is holomorphic if

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists for all $z_0 \in D$.

If f^{-1} exists and is continuous, and $f(D)$ is a domain, $f(z) \neq 0$ then f^{-1} is holomorphic, with $\left(f^{-1}\right)'(z) = \frac{1}{f'(f^{-1}(z))}$.

Exercise 1.1. $C = \partial B(0,1)$ be the unit circle, and $C \subseteq D$ with f holomorphic on D. Let $z_0 \in C$. What is the tangent at $f(z_0)$.

Proof. If $z_0 = e^{it_0}$, then take derivative of the map $t \mapsto f(e^{it})$, to get $f'(e^{it_0}) \cdot ie^{it_0} = f'(z_0) \cdot iz_0$. □

$f(z) = \sum_n a_n z^n$ power series, then the radius of convergence is

$$R = \liminf \frac{1}{|a_n|^{1/n}}.$$

If $|z| < R$, then $\sum a_n z^n$ converges. On $B(0,r)$ with $r < R$, f converges uniformly. If $|z| > R$, then the series diverges.

Proof. If $|z| > R$, then there are infinitely many n such that $|a_n z^n| > 1$.

If $|z| < R$, then for large enough n, we have $|a_n z^n| < q^n$ for some $q < 1$. Uniformly, because q is independent of z. □

Theorem 1.2. *$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ is holomorphic on $B(z_0, r)$ for $r < R$. Conversely, if f is holomorphic on $B(z_0, r)$, then $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$.*

To get the a_n's, take derivative and evaluate $z = z_0$, so

$$a_n = \frac{f^{(n)}(z_0)}{n!}.$$

Let $\gamma : [a,b] \to \mathbb{R}^2$ be a continuous curve of finite (or σ-finite) length.

$$\int_\gamma f(z)dz = \lim_i \sum f(s_i)(z_i - z_{i-1})$$

where z_i's partition γ, and s_i's are some point in the interval. If f is continuous and γ has finite length, then the integral exists, and

$$\left| \int_\gamma f(z)dz \right| \leq (\text{length of } \gamma) \cdot \max |f(z)|$$

If γ is piece-wise smooth, then

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\, \gamma'(t)dt.$$

D is simply connected, if for any closed $\gamma \subseteq D$, there is a continuous deformation mapping it to a point notation: $n(\gamma, z_0)$ is the number of times γ goes around z_0.

Theorem 1.3. $n(\gamma, z_0) = \frac{1}{2\pi i}\int_\gamma \frac{1}{z-z_0}dz$.

Theorem 1.4. *(Cauchy) If D is simply connected, and f is holomorphic on D, $\gamma \subseteq D$ is a closed curve, then*

$$\int_\gamma f(z)dz = 0.$$

Additionally,

$$f(z_0)n(\gamma, z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-z_0}dz.$$

Similarly,

$$f^{(n)}(z_0)n(\gamma, z_0) = \frac{n!}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_0)^{n+1}}dz.$$

Special case: γ is a circle around z_0, then $n(\gamma, z_0)$ is always 1, so

$$a_n = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_0)^{n+1}}dz.$$

Theorem 1.5. *If D is simply connected, and f is holomorphic, then*

$$\int_{z_1}^{z_2} f'(z) = f(z_2) - f(z_1)$$

Here, can integrate over any γ.

2. Local Behaviour of Holomorphic Functions

Suppose $f(z) \neq$constant, holomorphic on $B(z_0, r)$, then

$$f(z) = \sum_{n\geq 0} a_n(z-z_0)^n.$$

Let $m \geq 1$ be the smallest index such that $a_m \neq 0$. Then

$$f(z) - f(z_0) = \sum_m a_n(z-z_0)^n = (z-z_0)^m \sum_{n\geq 0} a_{n+m}(z-z_0)^n.$$

This means that $\frac{f(z)-f(z_0)}{(z-z_0)^m} \to a_m$ as $z \to z_0$ (need to prove that the sum on the right converges uniformly, not just term-wise).

Corollary 2.1. $\frac{|f(z)-f(z_0)|}{|z-z_0|^m} \to |a_m|$ *is called circle preserving.*

Basically, fixing a small circle of radius r around z_0, $f(z)$ is not far away from $f(z_0)$, with radius about $|a_m|\, r^m$.

Corollary 2.2. *We have*

$$Ang\,(f(z) - f(z_0)) - m \cdot Ang\,(z-z_0) \to Ang(a_m).$$

This is like angle preserving.

If we tend to z_0 along a half-line, then $Ang\,(f(z) - f(z_0)) \to m \cdot Ang\,(z-z_0) + Ang(a_m)$ where the RHS is constant. It's like we stay inside a cone. That is, the image has a tangent at $f(z_0)$ of this direction. If we tend to z_0 along a different half line, then the angle between the images is m times larger.

Theorem 2.3. *(Maximal Principle). f is holomorphic, non-constant on some domain D. Then $|f|$ cannot attend its supremum on D (domains are open).*

Definition 2.4. f is holomorphic on $B(0, r_0)$. Define the maximal modulus of f to be $M(r) = \sup_{B(0,r)} |f(z)|$, with $0 < r \leq r_0$.

Fact 2.5.

(1) $M(r)$ is increasing
(2) If $f \neq$constant, then $M(r)$ is strictly increasing
(3) If f is continuous on $cl(B(0,r))$, then $M(r) = \max_{z\in\partial B(0,r)} |f(z)|$.
(4) $M(r)$ is continuous (f can jump, but M is defined a sup of open sets, so we are fine. Need to check this for going up and below, cuz you are monotone. The other direction, take a sequence converging, take a converging subseuqnece of f(blah), since f continuous on it, it can't be too small.

Proposition 2.6. *Let $f : B(0,1) \to B(0,1)$ with $f(0) = 0$ and $a_0 = ... = a_n$. Then $\frac{M(r)}{r^{n+1}}$ is increasing. This is a generalization of the above.*

Corollary 2.7. *(Special Case). Schwartz Lemma. $f : B(0,1) \to B(0,1)$ with $f(0) = 0$ then $|f(z)| \leq |z|$. If equality holds at some z, then f is constant times z.*

Proof. Since $\frac{M(|z|)}{|z|}$ is increasing, just need to check at an endpoint. Well, f maps into $B(0,1)$, so it's bounded by 1.

$f(z) = z(a_1 z + a_2 z + ...)$ call the bracket $g(z)$. Then $M_f(r) = rM_g(r)$ for $r < 1$. Then $M_f(1) = M_g(1)$. Since $M_f(1) \leq 1$, so is $M_g(r)$, as required.

If equality holds, then g is constant. □

Theorem 2.8. *Hadamard's Three circle theorem. Take 3 circles, with $r^2 = r_1 r_2$ then $M(r)^2 \leq M(r_1)M(r_2)$. That is, $\log M$ is a convex function of $\log r$.*

Alternatively, f is bounded on a domain that contains (closed) annulus.

Proof. Suppose f is maximal on $|z| = r$ at z_0. Let

$$g(z) = f(z)f\left(\frac{z_0^2}{z}\right)$$

on the annulus (between the two circles). The $\frac{z_0^2}{z}$ is really mapping the annulus to itself.

Since g is bounded on the annulus, with

$$M(r)^2 \leq \left|f(z_0)^2\right| = |g(z_0)| \leq \max\left(M_g(r_1), M_g(r_2)\right).$$

Notice that $M_g(r_1) \leq M_f(r_1)M_f(r_2)$ and $M_g(r_2) \leq M_f(r_2) \cdot M_f(r_1)$. □

Proof. 2 proofs of maximal principle. Cauchy's theorem. write $z_0 \in D$, $f(z_0)$ as

$$f(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z - z_0}dz$$

where γ is a circle around 0. Then

$$|f(z_0)| \leq \frac{1}{2\pi} \cdot 2\pi r \cdot \max_{|z-z_0|=r} \frac{|f(z)|}{r} = \max|f(z)|.$$

□

By the same argument:

Theorem 2.9. *Louiville's theorem. f is holomorphic and bounded on $\mathbb{C}$ then f is constant. If it is bounded by a polynomial $|z|$ then it's a polynomial.*

Proof. Open mapping theorem to imples maximal principle. Given something in the image, and put a small disk around an image point, then there's a small neighbourhood in the domain mapping into it. □

Theorem 2.10. *(Quantative open mapping theorem). Let D be a domain, with $z_0 \in D$, f holomorphic non-constant. Let m be first number such that $m \geq 1$, $a_m \neq 0$. For every $\epsilon > 0$ small enough, there exists $\delta > 0$ such that for every $w \in B(f(z_0), \delta)$, there are exactly m points $z_1, ..., z_m \in B(z_0, \epsilon)$ such that $f(z_i) = w$.*

Proof. Choose ϵ small, so that $f(z_0)$ is not attained at any other point $B(z_0, \epsilon)$ and $f' \neq 0$, except possibly at z_0.

Let γ be $\partial B(z_0, \epsilon)$. Let

$$g(z) = \frac{f(z) - w}{f(z) - f(z_0)}.$$

Then g has a pole at z_0 of multiplicity m. Choose $\delta < \min_{z\in\gamma} |f(z) - f(z_0)|$. Then

$$|g(z) - 1| = \left| \frac{f(z_0) - w}{f(z) - f(z_0)} \right| < \frac{\delta}{\delta} = 1$$

for all $z \in \gamma$. Therefore, $g(z) \neq 0$ for all $z \in \gamma$ (that is, $n\left(g(\gamma), 0\right) = 0$).

By the argument principle, since g has a pole of multiplicity m, it must also have a zero of multiplicity m. Therefore, there are m-roots with multiplicity 1 because $f' \neq 0$. (argument principle says that

$$\int_\gamma \frac{f'(z)}{f(z)} dz = 2\pi i \cdot n\left(f \circ \gamma, 0\right) = 2\pi i(\text{zeroes} - \text{poles})$$

(cuz centered at 0). ◻

3. Conformal Mappings

Definition 3.1. Let $f : D_1 \to D_2$ is a bijection, f holomorphic and f^{-1} holomorphic.

D_1, D_2 are conformally equivalent if there exists such an f.

Theorem 3.2. *Riemann mapping theorem.*

Remark 3.3. THE MIDTERM WILL HAVE A QUESTION ON THE LEMMA OF THIS PROOF!!!!!!!!!

Example 3.4.

(1) $\mathbb{C} \to \mathbb{C}$ translation, rotation, dilation.
(2) $\log z$ conformal action on $\mathbb{C}\backslash\mathbb{R}^{\geq 0} = D_1$. For $z = re^{i\theta}$, $\log z = \log r + i\theta$. $D_2 = \mathbb{R} \times (-\pi, \pi)$.
(3) Half-plane to the disc.
Take $f(z) = \frac{i-z}{i+z}$. Since for every point of the upper halfplane, z is closer to i than $-i$, this is in $B(0, 1)$. This is holomorphic because $-i \notin \mathbb{H}$. Let $\frac{i-z}{i+z} = w$, then $z = i\frac{1-w}{1+w}$. This is a holomorphic function on the disk. Now, if $w = u + iv$, then $z = i\frac{1-u-iv}{1+u+iv}$. Just need the real part of the fraction is positive. The real part is $\frac{(1-u)(1+u)-v^2}{blah}$.

Exercise 3.5. Consider $f(z) = \frac{1-z}{1+z}$. Show that this is a conformal mapping between upper half disc and....

Proof. Notice that we can write

$$\frac{1-z}{1+z} = \frac{2}{1+z} - 1 = \frac{2(1+x)}{(1+x)^2+y^2} - i\frac{2y}{(1+x)^2+y^2} - 1.$$

From this, we see that the $Re(f(z)) > 0$ and $Im\left(f(z)\right) < 0$. Claim that we have the whole bottom right quadrant. Suppose

$$\frac{1-z}{1+z} = w \implies z = \frac{1-w}{1+w}$$

is holomorphic. To see that this lands in the upper half disk, we multiply by i, to rotate to the upper right plane. From $B(0, 1) \cong \mathbb{H}^n$ via $\frac{i-z}{i+z}$, just identify the parts we want. ◻

Definition 3.6. Fractional linear transform is $f(z) = \frac{az+b}{cz+d}$ on the Riemann sphere $\mathbb{C} \cup \{\infty\}$, with $\det \neq 0$.

Fact 3.7. *These form a group, have translations given by* $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ *and rotations* $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ *with* $|a| = 1$; *and* $\frac{1}{z}$ *given by* $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

This is conformal, because

$$\begin{aligned} z &\overset{\text{translation}}{\mapsto} z+\frac{d}{c} \overset{\text{dilation}}{\mapsto} \left|c^2\right|\left(z+\frac{d}{c}\right) \overset{1/z}{\mapsto} \frac{1}{\left|c^2\right|\left(z+\frac{d}{c}\right)} \\ &\overset{\text{dilate}}{\mapsto} \frac{|bc-ad|}{\left|c^2\right|\left(z+\frac{d}{c}\right)} \overset{\text{rotate}}{\mapsto} \frac{bc-ad}{c^2\left(z+\frac{d}{c}\right)} \overset{\text{translation}}{\mapsto} \frac{bc-ad}{c^2\left(z+\frac{d}{c}\right)}+\frac{a}{c} \\ &= \frac{az+b}{cz+d} \end{aligned}$$

Let $f(z)=\frac{az+b}{cz+d}$ conformal from $\mathbb{C}\backslash\left\{\frac{-d}{c}\right\} \to \mathbb{C}\backslash\left\{\frac{a}{c}\right\}$. Then define

$$f\left(-\frac{d}{c}\right)=\infty \text{ and } f(\infty)=\frac{a}{c}.$$

This makes sense geometrically (stereographic projection). Through the stereographic projection, we see that

line in $\mathbb{C} \leftrightarrow$ circle in sphere through the north pole.

Let $\bar{\mathbb{C}}=\mathbb{C}\cup\{\infty\}$. If $z\mapsto\infty$, then $z'\to N$ (on the sphere). So $\{|z|>r\}$ is a basis of the topology around ∞. If f is continuous at ∞, this means that $f(z)$ has a limit as $z\to\infty$.

A circle in $\mathbb{C}$ gets mapped to a circle in S (viewing lines as circles passing through ∞). Consider the equation

$$A\left(x^2+y^2\right)+Bx+Cy+D=0$$

of a circle in $\mathbb{R}^2$. Let

$$x'=\frac{x}{1+x^2+y^2},\ y'=\frac{y}{1+x^2+y^2},\ z'=\frac{x^2+y^2}{1+x^2+y^2}.$$

Divide the above equation by $1+x^2+y^2$, to get

$$(A-D)\,z'+Bx'+Cy'+D=0$$

is a plane in $\mathbb{R}^3$, and intersections our sphere in a circle (a planar section).

Theorem 3.8. *The image of a circle or a line through a fractional linear transform is a circle or a line.*

Proof. Since the group is generated by transformation, dilation, rotation and $\frac{1}{z}$ (the first few have the desired properties). Just need to check this for $\frac{1}{z}$. Enough to choose a chart on the sphere and check circle goes to circle.

If $rr^\star=1$, then if$|z|=r$ then . $r=\tan(stuff)$ and $r^\star=\tan(stuff)$. Point is $z\mapsto z^\star$ in $\mathbb{C}$ gives φ to $-\varphi$ reflecton on the sphere (φ is the angle). □

Fact 3.9. *Given z_2,z_3,z_4 distinct points, there exists unique fractional linear transform mapping to v_2,v_3,v_4 distinct points.*

Proof. Enough to check this for $v_2=1$, $v_3=0$ and $v_4=\infty$. Take

$$f(z)=\left(\frac{z-z_3}{z-z_4}\right)/\left(\frac{z_2-z_3}{z_2-z_4}\right).$$

□

Definition 3.10. The cross ratio $(z_1,...,z_4)\in\bar{\mathbb{C}}$ is the image of z_1 under the fractional linear transformation that maps z_2,z_3,z_4 to $1,0,\infty$.

Theorem 3.11. *Cross ratio is invariant under fractional linear transform.*

Proof. Suppose f is a fractional linear transform,

$$(z_1,...,z_4)=(f(z_1),...,f(z_4)).$$

Let g denote the fractional linear transform sending z_2,z_3,z_4 to $1,0,\infty$. Then $g(z_1)=\left(g\circ f^{-1}\right)(f(z_1))$. □

Corollary 3.12. *TFAE:*

(1) $(z_1, ..., z_4) \in \mathbb{R}$
(2) $z_1, ..., z_4$ *lie on a circle or a line*
(3) *stereographic image is on a circle*

Proof. Choose fractional linear transform f sending z_2, z_3, z_4 to $1, 0, \infty$. Then

$$(z_1, ..., z_4) = (f(z_1), 1, 0, \infty).$$

LHS is real iff $f(z_1) \in \mathbb{R}$ iff $f(z_1), 1, 0, \infty$ is on a line (the real line actually) iff (by taking f^{-1}) LHS is on a line or circle. □

Definition 3.13. Reflection. $z, z^\star$ are symmetric with respect to C (a circle or a line) if $\overline{(z, .., z_4)} = (z^\star, z_2, ..., z_4)$, $z_2, z_3, z_4 \in C$.

Remark 3.14. If $z_2, ..., z_4$ are real, then

$$\overline{(z_1, ..., z_4)} = (\bar{z}_1, z_2, ..., z_4).$$

This can be deduced from the f above.

Fact 3.15. *If C is a line, then $z \to z^\star$ is the usual reflection.*

If C is the unit circle (correponding to the equator of $\bar{\mathbb{C}}$) then $z^\star = \frac{1}{\bar{z}}$.

Symmetric points are preserved by fractional linear transformation

Every reflection maps (circle or line) to (circle or line)

Exercise 3.16. Two points are symmetric wrt C (circle or line) iff any circle or line C' through the two points is orthogonal to C.

Proof. Since fractional linear transform preserves symmetry and angles, can reduce to case where the reflection is over $\mathbb{R}$-axis. □

Claim 3.17. The conformal automorphisms of $B(0, 1)$ are of the form

$$f(z) = c\frac{z - z_0}{z\bar{z}_0 - 1}$$

with $z_0 \in B(0, 1)$ and $|c| = 1$. These all also conformal automorphisms.

Proof. First, assume that f is a conformal automorphism of $B(0, 1)$ and f is a fractional linear transform. Denote $z_0 = f^{-1}(0) \in B(0, 1)$.

Claim that $\frac{1}{\bar{z}_0} = f^{-1}(\infty)$. This is because 0 and ∞ are symmetric, and so $f^{-1}(\infty)$ must be symmetric (reflected through the unit circle, which is preserved by f). Then

$$f(1) = c\frac{1 - z_0}{\bar{z}_0 - 1}.$$

Since the top and bottom has the same modulus, $|c| = 1$ (have to map circle to circle).

Every f defined in the claim, is a conformal aut of $B(0, 1)$. Since z_0 and $\frac{1}{\bar{z}_0}$ are symmetric, then $0, \infty$ must be symmetric such that the image of the unit circle becomes a circle around 0 (from having to be perpendicular to the 2 circles by the exercise. By the formula, $|f(1)| = 1$, and so the image has radius 1.

Let g be an arbitrary automorphism. Need to show g has the desired form. Let $z_0 = g(0)$. Let f be the fractional linear transform sending z_0 to 0. Then $f(g(0)) = 0$ and $f \circ g$ fixes 0, $(f \circ g)^{-1}$ fixes 0.

Schwartz lemma implies that $|(f \circ g)(z)| \leq z$ and $\left|(f \circ g)^{-1} z\right| \geq z$ and so $f \circ g$ is a rotation. Rotation is a fractional linear transform and so is f, this implies so is g. By the above, g has the desired form. □

Example 3.18.

(1) $f(z) = z^{\alpha}$ upper half-plane to somewhere, with $0 < \alpha < 2$. (Want it to be injective, so $\alpha < 2$. Since highest angle on upper half plane is π, this maps to angle $\alpha\pi$ so still injective).
(2) $f(z) = \int_0^z \frac{1}{\sqrt{1-w^2}} dw$. For example, $f(1) = \int_0^1 \frac{1}{\sqrt{1-w^2}} dw = \frac{\pi}{2}$. When $z > 1$ (and real), then this becomes purely complex. The boundary then looks like

$\sqcup$-shape

(symmetry around 0, goes out to $\pm\frac{\pi}{2}$). This is conformal, because it's the inverse of $\sin z$.
(3) $f(z) = \int_0^z \frac{1}{\sqrt{1-w^2}\sqrt{1-c^2w^2}} dw$ with $0 < c < 1$. When $0 < w < 1$, both terms are positive, the image of $[0, 1]$ is then some interval $[0, k_1]$, where $k_1 = f(1)$. The image of $[1, \frac{1}{c}]$ is a vertical line from k_1 to $k_1 + ik_2$ (purely imaginary). After $\frac{1}{c}$, it becomes negative real, so we go back etc. Get a rectangle.
This is called an elliptic integral
(4) Schwarz-Christofell Integral
$$S(z) = \int_0^z \frac{1}{(w - a_1)^{\alpha_1} \ldots (w - a_n)^{\alpha_n}} dw$$
with $a_1 < \ldots < a_n$ real numbers, each $-1 < \alpha_i < 1$ and $1 < \sum \alpha_i < 2$. The above were special cases of this. These are not all conformal mapping of polygons. Claim will be that all are close to this form.

Claim 3.19. If f is a conformal mapping from upper half-plane to some P, then f is of the form $f(z) = c_1 S(z) + c_2$. (not iff)

Fact 3.20.

(1) It is holomorphic on $\mathbb{C} \setminus \{a_j + iy : y \leq 0\}$ (the half-lines).
(2) $(x - a_j)^{\alpha_j} = \begin{cases} (x - a_j)^{\alpha_j} & x > a_j \\ (a_j - x)^{\alpha_j} e^{i\pi\alpha_j} & x < a_j \end{cases}$ *since $\alpha_j < 1$. Integral exists up to the real line including the points a_j.*
$$\left| \frac{1}{\prod (w - a_j)^{\alpha_j}} \right| \leq c \, |w|^{-\sum \alpha_i}$$
where $\sum \alpha_j > 1$, $S(\infty)$ exists.
(3) What is the image of $\mathbb{R}$?
$$S'(x) = \frac{1}{(x - a_1)^{\alpha_1} \ldots (x - a_n)^{\alpha_n}}$$
Argument of $S'(x) = 0 + \ldots + 0 - \pi(\alpha_{i+1} + \ldots + \alpha_n)$. The image is then a polygon connecting $S(a_1), \ldots, S(a_n), S(\infty)$. The angle of segment $S(a_2)$ to $S(a_3)$ with the next segment (extend and outside) is $\pi\alpha_3$. So the same for $S(\infty)$ to $S(a_1)$ is $(2 - \sum \alpha_i)\pi$.

3.1. Main Lemmas.

Lemma 3.21. *(Area formula 1). $f : D \to f(D)$ conformal. Then area of $f(D) = \int \int_D |f'(z)|^2 dxdy$.*

Proof. We know
$$\text{area of } f(D) = \int \int_{f(D)} 1 dxdy = \int \int_D \det J_f dxdy.$$
Write $f = u + iv$, then we know that
$$\det J_f = \det \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial x} \right)^2.$$
For derivative, can look at any direction. Pick x direction, so $f' = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$. □

Lemma 3.22. *Caratheodort Extension. If f is conformal between $B(0, 1)$ and P, then it extends continuously to $cl\,(B(0, 1)) \to cl\,(P)$ homeomorphically (?)*

Proof. Two sequences that converges to $z \in B(0, 1)$, their image can not converge to different points w_1, w_2. Suppose $z_n, z'_n \in D$ converging to z, $f(z_n) \to w$ and $f(z'_n) \to w'$, not equal.

Let $d = \text{dist}(w, w')$.

Claim. We have curves γ and γ' from w to w' such that $\text{dist}(\gamma, \gamma') > \frac{d}{3}$, and γ contains infinity many of $f(z_n)$ and γ' contains infinitely many $f(z_n')$.

Proof. Just connect the points of the sequences. □

Take inverse images of the γ and γ'. Let r be small, and take a small circle around z. Let z_r and z_r' be any two points of intersection of the circle with the preimage of the curves.

$$\frac{d}{3} \leq |f(z_r) - f(z_r')| = \left| \int_{z_r}^{z_r'} f(s) ds \right|.$$

Can use any curve connecting z_i and z_i'. Pick $z_i + re^{i\theta}$. Then

$$\frac{d}{3} \leq \int_{\theta_1(r)}^{\theta_2(r)} |f'(z)| \, r d\theta \leq \left(\int |f'(z)|^2 \, r d\theta \right)^{1/2} \left(\int r d\theta \right)^{1/2} \text{ by Holder's}$$

But $\left(\int r d\theta \right)^{1/2} \leq \sqrt{2\pi r}$. This means that

$$\frac{d/3}{\sqrt{2\pi r}} \leq \left(\int |f'(z)|^2 \, r d\theta \right)^{1/2}$$

But

$$\infty = \int_0^r \frac{(d/3)^2}{2\pi r} dr \leq \int \int |f'(z)|^2 \, dx dy = \text{area of some domain} < \infty.$$

This is a contradiction.

Then extend via this. Need to check continuity holds. This is true by a sequence from inside to the boundary. But what about a sequence on the boundary. But I can just pick a sequence in the interior that approximates the sequence on the boundary. □

Corollary 3.23. *The upper halfplane instead of the ball, and we get a homeomorphism of the closed upper halfplane, including ∞ and $cl(P)$.*

Lemma 3.24. *Schwarz reflection. D domain, symmetric about the $\mathbb{R}$-axis, call the two pieces D^+ and D^-. Suppose f is holomorphic on D^+. Let $I = \mathbb{R} \cap D$. f extends to I continuously, with $f(I) \subseteq \mathbb{R}$. Then f can be extended to D as a holomorphic function.*

Proof. Define $f(\bar{z}) = \overline{f(z)}$. Claim that this is holomorphic.

If we write $f(z) = \sum a_n (z - z_0)^n$, then $f(\bar{z}) = \sum \bar{a}_n (\bar{z} - \bar{z}_0)^n$. Just need that this has non-zero radius of convergence. But the radii of convergences are the same. Now, have to check that f is holomorphic on I. Take a circle γ around the point. Need to check that $\int_\gamma = 0$.

Will integrate over an ϵ-away upper half of the circle, and ϵ-away bottom half of the circle. Since each half are strictly above or below $\mathbb{R}$, they are both zero. As $\epsilon \to 0$, converges to

$$\int_{\text{upper half circle}} + \int_{\text{line segment close to } \mathbb{R}}.$$

Now, suffices to show that the integral of line segments close to $\mathbb{R}$ coverge to integral on $\mathbb{R}$ as $\epsilon \to 0$. This follows from continuity. □

Corollary 3.25. *Instead of $\mathbb{R}$s, we can choose any circle/ or line.*

Proof. Fractional linear transforms preserve symmetry. □

Lemma 3.26. *Analytic continuation. Suppose D_1 and D_2 are two domains, with f_i holomorphic on D_i. $D = D_1 \cap D_2$ also a domain, with $f_1 = f_2$ on D (in fact, just need to agree in some ball in D). Then they extend to a holomorphic function on $D_1 \cup D_2$.*

Take the power series $1 + z + z^2 + ...$ converges in the unit disk. It does not converge outside, but it can be extended outside holomorphically. What happens if you extend, extend etc then come back. No reason we should have the same thing. If we assume that f_i holomorphic on D_i and $D = D_1 \cap D_2$ is a domain.

Let $f : \mathbb{H} \to P$ conformal. Claim that $f(z) = c_1 S(z) + c_1$.

Proof. First, choose α_j to be the exterior angles of the polygon, a_j's to be the inverse image of the vertices.

It is enough to show that $\left(\frac{f'}{S'}\right)' = 0$. This is

$$\frac{f''S' - f'S''}{(S')^2} = \frac{\frac{f''}{f'} - \frac{S''}{S'}}{\frac{(S')^2}{f'S'}}.$$

Thus, enough to show that $\frac{f''}{f'} = -\sum_j \frac{\alpha_j}{z-a_j}$.

Recall:

$$S'(x) = \frac{1}{(x-a_1)^{\alpha_1} ... (x-a_n)^{\alpha_n}}.$$

Now, show that $\frac{f''}{f'} + \sum \frac{\alpha_j}{z-a_j} = 0$. Want to show that the function is entire. Think about the domain to be strips (mapping to sections of the polygon). Use Schwarz reflection to reflect the line, to get things outside the polygon. Have to be careful with the vertices (corresponding to these division lines). Want some overlaps between the strips, to use analytic continuation. Want to reflect through strip of a_{k-1} to a_{k+1} (includes a_k). But the line of a_k does not map to line. Need to straighten it out.

$$h_k(z) = (f(z) - f(a_k))^{\frac{1}{1-\alpha_k}}.$$

The exterior angle is $\pi\alpha_k$, so the interior angle is $\pi(1-\alpha_k)$. This above will then map interval of a_{k-1} to a_k to a line segment, and so can apply Schwarz reflection. This extends H_k^+ to H_k^-.

Write $h_k = h$, $\alpha_k = \alpha$ and $a_k = a$.

$$\begin{aligned} h'(z) &= \frac{1}{1-\alpha}(f(z) - f(a))^{\frac{1}{1-\alpha}-1} f'(z) = \frac{1}{1-\alpha} h(z)^\alpha f'(z) \\ f' &= (1-\alpha)h'h^{-\alpha} \text{ on } H^+ \\ f''(z) &= -\alpha(1-\alpha)(h')^2 h^{-\alpha-1} + (1-\alpha)h''h^{-\alpha} \end{aligned}$$

Then

$$\frac{f''}{f'} = -\alpha h' h^{-1} + \frac{h''}{h'}.$$

Just need $\frac{h''}{h'}$ is holomorphic, which is $h' \neq 0$ on $H_\epsilon^+ \cup H_\epsilon^-$ (including boundary). Meanwhile, $\frac{h'}{h} = \frac{-\alpha}{z-a} +$holomrphic from h having a simple pole at a.

Hence, on H_k^+, we have $\frac{f''}{f'} = -\frac{\alpha}{z-a} +$holomorphic. Hence, $\frac{f''}{f'}$ is holomorphic, and can be extended to the union of the two strips. Therefore, $\frac{f''}{f'} + \sum_j \frac{\alpha_j}{z-a_j}$ is holomorphic on $H_k^+ \cup H_k^-$ for each k. These strips have non-trivial intersection. We don't know that they agree on the intersection of the bottom strip, but they agree in a ball of the upper half plane, and that suffices to apply analytic continuation. There's more work for H_1^+ and H_n^+ but it's not a big deal.

Now, we know the function is entire. Will show that $\frac{f''}{f'} + \sum \frac{\alpha_j}{z-a_j} \to 0$ as $z \to \infty$. Then it's bounded and must be the constant 0. Extend f to $\mathbb{C} \backslash ball$ from f^+ on $\mathbb{H}\backslash$ball. This is bounded because f is bounded on the upper half plane. □

Exercise 3.27. For every function f holomorphic at ∞, $\frac{f''}{f'}$ decays like $\frac{1}{z}$ as $z \to \infty$.

Proof. Can differentiate normally, cuz it's uniformly bounded at disk sufficiently large. Then we see that using $\frac{1}{z}$ expansion, we get it. If $a_1 = 0$, then just do the next, and continue. □

Now, suppose $f(z) = a_0 + a_1 z + ...$ Is it conformal. We can assume that $a_0 = 0$ and $a_1 = 1$.

Theorem 3.28. *If $\sum_{n=2}^\infty n|a_n| \le 1$, then f is conformal on $B(0,1)$.*

Proof. If that holds, each $n|a_n| \le 1$, and so $\sqrt[n]{|a_n|} \le \sqrt[n]{\frac{1}{n}} \to 1$ so this is holomorphic, with radius of convergence ≥ 1.

$$\begin{aligned} f(z_1)-f(z_2) &= (z_1-z_2)+a_2\left(z_1^2-z_2^2\right)+... \\ &= (z_1-z_2)\left(1+a_2(z_1+z_2)+a_3(z_1^2+2z_1z_2+z_2^2)+...\right). \end{aligned}$$

The $|a_2(z_1+z_2)|<2|a_2|$, $|a_3...|\le 3|a_3|$... etc (we are in the unit ball). Therefore,

$$|f(z_1)-f(z_2)|\ge |z_1-z_2|(1-2|a_2|-3|a_3|)\ge 0.$$

Basically, for $f=z+a_2z^2+...$ the sum $a_2z^2+....$ is less than z. □

Theorem 3.29. *If it is conformal, then $|a_2|\le 2$. (HWK, to check that this is sharp)*

Exercise 3.30. Sharp

Proof. $f(z)=z+2z^2+3z^3+...=z\left(1+2z+3z^2+...\right)=z\left(\frac{d}{dz}\frac{z}{1-z}\right)=\frac{z}{(1-z)^2}$. This is also $\frac{z}{1-z}+\left(\frac{z}{1-z}\right)^2$. Then we can write $f(z)=(g\circ h)(z)$ with $h(z)=\frac{z}{1-z}$ and $g(z)=z+z^2$. Then check.... □

Corollary 3.31. *(Koebe-Bierberbach) If $f(z)=z+a_2z^2+....$ is conformal, then $f(B(0,1))\supseteq B\left(0,\frac{1}{4}\right)$.*

Proof. Suppose $z_0\notin f(B(0,1))$. Define $g(z)=\frac{z_0f(z)}{z_0-f(z)}$ is holomorphic. Easy to check that this is injective.

From $z_0f(z)=(z_0-f(z))\,g(z)$, we see that

$$z_0z+a_2z_0z^2+... \quad = \quad \left(z_0-z-a_2z^2-a_3z^3+...\right)\left(z+\left(\frac{1}{z_0}+a_2\right)z^2+...\right)$$

This gives us the formula for g, and $|a_2|\le 2$ and $\left|\frac{1}{z_0}+a_2\right|\le 2$ so $\left|\frac{1}{z_0}\right|\le 4$. Therefore, $|z_0|\ge\frac{1}{4}$. □

Lemma 3.32. *(Area formula 2) $f(z)=\sum_{n=-\infty}^{\infty}a_nz^n$ holomorphic on a domain that contains the curve C_r =contain 0 with radius r. f is injective on C_r. Then the area enclosed by*

$$f(C_r)=\pi\left|\sum_{n=-\infty}^{\infty}n|a_n|^2r^{2n}\right|.$$

Proof. Write $f=u+iv$ and $z=re^{i\theta}$. The area enclosed is equal to

$$\int_0^{2\pi}u(\theta)v'(\theta)d\theta.$$

$u(\theta)=\frac{1}{2}\sum\left(a_ne^{in\theta}+\overline{a_n}e^{-in\theta}\right)r^n$. Similarly, $v(\theta)=\frac{1}{2i}\sum\left(a_ne^{in\theta}-\bar{a}_ne^{-in\theta}\right)r^n$.

$$v'(\theta)=\frac{1}{2i}\sum_m\left(a_me^{im\theta}+\overline{a_m}e^{-im\theta}\right)imr^m.$$

Therefore, the product is

$$uv(\theta)=\frac{1}{4}\sum_{m,n}(\cdot)\,mr^{n+m}.$$

When integrating from $\int_0^{2\pi}$, a lot are 0, and we get

$$\begin{aligned} \left|\int_0^{2\pi}\right| &= \left|\frac{1}{4}\cdot 2\pi\sum_m m\left(a_ma_{-m}+a_m\bar{a}_mr^{2m}+\bar{a}_ma_mr^{2m}+\overline{a_{-m}a_m}\right)\right| \\ &= \frac{\pi}{2}\sum_m 2m|a_m|^2r^{2m} \end{aligned}$$

which is what we wanted. Here, $\sum_m ma_ma_{-m}=0$ because m and $-m$ appears in the sum. □

Corollary 3.33. $f(z)=\frac{1}{z}+b_0+b_1z+b_2z^2+....$ *Suppose this is conformal on $B(0,1)\setminus\{0\}$ then $\sum_{n=0}^{\infty}n|b_n|^2\le 1$.*

Proof. If r is close to r, the area enclosed is $\pi\left|-\frac{1}{r^2}+\sum_{n=0}^{\infty}n|b_n|^2r^{2n}\right|$. When r is very small, the main contribution is $\frac{1}{r^2}$, and the inside is negative. For all $0<r<1$ is is negative, because it's continuity and never 0.

What happens as $r \to 1$. $1 - \sum n |b_n|^2 \leq 0$. We get our result. □

Suppose $f(z) = z + a_2 z^2 + a_3 z^3 + \ldots$. Then $\frac{f(z)}{z} = 1 + a_2 z + \ldots$ is still holomorphic on $B(0,1)$ and non-zero (because $f(z)$ can only have a simple root at 0). Therefore, we can take

$$g(z) = \sqrt{\frac{f(z)}{z}} = 1 + \frac{a_2}{2} z + \ldots$$

$h(z) = zg(z^2) = z + \frac{a_2}{2} z^3 + \ldots$. Then

$$\frac{1}{h(z)} = \frac{1}{z} - \frac{a_2}{2} z + \ldots$$

Claim 3.34. h is conformal.

Proof. Need to show injectivity. $h(z_1) = h(z_2)$. Sinze $h^2(z) = f(z^2)$ then $z_1^2 = z_2^2$ implies $z_1 = \pm z_2$. Then $z_1 = z_2$ because

$$h(z_1) = z_1 g(z_1^2) = z_2 g(z_2^2) = h(z_2)$$

which determines the sign. □

This means that for $\frac{1}{h(z)}$, $\frac{a_2}{2} \leq 1$.

Definition 3.35. f is called typically real if $f(z) \in \mathbb{R}$ iff $z \in \mathbb{R}$.

We will show that if f is $z + a_2 z^2 + a_3 z^3 + \ldots$ conformal on $B(0,1)$ and typically real, then $|a_n| \leq n$ for all n.

Claim 3.36. $z + a_2 z^2 + \ldots$ typically real and conformal on $B(0,1)$ then $|a_n| \leq n$ for all n

Proof. $f = u + iv$ with $z = re^{i\theta}$. Then

$$f(z) = \sum a_n (\cos n\theta + i \sin n\theta) r^n.$$

Hence, $v(re^{i\theta}) = \sum a_n r^n \sin n\theta$. Then

$$\int_0^\pi v\left(re^{i\theta}\right) \sin m\theta = \int_0^\pi \sum_n a_n r^n \sin(n\theta) \sin(m\theta) = \frac{\pi}{2} a_m r^m.$$

We know

$$\left|\frac{\pi}{2} a_m r^m\right| \leq m \int_0^\pi \left|v(re^{i\theta}) \sin\theta\right|.$$

We know $v(ie^{i\theta})$ and $\sin\theta$ don't change sign between $(0,\pi)$. Hence, the above is equal to $m\frac{\pi}{2}r$. This says

$$|a_m r^m| \leq mr.$$

This is true for all $r \in (0,1)$, and $r \to 1$, $|a_m| \leq m$. □

4. Entire Functions

- Where can it be 0? It can be constant 0 or discrete.

Claim 4.1. Given $z_1, z_2, \ldots \in \mathbb{C}$ with no accumulation points, there is an entire function that vanishes exactly at these points, with desired multiplicity.

- How does it grow at infinity.
- To what extend is it determined by its zeroes? Answer: unique up to a multiplicative factor, provided f has finite rate of growth

Definition 4.2. Suppopse f is entire. f has order of growth at most α if

$$|f(z)| \leq c_1 e^{c_2 |z|^\alpha}$$

for some c_1, c_2 constants. The order of growth is the infimum of the α's.

Example 4.3.

- e^z. We know
$$|e^z| = e^{Re(z)} \le e^{|z|}$$
and so the order of growth is 1. (note e^z is not always big. It's big if $Re(z)$ is big).
- $\sin z$ and $\cos z$ can be placed in an exponential, and get 1
- e^{e^z} is infinity

Notation: f entire function. Denote by $n(r)$ to be the number of roots in $B(0,r)$ (with multiplicity).

Theorem 4.4. *If f has order of growth $\rho < \alpha$ then $n(r) \le cr^\alpha$ for some constant c, for large enough r.*

Lemma 4.5. *(Jensen's formula). $f \neq 0$ on the circle $|z| = r$, and $\neq 0$ at the origin. Let $z_1, ..., z_n$ be roots in $B(0,r)$ with multiplicity. Then*
$$\log|f(0)| = \sum_{\ell=1}^{n} \log\frac{|z_\ell|}{r} + \frac{1}{2\pi}\int_0^{2\pi} \log\left|f(re^{i\theta})\right| d\theta.$$

Proof.

(1) If f has no roots, let $g(z) = \log f(z)$ exists.
$$|f(z)| = \left|e^{g(z)}\right| = e^{Re(g(z))}$$
Then
$$\log|f(0)| = Re\,(g(0)) = Re\left(\frac{1}{2\pi}\int_0^{2\pi} g\left(re^{i\theta}\right) d\theta\right) = \frac{1}{2\pi}\int_0^{2\pi} \log\left|f(re^{i\theta})\right| d\theta.$$

(2) Suppose $f = z - z_0$ with $z_0 \in B(0,r)$. We need to show that
$$\log r = \frac{1}{2\pi}\int_0^{2\pi} \log\left|re^{i\theta} - z_0\right| d\theta,$$
which is to say
$$0 \;=\; \frac{1}{2\pi}\int_0^{2\pi} \log\left|e^{i\theta} - \frac{z_0}{r}\right| d\theta = \frac{1}{2\pi}\int_0^{2\pi} \log\left|1 - \frac{e^{-i\theta}z_0}{r}\right| d\theta.$$
We see that $1 - \frac{e^{-i\theta}z_0}{r}$ is never zero. We can apply step 1, to $h(z) = 1 - \frac{z_0}{r}z$ to get
$$0 = \log(1) = \frac{1}{2\pi}\int_0^{2\pi} \log\left|1 - \frac{z_0}{r}e^{i\theta}\right| d\theta,$$
it's not $-\theta$, but going backwards doesn't change the desired result.

(3) General case, let $f(z) = (z - z_1)\ldots(z - z_k)\,\frac{f(z)}{(z-z_1)\ldots(z-z_k)}$. We proved it for every term of the product already. The product breaks into a sum, so we are done.

□

Exercise 4.6. f has order of growth ρ

Proof. of Theorem. Recall that
$$\left|\sum_{\ell=1}^{n} \log\left|\frac{z_\ell}{r}\right|\right| \le \left|\frac{1}{2\pi}\int_0^{2\pi} \log\left|f(re^{i\theta})\right| d\theta - \log|f(0)|\right| \le cr^\alpha + c.$$
Consider the ℓ such that $|z_\ell| < \frac{r}{2}$. Each contributes a $\log 2 \le$ in the LHS of the above. So get
$$\log 2 \cdot n\left(\frac{r}{2}\right) \le c_1 r^\alpha + c_2$$
and so
$$n(r) \le \frac{c_1(2r)^\alpha + c_2}{\log 2} \le cr^\alpha.$$

□

Corollary 4.7. *If f has order growth $\rho < \alpha$ and $z_1, z_2, ...$ roots with multiplicity. Then $\sum \frac{1}{|z_\ell|^\alpha} < \infty$.*

Proof. Choose $\rho < \beta < \alpha$. Then

$$\begin{aligned}\sum_i \sum_{2^i \le |z_\epsilon| \le 2^{i+1}} \frac{1}{|z_\epsilon|^\alpha} &\le c \sum_j \frac{1}{2^{j\alpha}} n(2^{j+1}) \le c \sum_j \frac{1}{2^{j\alpha}} c' 2^{(j+1)\beta} \\ &\le c \sum_j 2^{j(\beta-\alpha)+\beta} < \infty.\end{aligned}$$

This is because $n(2^{j+1}) \le c' 2^{(j+1)\beta}$. □

Suppose $z_1, z_2, ... \in \mathbb{C}$ have no accumulation points. Want to find an entire function with exactly these roots. Define

$$f(z) = (z - z_1)(z - z_2) ...$$

What happens with infinite products.

$$\prod_{n=1}^\infty z_n = \lim_{N\to\infty} \prod_{n=1}^N z_n$$

if it exists. If $z_n = 0$ for any then it exists and is 0. Assume it's not the case. If the limit exists, then $z_n \to 1$. Write $z_n = 1 + w_n$.

Fact 4.8. *If $\sum |w_n| < \infty$ then the product exists, and $\prod z_n \neq 0$, unless $z_n = 0$ for some n.*

Proof. $\prod_{n=1}^N (1 + w_n) = \prod_{n=1}^N e^{\log(1+w_n)} = e^{\sum_{n=1}^N \log(1+w_n)}$. We know that $|\log(1 + w_n)| \le 2|w_n|$ if $|w_n| < \frac{1}{2}$. This will happen, since our thing absolutely converges. This shows that $\sum_{n=1}^N \log(1 + w_n)$ has a limit as $N \to \infty$. □

Claim 4.9. Suppose $f_1, f_2, ...$ holomorphic on D with $|1 - f_n(z)| \le c_n$ for every $z \in D$, with $\sum c_n < \infty$ then $\prod_{n=1}^\infty f_n$ converges uniformly to holomorphic f on D. (something about non-zero too).

Proof. We can assume that $c_n \le \frac{1}{2}$.

$$\prod_{n=1}^N f_n(z) = e^{\sum \log(1+g_n(z))} \to e^{g(z)}$$

uniformly on D. Weierstrass implies $g(z)$ is holomorphic. □

Remark 4.10. $\sum \log f_n(z) \to \log f(z)$. Similarly, $\sum \frac{f_n'}{f_n} \to \frac{f'}{f}$ on the set where $f \neq 0$.

Now, back to the $f(z) = (z - z_1)(z - z_2)... = \prod_{n=1}^\infty \left(1 - \frac{z}{z_n}\right)$. This is okay if $\sum \left|\frac{z}{z_n}\right| < \infty$. Will have to do this on a disk. Suppose $\sum \left|\frac{1}{z_n}\right| < \infty$. In particular, this is good if the order of growth is < 1.

An example when it doesn't work is $\sin(nz)$ with integer roots.

$$z \prod_{n\in\mathbb{Z}\setminus 0} \left(1 - \frac{z}{n}\right),$$

we have $\sum \frac{1}{|n|} = \infty$. However, $\left(1 - \frac{z}{n}\right)\left(1 + \frac{z}{n}\right) = \left(1 - \frac{z^2}{n^2}\right)$. So view

$$z \prod_{n=1}^\infty \left(1 - \frac{z^2}{n^2}\right) = f(z)$$

which does converge and is an entire function, since $\sum \left|\frac{1}{n^2}\right| < \infty$, so bounded on each disk. Is this $\sin(\pi z)$? No, but close.

$$f'(0) = 1 \neq \pi.$$

Theorem 4.11. *However, $\pi z \prod_{n=1}^\infty \left(1 - \frac{z^n}{n^2}\right) = \sin(\pi z)$.*

Proof. We have

$$f(z) = \pi z \left(1 - \frac{z^2}{1}\right)\left(1 - \frac{z^2}{4}\right)\left(1 - \frac{z^2}{9}\right) ...$$

call the terms f_i. Consider

$$\sum \frac{f_n'}{f_n} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} \to \frac{f'(z)}{f(z)}.$$

Meanwhile,

$$\frac{(\sin(\pi z))'}{\sin(\pi z)} = \pi \cot(\pi z).$$

Note, that if $\frac{f'}{f} = \frac{g'}{g}$ then $f = constant \cdot g$. Enough to show that

$$\frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} = \pi \cot(\pi z).$$

- Both holomorphic on $\mathbb{C} \backslash \mathbb{Z}$
- simple pole at 0 (because it's $\frac{1}{z}$+holomorphic)
- Both are odd functions $f(-z) = -f(z)$.
- Both are periodic (because

  $$\sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} = \sum \left(\frac{1}{z-n} + \frac{1}{z+n} \right)$$

  and do stuff.
- Bounded on the set

  $$\left\{ |x| \leq -\frac{1}{2}, |y| \geq 1 \right\}$$

  where $z = x + iy$.

Difference is an entire function, bounded (because bounded in the strip, bounded in the reflection because of odd, bounded in the middle cuz bounded domain, so bounded everywhere by periodicity). It's odd, so the only constant possible is 0. $\square$

Can use $z = \frac{1}{2}$, to get

$$\frac{\sin \frac{\pi}{2}}{\pi/2} = \prod_{n=1}^{\infty} \left(1 - \frac{1}{4n^2}\right) = \prod \left(\frac{4n^2 - 1}{4n^2} \right) = \prod \left(\frac{2n-1}{2n} \cdot \frac{2n+1}{2n} \right).$$

Reciprocate to get

$$\frac{\pi}{2} = \prod_{n=1}^{\infty} \left(\frac{2n}{2n-1} \frac{2n}{2n+1} \right) = \frac{2}{1} \frac{2}{3} \frac{4}{3} \frac{4}{5} ...$$

is called Wallis product.

Natural questions:

(1) Given a sequence $a_1, ... \in \mathbb{C}$, is there an entire function with zeroes and precisely these points.
(2) Given an entire function, can we factor it based on its zeroes

1 is true if $|a_n| \to \infty$ (no accumulation point). We can try $\prod (z - a_i)$, but will not converge. Try $\prod \left(1 - \frac{z}{a_i}\right)$, will only work if $\sum \frac{1}{|a_n|} < \infty$ converges, but not in general. Call $E_0(z) = 1 - z$, which satisfies

(1) $E_0(1) = 0$
(2) $|1 - E_0(z)| \leq |z|$

Definition 4.12. Define Weierstrass canonical factors

$$E_k(z) = (1 - z) \exp \left(z + \frac{z^2}{2} + ... + \frac{z^k}{k} \right).$$

The inside is the Taylor polynomial for $-\log(1 - z)$. This is an entire function with order of growth k.

Observe that if $|z| < \frac{1}{2}$, then

$$\begin{aligned} E_k(z) &= \exp\left(\log(1-z) + z + \frac{z^2}{2} + ... + \frac{z^k}{k}\right) \\ &= \exp\left(-\frac{z^{k+1}}{k+1} - \frac{z^{k+2}}{k+2} - ...\right). \end{aligned}$$

Call the inside W. Then

$$|W| \le \frac{|z|^{k+1}}{k+1}\left(1 + |z| + |z|^2 + ...\right) \le \frac{|z|^{k+1}}{2} \cdot 2 = |z|^{k+1}.$$

This means that

$$|1 - E_k(z)| = \left|1 - e^W\right| \le \frac{|W|}{1} + \frac{|W|^2}{2!} + ... \le 2|W| \le 2|z|^{k+1}.$$

Then $E_k(z)$ satisfies

(1) $E_k(1) = 0$
(2) $|1 - E_k(z)| \le 2|z|^{k+1}$ for $|z| < \frac{1}{2}$

Theorem 4.13. *Weierstrass factorization theorem.*

(1) Given $a_1, a_2, ... \in \mathbb{C}$ such that $|a_n| \to \infty$ with $a_n \neq 0$ for all n. The function

$$h(z) = \prod_{n=1}^{\infty} E_n\left(\frac{z}{a_n}\right)$$

is an entire function with zeroes precisely at $\{a_n\}$. (multiply z in the front if you want zero at zero)

(2) If f is an entire function. Let m be the order of zero of f at 0. Let a_n be the other zeroes. Then there exists an entire g such that

$$f(z) = e^{g(z)} z^m \prod_{n=1}^{\infty} E_n\left(\frac{z}{a_n}\right).$$

Proof.

(1) Fix $R > 0$. Will show that $h(z)$ is holomorphic on the disk of radius R.

$$h(z) = \prod_{n, \left|\frac{R}{a_n}\right| \le \frac{1}{2}} E_n\left(\frac{z}{a_n}\right) \prod_{\left|\frac{R}{a_n}\right| > \frac{1}{2}} E_n(\frac{z}{a_n}).$$

For the second product, $2|z| > |a_n|$. This means that this is a finite product (because $|a_n| \to \infty$ and these are just the bounded ones).

For the other one, consider

$$\sum_{\left|\frac{R}{a_n}\right| \le \frac{1}{2}} \left|1 - E_n(\frac{z}{a_n})\right| \le 2\sum \left|\frac{z}{a_n}\right|^{n+1} \le 2\sum \left(\frac{1}{2}\right)^{n+1} < \infty.$$

Hence, the first also converges absolutely.

(2) Consider

$$\frac{f(z)}{z^m \prod_{n=1}^{\infty} E_n(\frac{z}{a_n})}$$

is an entire function with no zeroes. Therefore, it's equal to $e^{g(z)}$ for some entire g.

□

If f has finite order of growth, can we say more. Let f be entire, p_0 =order of growth of f.

$$|f(z)| \le Ce^{C|z|^{\rho_0 + \epsilon}}$$

for all $\epsilon > 0$. Let $\{a_n\}$ be the non-zero zeroes of f.

For which $k \geq 0$, is it true that $\prod_{n=1}^{\infty} E_k(\frac{z}{a_n})$ is entire. Recall that in the proof, we need a bound on

$$\sum_{\frac{R}{|a_n|} \leq \frac{1}{2}} \left|1 - E_k(\frac{z}{a_n})\right| \leq \sum \left(\frac{|z|}{|a_n|}\right)^{k+1} \leq R^{k+1} \sum_n \frac{1}{|a_n|^{k+1}}.$$

This converges if $k+1 > \rho_0$ (not iff).

Theorem 4.14. *Hadamard Factorization theorem. Let f be entire. ρ_0 the order of growth of f. Let $\{a_n\}$ be the non-zero zeroes. Let m be the order of f at $z = 0$. Then there exists a polynomial P with degree $k = \lfloor \rho_0 \rfloor$ such that*

$$f(z) = e^{P(z)} z^m \prod_{n=1}^{\infty} E_k\left(\frac{z}{a_n}\right).$$

Remark 4.15. Apply this to sin, which has order of growth 1. We have

$$\sin \pi z = e^{az+b} z \prod_{m=1}^{\infty} \left(E_1\left(\frac{z}{m}\right) E_1\left(\frac{z}{-m}\right)\right).$$

We know that

$$E_1\left(\frac{z}{m}\right) E_1\left(\frac{z}{-m}\right) = \left(1 - \frac{z}{m}\right) e^{\frac{z}{m}} \left(1 - \frac{z}{-m}\right) e^{\frac{z}{-m}} = 1 - \frac{z^2}{m^2}.$$

Hense,

$$\sin \pi z = e^{az+b} z \prod_{n=1}^{\infty} \left(1 - \frac{z^n}{n^2}\right).$$

Since $\sin \pi z$ is odd, and $\prod$ is even, z is odd. Hence, e^{az+b} is even. That is, $e^{az+b} = e^{-az+b}$ hence $a = 0$. To see $e^b = \pi$, take derivative and evaluate at 0.

Proof. Based on everything,

$$\frac{f(z)}{z^m \prod_{n=1}^{\infty} E_k(\frac{z}{a_n})} = e^{g(z)}$$

for some entire function g. Outline, $E(z) = z^m \prod_{n=1}^{\infty} E_k(\frac{z}{a_n})$ is not too small, so that $\left|\frac{f(z)}{E(z)}\right| \leq Ce^{C|z|^s}$ for most z, and for all $s > \rho_0$. Then

$$Ce^{c|z|^s} \geq e^{Re(g(z))}$$

so that $Re\,(g(z)) \leq C\,|z|^s$ for most s. □

Remark 4.16. Exam solution. $f(z) = \frac{1}{z} + a_0 + a_1 z + ...$ conformal on $\mathbb{D} - \{0\}$ and avoids z_0, z_1. Assume $z_0 = 0$. Let $g(z) = \frac{1}{f(z)}$. Then $\frac{1}{|z_0 - z_1|} \geq \frac{1}{4}$ and so $4 \geq |z_2 - z_1|$.

We had $E_0(z) = 1 - z$ and $E_k(z) = (1-z)e^{z + \frac{z^2}{2} + ... + \frac{z^k}{k}} = c^w$. We had $|w| \leq c\,|z|^{k+1}$ if $|z| \leq \frac{1}{2}$.

Exercise 4.17. Do this for cos.

Lemma 4.18. *We have*

$$|E_k(z)| \geq \begin{cases} e^{-c|z|^{k+1}} & \text{if } |z| \leq \frac{1}{2} \\ (1-z)\, e^{-c|z|^k} & \text{if } |z| \geq \frac{1}{2} \end{cases}.$$

Proof. The first already follows from the fact that if $E_k(z) = e^w$ then $|w| \leq c\,|z|^{k+1}$ if $|z| \leq \frac{1}{2}$.

If $|k| > \frac{1}{2}$. Then

$$E_k(z) = (1-z)e^{z + \frac{z^2}{2} + ... + \frac{z^k}{k}} = (1-z)\, e^w$$

where $|w| \leq c\,|z|^k$. □

Corollary 4.19. *We have*

$$\left|\prod E_k\left(\frac{z}{z_n}\right)\right| \geq e^{-c|z|^s}$$

where $\rho < s < k+1$, $z \notin B\left(z_n, \frac{1}{|z_n|^{k+1}}\right)$.

Proof. We have

$$\prod_{n:\,|z_n|\le 2z} E_k\left(\frac{z}{z_n}\right)\prod_{n:\,|z_n|\ge 2z} E_k\left(\frac{z}{z_n}\right)$$

The second product is easier. This is

$$\ge \prod_n e^{-c\left|\frac{z}{z_n}\right|^{k+1}} = e^{-c|z|^{k+1}\sum_n \frac{1}{|z_n|^{k+1}}}.$$

We get

$$\frac{1}{|z_n|^{k+1}} = \frac{1}{|z_n|^s}\frac{1}{|z_n|^{k+1-s}} \le \frac{1}{|z_n|^s}\frac{c}{|z|^{k+1-s}}$$

where $|z_n| \ge 2|z|$. Second product is $\ge e^{-c|z|^s}$ (by the lemma before). Also, $\sum \frac{1}{|z_n|^s} < \infty$.

The first product is

$$\ge \prod_{n:|z_n|\le 2|z|}\left(1-\frac{z}{z_n}\right)\prod_{n:\,\dots} e^{--c\left|\frac{z}{z_n}\right|^k}.$$

The second of these is $\frac{1}{|z_n|^k} = \frac{|z_n|^{s-1}}{|z_n|} \le c\frac{|z|^{s-k}}{|z_n|^s}$. Hence, the second product is $\ge e^{-c|z|^s}$.

First part is

$$\begin{aligned} &= \prod\left|\frac{z_n-z}{z_n}\right| \ge \prod\frac{\frac{1}{|z_n|^{k+1}}}{|z_n|} = \prod\frac{1}{|z_n|^{s+2}} \ge \left(\frac{1}{c|z|}\right)^{c|s|^{s-\epsilon}} \\ &= e^{-c|z|^{s-\epsilon}\log(c|z|)} \ge e^{-c|z|^s}. \end{aligned}$$

□

$f(z) = e^{g(z)}z^m\prod E_k\left(\frac{z}{z_n}\right)$. Need to show that g is a polynomial of degree $\le t$. We have

$$e^{Re(g)} \le \left|e^{g(z)}\right| \le e^{c|z|^s}$$

provided $z \notin \cup B\left(z_n, \frac{1}{|z_n|^{s+1}}\right)$. Take log, to get

$$Re(g) \le c|z|^s.$$

Almost done, but we have this bound outside of some balls. The sum of the radii are finite, so we can find and arbitrarily large R so that the circle does not touch any of the balls.

Recall $|Re(g(z))| \le c|z|^s$ for $\epsilon < s < \epsilon+1$ on a circle of radius r for any large r. Does this mean g is a polynomial of degree $\le s$. Let $z = re^{i\theta}$, $g(z) = \sum a_n z^n$.

$$\frac{1}{2\pi}\int_0^{2\pi} g\left(ie^{i\theta}\right)e^{-in\theta} = \begin{cases} a_n r^n & n \ge 0 \\ 0 & n < 0 \end{cases}.$$

Also,

$$\frac{1}{2\pi}\int \overline{g\left(re^{i\theta}\right)}e^{-in\theta}d\theta = 0$$

if $n > 0$.

$$\frac{1}{2\pi}\int 2Re\left(g(ie^{i\theta})\right)e^{-in\theta}d\theta = a_n r^n \quad n > 0.$$

Hence, $|a_n| \le \frac{1}{\pi r^n}\int_0^{2\pi}\left|Re\left(g(ie^{i\theta})\right)\right|d\theta \le cr^{s-n}$ for $n > s$ and $|a_n| = 0$ for $n > s$. This prove that g is a polynomial.

Theorem 4.20. *Little Picard Theorem. f is non-constant entire function of order of growth ρ.*

(1) If $\rho \notin \mathbb{N}$, then f attains every complex number at ∞-ly many points (counter, e^z is never zero)
(2) If $\rho \in \mathbb{N}$, then if f misses a value, then it attains every other value at infinitely many points

Proof. If $f(z) - w$ has only finitely many roots then it's $e^{p(z)}q(z)$ for some polynomials p, q. If $e^{p(z)}q(t)$ has no roots then q is constant. p attains every value and so $e^{p(z)}$ attachs anything except 0 (and infinitely many times). □

Theorem 4.21. *Picard theorem. If f is a non-constant entire function, then f can miss at most one value.*

Proof. Suppose f misses two points $w_0 \neq w_1$. Then

$$\frac{f(z)-w_0}{w_1-w_0}$$

misses 0 and 1. Let $g(z)=\frac{1}{2\pi i}\log\left(\frac{f(z)-w_0}{w_1-w_0}\right)$ misses $\mathbb{Z}$. Let

$$h(z)=\sqrt{g(z)}-\sqrt{g(z)-1}$$

misses all $\sqrt{n}\pm\sqrt{n-1},0$.

Proof. If we get 0,

$$\sqrt{g(z)}=\sqrt{g(z)-1}\implies g(z)=g(z)-1$$

which is a contradiction.

$$\begin{aligned}\sqrt{g(z)}-\sqrt{g(z)-1} &= \sqrt{m}\pm\sqrt{m-1}\\ \sqrt{g(z)}+\sqrt{g(z)-1}=\frac{1}{\sqrt{g(z)}-\sqrt{g(z)-1}} &= \frac{1}{\sqrt{m}\pm\sqrt{m-1}}=\sqrt{m}\mp\sqrt{m-1}\end{aligned}$$

Then

$$2\sqrt{g(z)}=\sqrt{m}.$$

□

$\log h(z)$ misses $\log\left(\sqrt{m}\pm\sqrt{m-1}\right)+2\pi\epsilon$ for all n,ϵ.

Proof. $\lim_{m\to\infty}\log\left(\sqrt{m}\pm\sqrt{m-1}\right)-\log\left(\sqrt{m-2}\pm\sqrt{m-2}\right)=0$. □

It is enough to idnetify that the image of every entire function contains an arbitrarily large disc. Suppose this is false for an entire function f non-constant. Then $f'\neq 0$. Choose z_0 such that $f'(z_0)\neq 0$. Let

$$g(z)=\frac{f(z)-f(z_0)}{f'(z_0)}\leq z+a_2z^2+...$$

$g_\epsilon(z)=\epsilon g\left(\frac{z}{\epsilon}\right)$. $g_\epsilon(0)=0$ and $g_\epsilon'(0)=1$. If $Im\,(g)$ does not contain any disc of radius R, then g_ϵ of radius ϵR.

Enough to show that the following theorem. □

Theorem 4.22. *For some $c>0$, the image of entire function$f(z)=z+a_2z^2+...$ contain a disc of radius c*

Proof. Consider $(1-r)\,M_{f'}(r)=w(r)$ continuous. $w(0)=1$ and $w(1)=0$. Let $r_0\geq 0$ be the largest with $w(r_0)=1$. Let $|z_0|=r_0$, $|f'(z_0)|=\frac{1}{1-r_0}$.

Then $B\left(z_0,\frac{1-r_0}{2}\right)\subseteq B\left(0,r_0+\frac{1-r_0}{2}\right)=B\left(0,\frac{1+r_0}{2}\right)\subseteq B\,(0,1)$. In the smallest ball, $|f'(z)|\leq M_{f'}\left(\frac{1+r_0}{2}\right)\leq\frac{1}{1-\frac{1+r_0}{2}}=\frac{2}{1-r_0}$.

Let $\rho=\frac{1-r_0}{2}$, then $f'(z_0)=\frac{1}{2\rho}$. Therefore,

$$|f(z)-f(z_0)|\leq 1$$

on $B\,(z_0,\rho)$.

Let $g(z)=f(z+z_0)-f(z_0)$, then $g(0)=0$. $|g'(0)|=\frac{1}{2\rho}$, $|g(z)|\leq 1$ on $B\,(0,\rho)$. $h(z)=2g(\rho z)$ on $B(0,1)$. Then $h(0)=0$, $h'(0)=1$, $|h(z)|\leq z$ on $B(0,1)$. □

Lemma 4.23. *$f(z)=z+a_2z^2+...$ holomorphic on $B(0,1)$ and bounded by M on $B(0,1)$ (for us, it's 2). Then the image of f contains a ball of radius $\frac{c}{M}$.*

Proof. $|a_n|\leq M$ for each M for each n, $a_n=1$ so $M\geq 1$. For $0<r<1$,

$$\begin{aligned}|f(z)| &\geq |z|-|f(z)-z|\geq r-M\left(r^2+r^3+...\right)\\ &= r-\frac{Mr^2}{1-r}.\end{aligned}$$

Choose $r = \frac{1}{4M}$. Then $|f(z)| \geq \frac{1}{4M} - \frac{M\frac{1}{16M^2}}{1-\frac{1}{4m}} \geq ... \geq \frac{1}{6M}$. Therefore, $f(z)$ and $f(z) - \frac{1}{6M}$ have the same number of roots. Since $f(z) = 0$ at 0, so does $f(z) - \frac{1}{6M}$. Can put and $|w| \leq \frac{1}{6M}$. Therefore, $f(z) - w$ has at least 1 root. That is,

$$im\,(f) \supseteq \left\{ w : \, |w| \leq \frac{1}{6M} \right\}.$$

□

Let $f(z) = a_0 + a_1 z + ...$ be an entire function. The radius of convergence is infinite, so $\sqrt[n]{|a_n|} \to 0$.

Theorem 4.24.

(1) *If* $\sqrt[n]{|a_n|} \leq \frac{c}{n^\alpha}$ *for some* c *for large enough* n *then* ρ *=order of growh* $\leq \frac{1}{\alpha}$

(2) *Conversely, if* $\rho < \frac{1}{\alpha}$ *then* $\sqrt[n]{|a_n|} \leq \frac{c}{n^\alpha}$ *for some* c *and large enough* n.

Proof.

(1) We can assume $n \geq a_0$ and $\alpha > 0$. Choose n_0 so large, so that $n_0 > \frac{1}{\alpha}$. Claim that

$$g(z) = \sum_{n=n_0}^{\infty} a_n z^n$$

has order of growth $\leq \frac{1}{\alpha}$.

$$\begin{aligned} |g(z)| &\leq \sum_{n=n_0}^{\infty} |a_n| \, |z|^n \leq \sum_{n=n_0}^{\infty} \frac{c^n}{n^{\alpha n}} |z|^n = \sum \left(\frac{\alpha c^{1/\alpha} |z|^{1/\alpha}}{\alpha n} \right)^{\alpha n} \\ &\leq \sum c \frac{\left(\alpha c^{1/\alpha} |z|^{1/\alpha}\right)^{k+1}}{k^k} \end{aligned}$$

there are at most $c = \frac{1}{\alpha} + 1$ many terms such that $\lceil \alpha n \rceil = k$. This is

$$\leq \sum_k c \, |z|^{1/\alpha} \, \frac{\left(c \, |z|^{1/\alpha}\right)^k}{k!}.$$

The $|z|^{1/\alpha}$ grows slower than exponential. That sum is basically a exponential in $|z|^{1/\alpha}$.

(2) We have

$$|a_n| \leq \frac{M(r)}{r^n} \leq c \frac{e^{cr^{1/\alpha}}}{r^n}$$

for all r. Take derivative and set to zero to find optimal r. To find $r = (\alpha n)^\alpha$ is the minimum. Then

$$|a_n| \leq c \frac{e^{c\alpha n}}{(\alpha n)^{\alpha n}} \leq \frac{c^n}{n^{\alpha n}}.$$

□

5. Prime number Theorem

$\mathfrak{p}$ be the set of prime numbers. If we write $\sum_{p \leq x}$ means some over all primes $\leq x$.

Definition 5.1. Define $\pi(x) = \sum_{p \leq x} 1$.

Notation: $f(x) \sim g(x)$ means that $\frac{f(x)}{g(x)} \to 1$ as $x \to \infty$.

Prime number theorem, says $\pi(x) \sim \frac{x}{\log x} \sim \frac{x}{\log x - 1}$.

Logarithmic integral

$$\ell_i(x) = \int_2^x \frac{1}{\log t} dt.$$

Then $\pi(x) \sim \ell_i(x)$.

Arithmetic formula on $a(n)$, $A(x) = \sum_{n \leq x} a(n)$.

Example 5.2. $a(n) = \chi_{prime}$

$d(n)$= number of divisors of n

$w(n)$ the number of prime factors

$\Omega(n)$=number of prime factors with multiplicity

Example 5.3. $\sum_{n\leq x} d(n) = \sum_{j\leq x}\lfloor \frac{x}{j}\rfloor$

$\sum_{n\leq x} w(n) = \sum_{p\leq x}\lfloor \frac{x}{p}\rfloor$.

Definition 5.4. $a(n)$ is multiplicative if

$$a(mn) = a(m)a(n)$$

if $(m, n) = 1$ (completely multiplicative if it holds regardless).

Theorem 5.5. *Abel summation, Dirichlet Test*

(1) *Discrete series, write $\sum a(n)f(n)$ in terms of $A(x)$ and differences of $f's$. Integration by parts give*

$$\sum_{k=m+1}^{n} a(k)f(k) = \sum_{k=m}^{n-1} A(k)\left(f(k) - f(k+1)\right) + A(n)f(n) - A(m)f(m).$$

Proof. The first follows from $a(k) = A(k) - A(k-1)$. □

Corollary 5.6. *If $f(n)$ is real and ≥ 0, decreasing and $\sum_{n\leq x} a(n) \leq c\sum_{n\leq x} b(n)$ for all x, then*

$$\sum_{n\leq x} a(n)f(n) \leq c\sum_{n\leq x} b(n)f(n)$$

for all x.

Corollary 5.7. *If $A(n)f(n) \to 0$ then $\sum_{n=1}^{\infty} a(n)f(n) = \sum_{n=1}^{\infty} A(n)\left(f(n) - f(n-1)\right)$ in the sense that if one of the sums converges, then so does the other.*

Theorem 5.8. *(Dirichlet)*

(1) *$|A(n)| \leq c$ for all n*
(2) *$f(n) \to 0$*
(3) *$\sum_{n=1}^{\infty}\left(f(n) - f(n+1)\right)$ converges absolutely*

then $\sum a(n)f(n)$ converges and

$$\left|\sum a(n)f(n)\right| \leq C\sum |f(n) - f(n+1)|$$

Example 5.9. $\sum \frac{a(n)}{n^s}$, $f(t) = \frac{1}{t^s}$ we need $|A(n)| \leq C$, $Re(s) > 0$, $a(n) = \frac{1}{n^{1+\epsilon}}$ then $\sum \frac{1}{n^s}$ converges absolutely if $Re(s) > 1$.

Continuous version.

$a(n)$ arithmetic function and $f : \mathbb{R} \to \mathbb{R}$ or $\mathbb{C}$ continuously differentiable on $(y, x]$. Then

$$f(n+1) - f(n) = \int_{n}^{n+1} f'(t)dt.$$

Abel:

$$\sum_{y<n\leq x} a(n)f(n) = A(x)f(x) - A(y)f(y) - \int_{y}^{x} A(t)f'(t)dt.$$

Theorem 5.10. *Dirichlet*

(1) *$|A(n)| \leq C$*
(2) *$f \to 0$*
(3) *$\int_0^{\infty} |f'(t)|\, dt$ converges*

Then $\sum a(n)f(n)$ *is convergent. Additionally,*

$$\left|\sum a(n)f(n)\right| \leq c\int_1^\infty |f'(t)|\,dt.$$

Example 5.11.

(1) Given that

$$\left|\sum \frac{a(n)}{n}\right| \leq C,$$

how to estimate $A(x)$. Let $b(n) = \frac{a(n)}{n}$, then $|B(n)| \leq C$ for all n,

$$\begin{aligned} A(n) &= b(1) + 2b(2) + 3b(3) + ...nb(n) \\ &= nB(n) - B(1) - ... - B(n-1) \end{aligned}$$

and $|A(n)| \leq (2n-1)C$

(2) Given that $|A(x)| \leq Cx$, what about $\sum \frac{a(n)}{n}$? Well

$$\left|\sum_{n\leq x} \frac{a(n)}{n}\right| \leq \left|\frac{A(x)}{x}\right| + \left|\int_1^x \frac{A(x)}{t^2}dt\right| \leq C + \int_1^x \frac{C}{t}dt = C\left(\log x + 1\right).$$

Exercise 5.12. Show that

(1) $\sum_{n\leq x} \frac{a(n)}{n} = \frac{A(x)}{x} + \int_1^x \frac{A(t)}{t^2}dt$

Proof. This is directly from Abel, using $f(t) = \frac{1}{t}$ (use $1-\epsilon$ for your y and let $\epsilon \to 0$). □

(2) Write an expression for $\sum_{y<n\leq x} a(n)\log n =$...

Proof. This is

$$A(x)\log x - A(y)\log y - \int_y^x \frac{A(t)}{t}dt$$

□

(3) $\sum_{y<n\leq x} na(n)$

Proof. This is

$$xA(x) - yA(y) - \int_y^x A(t)dt = xA(x) - yA(x) - \sum_{n=\lceil y\rceil}^{\lfloor x\rfloor} nA(n)$$

□

(4) If $a(1) = 0$ then $\sum_{n=2}^x \frac{a(n)}{\log n} = \frac{a(x)}{\log x} + \int_2^x \frac{A(t)}{t\log^2 t}dt$

Proof. This is exactly by definition.... □

Let $F(x) = \sum_{n\leq x} f(x)$ and $I(x) = \int_1^x f$. Then $F(n) - f(1) \leq I(n) \leq F(n-1)$ if f is decreasing.

If $f \geq 0$ is decreasing,

$$I(x) \leq F(x) \leq I(x) + f(1)$$

and increasing then

$$F(x) = I(x) + r(x)$$

with $|r(x)| \leq f(x)$.

Example 5.13. We have

(1) $\log x \leq \sum_{n\leq x} \frac{1}{n} \leq \log x + 1$

(2) $\left|\sum_{n\leq x} \log n - x\log x + x - 1\right| \leq \log x$. The sum is like $\log(x!)$, then $x\log x = x^x$... Gives us

$$m! \approx \left(\frac{m}{e}\right)^m.$$

Corollary 5.14. *If $f \geq 0$ decreasing, then $\sum_{n=1}^{\infty} f(n)$ converges iff $\int f(t)$ converges.*

$$\int_1^\infty f(t) \leq \sum_{n=1}^{\infty} f(n) \leq \int_1^\infty f(t) + f(1).$$

Example 5.15. Have many divisors does a random number have

Interested in $\frac{1}{x}\sum_{n\leq x} d(n) = \frac{1}{x}\sum_{n\leq x}\lfloor\frac{x}{n}\rfloor$. We have

$$\frac{x}{n} - 1 \leq \lfloor\frac{x}{n}\rfloor \leq \frac{x}{n}.$$

Therefore, the average is about $\log x$.

5.1. Euler Summation.

Proposition 5.16. *Euler's Summation, version 1. Let $f : [m,n] \to \mathbb{C}, \mathbb{R}$ continuously differentiable.*

$$\sum_{k=m+1}^{n} f(k) - \int_m^n f(t)dt = \int_m^n (t - \lfloor t\rfloor) f'(t)dt.$$

Proof. We have

$$\int_{k-1}^{k} (t - \lfloor t\rfloor) f'(t)dt = \int_{k-1}^{k} (t - k + 1) f'(t)dt$$

then use integration by parts to get

$$= f(t)(t-k+1)\,|_{k-1}^{k} - \int_{k-1}^{k} f(t)dt = f(k) - \int_{k-1}^{k} f(t).$$

The result follows. □

Proposition 5.17. *Version 2.*

$$\frac{f(m)+f(n)}{2} + \sum_{m+1}^{n-1} f(k) - \int_m^n f(t)dt = \int_m^n \left(t - \lfloor t\rfloor - \frac{1}{2}\right) f'(t)dt.$$

Proof. We have

$$\begin{aligned}\int_{k-1}^{k}\left(t - \lfloor t\rfloor - \frac{1}{2}\right) f'(t)dt &= (t - k + \frac{1}{2})f(t)\,|_{k-1}^{k} - \int_{k-1}^{k} f(t)dt \\ &= \frac{f(k)+f(k-1)}{2} - \int_{k-1}^{k} f(t)dt.\end{aligned}$$

□

Proposition 5.18. *Version 3. We have*

$$\sum_{m<k\leq x} f(k) - \int_m^x f(t)dt = \int_m^x (t - \lfloor t\rfloor) f'(t)dt - (x - \lfloor x\rfloor) f(x).$$

If f is continuously differentiable on $(1,\infty)$, and both $\sum f(n)$ and $\int f(t)$ converges, then

$$\begin{aligned}\sum_{n=1}^{\infty} f(n) - \int_1^\infty f(t)dt &= f(1) + \int_1^\infty (t - \lfloor t\rfloor) f'(t)dt\text{version 1} \\ &= \frac{f(1)}{2} + \int_1^\infty \left(t - \lfloor t\rfloor - \frac{1}{2}\right) f'(t)dt \qquad \text{version 2.}\end{aligned}$$

Even if they don't converge, we can still estimate the difference. If $f \to 0$ then $F(x) - I(x) \to L$ finite limit as $x \to 0$. Here,

$$L = f(1) + \int_1^\infty (t - \lfloor t\rfloor) f'(t)dt,$$

$0 \le L \le f(1)$. We can estimate

$$|F(x) - I(x) - L| = \left| \int_x^\infty (t - \lfloor t \rfloor) f'(t) dt \right| \le f(x)$$

for all x.

Example 5.19. $\sum_{n \le x} \frac{1}{n} - \log x$ has a limit, called γ the Euler constant.

$$\gamma = 1 - \int_1^\infty \frac{t - \lfloor t \rfloor}{t^2} dt$$

with $0 < \gamma < 1$. Look at

$$\left| \sum_{n \le x} \frac{1}{n} - \log x - \gamma \right| \le \frac{1}{x}.$$

Want $\ell_i(x) = \int_2^x \frac{1}{\log t} dt \sim \frac{x}{\log x}$. With integration by parts,

$$\int_e^x \frac{1}{\log t} dt = \left(\frac{t}{\log t} \right)_e^x + \int_e^x \frac{1}{\log^2 t} = \frac{x}{\log x} - e + \int_e^x \frac{1}{\log^2 t}.$$

If we let $I_n(x) = \int_e^x \frac{1}{(\log t)^n}$ then

$$I_n(x) = \frac{x}{(\log x)^n} - e + nI_{n+1}(x).$$

Claim 5.20. $I_n(x) \sim \frac{x}{(\log x)^n}$.

Proof. Need to show that $\frac{I_{n+1}(x)}{\frac{x}{(\log x)^n}} \to 0$ as $x \to 0$. This will imply that $I_{n+1}(x) \sim \frac{x}{(\log x)^{n+1}}$.

$$I_{n+1}(x) = \int_e^{\sqrt{x}} \frac{1}{(\log t)^{n+1}} + \int_{\sqrt{x}}^x \frac{1}{(\log t)^{n+1}}.$$

The first is $\le \sqrt{x}$, the second is $\le \frac{x}{\left(\frac{1}{2}\log x\right)^{n+1}} = \frac{2^{n+1} x}{(\log x)^{n+1}}$. □

Recall that

$$I_n(x) = \int_e^x \frac{1}{(\log t)^n} \sim \frac{x}{(\log x)^n}$$

and

$$I_n(x) = \frac{x}{(\log x)^n} - e + nI_{n+1}(x).$$

Also

$$\pi(x) \sim \ell_i(x) \sim \frac{x}{\log x}.$$

We have

$$\ell_i(x) = \frac{x}{\log x} + r(x)$$

with $r(x) \sim \frac{x}{\log(x)^2}$.

Claim 5.21. $\ell_i(x) = \frac{x}{\log x - 1} + q(x)$ with $q(x) \sim \frac{x}{\log(x)^3}$.

Proof. We have

$$\begin{aligned} q(x) &= I_1(x) + const - \frac{x}{\log x - 1} = \frac{x}{\log(x)} + const + 2I_2(x) - \frac{x}{\log x - 1} \\ &\sim x\left(\frac{1}{\log x} + \frac{1}{(\log x)^2} - \frac{1}{\log x - 1} \right) \\ &= x\left(\frac{\log x (\log x - 1) + \log x - 1 - \log^2 x}{\log^2 x (\log x - 1)} \right). \end{aligned}$$

□

By the same method,

$$\ell_i(x) = \frac{x}{\log x} + \frac{x}{\log^2 x} + ... + (n-1)!\frac{x}{(\log x)^n} + r_{n+1}(x).$$

Here, $r_{n+1}(x) \sim n!\frac{x}{(\log x)^{n+1}}$.

$$|\pi(x) - \ell_i(x)| \ll \frac{x}{(\log x)^m} \text{ for all } m.$$

Definition 5.22. Chebyshev's function.

$$\theta(x) = \sum_{p \leq x} \log p.$$

We have

$$\theta(x) = \sum_{n \leq x} \chi_p(n) \log n = \pi(x) \log x - \int_2^x \frac{\pi(t)}{t} dt$$

because $\pi(x) = \sum_{p \leq x} \chi_p$. We can also write

$$\pi(x) = \sum_{n \leq x} \frac{\chi_p \log n}{\log n} = \frac{\theta(x)}{\log x} + \int_2^x \frac{\theta(t)}{t \log^2(t)} dt.$$

These relations show that

$$\pi(x) \sim \frac{x}{\log x} \qquad \Longleftrightarrow \qquad \theta(x) \sim x.$$

Chebyshev showed that $c_1 x \leq \theta(x) \leq c_2 x$ but the constants are not close to 1. However, as a corollary:

$$c_1 \frac{x}{\log x} \leq \pi(x) \leq c_2 \frac{x}{\log x}.$$

Theorem 5.23. $\theta(x) \leq (\log 4)\, x$.

Proof. Assume x is an integer. Then

$$N = \binom{2n+1}{n} = \frac{(2n+1)!}{n!\,(n+1)!}$$

is divisible by the product of all primes between $n+1$ and $2n+1$.

$$\log N \geq \sum_{n+1}^{2n+1} \log p = \theta(2n+1) - \theta(n+1).$$

We know that

$$\sum_{k=0}^{2n+1} \binom{2n+1}{k} = 2^{2n+1}$$

so

$$n \log 4 \geq \log N.$$

Claim. By induction, $\theta(k) \leq klog(4)$.

Proof. Check this for $k = 1, 2$. Assume true up to $2n$. Then

$$\begin{aligned} \theta\,(2n+2) &\leq n \log(4) + \theta(n+1) \leq n \log 4 + (n+1) \log 4 \\ &= (2n+1) \log 4 \leq (2n+2) \log 4. \end{aligned}$$

□

□

Exercise 5.24. Calculate

$$\sum_{p \leq x} \frac{1}{p} \leq c \log \log x.$$

Proof. m □

5.2. **Dirichlet Series.**

Definition 5.25. Dirichlet series.

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^s}.$$

Converges for some half-plane.

Claim 5.26. There exists σ_c, σ_a where the series absolutely converges if $Re(s) > \sigma_a$ and no absolute convergence if $Re(s) < \sigma_c$ (possibly infinity).

Clearly, $\sigma_c \leq \sigma_a$ and in fact, $\sigma_a \leq \sigma_c + 1$.

Example 5.27. $a(n) = (-1)^n$, then $\sigma_a = 1$, $\sigma_c = 0$.

Theorem 5.28. *If $s = \sigma + it \in \mathbb{R}$, then*

(1) $\zeta(\sigma)$ converges for all $\sigma > 1$
(2) $\zeta(\sigma) > 1$ and is decreasing
(3) $\frac{1}{\sigma-1} \leq \zeta(s) \leq \frac{1}{\sigma-1} + 1$ (comparison with integral)
(4) $\zeta(\sigma) \to 1$ as $\sigma \to \infty$ and $\zeta(\sigma) \to \infty$ as $\sigma \to 1^+$
(5) $(\sigma - 1)\,\zeta(\sigma) \to 1$ as $\sigma \to 1^+$

Let $A(x) = \sum_{n \leq x} a(n)$. Then

$$\sum_{n \leq x} \frac{a(n)}{n^s} = \frac{A(x)}{x^s} + s \int_1^x \frac{A(t)}{t^{s+1}} dt.$$

If $s \neq 0$ and $\frac{A(x)}{x^s} \to 0$ as $x \to \infty$, then

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^s} = s \int_1^{\infty} \frac{A(t)}{t^{s+1}} dt$$

if they both converge (if one converges then so does the other).

$$\sum_{n > x} \frac{a(n)}{n^s} = -\frac{A(x)}{x^s} + s \int_x^{\infty} \frac{A(t)}{t^{s+1}} dt.$$

Suppose $f(t) = O(t^{\alpha})$ for some $\alpha \geq 0$, $t \geq 1$. The integral $I(s) = \int_1^{\infty} \frac{f(t)}{t^{s+1}} dt$ is called the Dirichlet integral of f, which converges for $Re(s) = \sigma > \alpha$. Similarly, define $I_x(s) = \int_1^x \frac{f(t)}{t^{s+1}}$. Since $|I(s)| \leq \frac{c}{\sigma - \alpha}$,

$$|I(s) - I_x(s)| \leq \frac{c}{\sigma - \alpha} \frac{1}{x^{\sigma - \alpha}}.$$

Theorem 5.29. *If $A(x) = \mathcal{O}(x^{\alpha})$ for some $\alpha \geq 0$, then*

$$F(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$$

converges for $\sigma > \alpha$. Additionally,

$$F_x(s) = \sum_{n \leq x} \frac{a(n)}{n^s}$$

satisfies $|F_x(s)| \leq c\frac{|s|}{\sigma - \alpha}$, and $|F(s) - F_x(s)| \leq \frac{c}{x^{\sigma - \alpha}} \left(\frac{|s|}{\sigma - \alpha} + 1\right)$.

Proof. of existence of σ_c.

Need to show that if $\sum \frac{a(n)}{n^{\alpha}}$ converges for some $\alpha \in \mathbb{R}$, then $\sum \frac{a(n)}{n^s}$ converges for every s with $\sigma > \alpha$.

Let $b(n) = \frac{a(n)}{n^{\alpha}}$, and $B(n) = \sum_{n \leq x} b(n)$, $B(x) = \mathcal{O}\left(x^0\right)$. Then

$$\sum \frac{b(n)}{n^s} = \sum \frac{a(n)}{n^{s+\alpha}}$$

converges if $Re(s) > 0$. □

Theorem 5.30.

(1) *If* $\sum \frac{a(n)}{n^\sigma}$ *converges for some* $\sigma > 0$*, then* $A(x) = \mathcal{O}(x^\sigma)$

(2) *Let* α *be the inf of these* σ *for which* $A(x) = \mathcal{O}\left(x^\sigma\right)$*, if* $\alpha > 0$ *then* $\sigma_c = \alpha$.

Proof.

(1) $f(n) = \sum \frac{a(n)}{n^\sigma}$ and $\sum b(n)$ converges with $B(x)$ bounded.

$$A(x) = \sum_{n \le x} n^\sigma b(n) = x^\sigma B(x) - \sigma \int_1^x t^{s-1} B(t) dt.$$

□

Exercise 5.31. Let $I = \inf\{\sigma \in \mathbb{R} : |A(x)| \le \mathcal{O}(x^\sigma)\}$. If $I > 0$ then $\sigma_c = I$. What happens when I is not > 0.

Proof. Let $b(n) = \frac{a(n)}{n^\sigma}$, which is bounded so $|B(x)| \le O(1)$.

$$A(x) = \sum_{n \le x} b(n) n^\sigma = B(x) x^\sigma - \sigma \int_1^x B(t) t^{\sigma - 1} dt.$$

The $B(x)x^\sigma = O(x^\sigma)$ and the integral is $O(x^\sigma) - O(1)$. Need to analyze the tail of $A(x)$. Let $L = \sum_{n \ge 1} a(n)$ if it exists and 0 else. Then for $\sigma < 0$.

$$\sum_{y < n \le x} a(n) = \sum_{y < n \le x} b(n) n^\sigma = B(x) x^\sigma - B(y) y^\sigma - \sigma \int_y^x B(t) t^{\sigma - 1} dt$$

Still assuming that $\sum b(n)$ converges, so $|B(x)| \le \mathcal{O}(1)$. As $x \to \infty$, LHS is $L - A(y)$, RHS is

$$-B(y) y^\sigma - \sigma \int_y^\infty B(t) t^{\sigma - 1} dt = \mathcal{O}(y^\sigma) + \mathcal{O}(y^\sigma) = \mathcal{O}(y^\sigma).$$

Therefore, the change of the statement should be let L be as above, $\sum \frac{a(n)}{n^\sigma}$ converges, then $|A(x) - L| \le \mathcal{O}(x^\sigma)$,

$$I = \inf\left\{\sigma \in \mathbb{R} : |A(x) - L| \le \mathcal{O}\left(x^\sigma\right)\right\}$$

then $\sigma_c = I$. This is because for all $r \in (I, Re(s))$,

$$\sum_{n \le x} \frac{a(n)}{n^s} = \frac{A(x) - L}{x^s} + \sum \frac{L}{x^s} + s \int_1^x \frac{A(T) - L + L}{t^{\sigma + 1}} dt$$

then analyze this. □

Theorem 5.32. $F(s) = \sum \frac{a(n)}{n^s}$ *is a holomorphic function on* $\{\sigma > \sigma_c\}$*, then* $F'(s) = -\sum \frac{a(n) \log n}{n^s}$.

Proof. Need to show some local uniform convergence of the sum.

For $\alpha > \sigma_c$, $b(n) = \frac{a(n)}{n^\alpha}$, $|B(x)| \le C$. $G(s) = \sum \frac{b(n)}{n^s}$, $F(s) = G(s - \alpha)$.

Claim. G is holomorphic on $|Re(s) > 0|$.

Proof. We have

$$|G(s) - G_n(s)| \le \frac{c}{n^\sigma}\left(\frac{|s|}{\sigma} + 1\right)$$

for $s = \sigma + it$. Then $G_n(s) \to G(s)$ locally uniformally on $\{\sigma > 0\}$. □

□

Example 5.33. $\zeta(s)$ is holomorphic on $\{Re(s) > 1\}$.

Arithmetic function $a(n)$ for $F_a(s)$.

- $a = 1$, then $F_1(s) = \zeta(s)$
- $a = \chi_p$, then $F_a(s) = \sum \frac{1}{p^s}$

- a =square modulus, then $F_a(s) = \zeta(2s)$

Fact 5.34. *If things converges nicely,*

$$F_a(s)F_b(s) = \left(\sum \frac{a(k)}{k^s}\right)\left(\sum \frac{b(\ell)}{\ell^s}\right) = \sum_n \frac{c(n)}{n^s}$$

where

$$c(n) = (a \star b)(n) = \sum_{n=kl} a(k)b(\ell)$$

is the convolution of a and b. Then

$$F_aF_b = F_{a\star b}.$$

If LHS absolutely converges on $\{Re(s) \geq \sigma\}$ then so does RHS. We have product normula, $\ell(n) = \log n$, then

$$\begin{aligned} \ell(a \star b) &= (\log n)\sum_{n=k\ell} a(k)b(\ell) = \sum_{n=k\ell} (\log k + \log \ell)\, a(k)b(\ell) \\ &= \sum_{n=k\ell} (\log k)\, a(k)b(\ell) + \sum_{n=k\ell} a(k)(\log \ell)\, b(\ell) \\ &= (\ell a) \star b + a \star (\ell b) \end{aligned}$$

Example 5.35. $\delta_i \star \delta_j = \delta_{ij}$, where $\delta_i(h) = \begin{cases} 1 & i = h \\ 0 & i \neq h \end{cases}$, then $1(n) = 1$ for every n.

$\delta_i \star 1 = \begin{cases} 1 & if\ i \mid n \\ 0 & \text{else} \end{cases}$ and $a \star 1 = \sum_{j|n} a(j)$.
$1 \star 1(n) = d(n)$ and $\chi_p \star 1(n) = \omega(p)$.
$\zeta(s)^2 = \sum_{n=1}^{\infty} \frac{d(n)}{n^s}$ with $Re(s) > 1$. $\zeta(s) = \sum_p \frac{1}{p^s} = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$.

$$\left(\sum_{n=1}^{\infty} a(n)\right)\left(\sum_{n=1}^{\infty} b(n)\right) = \sum_n (a \star b)(n)$$

if they converges absolutely.

Let $A(x) = \sum_{n \leq x} a(n)$ and $B(x) = \sum_{n \leq x} b(n)$. We see that

$$\sum_{n \leq x} (a \star b)(n) = \sum_{j,k \leq x} a(j)b(k) = \sum_j a(j)B(\frac{x}{j}) = \sum_k b(k)A(\frac{x}{k}).$$

Additionally,

$$\sum_{n \leq x} (a \star 1)(n) = \sum_{j \leq x} a(j)\lfloor \frac{x}{j} \rfloor.$$

From $d = 1 \star 1$, we have the formula:

$$\sum_{n \leq x} d(n) = \sum_{j \leq x} \lfloor \frac{x}{j} \rfloor.$$

Proposition 5.36. *(Dirichlet's Hyperbola Identity) [Draw a picture to see this]*

$$\sum_{n \leq x} (a \star b)(n) = \sum_{j \leq y} a(j)B\left(\frac{x}{j}\right) + \sum_{k \leq \frac{x}{y}} b(k)A\left(\frac{x}{k}\right) - A(y)B(\frac{x}{y}).$$

Proposition 5.37.

(1) Suppose a, b are multiplicative, then $a \star b$ is completely multiplicative
(2) a, b are arbitrary, c is completely multiplicative, then

$$(ac) \star (bc) = (a \star b)c.$$

(3) The identity is δ_1 and a has an inverse iff $a(1) \neq 0$ <—– exercise

Euler Product. $a(n)$ completely multiplicative, $|a(p)|<1$ for each prime p (absolute convergence of $\sum a(n)$ will give this), then

$$\prod_p \frac{1}{1-a(p)} = \prod_p \left(1+a(p)+a(p^2)+...\right) = \sum_{n\geq 1} a(n).$$

Proof. Of exercise. Suppose $a(1)\neq 0$, then to get inverse, easy check that $b(1)=\frac{1}{a(1)}$, then try $(a\star b)(2)$... □

Example 5.38. $\zeta(s)=\prod \frac{1}{1-p^{-s}}$.

$\prod_{p\leq N}\left(1+a(p)+a(p^2)...\right)=\sum_n a(n)$ where the sum is over all n's whose prime factors are $\leq N$. Therefore,

$$\left|\sum_n a(n) - \prod_{p\leq N} ...\right| = \left|\sum_{p|n \text{ with } p>N} a(n)\right| \leq \sum_{n\geq N} |a(n)| \to 0$$

if $\sum a(n)$ is absolutely convergent. This is because $|a(p)|<1$ for every p.

Corollary 5.39. *If $a(n)$ is completely multiplicative, $\sum a(n)$ absolutely convergent, then $\sum a(n)\neq 0$. In particular, $\zeta(s)\neq 0$ and $Re(s)>1$.*

Exercise 5.40. Calculate $A=$ all integers of the form $2^n 3^m$, $\sum_{n\in A}\frac{1}{n^2}$.

Proof. Let $a(n)=\chi_A(n)\frac{1}{n^2}$, $\sum_{n\in A}\frac{1}{n^2}=\prod_p \frac{1}{1-a(p)} = \frac{1}{1-\frac{1}{4}}\frac{1}{1-\frac{1}{3}}=\frac{3}{2}$. □

Theorem 5.41. *We have $\sum_{p\leq x}\frac{1}{p}\geq \log\log x - \frac{1}{2}$.*

Proof. We have

$$\prod_{p\leq N}\frac{1}{1-p^{-1}} = \sum_{n \text{ with prime factors } p\leq N}\frac{1}{n} \geq \sum_{n=1}^{N}\frac{1}{n} > \log N.$$

Let $s_N=\sum_{p\leq N}\frac{1}{p}$ then

$$\left|\log\left(\prod_{p\leq N}\frac{1}{1-\frac{1}{p}}\right) - s_N\right| \leq \frac{1}{2}.$$

□

From the Euler product,

$$\sum \mu(n)a(n) = \prod_p (1-a(p)) = \frac{1}{\sum a(n)}.$$

This $\mu(n)$ is called the mobius function, and is

$$\mu(n)=\begin{cases} 0 & \text{if } n \text{ is not square free} \\ (-1)^k & k \text{ number of primes}\end{cases}.$$

Then $\frac{1}{\zeta(s)}=\sum\frac{\mu(n)}{n^s}$.

Theorem 5.42. *If $\sum a(n)$ is absolutely convergent, $a(n)$ completely multiplicative then*

$$\frac{1}{\sum_{n\geq 1} a(n)} = \sum_{n\geq 1}\mu(n)a(n).$$

Remark 5.43. We have $\frac{1}{\zeta(s)}\cdot\zeta(s)=1$ then

$$F_\mu \cdot F_1 = F_{\delta_1}.$$

This suggestes $\mu\star 1=\delta_1$.

Indeed,

$$(\mu\star 1)(n) = \sum_{j|n}\mu(j) = \sum (-1)^i\binom{r}{i} = (1-1)^r = 0$$

if $r\neq 0$, where r is the number of prime factors of n.

Corollary 5.44. *$a\star 1=b$ iff $a=b\star\mu$. (just convolute with μ on both sides).*

If a is completely multiplicative, the inverse of a is μa

Proof. $1\star\mu=\delta_1$, then $a\star\mu a=a\delta_1=\delta_1$. □

Corollary 5.45. *Suppose $a\star 1=b$ (iff $a=b\star\mu$), then*

$$B(x)=\sum_{n\leq x}(a\star 1)(n)=\sum_{n\leq x}A\left(\frac{x}{n}\right).$$

Similarly,

$$A(x)=...=\sum_{j\leq x}\mu(j)B(\frac{x}{j}).$$

Theorem 5.46. *Mobius Inversion. Let $F:\mathbb{R}^+\to\mathbb{R}$ or $\mathbb{C}$. $F(x)=0$ if $x<\epsilon$ define $G(x)=\sum_{n=1}^{\infty}F(\frac{x}{n})$ finite sum. Then*

$$F(x)=\sum_{n=1}^{\infty}\mu(n)G(\frac{x}{n}).$$

Proof. We have

$$\begin{aligned}F(x) &= \sum_{j=1}^{\infty}\delta_1(j)F(\frac{x}{j})=\sum_{j}F(\frac{x}{j})\sum_{i|j}\mu(i)=\sum_{i}\mu(i)\sum_{k}F(\frac{x}{k}),\text{ where } j=ik\\ &= \sum_{i}\mu(i)G(\frac{x}{i}).\end{aligned}$$

□

Remark 5.47. We have

$$\frac{1}{\zeta(s)}=\sum\frac{\mu(n)}{n^s}\qquad \mathrm{Re}(s)>1.$$

If $s\to 1$, $\frac{1}{\zeta(s)}\to 0$. This suggests that $\sum\frac{\mu(n)}{n}=0$. This is in fact true, but there's no simple proof. It is actually equivalent to prime number theorem.

Easier: $\left|\sum\frac{\mu(n)}{n}\right|\leq 1$.

Proof. We have

$$1=\sum_1^N(1\star\mu)(n)=\sum_1^N\mu(n)\lfloor\frac{N}{n}\rfloor=N\sum_1^N\frac{\mu(n)}{n}-\sum_1^N\mu(n)\left\{\frac{N}{n}\right\}.$$

The latter term has $|\cdot|\leq N-1$. □

Definition 5.48. $\varphi(n)$ =number of numbers between 1 and n coprime to n.

Then given $n=\prod\mathfrak{p}_i^{n_i}$ then

$$\varphi(n)=n\prod_{i=1}^{k}\left(1-\frac{1}{p^i}\right)=n\sum_{k|n}\frac{\mu(k)}{k}.$$

This means that $\varphi=id\star\mu$ and so $\varphi\star 1=id$. Therefore,

$$\sum_{j|n}\varphi(j)=n$$

for all $n\neq 0$.

$$\zeta(s)=\sum\frac{1}{n^s}=\prod\frac{1}{1-\frac{1}{p^s}}$$

then

$$\log\zeta(s)=\sum_m\sum_p\frac{1}{mp^{ms}}=\sum_{n=1}^{\infty}\frac{c(n)}{n^s}$$

then

$$c(n) = \begin{cases} \frac{1}{n} & n = p^m \\ 0 & else \end{cases}.$$

Often unclear,

$$\left| \log \zeta(s) - \sum \frac{1}{p^s} \right| < \frac{1}{2}.$$

We also have

$$\frac{\zeta'(s)}{\zeta(s)} = \sum \frac{-\log(n)c(n)}{n^s} = -\sum \frac{\Lambda(n)}{n^s}$$

and this $\Lambda(n)$ is called the Mangolds function. It is

$$\begin{cases} \log p & n = p^m \\ 0 & \text{else} \end{cases}.$$

We know that

$$\frac{\zeta'(s)}{\zeta(s)} \zeta(s) = \zeta'(s),$$

which suggests that $\Lambda \star 1 = \ell$ $(\ell(n) = \log n)$.

Proof. When $n = 1$, this is good.

$n = p_1^{\alpha_1} ... p_k^{\alpha_k}$, then

$$(\Lambda \star 1)(n) = \sum \alpha_j \log p_j = \log n.$$

□

Knowing this, we now know that $\ell \star \mu = \Lambda$.

Recall that

$$\pi(x) \sim \frac{x}{\log x} \iff \theta(x) \sim x.$$

Let $\psi(x) = \sum_{n \leq x} \Lambda(n)$. We know that $\psi(x) \geq \theta(x)$. Meanwhile,

$$\psi(x) \leq \theta(x) + \theta(\sqrt{x}) + ...\theta\left(\sqrt[m]{x}\right) \sim \theta(x) + c\log x \leq \theta(x) + \sqrt{x}.$$

Since $\sqrt{x}\log x << x$, we know that

$$\theta(x) \sim x \iff \psi(x) \sim x.$$

We already know that $\psi(x) \leq cx$, just need the reverse.

Notation:

$$\nu = \delta_1 - 2\delta_2 = \begin{cases} 1 & \text{if } n = 1 \\ -2 & \text{if } n = 2 \\ 0 & \text{else} \end{cases}$$

and

$$(\nu \star 1)(n) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ -1 & \text{if } n \text{ is even} \end{cases}.$$

Define

$$E(x) = \sum_{n \leq x} (\nu \star 1)(n) = \begin{cases} 1 & \text{if } \lfloor x \rfloor \text{ is odd} \\ 0 & \text{if } \lfloor x \rfloor \text{ is even} \end{cases}$$

and

$$\begin{aligned}\sum \Lambda(j)E(\frac{x}{j}) &= \sum_{n\le x}(\Lambda\star(\nu\star 1))(n) = \sum_{n\le x}(\ell\star\nu)(n) = \sum_j \nu(j)\sum_{k\le\frac{x}{j}}\log k\\ &= \sum_{k\le x}\log k - 2\sum_{k\le\frac{x}{2}}\log k.\end{aligned}$$

When $x = 2n$,

$$\psi(2n) = \sum_{j\le 2n}\Lambda(j) = \sum_{j\le 2n}\Lambda(j)E(\frac{2n}{j}) = \log\binom{2n}{n} \ge \log\left(\frac{4^n}{2n+1}\right) \ge n\log 4 - \log(2n+1).$$

Use monotonicity to finish.

Theorem 5.49. *The probability that two random numbers are co-prime are* $\frac{6}{\pi^2}$.

Proof. We have

$$\frac{2}{x^2}\sum_{n\le x}\varphi(n) \to ?.$$

The reason this is what we want, is because the set of (k,n) with $k\le n$ where they are coprime is just $\varphi(n)$. The 2 is just with $k \ge n$. Now,

$$\sum_{n\le x}\varphi(n) = \sum_k \mu(k)\sum_{j\le\frac{x}{k}} j.$$

Meanwhile,

$$\sum_{j\le\frac{x}{k}} j = \int_0^{x/k} tdt + error = \frac{x^2}{2k^2} + error.$$

The error is $c\frac{x}{k}$. So

$$\sum_{n\le x}\varphi(n) = \frac{1}{2}x^2\sum_{k\le x}\frac{\mu(k)}{k^2} + \sum error.$$

The error is $\sim x\log x$. Also,

$$\sum_{k\le x}\frac{\mu(k)}{k^2} = \sum_1^\infty\frac{\mu(k)}{k^2} - \sum_{k+1}^\infty\frac{\mu(k)}{k^2}$$

the first is bounded by $\frac{1}{\zeta(2)}$, and

$$\left|\sum_{k>x}\frac{\mu(k)}{k^2}\right| \le \sum_{k>x}\frac{1}{k^2} \le \frac{1}{x} \to 0.$$

□

6. Riemann Zeta Function $\zeta(s)$

We can extend $\zeta(s)$ to a holomorphic function on $\{Re(s) > 0\}$ with simple pole at $s = 1$.

Euler summation:

$$\zeta(s) = \frac{1}{s-1} + 1 - s\int_1^\infty \frac{t-\lfloor t\rfloor}{t^{s+1}}dt$$

makes sense if the integral is holomorphic in $Re(s) > 0$. Look at partial sums:

$$\sum_{n=2}^N f(n) = \int_1^N f(t)dt + \int_1^N (t-\lfloor t\rfloor)f'(t)dt.$$

For us, $f(t) = \frac{1}{t^s}$, and

$$\sum_{n=1}^N \frac{1}{n^s} = 1 + \frac{1}{s-1} + \frac{N^{1-s}}{s-1} - s\int_1^N \frac{t-\lfloor t\rfloor}{t^{s+1}}dt.$$

Then

$$\zeta(s) = \sum_{n=1}^N \frac{1}{n^s} + \frac{N^{1-s}}{s-1} - s\int_N^\infty \frac{t-\lfloor t\rfloor}{t^{s+1}}dt.$$

The integral $\to 0$ as $N \to \infty$. Then

$$\zeta(s) = \lim_{N\to\infty} \left(\sum_{n=1}^{N} \frac{1}{n^s} + \frac{N^{1-s}}{s-1} \right)$$

for $Re(s) > 0$. Each of these do not converge on its own, but does together.

Claim 6.1. $\zeta(s) = \frac{1}{s-1} + a_0 + a_1(s-1) + a_2(s-1)^2 + ...$

Proof. We have

$$\zeta(s) - \frac{1}{s-1} = 1 - s\int_1^\infty \frac{t - \lfloor t \rfloor}{t^{s+1}} dt.$$

As $s \to 1$,

$$a_0 = 1 - \int_1^\infty \frac{t - \lfloor t \rfloor}{t^{s+1}} dt$$

is the same you get from Euler summation of $\sum \frac{1}{n}$. □

From $\zeta(s) = \frac{1}{s-1} + \gamma + a_1(s-1) + a_2(s-1)^2 + ...$ get

$$\frac{1}{\zeta(s)} = (s-1) - \gamma(s-1)^2 + ...$$

and

$$\frac{\zeta'(s)}{\zeta(s)} = -\frac{1}{s-1} + \gamma + ...$$

We know

$$|\zeta(\sigma + it)| \leq \zeta(\sigma)$$

but this does not hold for $\sigma < 1$.

Theorem 6.2. *We have instead*

$$|\zeta(\sigma + it)| \leq \log t + 4$$

for $\sigma \geq 1$, $t \geq 2$. *(*$\sigma = 1$ *is allowed).*

Proof. We compute

$$\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} + \frac{N^{1-s}}{s-1} + r_N(s).$$

The error term is

$$|r_N(s)| \leq \frac{|s|}{\sigma N^\sigma} \leq \left(1 + \frac{t}{\sigma}\right)\frac{1}{N^\sigma}.$$

Choose $N = \lfloor t \rfloor$. Then

$$|r_N(s)| \leq \frac{1+t}{N} \leq 2.$$

Meanwhile,

$$\left| \sum_{n=1}^{N} \frac{1}{n^s} \right| \leq \sum_{n=1}^{N} \frac{1}{n} \leq \log N + 1 \leq \log t + 1.$$

Finally,

$$\left| \frac{N^{1-s}}{s-1} \right| \leq \frac{1}{t} \leq \frac{1}{2}.$$

□

Theorem 6.3. *Same assumptions,*

$$|\zeta'(\sigma + it)| \leq \frac{1}{2}(\log t + 3)^2$$

for $\sigma \geq 1$, $t \geq 2$.

Proof. From

$$\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} + \frac{N^{1-s}}{s-1} - s\int_N^\infty \frac{t - \lfloor t \rfloor}{t^{s+1}} dt,$$

for $N = \lfloor t \rfloor$, we get

$$\begin{aligned} |\zeta'(s)| &\leq \left|\sum_{n=1}^{N} \frac{-\log n}{n^s}\right| + \left|\frac{N^{1-s}\log N}{s-1}\right| + \left|\frac{N^{1-s}}{(s-1)^2}\right| \\ &+ \left|\int_N^\infty \frac{t-\lfloor t \rfloor}{t^{s+1}}dt\right| + \left|s\int_N^\infty \frac{(t-\lfloor t \rfloor)\log t}{t^{s+1}}dt\right|. \end{aligned}$$

We know that

$$\left|\sum_{n=1}^{N} \frac{\log n}{n}\right| \leq \frac{\log 2}{2} + \frac{\log 3}{3} + \int_3^N \frac{\log t}{t}dt < \frac{1}{2}\log^2 6 + \frac{1}{8}.$$

Also,

$$\left|\frac{N^{1-s}\log N}{s-1}\right| \leq \frac{\log N}{t} \leq \frac{\log t}{t} \leq \frac{1}{e} < \frac{1}{2}.$$

Next,

$$\left|\int_N^\infty \frac{t-\lfloor t \rfloor}{t^{s+1}}dt\right| \leq \int_N^\infty \frac{1}{t^2} = \frac{1}{N} \leq \frac{1}{2}.$$

Finally,

$$\left|\int_N^\infty \frac{t-\lfloor t \rfloor}{t^{s+1}}\log t dt\right| \leq \left|\int_N^\infty \frac{\log t}{t^{s+1}}dt\right| = \frac{\log t}{\sigma N^\sigma} + \frac{1}{\sigma^2 N^\sigma}.$$

We wanted to bound s times that, so

$$\left|s\int\right| \leq \left(1+\frac{t}{\sigma}\right)\frac{\log N + 1}{N} \leq \frac{1+t}{N}(\log t + 1) \leq 2(\log t + 1).$$

□

Next goal: $\zeta(s) \neq 0$ when $Re(s) = 1$.

Claim 6.4. If $a(n) \geq 0$ for all n,

$$\sum \frac{a(n)}{n^s}$$

converges to $f(s)$ for $Re(s) > \sigma$ then

$$Re\left(3f(\sigma) + 4f(\sigma+it) + f(\sigma+2it)\right) \geq 0.$$

Proof. Write $f(s) = \sum \frac{a(n)}{n^s}$. Then

$$(\star) = \sum_n \frac{a(n)}{n^\sigma}\left(3 + \frac{4}{n^{it}} + \frac{1}{n^{2it}}\right)$$

We have

$$Re\left(3 + \frac{4}{n^{it}} + \frac{1}{n^{2it}}\right) = 3 + 4\cos(\alpha) + \cos(2\alpha).$$

Recall that $\cos(2\alpha) = 2\cos^2(\alpha) - 1$ and so

$$Re\left(3 + \frac{4}{n^{it}} + \frac{1}{n^{2it}}\right) = 2\cos^2\alpha + 4\cos\alpha + 2 = 2(\cos\alpha + 1)^2.$$

□

Corollary 6.5. $\left|\zeta(\sigma)^3\zeta(\sigma+it)^4\zeta(\sigma+2it)\right| \geq 1$

Proof. Let $f(s) = \log\zeta(s)$. □

Now, assume $\zeta(1+it) = 0$ for some t.

$$\lim_{\sigma\to 1^+} \frac{\zeta(\sigma+it) - \zeta(1+it)}{\sigma - 1} \to \zeta'(\sigma+it).$$

We have

$$\left|\left[(\sigma-1)^3\zeta(\sigma)^3\right]\left[\frac{\zeta(\sigma+it)}{\sigma-1}\right]^4\zeta(\sigma+2it)(\sigma-1)\right| \geq 1.$$

However, the first $[\cdot] \to 1$, the second $\zeta'(1+it)$, the next term $\zeta(1+2it)$ and the last 0. This is a contradiction.

Theorem 6.6. *$\zeta(s) \neq 0$ when $Re(s) = 1$.*

Lemma 6.7. *(Quantative version)*

$$|\zeta(\sigma + 2it)| \leq M_1(t) \text{ and } |\zeta'(\sigma + it)| \leq M_2(t).$$

These imply that $|\zeta(\sigma+it)| \geq \frac{1}{32M_1(t)M_2(t)^3}$ for $\sigma \geq 1$, $t \geq t_0$.

Corollary 6.8. $|\zeta(\sigma + it)| \geq c\,|\log t + c|^2$.

Suppose

$$\begin{aligned} |\zeta(\sigma + 2it)| &\leq M_1(t) \\ |\zeta'(\sigma + it)| &\leq M_2(t) \end{aligned}$$

for all $t \geq t_0 \geq 1$ and $\sigma \geq 1$ $(M_1(t), M_2(t) \geq 1)$.. Then for any such σ and t,

$$|\zeta(\sigma + it)| \geq \frac{1}{32M_1(t)M_2(t)^3}.$$

Proof. By continuity, can assume that $\sigma > 1$. Then $\frac{1}{|\zeta(s)|} \leq |\zeta(\sigma)| \leq \frac{\sigma}{\sigma-1}$.

Case 1. $\sigma > \frac{5}{4}$, then $\frac{\sigma}{\sigma-1}$ is already a better bound than the lemma.

Case 2. $\sigma < \sqrt[3]{2}$, then $|\zeta(\sigma)| \leq \frac{2^{1/3}}{\sigma-1}$.

Then for a fixed t,

$$\frac{2}{|\sigma-1|^3}\,|\zeta(\sigma+it)|^4\,M_1(t) \geq 1$$

from last time. Therefore,

$$|\zeta(\sigma+it)| \geq \frac{(\sigma-1)^{3/4}}{2^{1/4}M_1(t)^{1/4}} = f(\sigma) \qquad (\star).$$

Choose q such that $f(q) = 2M_2(t)(q-1)$. Then

$$q - 1 = \frac{1}{2^5 M_1(t) M_2(t)^{1/4}} \leq \frac{1}{4}$$

not only $\sigma < \frac{5}{4}$ but $q < \frac{5}{4}$. Hence, $(\star)$ holds for q.

If $\sigma < q$, then

$$|\zeta(\sigma+it) - \zeta(q+it)| \leq \int_\sigma^q \zeta'(\alpha+it)d\alpha \leq M_2(t)(q-\sigma) \leq M_2(t)(q-1).$$

Then

$$|\zeta(\sigma+it)| \geq M_2(t)(q-1) = \frac{1}{2^5M_1(t)M_2(t)^3}.$$

If $\sigma > q$ then

$$|\zeta(\sigma+it)| \geq f(\sigma) \geq f(q) = \frac{1}{2^5M_1(t)M_2(t)^3}.$$

□

Notation. $E(x) = \begin{cases} 1 & \text{if } x \geq 1 \\ 0 & \text{else} \end{cases}$, and $\int_{L_c}$ means integrating along the vertical line $Re(s) = c$.

Lemma 6.9. *We have*

$$\frac{1}{2\pi i}\int_{L_c} \frac{x^s}{s^2} = E(x)\log x$$

for any $x > 0$ and $c > 0$.

Proof. We have
$$\frac{x^s}{s^2} = \frac{e^{s\log x}}{s^2} = \frac{1}{s^2}\left(1 + s\log x + s^2\log^2 x + ...\right).$$
Call the circle γ. Then
$$\frac{1}{2\pi i}\int_\gamma = \begin{cases} \log x & \text{simple curve } \gamma \text{ going around } 0 \\ 0 & \text{else} \end{cases}$$
(just look at the residue). Call C_1 the circle cut by the the left side of γ and C_2 the right (middle line is L_c). Then
$$\left|\frac{1}{2\pi i}\int_{C_1}\frac{x^s}{s^2}\right| \le \frac{1}{2\pi}\frac{x^2}{R^2}\cdot 2\pi R \to 0$$
as $R\to\infty$, for $|x^s| \le x^c$ if $x \ge 1$ and $c > Re(s)$.
$$\left|\frac{1}{2\pi i}\int_{C_2}\frac{x^s}{s^2}\right| \le \frac{1}{2\pi}\frac{x^c}{R^2}2\pi R \to 0$$
as $R\to\infty$, if $x < 1$ and $c < Re(s)$. ∎

Lemma 6.10. *We have*
$$\frac{1}{2\pi i}\int_{L_c}\frac{x^s}{s(s-1)} = E(x)(x-1)$$
for $x > 0$ and $c > 1$.

Proof. Assume as because,
$$\frac{1}{2\pi i}\int_{L_c}\frac{x^s}{s-1} - \frac{1}{2\pi i}\int_{L_c}\frac{x^s}{s} = \begin{cases} x-1 \\ 0 \end{cases}.$$
The estimate we use, is
$$\left|\frac{x^s}{s(s-1)}\right| \le \frac{x^c}{R(R-1)}$$
and so both $C_1, C_2 \to 0$. ∎

If $f(s) = \sum_{n=1}^\infty \frac{a(n)}{n^s}$ for $Re(s) > 1$, integration term-wise can be justified. Then
$$\begin{aligned}\frac{1}{2\pi i}\int_{L_c}\frac{x^s}{s(s-1)}f(s) &= \sum_{n\ge 1}\frac{a(n)}{2\pi i}\int_{L_c}\frac{1}{s(s-1)}\left(\frac{x}{n}\right)^s = \sum_{n\ge1} a(n)E(\frac{x}{n})\left(\frac{x}{n}-1\right) \\ &= \sum_{n\le x} a(n)\left(\frac{x}{n}-1\right).\end{aligned}$$
Also,
$$\frac{1}{2\pi i}\int_{L_c}\frac{x^{s-1}}{s(s-1)}f(s) = \sum_{n\le x} a(n)\left(\frac{1}{n}-\frac{1}{x}\right) \underset{\text{Abel}}{=} \int_1^x \frac{A(\xi)}{\xi^2}d\xi.$$
Notice that the last expression does not depend on c. This is like Cauchy's theorem, that it does not matter which circle you choose.

Theorem 6.11. *Suppose $f(s) = \sum_{n\ge1}\frac{a(n)}{n^s}$ is absolutely convergent on $Re(s) > 1$. Let $A(x) = \sum_{n\le x} a(n)$. Then*
$$\frac{1}{2\pi i}\int_{L_c}\frac{x^{s-1}}{s(s-1)}f(s)ds = \sum_{n\le x} a(n)\left(\frac{1}{n}-\frac{1}{x}\right) = \int_1^x \frac{A(\xi)}{\xi^2}d\xi$$
for $x > 1$ and $c > 1$.

Proof. We have
$$x^s f(s) = G(s) + H(s)$$
where
$$G(s) = \sum_{n\le x} a(n)\left|\frac{x}{n}\right|^s$$

finite sum.

$$H(s) = \sum_{n>x} a(n) \left|\frac{x}{n}\right|^s$$

is an absolutely convergent sum and so is bounded, by say M.

$$\left|\frac{1}{2\pi i}\int_{C_2} \frac{H(s)}{s(s-1)}\right| \leq \frac{1}{2\pi} \cdot \frac{M}{R(R-1)} 2\pi R \to 0$$

as $R \to \infty$. □

Theorem 6.12. *(Main Theorem).*

(1) If $f(s) = \sum_{n\geq 1} \frac{a(n)}{n^s}$ absolutely converges on $Re(s) > 1$, holomorphic in a neighbourhood of $Re(s) \geq 1$ except possibly a simple pole at $s = 1$

(2) $f(s) = \frac{\alpha}{s-1} + \alpha_0 + (s-1)h(s)$, $h(s)$ is holomorphic in a neighbourhood of $Re(s) \geq 1$

(3) There is $P(t)$ such that $|f(\sigma \pm it)| \leq P(t)$ for $\sigma \geq 1$, $t \geq t_0 \geq 1$ and $\int \frac{P(t)}{t^2} < \infty$.

Then,

$$\int_1^\infty \frac{A(x) - \alpha x}{x^2} dx = \alpha_0 - \alpha.$$

Remark 6.13. This applies to $f(s) = \zeta(s)$, and PNT follows.

Lemma 6.14. *If $\varphi \in L^1$, then $\int_{-\infty}^{\infty} e^{i\lambda t}\varphi(t)dt \to 0$ as $\lambda \to \pm\infty$.*

Theorem 6.15. *We know that*

$$f(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$$

converges absolutely on $Re(s) > 1$, $= \frac{\alpha}{s-1} + \alpha_0 + (s-1)h(s)$ where $h(s)$ is holomorphic in a neighbourhood of $Re(s) \geq 1$. Additionally, $|f(\sigma \pm it)| \leq P(t)$ for all $\sigma \geq 1$, $t \geq t_0 \geq 1$ with $\int \frac{P(t)}{t^2} < \infty$, THEN

$$\int_1^\infty \frac{A(x) - \alpha x}{x^2} dx = \alpha_0 - \alpha.$$

We know that $f(s) = \zeta(s)$ satisfies this. $\int_1^\infty \frac{\zeta(x) - x}{x^2} dx = \gamma - 1$.

Recall, *PNT* iff $\psi(x) \sim x$. Let $f(x) = \sum_{n=1}^\infty \frac{\Lambda(s)}{n^s} = -\frac{\zeta'(s)}{\zeta(s)}$. $P(t) \leq c\,(\log t + c)^9$, satisfies the theorem. Conclusion:

$$\int_1^\infty \frac{\psi(x) - x}{x^2}$$

converges. This implies that $|\psi(x) - x| < \epsilon x$ for every ϵ most of the time.

Proof. We see that

$$\varphi(s) = \frac{h(s)}{s} = \frac{f(s)}{s(s-1)} - \frac{\alpha}{(s-1)^2} - \frac{\alpha_0 - \alpha}{s(s-1)}.$$

So we have, for $c > 1$,

$$\begin{aligned} \frac{1}{2\pi i}\int_{L_c} x^{s-1}\varphi(s)ds &= \underbrace{\frac{1}{2\pi i}\int_{L_c} \frac{x^{s-1}}{s(s-1)} f(s)ds}_{\int_1^x \frac{A(y)}{y^2}dy} - \underbrace{\frac{\alpha}{2\pi i}\int_{L_c} \frac{x^{s-1}}{(s-1)^2}ds}_{\alpha \log x} - \underbrace{\frac{\alpha_0 - \alpha}{2\pi i}\int_{L_c} \frac{x^{s-1}}{s(s-1)}ds}_{(\alpha_0 - \alpha)(1 - \frac{1}{x})} \\ &= \int_1^x \frac{A(y) - \alpha y}{y^2} - (\alpha_0 - \alpha)\left(1 - \frac{1}{x}\right). \end{aligned}$$

Goal is to show that LHS goes to 0 as $x \to \infty$.

(1) The same is true for $c = 1$

Goal is to show that for c sufficiently close, integral of L_c and L_1 are close. First, for $c < 2$,

$$\left|x^{s-1}\varphi(s)\right| \leq |x\varphi(s)| \leq x\frac{P(t)}{t^2}.$$

This shows that the difference of $\int_{|t|>N} x^{s-1}\varphi(s)$ between $Re(s) = 1$ or c differs by $< \epsilon$ for N large enough.

(2) When $c = 1$,

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} x^{i\lambda}\varphi(1+i\lambda)d\lambda = \frac{1}{2\pi i}\int_{-\infty}^{\infty} x^{i\lambda \log x}\varphi(1+i\lambda)d\lambda \to 0$$

(because this is the Fourier transform of φ), as long as we show $\varphi \in L^1$.

(3) $\varphi \in L^1$, because

$$t^2\varphi(s) \le |s(s-1)\varphi(s)| \le P(t) + const$$

for $s = \sigma + it$. So $\varphi(s) \le \frac{P(t)}{t^2} \in L^1$.

□

Finally, we will check that if

$$\int_1^\infty \frac{A(x) - \alpha x}{x^2}$$

converges, with $A \ge 0$ and increasing, then $\frac{A(x)}{x} \to \alpha$ as $x \to \infty$.

Proof. Do this in case.

Case 1. $\alpha = 0$.

For all ϵ there exists N where the following holds. Since $\int_1^\infty \frac{A(x)}{x^2}$ converges, $A(x) \le \epsilon x$ for $x > N$. Suppose $A(x_0) > \epsilon x_0$,

$$\int_{x_0}^\infty \frac{A(x)}{x^2} \ge \epsilon x_0 \int_{x_0}^\infty \frac{1}{x^2} = \epsilon.$$

Case 2. $\alpha = 1$.

If $A(x_0) > (1+\epsilon)x_0$,

$$\int_{x_0}^{x_1} \frac{A(x) - x}{x^2} \ge (1+\epsilon)x_0 \int_{x_0}^{x_1} \frac{1}{x^2} - \int_{x_0}^{x_1} \frac{1}{x}.$$

We can choose $x_1 = (1+\epsilon)\, x_0$,

$$x_1\left(\frac{1}{x_0} - \frac{1}{x_1}\right) - \log\frac{x_1}{x_0} = \epsilon - \log(1+\epsilon) \sim \frac{\epsilon^2}{\epsilon} > 0.$$

If $A(x_0) < (1-\epsilon)x_0$,

$$\int_{x_2}^{x_0} \frac{A(x) - x}{x^2} \le (1-\epsilon)x_0 \int_{x_2}^{x_0} \frac{1}{x_2} - \int_{x_2}^{x_0} \frac{1}{x}.$$

If we let $x_2 = (1-\epsilon)x_0$,

$$x_2\left(\frac{1}{x_2} - \frac{1}{x_0}\right) - \log\frac{x_0}{x_2} = \epsilon + \log(1-\epsilon) \le -\frac{\epsilon^2}{2} < 0.$$

□

Recall,

$$\sum_{n=1}^\infty \frac{\mu(n)}{n} = 0.$$

That's the next goal.

$f(s) = \sum \frac{a(n)}{n^s}$ absolutely converges in $Re(s) > 1$, $= \frac{\alpha}{s-1} + \alpha_0 + (s-1)h(s)$.

$$|f(\sigma + it)| \le ...$$

Prime number theorem then tells us that

(1) Integral version $\int \frac{A(x) - \alpha x}{x^2}dx = \alpha_0 - \alpha$
(2) Limit version $\frac{A(x)}{x} \to \alpha$ (provided $A \ge 0$ increasing. This, we can also do $A = B - C$ where B, C are monotone and satisfy all the other conditions).
(3) Series version

$\sum_{n \le x} \frac{a(n)}{n} - \alpha \log x \to \alpha_0$.

Proof. This is
$$= \frac{A(x)}{x} + \int_1^x \frac{A(y)}{y^2} dy - \alpha \int_1^x \frac{1}{y} dy \to \alpha + (\alpha_0 - \alpha) = \alpha_0.$$
□

- Proof of $\sum_{n\geq 1} \frac{\mu(n)}{n} = 0$, use $\frac{1}{\zeta(s)} = \sum_{n\geq 1} \frac{\mu(n)}{n^s}$. $\alpha = \alpha_0 = 0$.
  Integral version, $M(x) = \sum_{n\leq x} \mu(n)$, $\int \frac{M(x)}{x^2} = 0$.
  Series version: $\sum_{n\geq 1} \frac{\mu(n)}{n} = 0$.
  Note, for us, $\mu = 1 - (1-\mu)$ (which are the B and C). $1 \geq 0$ and gives $\zeta(s)$, the other is also ≥ 0 and gives $\zeta(s) - \frac{1}{\zeta(s)}$.
- Proof of $|\psi(x) - x| = O(x^\alpha)$ then ζ has no roots $\alpha < Re(s)$.
  $\frac{1}{\zeta(s)} = s\int_1^\infty \frac{M(x)}{x^{s+1}} dx$, $Re(s) > 1$, $Re(s) > \alpha$. This implies that
  $$\zeta(s) \neq 0$$
  if $Re(s) > \alpha$.
  $$s \int \frac{x - \psi(x)}{x^{s+1}} = \frac{s}{s-1} + \frac{\zeta'(s)}{\zeta(s)}$$
  implies that RHS is holomorphic when $Re(s) > \alpha$.
  (GET PICTURE FROM ADAN, or PALLAV). From this,
  $$|\psi(x) - x| \leq ce^{-c\sqrt{\log x}}.$$
  Which is equivalent to $|\pi(x) - \ell_i(x)|$ having some bound.
  $$\frac{1}{x^\epsilon} \ll e^{-\sqrt{\log x}} \ll \frac{1}{(\log x)^2}$$
  and $I_n(x) = \int \frac{1}{(\log t)^n}$.