Lecture III. Topological K-theory and Algebraic Geometry

In this lecture, we will develop some of the machinery which makes topological K-theory both useful and computable. Not only is this of considerable interest in its own right, but also much of current research by algebraic K-theorists centers around developing similar results for algebraic K-theory.

The topological group $O(n) \subset M_n(\mathbf{R}) \simeq \mathbf{R}^{n^2}$ consists of those real matrices A with $A \cdot A^t = 1$; the topological group $U(n) \subset M_n(\mathbf{C}) \simeq \mathbf{C}^{n^2}$ consists of those complex matrices A with $A \cdot \overline{A}^t = 1$. Stabilizing with respect to n in the usual way, we obtain

$$O = \bigcup_{n \geq 1} O(n), \quad U = \bigcup_{n \geq 1} U(n).$$

Our analysis of classifying spaces for G-torsors with $G = O(n)$ or $U(n)$ is the major ingredient in the proof of the following theorem.

Theorem. *Let X be a finite dimensional C.W. complex. Then*

$$KO^0_{top}(X) = [X, BO \times \mathbf{Z}], \quad K^0_{top}(X) = [X, BU \times \mathbf{Z}].$$

Of fundamental importance in the study of topological K-theory is the following theorem of Raoul Bott. Recall that if (X, x) is pointed space, then the **loop space** ΩX is the function complex (with the compact-open topology) of continuous maps from (S^1, ∞) to (X, x).

Theorem: Bott Periodicity.

(a.) There is a homotopy equivalence

$$BO \times \mathbf{Z} \simeq \Omega^8(BO \times \mathbf{Z})$$

from $BO \times \mathbf{Z}$ to its 8-fold loop space. Moreover, the homotopy groups $\pi_i(BO \times \mathbf{Z})$ are given by $\mathbf{Z}, \mathbf{Z}/2, \mathbf{Z}/2, 0, \mathbf{Z}, 0, 0, 0$ depending upon whether i is congruent to $0, 1, 2, 3, 4, 5, 6, 7$ modulo 8.

(b.) There is a homotopy equivalence

$$BU \times \mathbf{Z} \simeq \Omega^2(BU \times \mathbf{Z})$$

from $BU \times \mathbf{Z}$ to its 2-fold loop space. Moreover, $\pi_i(BU \times \mathbf{Z})$ is $\mathbf{Z}$ if i is even and equals 0 if i is odd.

Thus, both $BO \times \mathbf{Z}$ and $BU \times \mathbf{Z}$ are Ω-spectra as defined as follows.

Definition III.1. *An Ω-spectrum $\underline{E}$ is a sequence of pointed spaces $\{E^0, E^1, \dots\}$, each of which has the homotopy type of a pointed C.W. complex, together with homotopy equivalences relating each E^n with the loop space of E^{n+1}; in other words, a sequence of pointed homotopy equivalences*

$$E^0 \xrightarrow{\simeq} \Omega E^1 \xrightarrow{\simeq} \Omega^2 E^2 \xrightarrow{\simeq} \cdots \xrightarrow{\simeq} \Omega^n E^n \to \cdots$$

Theorem. *(cf. [Spanier]) Let $\underline{E}$ be an Ω-spectrum. Then for any topological space X with closed subspace $A \subset X$, set*

$$h^n_{\underline{E}}(X,A) = [(X,A), E^n], \quad n \geq 0$$

Then $(X,a) \mapsto h^*_{\underline{E}}(X,A)$ is a generalized cohomology theory which satisfies all of the Eilenberg-Steenrod axioms except that its value at a point (i.e., $(*,\emptyset)$) may not be that of ordinary cohomology:

(a.) $h^*_{\underline{E}}(-)$ is a functor from the category of pairs of spaces to graded abelian groups.
(b.) for each $n \geq 0$ and each pair of spaces (X,A), there is a functorial connecting homomorphism $\partial : h^n_{\underline{E}}(A) \to h^{n+1}_{\underline{E}}(X,A)$.
(c.) the connecting homomorphisms of (b.) determine long exact sequences for every pair (X,A).
(d.) $h^*_{\underline{E}}(-)$ satisfies excision: i.e., for every pair (X,A) and every subspace $U \subset A$ whose closure lies in the interior of A, $h^*_{\underline{E}}(X,A) \simeq h^*_{\underline{E}}(X-U, A-U)$.

Observe that in the above definition we use the notation $h^*_{\underline{E}}(X)$ for $h^*_{\underline{E}}(X,\emptyset) = h^*_{\underline{E}}(X_+,*)$, where X_+ is the disjoint union of X and a point $*$.

Definition III.2. *The (periodic) topological K-theories $KO^*_{top}(-), K^*_{top}(-)$ are the generalized cohomology theories associated to the Ω-spectra given by $BO \times \mathbf{Z}$ and $BU \times \mathbf{Z}$ with their deloopings given by Bott periodicity. In particular,*

$$K^{2j}_{top}(X) = [X, BU \times \mathbf{Z}], \quad K^{2j-1}_{top}(X) = [X, U],$$

so that we recover our definition of $K^0_{top}(X)$ (and similarly $KO^0_{top}(X)$) whenever X is a finite dimensional C.W. complex.

Tensor product of vector bundles induces a multiplication

$$K^0_{top}(X) \otimes K^0_{top}(X) \to K^0_{top}(X)$$

for any finite dimensional C.W. complex X. This can be generalized by observing that tensor product induces group homomorphisms $U(m) \times U(n) \to U(n+m)$ and thereby maps of classifying spaces

$$BU(m) \times BU(n) \to BU(n+m).$$

With a little effort, one can show that these multiplication maps are compatible up to homotopy with the standard embeddings $U(m) \subset U(m+1), U(n) \subset U(n+1)$ and thereby give us a pairing

$$(BU \times \mathbf{Z}) \times (BU \times \mathbf{Z}) \to BU \times \mathbf{Z}$$

(factoring through the smash product). In this way, $BU \times \mathbf{Z}$ has the structure of an H-space which induces a pairing of spectra and thus a multiplication for the generalized cohomology theory $K^*(-)$. (A completely similar argument applies to $KO^*(-)$).

As an example of how topological K-theory inspired even the early effort in algebraic K-theory we mention the following classical theorem of Hyman Bass. The analogous result in topological K-theory for rank e vector bundles over a finite dimension C.W. complex of dimension $d < e$ can be readily proved using the standard method of "obstruction theory".

Theorem: Bass stability theorem. *Let A be a commutative, noetherian ring of Krull dimension d. Then for any two projective A-modules P, P' of rank $e > d$, if $[P] = [P'] \in K_0(A)$ then P must be isomorphic to P'.*

We next describe the close relationship between $K^*_{top}(X)$ and the integral cohomology $H^*(X) = H^*(X, \mathbf{Z})$ of X. Indeed, we give below the general form of the spectral sequence for any generalized cohomology theory, and observe that this is particularly nice for $K^*_{top}(-)$.

Theorem: Atiyah-Hirzebruch spectral sequence. *For any generalized cohomology theory $h^*_{\underline{E}}(-)$ and any topological space X, there exists a right half-plane spectral sequence of cohomological type*

$$E_2^{p,q} = H^p(X, h^q(*)) \Rightarrow h^{p+q}_{\underline{E}}(X).$$

In the special case of $K^*_{top}(-)$, this takes the following form

$$E_2^{p,q} = H^p(X, \mathbf{Z}(q/2)) \Rightarrow K^{p+q}_{top}(X)$$

where $\mathbf{Z}(q/2) = \mathbf{Z}$ if q is an even non-negative integer and 0 otherwise.

What is a spectral sequence of cohomological type? This is the data of a 2-dimensional array $E^{p,q}_r$ of abelian groups for each $r \geq r_0$ (typically, r_0 equals 0, or 1 or 2; in our case $r_0 = 2$) and homomorphisms

$$d^{p,q}_r : E^{p,q}_r \to E^{p+r,q-r+1}_r$$

such that the next array $E^{p,q}_{r+1}$ is given by the cohomology of these homomorphisms:

$$E^{p,q}_{r+1} = ker\{d^{p,q}_r\}/im\{d^{p-r,q+r-1}_r\}.$$

To say that the spectral sequence is "right half plane" is to say $E^{p,q}_r = 0$ whenever $p < 0$. We say that the spectral sequence **converges to the abutment** E^*_∞ (in our case $h^*_{\underline{E}}(X)$) if at each spot (p,q) there are only finitely many non-zero homomorphisms going in and going out and if there exists a decreasing filtration $\{F^p E^n_\infty\}$ on each E^n_∞ so that

$$E^n_\infty = \bigcup_p F^p E^n_\infty, \ 0 = \bigcap_p F^p E^n_\infty,$$

$$F^p E^n_\infty / F^{p+1} E^n_\infty = E^{p,n-p}_R \quad R >> 0.$$

If X is a C.W. complex then we can define its **p-skeleton** $sk_p(X)$ for each $p \geq 0$ as the subspace of X consisting of the union of those cells of dimension $\leq p$. Then the filtration on $h^*_{\underline{E}}(X)$ for the Atiyah-Hirzebruch spectral sequence is given by

$$F^p E^*_\infty = ker\{h^*_{\underline{E}}(X) \to h^*_{\underline{E}}(sk_p(X)\}.$$

While we are discussing spectral sequences, we should mention the following:

Theorem: Serre spectral sequence. *Let (B, b) be a connected, pointed C.W. complex. For any fibration $p : E \to B$ of topological spaces with fibre $F = p^{-1}(b)$ and for any abelian group A, there exists a convergent first quadrant spectral sequence of cohomological type*

$$E_2^{p,q} = H^p(B, H^q(F, A)) \Rightarrow H^{p+q}(E, A)$$

provided that $\pi_1(B, b)$ acts trivially on $H^*(F, A)$.

The following theorem tells us that topological K-theory tensor the rational numbers is essentially rational cohomology $H^*(-, \mathbf{Q})$.

Theorem (Atiyah-Hirzebruch). *Let X be a C.W. complex. Then there exists a homomorphism of graded rings*

$$ch : K_{top}^*(X) \otimes \mathbf{Q} \to H^*(X, \mathbf{Q})$$

which restricts to isomorphisms

$$K_{top}^0(X) \otimes \mathbf{Q} \simeq H^{ev}(X, \mathbf{Q}), \quad K_{top}^1(X) \otimes Q \simeq H^{odd}(X, \mathbf{Q}).$$

In Lecture V, we shall discuss operations on both topological and algebraic K-theory. These operations originate from the observation the the exterior products $\Lambda^i(P)$ of a projective module P are likewise projective modules and the exterior products $\Lambda^i(E)$ of a vector bundle E are likewise vector bundles. The other simple fact that we use is the following relationship

$$\Lambda^n(V \oplus W) = \oplus_{i+j=n} \Lambda^i(V) \otimes \Lambda^j(W)$$

which one can first check for vector spaces, then extend to either projective modules or vector bundles.

In particular, J. Frank Adams introduced operations

$$\psi^k(-) : K_{top}^0(-) \to K_{top}^0(-), \quad k > 0$$

(called **Adams operations**) which have many applications. We shall use these Adams operations in algebraic K-theory, but here is a short list of famous theorems of Adams using topological K-theory and Adams operations:

Applications (Adams)

(1.) Determination of the number of linearly independent vector fields on the n-sphere S^n for all $n > 1$.

(b.) Determination of the only dimensions (namely, $n = 1, 2, 4, 8$) for which $\mathbf{R}^n$ admits the structure of a division algebra. (The examples of the real numbers $\mathbf{R}$, the complex numbers $\mathbf{C}$, the quaternions, and the Cayley numbers gives us structures in these dimensions.)

(c.) Determination of those (now well understood) elements of the homotopy groups of spheres associated with $KO_{top}^0(S^n)$.

In the remainder of this lecture, I shall review some of the basic concepts of algebraic geometry.

Definition III.3. *A* **sheaf** *of sets F on a topological space X is a contravariant functor*

$$F : (\text{open sets of } X, \text{ inclusions}) \to (\text{sets})$$

satisfying the sheaf axiom: for any open set U, any open covering $\{U_i \to U\}_{i\in I}$,

$$F(U) = \text{equalizer } \{\prod_{i\in I} F(U_i) \rightrightarrows \prod_{i,j\in I} F(U_i \cap U_j)\}.$$

In other words, an element $s \in F(U)$ (called a "section of F on U") is uniquely determined by elements $s_i \in F(U_i)$ which agree on intersections $s_{i|U_i\cap U_j} = s_{j|U_i\cap U_j}$ for all $i, j \in I$, where $s_{j|U_i\cap U_j}$ denotes the image of $s_i \in F(U_i)$ under the functorially given map $F(U_i) \to F(U_i \cap U_j)$.

For any $x \in X$, the **stalk** *of F_x at x is the following set: $\text{colimit}_{x\in U} F(U)$.*

If the functor F is actually a functor to (groups)(respectively, (abelian groups); resp., (rings);etc.), then we say that F is a sheaf of groups (resp., sheaf of abelian groups; resp., sheaf of rings;etc).

Basic Example Let $p : E \to X$ be a continuous map. Define an associated sheaf F by sending an open subset $U \subset X$ to the set of continuous functions $s : U \to E$ with the property that $p \circ s : U \to X$ is the inclusion $U \subset X$.

Recall that if A is a commutative ring we denote by $SpecA$ the set of prime ideals of A. The set $X = SpecA$ is provided with a topology, the **Zariski topology** defined as follows: a subset $Y \subset X$ is closed if and only if there exists some ideal $I \subset A$ such that $Y = \{p \in X; I \subset p\}$. We define the **structure sheaf** $\mathcal{O}_X$ of commutative rings on $X = SpecA$ by specifying its value on the basic open set $X_f = \{p \in SpecA, f \notin p\}$ for some $f \in A$ to be the ring A_f obtained from A by adjoining the inverse to f. (Recall that $A \to A_f$ sends to 0 any element $a \in A$ such that $f^n \cdot a = 0$ for some n). We now use the sheaf axiom to determine the value of $\mathcal{O}_X$ on any arbitrary open set $U \subset X$, for any such U is a finite union of basic open subsets. The stalk $\mathcal{O}_{X,p}$ of the structure sheaf at a prime ideal $p \subset A$ is easily computed to be the local ring $A_p = \{f \notin p\}^{-1}A$.

Thus, $(X = SpecA, \mathcal{O}_X)$ has the structure of a **local ringed space**: a topological space with a sheaf of commutative rings each of whose stalks is a local ring. A map of local ringed spaces $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is the data of a continuous map $f : X \to Y$ of topological spaces and a map of sheaves $O_Y \to f_*O_X$ on Y, where $f_*O_X(V) = O_X(f^{-1}(V))$ for any open $V \subset Y$.

If M is an A-module for a commutative ring A, then M defines a sheaf $\tilde{M}$ of $\mathcal{O}_X$-modules on $X = SpecA$. Namely, for each basic open subset $X_f \subset X$, we define $\tilde{M}(X_f) \equiv A_f \otimes_A M$. This is easily seen to determine a sheaf of abelian groups on X with the additional property that for every open $U \subset X$, $\tilde{M}(U)$ is a sheaf of $\mathcal{O}_X(U)$-modules with structure compatible with restriction to smaller open subsets $U' \subset U$.

Definition III.4. *A local ringed space $(X, \mathcal{O}_X)$ is said to be an* **affine scheme** *if it is isomorphic (as local ringed spaces) to $(X = SpecA, \mathcal{O}_X)$ as defined above. A* **scheme** *$(X, \mathcal{O}_X)$ is a local ringed space for which there exists a finite open covering $\{U_i\}_{i\in I}$ of X such that each $(U_i, \mathcal{O}_{X|U_i})$ is an affine scheme.*

*If k is a field, a k-***variety** *is a scheme $(X, \mathcal{O}_X)$ with the property there is a finite open covering $\{U_i\}_{i\in I}$ by affine schemes with the property that each $(U_i, \mathcal{O}_{X|U_i}) \simeq (SpecA_i, \mathcal{O}_{SpecA_i})$ with A_i a finitely generated k-algebra without nilpotents.*

Example The scheme $\mathbf{P}^1_{\mathbf{Z}}$ is a non-affine scheme defined by patching together two copies of the affine scheme $Spec\mathbf{Z}[t]$. So $\mathbf{P}^1_{\mathbf{Z}}$ has a covering $\{U_1, U_2\}$ corresponding to rings $A_1 = \mathbf{Z}[u], A_2 = \mathbf{Z}[v]$. These are "patched together" by identifying the open subschemes $Spec(A_1)_u \subset SpecA_1,\ Spec(A_2)_v \subset SpecA_2$ via the isomorphism of rings $(A_1)_u \simeq (A_2)_v$ which sends u to v^{-1}.

Note that we have used $SpecR$ to denote the local ringed space $(SpecR, \mathcal{O}_{SpecR})$; we will continue to use this abbreviated notation.

Definition. *Let $(X, \mathcal{O}_X)$ be a scheme. We denote by $\mathcal{P}_X$ the exact category of sheaves F of $\mathcal{O}_X$-modules with the property that there exists a finite open covering $\{U_i\}$ of X by affine schemes $U_i = SpecA_i$ and free, finitely generated A_i-modules M_i such that the restriction $F_{|U_i}$ of F to U_i is isomorphic to the sheaf $\tilde{M}_i$ on $SpecA_i$.*

We define the algebraic K-theory of the scheme X by setting

$$K_*(X) = K_*(\mathcal{P}_X).$$

In our last two lectures, we will need a generalization of the concept of a topology on a set, a generalization introduced by Grothendieck for which sheaf theory is still feasible.

Definition III.5. *A* **Grothendieck site** *on a scheme $(X, \mathcal{O}_X)$ consists of a class $\mathcal{C}/X$ of schemes over X (i.e., each provided with a specified morphism to X) closed under fibre products (i.e., if $Y_1 \to X, Y_2 \to X \in \mathcal{C}/X$, then $Y_1 \times_X Y_2 \to X \in \mathcal{C}/X$) together with a class of morphisms $\mathcal{E}$ such that*

(a.) all isomorphisms of $\mathcal{C}/X$ are in $\mathcal{E}$
(b.) $\mathcal{E}$ is closed under composition
(c.) if $V \to U, V' \to U \in \mathcal{E}$, then $V \times_U V' \to V' \in \mathcal{E}$.
(d.) if $Y \to X \in \mathcal{C}/X, U \to Y \in \mathcal{E}$, then the composition $U \to X \in \mathcal{C}/X$.

An $\mathcal{E}$-covering $\{U_i \to U\}_{i \in I}$ consists of morphisms in $\mathcal{E}$ with the property that the union of their images equals U.

A sheaf on such a site is a contravariant functor $F : \mathcal{C}/X \to (sets)$ satisfying the sheaf axiom: for all $\mathcal{E}$-coverings $\{U_i \to U\}_{i \in I}$,

$$F(U) = equalizer\ \{\prod_{i \in I} F(U_i) \rightrightarrows \prod_{i,j \in I} F(U_i \times_U U_j)\}.$$

The **(Grothendieck) $\mathcal{E}$-topology** on the site $(\mathcal{C}/X, \mathcal{E})$ is the collection of all $\mathcal{E}$-coverings for this site. In our next lecture, we will be particularly interested in the **etale topology** in which $\mathcal{E}$ consists of all etale maps between schemes.